Zero temperature convergence of Gibbs measures for a locally finite potential in a 2-dimensional lattice

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Summary

I.1. Zero-temperature chaotic convergence

We want to understand whether some spin systems exhibit a phenomenon called *zero-temperature chaotic convergence* introduced by van Enter and Ruszel (2007).

Definition

Let $\Sigma^d(\mathscr{A}) = \mathscr{A}^{\mathbb{Z}^d}$ be a spin system over a finite alphabet $\mathscr{A}$ and $\varphi : \Sigma^d(\mathscr{A}) \to \mathbb{R}$ be a continuous function (potential). Let $\mu_{\beta\varphi}$ be any invariant Gibbs measure for the potential $\beta\varphi$.

The zero-temperature chaotic convergence is a phenomenon where there exists a sequence $(\beta_k)_{k\geqslant 0}$, $\beta_k \to +\infty$, and two disjoint invariant compact sets $K_0, K_1 \subseteq \Sigma^d(\mathscr{A})$ such that if $\mu_{\beta_k\varphi}$ is any invariant Gibbs measure,

- any weak$*$ limit of $(\mu_{\beta_{2k}\varphi})_{k\geqslant 0}$ is supported in K_0
- any weak$*$ limit of $(\mu_{\beta_{2k+1}\varphi})_{k\geqslant 0}$ is supported in K_1

I.2. Zero-temperature chaotic convergence

Remark

- By compactness argument, some subsequences $(\mu_{\beta_k})_{k\geqslant 0}$ are converging and are not chaotic. So the chaotic convergence cannot be expected for all subsequences.
- Coronel, Rivera-Letelier (2015) introduced a stronger notion of *sensitive dependence of the chaotic convergence*: for every sequence $\beta_k \to +\infty$, for every $\epsilon > 0$, there exists $\|\psi - \phi\|_\infty < \epsilon$ and a subsequence $(\beta_{\sigma(k)})_{k\geqslant 0}$ such that $(\mu_{\beta_{\sigma(k)}\psi})_{k\geqslant 0}$ has a chaotic convergence at zero temperature.
- We will not discuss that notion, but van Enter's method is robust and it is likely that our results are also true in that case.

I.3. General setting

Notation

- *A spin system*: $\mathscr{A}$ a finite alphabet, $\Sigma^d(\mathscr{A}) = \mathscr{A}^{\mathbb{Z}^d}$, the full shift
- The group of *space translations* $\sigma^k : \Sigma^d(\mathscr{A}) \to \Sigma^d(\mathscr{A})$, $k \in \mathbb{Z}^d$
- The *Hamiltonian* is given by a Lipschitz function $\varphi : \Sigma^d(\mathscr{A}) \to \mathbb{R}$

$$H_\Lambda(x) = \sum_{k \in \Lambda} \varphi \circ \sigma^k(x), \quad \Lambda \Subset \mathbb{Z}^d \tag{1}$$

Remark

We will be interested in studying *short range interactions* $\Phi = (\Phi_X)_X$ where $X = k + [\![1, D]\!]^d$ is any square of fixed size D. Our Hamiltonian is equivalent to the one defined by summing over all interactions

$$H_\Lambda^\varnothing(x) = \sum_{X \subseteq \Lambda} \Phi_X(x), \quad \varphi(x) = \frac{1}{D^2} \sum_{0 \in X} \Phi_X(x) \tag{2}$$

I.4. Equilibrium measures/Gibbs measures

Definition

An *equilibrium measure* is a shift invariant probability measure $\mu_{\beta\varphi}$ solution of the variational principle: $\mu_{\beta\varphi}$ minimizes the *free energy*

$$F_\beta(\varphi) := \inf \Big\{ \int \varphi \, d\mu - \beta^{-1}\mathrm{Ent}(\mu) : \mu \text{ shift invariant probability } \Big\}$$

- *shift invariance*: $\sigma_\sharp^k(\mu) = \mu, \quad \forall\, k \in \mathbb{Z}^d$,
- *Kolmogorov-Sinai entropy*: $\mathscr{P}$ the canonical generating partition

$$\mathrm{Ent}(\mu) = \lim_{n\to+\infty} \frac{1}{n^d}\mathrm{Ent}\Big(\mu, \bigvee_{k\in\, [\![1,n]\!]^d} \sigma^{-k}\mathscr{P}\Big)$$

Remark

If $\varphi : \Sigma^d(\mathscr{A}) \to \mathbb{R}$ is Lipschitz, *shift invariant Gibbs measures* defined by the DLR procedure and equilibrium measures are the same notions

I.5. Main question

Let $\mathscr{G}_\beta(\varphi)$ be the set of equilibrium measures or Gibbs measures.

Remark

- Thanks to Dobrushin's argument, $\mathscr{G}_\beta(\varphi)$ is a single element at large temperaure
- For simple systems (at least for short range φ), $\mathscr{G}_\beta(\varphi)$ may have several pure states at small temperature. For the Ising model in $\mathbb{Z}^2$

$$\forall\, \beta < \beta_c,\ \operatorname{card}(\mathscr{G}_\beta(\varphi)) = 1, \quad \forall\, \beta > \beta_c,\ \mathscr{G}_\beta(\varphi) = [\mu_\beta^+, \mu_\beta^-] \tag{1}$$

Question

What are the limits of Gibbs measures as the temperature goes to zero? More precisely what are the limits of the whole set $\mathscr{G}_\beta(\varphi)$ as $\beta \to +\infty$? For the Ising model

$$[\mu_\beta^+, \mu_\beta^-] \to [\mu^+, \mu^-], \quad \text{there is no chaotic convergence} \tag{2}$$

I.6. Minimizing measures

What are the possible weak limits of $(\mu_\beta)_{\beta\to+\infty}$?

$$\int \varphi\, d\mu_\beta - \beta^{-1}\mathrm{Ent}(\mu_\beta) = F_\beta(\varphi) \tag{3}$$

Definition

- *The ground energy level* (by freezing the system $\beta \to +\infty$)

$$F_\infty(\varphi) := \inf\Big\{ \int \varphi\, d\mu : \mu \text{ shift invariant probability } \Big\}$$

- A *minimizing measure* μ_{min} is a shift invariant probability measure satisfying

$$\int \varphi\, d\mu_{min} = F_\infty(\varphi) \tag{4}$$

Theorem (Obvious)

- $F_\beta(\varphi) \to F_\infty(\varphi)$
- *Any accumulation point of* $(\mu_\beta)_{\beta\geqslant 0}$ *is a minimizing measure that maximizes the entropy of all minimizing measures.*

I.7. The set K of ground configurations

Observation

- Assume $X := \{\varphi = \min \varphi\}$ is shift invariant, then
  - ⋆ any weak limit of $\mu_\beta \to \mu_{min}$ satisfies $\mathrm{supp}(\mu_{min}) \subseteq X$
  - ⋆ but is it is *not true in general* that any invariant measure supported on X is a candidate to be a limit of a Gibbs measure.
- If in addition X has a unique invariant measure μ_{min}, then $\mathscr{G}_\beta(\varphi) \to \{\mu_{min}\}$ (no chaotic convergence)
- In general $\varphi^{-1}(\min \varphi)$ is not invariant. The existence or not of a chaotic convergence will depend strongly on the complexity of

$$K = \bigcup \Big\{ \mathrm{supp}(\mu) : \ \mu \text{ is minimizing: } \int \varphi \, d\mu = F_\infty(\varphi) \ \Big\}$$

Hypothesis (minimal requirements)

- $\varphi : \Sigma^d(\mathscr{A}) \to \{0, 1\}$ has finite range and depends on a finite number of coordinates: we say φ *is locally finite*
- K is a *computable subshift* (or effectively closed subshift)

I.8. Subshift of finite type

Definition

- A function $\varphi : \Sigma^d(\mathscr{A}) \to \{0,1\}$ is said to be *locally finite* if there exists $D \geqslant 1$ such that, for every $x, y \in \Sigma^d(\mathscr{A})$

$$x_{[\![1,D]\!]^d} = y_{[\![1,D]\!]^d} \quad \Rightarrow \varphi(x) = \varphi(y).$$

- A *subshift* X is a closed shift invariant subspace of $\Sigma^d(\mathscr{A})$ that is defined by a countable set $\mathscr{F}$ of *forbidden patterns (words)*
  - ⋆ $\mathscr{F} \subseteq \bigsqcup_{n\geqslant 1} \mathscr{A}^{[\![1,n]\!]^d}$
  - ⋆ $X = \Sigma^d(\mathscr{A}, \mathscr{F}) := \{x \in \Sigma^d(\mathscr{A}) : \forall\, w \in \mathscr{F},\ w \not\Subset x\}$
- A subshift X is *computable* if $\mathscr{F}$ is enumerated by a *Turing machine* $\mathbb{M}$ by increasing size
- X is of *finite type* if there exists $D \geqslant 1$ such that $\mathscr{F} \subseteq \mathscr{A}^{[\![1,D]\!]^d}$

I.9. Turing machine

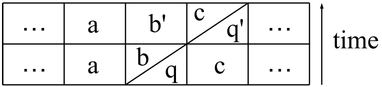
Definition

A *Turing machine* is given by $(\mathscr{A}, \{\sharp\}, \mathscr{Q}, \delta)$ where

- $\mathscr{A}$ is a finite alphabet
- $\{\sharp\}$ is an extra symbol
- $\mathscr{Q} = \{q_1, \ldots, q_n\} \sqcup \{q_{\mathtt{ini}}, q_{\mathtt{fin}}\}$
- $\delta : (\mathscr{A} \sqcup \{\sharp\}) \times \mathscr{Q} \to (\mathscr{A} \sqcup \{\sharp\}) \times \mathscr{Q} \times \{+, -\}$ (a transition function)
- an infinite ribbon where finite words of the form

$$(\cdots, \sharp, \sharp, w_1, \ldots, w_n, \sharp, \sharp, \cdots)$$

In a schematic way:

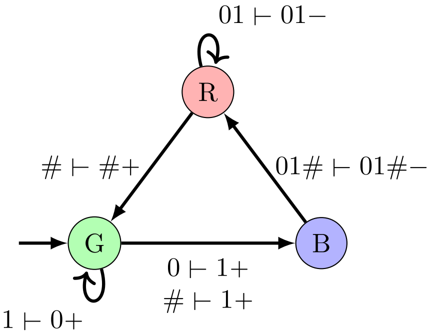


$$\delta(b, q) = (b', q', +)$$

I.10. Turing machine

#	0	0	1	#
#	0	0	#	#
#	0	1	#	#
#	1	1	#	#
#	1	1	#	#
#	1	1	#	#
#	1	1	#	#
#	0	1	#	#
#	0	1	#	#
#	0	1	#	#
#	0	1	#	#
#	0	1	#	#
#	0	#	#	#
#	1	#	#	#
#	1	#	#	#
#	1	#	#	#
#	1	#	#	#
#	0	#	#	#

Example: odometer



The notations $\boxed{a \vdash a'+}$ means the head reads a, then writes a', and moves forward.

$\Longrightarrow$ **A Turing machine is equivalent to a tiling defined by a finite number of local constraints**

I.11. Conclusion

- We want to understand whether a system is chaotic at zero temperature: do there exists a subsequence $\beta_k \to +\infty$ and $K_0, K_1 \subseteq \Sigma^d(\mathscr{A})$, compact and disjoint, such that
  - ⋆ $\mathscr{G}_{\beta_{2k}}(\varphi) \to$ measures supported on K_0
  - ⋆ $\mathscr{G}_{\beta_{2k+1}}(\varphi) \to$ measures supported on K_1
- we want an example of potential φ as simple as possible:

$$\varphi = \mathbb{1}_{[\mathscr{F}]}, \quad \mathscr{F} \subseteq \mathscr{A}^{[\![1,D]\!]^d}, \quad [\mathscr{F}] \text{ denotes a cylinder}$$

⋆ we verify that ground configurations do exist

$$X = \Sigma^d(\mathscr{A}, \mathscr{F}) = \{x \in \Sigma^d(\mathscr{A} : \varphi \circ \sigma^k(x) = 0, \ \forall\, k \in \mathbb{Z}^d\} \neq \varnothing$$

obviously the ground energy and ground measures satisfy

$$F_\infty(\varphi) = 0 \ \text{ and } \ \mathrm{supp}(\mu_{min}) \subseteq X$$

⋆ we construct two disjoint compact invariant sets

$$K_0 \bigsqcup K_1 \subseteq X$$

⋆ we want to work in dimension $d = 2$: our main result is an extension of Chazottes-Hochman (2010) in $d \geqslant 3$

Summary

- I. Position of the problem
- **II. Some old results in dimension 1**
- III. A new result in dimension 2

II.1. Transfer matrix method

Assume $d = 1$, $\mathscr{A} = \{1, 2, \cdots, n\}$, $\varphi : \mathscr{A}^{\mathbb{Z}} \to \mathbb{R}$ is locally finite

$$\varphi(x) = \varphi(i_0, i_1), \quad \forall\, x = (\ldots, i_{-1} \mid i_0, i_1, \ldots) \in \mathscr{A}^{\mathbb{Z}} \tag{1}$$

Lemma (Transfer method)

Gibbs measures are built using the following procedure

- $M_\beta(i,j) = e^{-\beta\varphi(i,j)}$
- $\lambda_\beta =$ *the largest eigenvalue of* $M_\beta = (M_\beta(i,j))_{1 \leqslant i,j \leqslant n}$
- L_β, R_β *are the left and right eigenvectors*
- *normalization:* $\displaystyle\sum_{i=1}^{n} L_\beta(i) R_\beta(i) = 1$
- *the unique Gibbs measure at temperature* β^{-1} *is*

$$\mu_\beta([i_0, \ldots, i_n]) = \frac{L_\beta(i_0) R_\beta(i_n)}{\lambda_\beta^n} \exp\Big(-\beta \sum_{k=1}^{n} \varphi(i_{k-1}, i_k) \Big) \tag{2}$$

II.2. Minimizing cycles

Assume $\varphi : \mathscr{A}^{\mathbb{Z}} \to \mathbb{R}$ has the form: $\varphi(x) = \varphi(i_0, i_1)$. Recall

$$F_\infty(\varphi) = \inf \left\{ \int \varphi \, d\mu : \mu \text{ shift invariant probability } \right\} \tag{1}$$

- $F_\infty(\varphi) = \lim_{n \to +\infty} \inf_{x \in \mathscr{A}^{\mathbb{Z}}} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ \sigma^k(x)$
- A minimizing cycle is a τ-periodic path $(i_0, i_1, \ldots, i_{\tau-1})$ such that

$$\frac{1}{\tau} \sum_{k=0}^{\tau-1} \varphi(i_k, i_{k+1}) = F_\infty(\varphi) \tag{2}$$

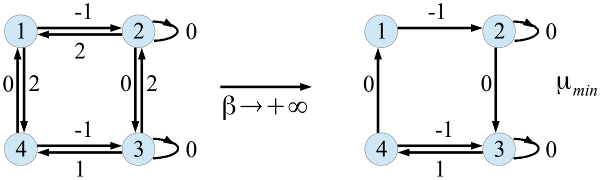
- A transition is forbidden if it belongs to no minimizing cycle
- If $\mathscr{F} = \{$ forbidden transitions $\}$ then

$$K = \bigcup \left\{ \text{supp}(\mu) : \ \mu \text{ is minimizing } \right\} = \Sigma^1(\mathscr{A}, \mathscr{F}) \tag{3}$$

II.3. Example of convergence

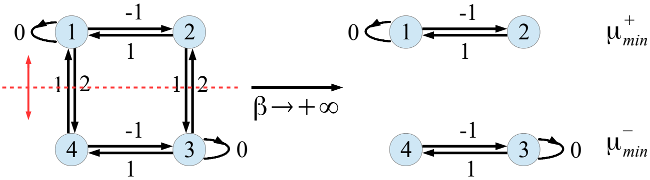
Example 1: $\mathscr{A} = \{1,2,3,4\}$, $\varphi(1,2) = -1$, $\varphi(2,2) = 0$

$\mu_\beta \to \mu_{min}$ (1 measure of maximal entropy)



Example 2:

$\mu_\beta \to \frac{1}{2}(\mu^+_{min} + \mu^-_{min})$ (2 measures of maximal entropy)



II.4. A exact result of convergence

Theorem (Brémont (2003))

Let $d = 1$, $\varphi : \Sigma^1(\mathscr{A}) \to \mathbb{R}$ be a locally finite potential. Then

(1) $\lim_{\beta\to+\infty} \mu_\beta = \mu^*_{min}$ *(without taking a subsequence)*

$\hat{U}$*(2)* μ^*_{min} *is a minimizing measure (possibly non ergodic)*

(3) $\mathrm{Ent}(\mu^*_{min}) = \sup\{\mathrm{Ent}(\mu) : \ \mu$ *is minimizing* $\}$

(4) μ^*_{min} *is a barycenter of minimizing measures of maximal entropy supported on disjoint SFTs. (The coefficients of the barycenter are algebraic numbers).*

Remark

(1) The proof uses tools in semi-algebraic theory.

(2) The set $K := \bigcup\{\mathrm{supp}(\mu) : \ \mu \text{ is minimizing } \}$ that supports all minimizing measures has a simple description: a subshift of finite type.

II.5. Examples of chaotic convergence

If the set $K := \bigcup\{\mathrm{supp}(\mu) : \ \mu \text{ is minimizing } \}$ has a large complexity, one may expect a chaotic convergence at zero temperature.

Theorem (Chazottes-Hochman (2010))

There exists an invariant compact set $K \subset \Sigma^1(\mathscr{A})$ such that the potential $\varphi(x) = d(x, K)$ is chaotic at zero temperature.

But the set of minimizing measures can be as simple as possible.

Theorem (Garibaldi, Bissacot, Th. (2018))

There exists a Lipschitz potential $\varphi : \{0,1\}^{\mathbb{Z}} \to [0, +\infty)$ such that

- *δ_{0^∞} and δ_{1^∞} are the only two ergodic minimizing measures*
- *φ is chaotic at zero temperature*
- *one defines an energy barrier $h : \Sigma^1 \times \Sigma^1 \to [0, +\infty)$ and in order to have chaotic convergence, we must have*

$$h(0^\infty, 1^\infty) = h(1^\infty, 0^\infty) = 0$$

Summary

III.1. Main result

Theorem (Barbieri, Bissacot, Dalle Vedove, Th. (2022))

There exists a locally finite potential $\varphi : \Sigma^2(\mathscr{A}) \to \mathbb{R}$ that is chaotic at zero temperature.

Remark

- Chazottes, Hochman (2010) proved the previous result for $d \geqslant 3$
- Simultaneously to us, Chazottes, Shinoda (2021), extended their previous result to $d = 2$. Their proof is different: Kleene fixed-point theorem.

Theorem (Gayral, Sablik, Taati (2023))

If $\mathscr{K}$ is a finite simplex of periodic measures (or Π_2-computable simplex), there exists $\varphi : \Sigma^2(\mathscr{A}) \to \mathbb{R}$ locally finite such that

- $\mathrm{diam}(\mathscr{G}_\beta(\varphi)) \to 0$
- *any $\mu \in \mathscr{K}$ is an accumulation point of a choice of $(\mu_{\beta\varphi})_{\beta\to+\infty}$*

III.2. General strategy

- Find a set of forbidden patterns $\mathscr{F} \subset \mathscr{A}^{[\![1,D]\!]^2}$ of size $D \geqslant 1$,
- Define the potential $\varphi = \mathbb{1}_{[\mathscr{F}]}$
- Make sure that

$$X = \Sigma^2(\mathscr{A}, \mathscr{F}) = \{x \in \Sigma^2(\mathscr{A}) : \varphi \circ \sigma^k(x) = 0, \ \forall\, k \in \mathbb{Z}^2\} \neq \emptyset$$

- Find a special cooling sequence $\beta_k \to +\infty$
- Take any Gibbs measure

$$\int \beta_k \varphi \, d\mu_{\beta_k} - \mathrm{Ent}(\mu_{\beta_k}) = \inf \Big\{ \int \beta_k \varphi \, d\mu - \mathrm{Ent}(\mu) : \ \mu \text{ is invariant } \Big\}$$

- Show that

$$\mu_{\beta_{2k+1}} \to \mu^1_{min}, \quad \mu_{\beta_{2k}} \to \mu^2_{min}, \quad \mu^1_{min} \neq \mu^2_{min}$$

- Notice that

$$\mathrm{supp}(\mu^1_{min}),\ \mathrm{supp}(\mu^2_{min}) \subset X$$

II.3. Step 1/6

We construct a *1d set of forbidden words* with alternating complexity

- $\tilde{\mathscr{A}} = \{0, 1, 2\}$, 0 is a marker,
- $\mathscr{A}^{(1)} = \{0, 1\}$, $\mathscr{A}^{(2)} = \{0, 2\}$
- construct inductively two languages of words of length ℓ_k

$$\mathscr{A}_k^{(1)} = \{1^{\ell_k}, a_k^{(1)}\}, \quad \mathscr{A}_k^{(2)} = \{2^{\ell_k}, a_k^{(2)}\}$$

- choose a finite set $\mathscr{F}_k$ such that

$$\tilde{X}_k := \Sigma(\mathscr{A}, \mathscr{F}_k) = \begin{cases} \text{bi-infinite configurations obtained as} \\ \text{concatenation of words in } \mathscr{A}_k^{(1)} \text{ and } \mathscr{A}_k^{(2)} \end{cases}$$

- construct similarly

$$\tilde{X}_k^{(1)} := \Sigma^1(\mathscr{A}, \mathscr{F}_k^{(1)}) \text{ and } \tilde{X}_k^{(2)} := \Sigma^1(\mathscr{A}, \mathscr{F}_k^{(2)})$$

- $\tilde{\mathscr{F}} = \bigsqcup \mathscr{F}_k$, the *first subshift* we construct

$$\tilde{X} = \Sigma^1(\tilde{\mathscr{A}}, \tilde{\mathscr{F}}) = \bigcap_{k \geqslant 0} \tilde{X}_k$$

II.4. Step 2/6

- we summarize $\tilde{X} = \bigcap_{k\geqslant 0} \downarrow \tilde{X}_k$

$$\tilde{X}_k \supset \tilde{X}_k^{(1)} \sqcup \tilde{X}_k^{(2)}, \quad \tilde{X}_k^{(1)} \subset \{0,1\}^{\mathbb{Z}},\ \tilde{X}_k^{(2)} \subset \{0,2\}^{\mathbb{Z}}$$

- impose an alternating complexity

$$\begin{cases} \text{for } k \text{ even}, & \mathrm{Freq}(\tilde{X}_k^{(1)}) \ll \mathrm{Freq}(\tilde{X}_k^{(2)}) \\ \text{for } k \text{ odd}, & \mathrm{Freq}(\tilde{X}_k^{(2)}) \ll \mathrm{Freq}(\tilde{X}_k^{(1)}) \end{cases}$$

- start with $a_0^{(1)} = 01$, $a_0^{(2)} = 02$, build

$$\mathscr{A}_0^{(1)} = \{11, a_0^{(1)}\}, \quad \mathscr{A}_0^{(2)} = \{22, a_0^{(2)}\},$$

- build by induction $\mathscr{A}_{k+1}^{(1)} = \{1^{\ell_k}, a_{k+1}^{(1)}\}$, assume k even

$$a_{k+1}^{(1)} = \underbrace{a_k^{(1)} a_k^{(1)} \cdots a_k^{(1)}}_{N_{k+1} \text{ times}}, \qquad a_{k+1}^{(2)} = a_k^{(2)} \underbrace{2^{\ell_k} \cdots\cdots 2^{\ell_k}}_{(N_{k+1}-2)\ell_k \text{ times}} a_k^{(2)}$$

- assume k is odd, permute (1) and (2)

II.5. Step 3/6

- assume k even

$$a_{k+1}^{(1)} = \underbrace{a_k^{(1)} a_k^{(1)} \cdots a_k^{(1)}}_{N_{k+1}\ \text{times}}, \qquad a_{k+1}^{(2)} = a_k^{(2)} \underbrace{2^{\ell_k} \cdots\cdots 2^{\ell_k}}_{(N_{k+1}-2)\ell_k\ \text{times}} a_k^{(2)}$$

- define $f_k^{(i)}$ to be the frequency of 0 in the word $a_k^{(i)}$
- because each a_k contains at least one 0

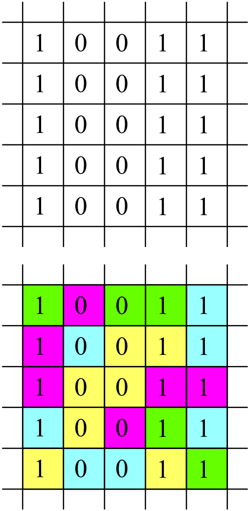
$$\begin{cases} \text{for } k \text{ even}, & f_k^{(1)} \ll f_k^{(2)} \\ \text{for } k \text{ odd}, & f_k^{(2)} \ll f_k^{(1)} \end{cases}$$

- the complexity could have been measured by the entropy (but as we will see, entropy is not the right notion)

Observation

$\mathscr{F}$ can be constructed *recursively* (provided $(N_k)_{k\geqslant 0}$ is also recursive). That is, there exists a Turing machine $\mathbb{M}$ that enumerates the words of $\mathscr{F}$ by increasing size and polynomial time enumeration $T_{\mathbb{M}}$ function;

II.6. Step 4/6



- Embed a 1d subshift into a 2d subshit by vertically repeating the symbols

$$\tilde{\tilde{X}} = \Sigma^2(\tilde{\mathscr{A}}, \tilde{\tilde{\mathscr{F}}})$$

- notice

$$\mathrm{Ent}(\tilde{\tilde{X}}) = 0$$

- Extend the 2d subshift by adding additional colors

$$\Pi : \hat{X} \to \tilde{\tilde{X}}$$

- Find a finite set of local constraints between the colors and the digits so that the vertically aligned subshift $\tilde{\tilde{X}}$ is revealed by erasing the colors

II.7. Step 5/6

Theorem (Aubrun, Sablik (2013))

Let $\tilde{\mathscr{F}}$ be a 1d computable set of forbidden words on the alphabet $\tilde{\mathscr{A}}$. Let $\tilde{X}$ the corresponding subshift and $\tilde{\tilde{X}}$ the vertically aligned extended subshift. Then $\tilde{\tilde{X}}$ is sofic.

- *One can decorate the original symbols:* $\hat{\mathscr{A}} = \tilde{\mathscr{A}} \times \mathscr{B}$
- *There exists $D \geqslant 1$ and $\hat{\mathscr{F}} \subset \hat{\mathscr{A}}^{[\![1,D]\!]^2}$ such that $\hat{X} := \Sigma^2(\hat{\mathscr{A}}, \hat{\mathscr{F}})$ is a shift equivariant extension of $\tilde{\tilde{X}}$:*
  - ⋆ *There exists a commuting diagram*

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\sigma} & \hat{X} \\ \downarrow \Pi & & \downarrow \Pi \\ \tilde{\tilde{X}} & \xrightarrow{\sigma} & \tilde{\tilde{X}} \end{array}$$

  - ⋆ Π *is surjective and is defined by erasing the decorations, by using a one-bloc factor map $\pi : \hat{\mathscr{A}} \to \tilde{\mathscr{A}}$, (the first projection)*

II.8. Step 6/6

The entropy of Aubrun-Sablik has also zero entropy. For the purpose of the rest of the proof we need alternating subshifts of large and small entropies.

1	0''	0''	1	1
1	0'	0''	1	1
1	0'	0'	1	1
1	0'	0''	1	1
1	0''	0'	1	1

- Duplicate the symbol 0

$$\hat{\mathscr{A}} = \{0,1,2\} \times \mathscr{B} \;\to\; \mathscr{A} = \{0',0'',1,2\} \times \mathscr{B}$$

- Duplicate the forbidden words

$$\hat{\mathscr{F}} \;\to\; \mathscr{F} \subset \mathscr{A}^{[\![1,D]\!]^2}$$

- We constructed successively

$$\tilde{X} \leftarrow \tilde{\tilde{X}} \leftarrow \hat{X} \leftarrow X$$

$$\tilde{X} = \bigcap_{k\geqslant 0} \tilde{X}_k, \;\; \tilde{X}_k \supset \tilde{X}_k^{(1)} \bigsqcup \tilde{X}_k^{(2)}$$

Entropy estimate: $\mathrm{Ent}(X_k^{(1)}) = \log(2) f_k^{(1)}$

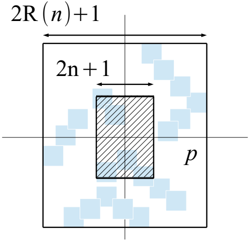
II.9. Our contribution

Our contribution in that problem is to recognize two estimates in Chazottes-Hochman that were not explicitly written in their proof.

Definition (Shift reconstruction function)

Let $\mathscr{F}$ be a set of forbidden patterns and $X = \Sigma^2(\mathscr{A}, \mathscr{F})$ be the corresponding subshift.

- A finite pattern $p \in \mathscr{A}^{[\![1,n]\!]^2}$ is *locally admissible* if no forbidden pattern appears in p.
- A finite pattern is *globally admissible* if it is a sub-pattern of an (infinite) configuration $x \in X$



- The reconstruction function is the function that associates for every $n \geqslant 1$, the smallest size $R \geqslant n$ such that if $p_{[\![-R(n),R(n)]\!]^2}$ is any locally admissible pattern, then $p_{[\![-n,n]\!]^2}$ is globally admissible.

II.10. Reconstruction function

Lemma

Let $\tilde{\mathscr{F}}$ be a 1d computable set of forbidden patterns. Assume

- *The time enumeration function* $T_{\tilde{\mathscr{F}}} = \mathscr{O}(P(n)|\mathscr{A}|^n)$
- *The reconstruction function* $R_{\tilde{\mathscr{F}}}(n) = \mathscr{O}(n)$

Then the Aubrun-Sablik extension $\hat{X} = \Sigma^2(\hat{\mathscr{A}}, \hat{\mathscr{F}})$ *satisfies*

$$\limsup_{n \to +\infty} \frac{1}{n} \log R_{\hat{\mathscr{F}}}(n) < +\infty$$

Notice that we don't say that the growth of the reconstruction function is computable

Remark

The set $\tilde{\mathscr{F}}$ constructed before satisfies the hypothesis of the lemma

II.11. Relative complexity function

Definition

Let $\Pi : \hat{X} = \Sigma^2(\hat{\mathscr{A}}, \hat{\mathscr{F}}) \to X = \Sigma^2(\mathscr{A}, \mathscr{F})$ be an extension with a one-bloc factor map $\pi : \hat{\mathscr{A}} \to \mathscr{A}$. For every $n \geqslant 1$, for every globally admissible pattern $p \in \mathscr{A}^{[\![1,n]\!]^2}$, let $\mathscr{L}(n,p)$ be the set of globally admissible patterns $\hat{p} \in \hat{\mathscr{A}}^{[\![1,n]\!]^2}$ that project onto p.

The relative complexity function is

$$C_{\hat{\mathscr{F}}}(n) := \sup_p \operatorname{card}(\mathscr{L}(n,p))$$

The Aubrun-Sablik extension is more than a zero-entropy extension.

Lemma

The Aubrun-Sablik extension satisfies

$$\lim_{n \to +\infty} \frac{1}{n} \log C_{\hat{\mathscr{F}}} = 0.$$

II.12. Final remarks

- We use Aubrun-Sablik as a black box: we don't really understand the reason in general why some tiling exhibits a chaotic convergence. How to recognize that a 2d tiling with a finite set of rules contains a hidden 1d subsystem that is chaotic?
- All the quantities we use are computable (obtained by an explicit algorithm). But there is a countable number of such objects. For instance the sequence of temperature $(\beta_k)_{k\geqslant 0}$ is also computable.
- What can we say for general sequence of temperatures? These sequences form an uncountable set of sequences and are not therefore computable.
- The alphabet of the Aubrun-Sablik extension is too large and there is no possible experiment to be done.
- What can we say for non-invariant Gibbs measures?
- What can we say for positive temperature?

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