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Synchronization hypothesis in the Winfree model

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ABSTRACT

We consider N oscillators coupled by a mean field as in the Winfree model. The model is governed by two parameters: the coupling strength κ and the spectrum width γ of the frequencies of each oscillator centred at 1. In the uncoupled regime, $\kappa = 0$, each oscillator possesses its own natural frequency, and the difference between the phases of any two oscillators grows linearly in time. In the zero-width regime for the spectrum, the oscillators are simultaneously in the death state if and only if κ is above some positive value κ_* . We say that N oscillators are synchronized if the difference between any two phases is uniformly bounded in time. We identify a new hypothesis for the existence of synchronization. The domain in (γ, κ) of synchronization contains $\{0\} \times [0, \kappa_*]$ in its closure. Moreover, the domain is independent of the number of oscillators and the distribution of the frequencies. We show numerically, on a specific family of Winfree models, that the above hypothesis seems to be a bifurcation criterion for the existence of synchronization domain. The transition is not, however, mathematically sharp.

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1. Introduction

In 1967, Winfree [1] proposed a model describing the synchronization of a population of organisms or *oscillators* that interact simultaneously. We assume that the state of each oscillator is described by a real number, the phase, that corresponds to a point on the circle. We call natural frequency, the frequency of each oscillator, as if it were isolated from the others. The natural frequencies are localized inside an interval $[1 - \gamma, 1 + \gamma]$ for some constant γ that we call *spectrum width*. The interaction of the surrounding oscillators on a particular oscillator has a shape independent of this oscillator. The term *mean field* is used for this kind of interaction or coupling. We denote by N the number of oscillators and by κ a parameter called *coupling strength* that measures the coupling between these oscillators.

A numerical investigation and a mathematical analysis of the transitions between the phases show a diagram of four major states (see [2]): a phase called *death state* where each oscillator is at rest, a phase called *locking state* where all the oscillators possess the same non-zero frequency, a phase called *incoherence state* where any two oscillators have different frequencies, and a phase called *partial death state* which is a mixed state where a part of

the oscillators are in death state and a part is in incoherence state. For small values of κ , the Winfree model is reduced to a different model, the Kuramoto model that will not be studied in this article. For large values of κ , the oscillators are in the death state. We denote by κ_* the bifurcation parameter for the death state in the case $\gamma = 0$.

We say that N oscillators are synchronized if $\max_{1 \leq i, j \leq N} |x_i(t) - x_j(t)|$ is bounded from above uniformly in time. Notice that we do not consider the phase of the oscillators but rather their lifts. For example, in the case $\gamma = 0$, our definition implies that N oscillators are always synchronized, independently of their initial conditions, thanks to the uniqueness property of periodic solutions of an ordinary differential equation (ODE). We identify a new hypothesis of synchronization that produces a domain of parameters (γ, κ) containing $\{0\} \times [0, \kappa_*]$ in its closure. We show in addition in a numerical example that the existence of synchronization seems to disappear when the hypothesis is not anymore realized.

The Winfree model is given by the following differential equation:

$$\dot{x}_i = \omega_i - \kappa \frac{1}{N} \sum_{j=1}^N P(x_j)R(x_i), \tag{1}$$

where P and R are C^2 2π -periodic functions, $x_i(t)$ is the phase of the i -th oscillator, and $X(t) = (x_1(t), \dots, x_N(t))$ is the global state of the system. Although $x_i(t)$ represents a scalar in the torus $\mathbb{R}/2\pi\mathbb{Z}$, we actually consider its unique continuous lift in \mathbb{R} , that we continue to call it $x_i(t)$. When $N = 1$ and $\omega_1 = 1$, the Winfree model reduces to the equation

$$\dot{x} = 1 - \kappa P(x)R(x). \tag{2}$$

We call *locking bifurcation critical parameter*, the coupling critical parameter κ_* which separates the death state and the locking state in the reduced Winfree model:

$$\kappa_* := \max\{\kappa > 0 : 1 - \kappa P(x)R(x) > 0, \forall x \in \mathbb{R}\}. \tag{3}$$

(Notice that $\kappa_* = +\infty$ if $\max_x P(x)R(x) \leq 0$.) We assume throughout this work the following hypotheses:

- H1:** the coupling strength κ is taken in the interval $(0, \kappa_*)$,
- H2:** the natural frequencies ω_i are chosen in $(1 - \gamma, 1 + \gamma)$ with $\gamma \in (0, 1)$,
- H3:** synchronization hypothesis: $\int_0^{2\pi} \frac{P(s)R'(s)}{1 - \kappa P(s)R(s)} ds > 0, \forall \kappa \in (0, \kappa_*)$.

For numerical results, we use a simplified version of the Winfree model

$$\dot{x}_i = \omega_i - \frac{\kappa}{N} \sum_{j=1}^N P_\beta(x_j) \sin(x_i), \text{ where } P_\beta(x) = 1 + \cos(x + \beta), \tag{4}$$

and $\beta \in [0, \pi]$. It is important to notice that we do not make any assumptions on the number N of oscillators; in particular, we do not assume that $N \rightarrow +\infty$. We do not assume either that the natural frequencies are distributed according to a particular law as it is done in [2]. Our main objective is to exhibit a domain in the parameter space (γ, κ) independent

of N and any choice of the natural frequencies satisfying H2 such that the oscillators stay synchronized for all time and for any initial conditions sufficiently packed.

Synchronization and locking may have several meanings or definitions depending on the authors. We choose the following definitions.

Definition 1.1 (Synchronization): We say that the oscillators $x_i(t)$ are synchronized if $\sup_{1 \leq i, j \leq N} |x_i(t) - x_j(t)|$ is bounded from above uniformly in time $t \geq 0$.

Definition 1.2 (Periodic locking): We say that the oscillators $x_i(t)$ are periodically locked to the frequency Ω if they are synchronized and if there exist $2\pi/\Omega$ -periodic functions $\Psi_i(t)$ such that

$$x_i(t) = \Omega t + \Psi_i(t), \quad \forall i = 1 \dots N, \quad \forall t \geq 0.$$

We notice that in the periodic locking case, the average frequency $\frac{x_i(t)}{t}$ admits the limit Ω as $t \rightarrow +\infty$ and that this limit is independent of the i -th oscillator. The fact that the limit does exist has never been addressed mathematically (except of course in the death state). Our main result is a partial result in that direction in the locking case when κ is any parameter in $(0, \kappa_*)$ and $\gamma \in (0, \Gamma(\kappa))$ for some $\Gamma : [0, \kappa_*] \rightarrow \mathbb{R}^+$.

Theorem 1.1: *We consider the Winfree model given by (1) and satisfying the hypotheses H1–H3. Then, there exists an open set U in the space of parameters $(\gamma, \kappa) \in (0, 1) \times (0, \kappa_*)$, independent of N , containing in its closure $\{0\} \times [0, \kappa_*]$, such that for every parameter $(\gamma, \kappa) \in U$, for every $N \geq 1$ and every choice of natural frequencies $(\omega_i)_{i=1}^N$,*

- (1) *There exists an open set $C_{\gamma, \kappa}^N$ invariant by the flow, and of the form,*

$$C_{\gamma, \kappa}^N := \left\{ X = (x_i)_{i=1}^N \in \mathbb{R}^N : \max_{i, j} |x_j - x_i| < \Delta_{\gamma, \kappa} \left(\frac{1}{N} \sum_{i=1}^N x_i \right) \right\}$$

where $\Delta_{\gamma, \kappa} : \mathbb{R} \rightarrow (0, 1)$ is a 2π -periodic C^2 function independent of N . In other words, for every parameter $(\gamma, \kappa) \in U$, the oscillators are synchronized for any initial conditions $x_i(0)$ with sufficiently small dispersion.

- (2) *There exists a particular initial condition $(x_i^*(0))_{i=1}^N \in C_{\gamma, \kappa}^N$, and a common frequency $\Omega_{\gamma, \kappa} > 0$ such that*

$$x_i^*(t) = \Omega_{\gamma, \kappa} t + \Psi_{i, \gamma, \kappa}^N(t), \quad \forall i = 1, \dots, N, \quad \forall t \geq 0,$$

where $\Psi_{i, \gamma, \kappa}^N : \mathbb{R}^+ \rightarrow \mathbb{R}$ are C^2 , $2\pi/\Omega_{\gamma, \kappa}$ -periodic functions, uniformly bounded with respect to N . In other words, the oscillators are periodically locked for a particular initial condition.

Notice that the simplified Winfree model (4) satisfies hypothesis H3 when $\beta = 0$ as it is proved in Section 3.

For small values of $\kappa \ll 1$, Quinn and Strogatz [3] use the Lindstedt’s method and find an approximation value of the tangent bifurcation curve of the synchronization sate. We

mention relative results about the Winfree model in [4–8]. Recently Ha et al. [9] studied the stability of the stationary state for large coupling.

2. Proof of Theorem 1.1

Let $X = (x_1, \dots, x_N)$ be a solution of (1). We call mean $\mu(t)$ and dispersion $\delta(t)$ of X the quantities

$$\mu(t) := \frac{1}{N} \sum_{j=1}^N x_j(t), \quad \delta(t) := \max_{i,j} |\delta_{i,j}(t)|, \quad \delta_{i,j}(t) := x_i(t) - x_j(t). \quad (5)$$

We want to build a space of parameters U of the form

$$U = \{(\gamma, \kappa) \in (0, 1) \times (0, \kappa_*) : 0 < \gamma < \Gamma(\kappa)\},$$

and a 2π -periodic function $\Delta_{\gamma,\kappa}$, called *dispersion curve* such that, if $X(t)$ is a solution of the Winfree model (1) satisfying H1–H3,

$$\delta(0) < \Delta_{\gamma,\kappa}(\mu(0)) \implies \delta(t) < \Delta_{\gamma,\kappa}(\mu(t)), \quad \forall t \geq 0.$$

The dispersion curve is obtained by solving a non-autonomous affine differential equation with periodic coefficients. The following lemma is standard.

Lemma 2.1: *We consider the affine differential equation*

$$\frac{d}{ds} \Delta(s) = \alpha - \beta(s) \Delta(s), \quad (6)$$

where $\alpha > 0$ and β is a C^1 , 2π -periodic function satisfying

$$\int_0^{2\pi} \beta(s) ds > 0.$$

Then, there exists a unique C^2 , positive and 2π -periodic function solution of (6) given by

$$\Delta(s) := \alpha \frac{\int_s^{s+2\pi} \exp\left(\int_s^t \beta(u) du\right) dt}{\exp\left(\int_0^{2\pi} \beta(u) du\right) - 1}.$$

Moreover,

$$\max_{s \in \mathbb{R}} \Delta(s) \leq \alpha 2\pi \frac{\exp\left(\int_0^{2\pi} \beta^-(u) du\right)}{1 - \exp\left(-\int_0^{2\pi} \beta(u) du\right)}$$

where $\beta^- := \max(0, -\beta)$, and $\beta^+ := \max(0, \beta)$.

Proof: The differential equation (6) is a first-order non-homogeneous linear differential equation. The general form of solution $z(t)$ of (6) with initial condition $z(s_0) \in \mathbb{R}$ at time

$s_0 \in \mathbb{R}$ is given by

$$z(s) = \exp\left(\int_{s_0}^s -\beta(u) du\right) z(s_0) + \alpha \int_{s_0}^s \exp\left(\int_t^s -\beta(u) du\right) dt.$$

If $z(s)$ is 2π -periodic, using the 2π -periodicity of β , one obtains a unique $z(s_0)$ given by

$$z(s_0) = \alpha \frac{\int_{s_0}^{s_0+2\pi} \exp\left(\int_t^{s_0+2\pi} -\beta(u) du\right) dt}{1 - \exp\left(\int_{s_0}^{s_0+2\pi} -\beta(u) du\right)}$$

and a unique 2π -periodic solution, positive, thanks to $\alpha > 0$, given by

$$z(s) = \alpha \frac{\int_s^{s+2\pi} \exp\left(\int_t^{s+2\pi} -\beta(u) du\right) dt}{1 - \exp\left(\int_0^{2\pi} -\beta(u) du\right)}.$$

The maximum of Δ is obtained using the following estimates:

$$\begin{aligned} \int_s^t \beta(s) ds &= \int_s^t (\beta^+ - \beta^-)(s) ds \leq \int_s^t \beta^+(s) ds \leq \int_0^{2\pi} \beta^+(s) ds, \\ &\leq \int_0^{2\pi} (\beta^+ - \beta^-)(s) ds + \int_0^{2\pi} \beta^-(s) ds, \\ &\leq \int_0^{2\pi} \beta(s) ds + \int_0^{2\pi} \beta^-(s) ds. \end{aligned}$$

■

The following lemma shows that, if the dispersion of a solution $X(t)$ of (1) is a priori uniformly bounded, $\delta(t) < D, \forall t \in [0, t_*]$, then each $\delta_{i,j}(t)$ is a sub-solution strict of an affine differential equation as in Lemma 2.1.

Lemma 2.2: *Let us assume that for some $(\gamma, \kappa) \in (0, 1) \times (0, \kappa_*)$, there exist $D > 0$ and $t_* > 0$ such that for every $t \in [0, t_*]$, the solution $X(t)$ of (1) satisfies $\delta(t) < D$. Then, for every $1 \leq i, j \leq N$, and $t \in [0, t_*]$*

$$\begin{aligned} \frac{d}{dt} \delta_{i,j} &< (2\gamma + C\kappa D^2) - \kappa P(\mu) R'(\mu) \delta_{i,j}, \\ \left| \frac{d}{dt} \delta_{i,j} \right| &\leq 2\gamma + C\kappa D^2 + \tilde{C}\kappa D, \end{aligned}$$

where $\delta_{i,j}(t) := x_i(t) - x_j(t)$, $\mu(t) = \frac{1}{N} \sum_{k=1}^N x_k(t)$,

$$C := \|P\|_\infty \|R''\|_\infty + \|P'\|_\infty \|R'\|_\infty, \text{ and } \tilde{C} := \|P'\|_\infty \|R\|_\infty + \|P\|_\infty \|R'\|_\infty.$$

Proof: By subtracting two equations (1), we obtain

$$\begin{aligned} \frac{d\delta_{i,j}}{dt} &= \omega_i - \omega_j - \frac{\kappa}{N} \sum_{k=1}^N P(x_k) [R(x_i) - R(x_j)] \\ &= \omega_i - \omega_j - \kappa P(\mu) R'(\mu) \delta_{i,j} + \frac{\kappa}{N} \sum_{k=1}^N E_{i,j,k} \end{aligned}$$

where $E_{i,j,k} = P(\mu)R'(\mu)(x_i - x_j) - P(x_k)[R(x_i) - R(x_j)]$ that we bound from above using the following estimates:

$$\begin{aligned} P(x_k)[R(x_i) - R(x_j)] &= P(\mu)[R(x_i) - R(x_j)] + \tilde{E}_{i,j,k}, \\ |\tilde{E}_{i,j,k}| &\leq \|P'\|_\infty \|R'\|_\infty |x_k - \mu| |x_i - x_j| < \|P'\|_\infty \|R'\|_\infty D^2, \\ P(\mu)[R(x_i) - R(x_j)] &= P(\mu)R'(\mu)(x_i - x_j) + \hat{E}_{i,j,k}, \\ |\hat{E}_{i,j,k}| &< \|P\|_\infty \|R'\|_\infty D^2, \end{aligned}$$

and $|\omega_i - \omega_j| \leq 2\gamma$. ■

We estimate in the following lemma the velocity of μ . We will later find conditions on (γ, κ) so that $\frac{d\mu}{dt} > 0$. The constant \tilde{C} is defined in Lemma 2.2.

Lemma 2.3: *Assuming the same hypotheses as in Lemma 2.2, we have*

$$\left| 1 - \kappa P(\mu)R(\mu) - \frac{d}{dt}\mu \right| \leq \gamma + \tilde{C}\kappa D, \quad \forall t \in [0, t_*].$$

Proof: We use indeed the estimates $|x_i - \mu| \leq D, |1 - \frac{1}{N} \sum_{k=1}^N \omega_k| \leq \gamma$,

$$\begin{aligned} \frac{d\mu}{dt} &= 1 - \kappa P(\mu)R(\mu) - \left[1 - \frac{1}{N} \sum_{k=1}^N \omega_k \right] + \frac{\kappa}{N} \sum_{k=1}^N E_k, \quad \text{where} \\ E_k &= \frac{1}{N} \sum_{i=1}^N [P(\mu)(R(\mu) - R(x_i)) - (P(x_k) - P(\mu))R(x_i)]. \end{aligned}$$

We now assume that (γ, κ) have been chosen so that

$$1 - \gamma - \tilde{C}\kappa D - \frac{\kappa}{\kappa_*} > 0. \tag{7}$$

Notice that the equation $\frac{dx_i}{dt} = 1 - \kappa P(x_i)R(x_i)$ corresponds to $\gamma = 0$ and identical initial conditions in Equation (1). By a small perturbation of $\gamma = 0$, and a small perturbation of the initial condition $x_1(0) = \dots = x_N(0)$, one obtain $\frac{d\mu}{dt} = 1 \pm \epsilon - \kappa P(\mu)R(\mu)$ and $\frac{d}{dt}(x_i - x_j) = \pm \epsilon_{i,j} - \kappa P(\mu)R'(\mu)(x_i - x_j)$. The synchronization hypothesis H3 may be interpreted as a stability hypothesis after changing the time variable t to $s = \mu(t)$.

We now proceed formally. Thanks to the definition of κ_* in (3), condition (7) implies $\frac{d\mu}{dt} > 0$ on $[0, t_*]$. We then consider the change of variable $s = \mu(t)$ from $[0, t_*]$ to $[s_0, s_*]$

whre $s_0 = \mu(0)$ and $s_* = \mu(t_*)$. Let τ be the inverse function

$$\tau : \begin{cases} [s_0, s_*] \rightarrow [0, t_*] \\ s \mapsto \tau(s) \end{cases} . \quad (8)$$

Define $x_i^*(s) = x_i \circ \tau(s)$, $X^*(s) = (x_1^*(s), \dots, x_N^*(s))$, $\delta_{i,j}^*(s) = x_i^*(s) - x_j^*(s)$. With respect to the new variable s , Lemma 2.2 admits the following equivalent form.

Lemma 2.4: *We assume the same hypotheses as in Lemma 2.2 and the new condition (7). Then,*

$$\frac{d}{ds} \delta_{i,j}^*(s) < \alpha(\gamma, \kappa, D) - \beta_\kappa(s) \delta_{i,j}^*(s), \quad \forall s \in [s_0, s_*],$$

where

$$\alpha(\gamma, \kappa, D) := \frac{2\gamma + C\kappa D^2}{1 - \kappa/\kappa_*} + \frac{(2\gamma + C\kappa D^2 + \tilde{C}\kappa D)(\gamma + \tilde{C}\kappa D)}{(1 - \gamma - \tilde{C}\kappa D - \kappa/\kappa_*)(1 - \kappa/\kappa_*)},$$

$$\beta_\kappa(s) := \frac{\kappa P(s)R'(s)}{1 - \kappa P(s)R(s)}.$$

Proof: The chain rule gives $\frac{d}{dt} \delta_{i,j} = \frac{d}{ds} \delta_{i,j}^* \frac{d}{dt} \mu$, or

$$\frac{d}{ds} \delta_{i,j}^*(1 - \kappa P(\mu)R(\mu)) = \frac{d}{dt} \delta_{i,j} + \frac{d}{ds} \delta_{i,j}^* \left(1 - \kappa P(s)R(s) - \frac{d}{dt} \mu \right).$$

The definition of κ_* implies $1 - \kappa P(\mu)R(\mu) \geq 1 - \kappa/\kappa_* > 0$. From Lemma 2.2, one obtains

$$\frac{d}{ds} \delta_{i,j}^* \leq \frac{(2\gamma + C\kappa D^2) + \frac{d}{ds} \delta_{i,j}^* \left(1 - \kappa P(s)R(s) - \frac{d}{dt} \mu \right)}{1 - \kappa P(\mu)R(\mu)} - \beta_\kappa(s) \delta_{i,j}.$$

From Lemma 2.3, one obtains

$$\frac{d}{dt} \mu \geq 1 - \kappa P(\mu)R(\mu) - \gamma - \tilde{C}\kappa D \geq 1 - \gamma - \tilde{C}\kappa D - \kappa/\kappa_*,$$

$$\left| \frac{d}{ds} \delta_{i,j}^* \right| \leq \frac{2\gamma + C\kappa D^2 + \tilde{C}\kappa D}{1 - \gamma - \tilde{C}\kappa D - \kappa/\kappa_*}.$$

Let $\Delta_{\gamma, \kappa, D}(s)$ be the unique C^2 2π -periodic solution of the equation

$$\frac{d\Delta}{ds} = \alpha(\gamma, \kappa, D) - \beta_\kappa(s) \Delta$$

given by Lemma 2.1. The following lemma gives sufficient conditions on (γ, κ, D) so that $\max_s \Delta_{\gamma, \kappa, D}(s) < D$.

Lemma 2.5: *There exists an open set U of parameters (γ, κ) whose closure contains $\{0\} \times [0, \kappa_*]$, defined in the following way:*

$$U := \{(\gamma, \kappa) \in (0, 1) \times (0, \kappa_*) : 0 < \gamma < \kappa D^2(\kappa)\}, \quad \text{where}$$

$$D(\kappa) := \min \left(1, \frac{L(\kappa)}{2(2+C)/(1-\kappa/\kappa_*) + 2\tilde{C}(1+\tilde{C})\kappa/(1-\kappa/\kappa_*)^2} \right),$$

$$L(\kappa) := \frac{1 - \exp(-\int_0^{2\pi} \beta_\kappa(s) ds)}{2\pi\kappa \exp(\int_0^{2\pi} \beta_\kappa^-(s) ds)},$$

such that, for every $(\gamma, \kappa) \in U$,

$$1 - \gamma - \tilde{C}\kappa D(\kappa) - \frac{\kappa}{\kappa_*} > 0, \quad \text{and} \quad \max_{s \in [0, 2\pi]} \Delta_{\gamma, \kappa, D(\kappa)}(s) < D(\kappa).$$

Proof: Thanks to Lemma 2.1, it is enough to check that

$$\frac{\alpha(\gamma, \kappa, D(\kappa))}{\kappa L(\kappa)} < D(\kappa).$$

We have

$$D(\kappa) < \frac{1 - \kappa/\kappa_*}{2\kappa(1 + \tilde{C})}, \quad \gamma + \tilde{C}\kappa D(\kappa) < \kappa D(\kappa)(1 + \tilde{C}) < \frac{1}{2} \left(1 - \frac{\kappa}{\kappa_*} \right),$$

$$1 - \gamma - \tilde{C}\kappa D(\kappa) - \kappa/\kappa_* > \frac{1}{2} \left(1 - \frac{\kappa}{\kappa_*} \right),$$

$$\alpha(\gamma, \kappa, D(\kappa)) < \frac{2(2\gamma + C\kappa D(\kappa)^2)}{1 - \kappa/\kappa_*} + \frac{\tilde{C}\kappa D(\kappa)(\gamma + \tilde{C}\kappa D(\kappa))}{\frac{1}{2}(1 - \kappa/\kappa_*)^2},$$

$$< \kappa D(\kappa)^2 \left[\frac{2(2+C)}{1 - \kappa/\kappa_*} + \frac{2\tilde{C}(1 + \tilde{C})\kappa}{(1 - \kappa/\kappa_*)^2} \right] < \kappa D(\kappa)L(\kappa). \quad \blacksquare$$

Definition 2.1: We call *dispersion curve* the periodic function

$$\Delta_{\gamma, \kappa}(s) := \Delta_{\gamma, \kappa, D(\kappa)}(s), \quad \forall s \in \mathbb{R}.$$

We prove the first part of Theorem 1.1. The set $C_{\gamma, \kappa}^N$ defined in Theorem 1.1 is obviously open and has the shape of a tubular neighbourhood about the line $(1, 1, \dots, 1)\mathbb{R}$ with bounded convex transverse section $\mu(X) = \mu_0$. We want to prove that $C_{\gamma, \kappa}^N$ is invariant by the flow of (1). We recall that U denotes the set of parameters (γ, κ) defined in Lemma 2.5.

Proof of Theorem 1.1 - Item 1: Let $X(0) \in C_{\gamma, \kappa}^N$ and

$$t_* := \sup\{t \geq 0 : \forall 0 < t' < t, \delta(t') < \bar{\Delta}_{\gamma, \kappa}(\mu(t'))\}.$$

Assume by contradiction that $t_* < +\infty$. We use the change of variable $s = \mu(t)$ for every $s \in [s_0, s_*]$, $s_0 = \mu(0)$ and $s_* = \mu(t_*)$. Then, there exist $1 \leq i_0, j_0 \leq N$ such that $\delta_{i_0, j_0}^*(s_*) =$

$\Delta_{\gamma,\kappa}(s_*)$. Since $\max \Delta_{\gamma,\kappa} < D(\kappa)$ and $1 - \gamma - \tilde{C}\kappa D(\kappa) - \kappa/\kappa_* > 0$, Lemma 2.4 implies

$$\begin{aligned} \frac{d}{ds} \delta_{i_0, j_0}^*(s_*) &< \alpha(\gamma, \kappa, D(\kappa)) - \beta_\kappa(s) \delta_{i_0, j_0}^*(s_*), \\ &= \alpha(\gamma, \kappa, D(\kappa)) - \beta_\kappa(s) \Delta_{\gamma,\kappa}(s_*) = \frac{d}{ds} \Delta_{\gamma,\kappa}(s_*). \end{aligned}$$

There exists $s < s_*$ close enough to s_* such that $\delta_{i_0, j_0}^*(s) > \Delta_{\gamma,\kappa}(s)$, or in other words, there exists $t < t_*$ close enough to t_* such that $\delta_{i_0, j_0}(t) > \Delta_{\gamma,\kappa}(\mu(t))$. We have obtained a contradiction. ■

We now prove the second part of Theorem 1.1. We show there exists a periodically locked solution $X(t) = (x_1(t), \dots, x_N(t))$ of the system (1) with some initial condition in $C_{\gamma,\kappa}^N$, that is, there exists a common rotation number $\Omega_{\gamma,\kappa} > 0$ such that $x_i(t) = \Omega_{\gamma,\kappa} t + \Psi_i(t)$, $\forall i = 1, \dots, N$ where $\Psi_i(t)$ are periodic functions of period $2\pi/\Omega_{\gamma,\kappa}$. Our strategy consists in constructing, by fixing the mean of $X(0)$, a compact and convex transverse section $\Sigma_{\gamma,\kappa}$ to the closure $C_{\gamma,\kappa}^N$, and a continuous Poincaré map $P_{\gamma,\kappa}: \Sigma_{\gamma,\kappa} \rightarrow \Sigma_{\gamma,\kappa}$ by waiting the first time the mean of $X(t)$ return to 0. We then use Brouwer fixed-point theorem to prove the existence of a fixed point of $P_{\gamma,\kappa}$. We denote by $\Phi^t(X)$ the flow of Equation (1).

Lemma 2.6: *Let $(\gamma, \kappa) \in U$ where U is defined in Lemma 2.5. Define*

$$\Sigma_{\gamma,\kappa} = \{X \in C_{\gamma,\kappa}^N : \mu(X) = 0\} \quad \text{and} \quad \mathbb{1} = (1, \dots, 1).$$

Then, there exist a C^2 map (the Poincaré map) $P_{\gamma,\kappa}: \Sigma_{\gamma,\kappa} \rightarrow \Sigma_{\gamma,\kappa}$ and a C^2 function (the return time map) $\theta_{\gamma,\kappa}: \Sigma_{\gamma,\kappa} \rightarrow \mathbb{R}^+$ such that,

$$\begin{aligned} \Phi^{\theta(X)}(X) &= P_{\gamma,\kappa}(X) + 2\pi \mathbb{1}, \quad \forall X \in \Sigma_{\gamma,\kappa}, \\ \frac{2\pi}{1 + \gamma + \kappa \|P\|_\infty \|R\|_\infty} &< \theta(X) < \frac{2\pi}{1 - \gamma - \tilde{C}\kappa D(\kappa) - \frac{\kappa}{\kappa_*}}. \end{aligned}$$

Proof: Let $X \in C_{\gamma,\kappa}^N$ such that $\mu(X) = 0$. Let $\mu(t) := \mu(\Phi^t(X))$ and $\tau(s)$ be the inverse function of $\mu(t)$ as it has been defined in (5) and (8). We prefer to write the explicit dependence on X : $\mu_X(t) = \mu(t)$ and $\tau_X(s) = \tau(s)$. Thanks to Lemma 2.3, we obtain

$$1 - \gamma - \tilde{C}\kappa D(\kappa) - \frac{\kappa}{\kappa_*} < \dot{\mu}_X(t) < 1 + \gamma + \kappa \|P\|_\infty \|R\|_\infty.$$

Define $\theta(X) := \tau_X(2\pi)$. Then, $\int_0^{\tau_X(2\pi)} \dot{\mu}_X(t) dt = 2\pi$ implies the second estimate of the lemma. Let $P_{\gamma,\kappa}(X) := \Phi^{\theta(X)}(X) - 2\pi \mathbb{1}$. Then,

$$\begin{aligned} \mu(P_{\gamma,\kappa}(X)) &= \mu_X(\theta(X)) - 2\pi = \mu_X \circ \tau_X(2\pi) - 2\pi = 0, \\ \delta(P_{\gamma,\kappa}(X)) &= \delta(\Phi^{\theta(X)}(X)) < \Delta_{\gamma,\kappa}(\mu_X(\theta(X))) = \Delta_{\gamma,\kappa}(2\pi) = \Delta_{\gamma,\kappa}(0). \end{aligned}$$

We have shown that $P_{\gamma,\kappa}$ is a map from $\Sigma_{\gamma,\kappa}$ into itself. ■

Corollary 2.1: *The Poincaré map $P_{\gamma,\kappa}$ defined in Lemma 2.6 admits a fixed point $X_* \in \Sigma_{\gamma,\kappa}$.*

Proof: $\bar{\Sigma}_{\gamma,\kappa}$ is compact and convex; $P_{\gamma,\kappa} : \bar{\Sigma}_{\gamma,\kappa} \rightarrow \bar{\Sigma}_{\gamma,\kappa}$ is continuous. By Brouwer fixed-point theorem, $P_{\gamma,\kappa}$ admits a fixed point in $X_* \in \bar{\Sigma}_{\gamma,\kappa}$. We claim that $X_* \notin \partial\bar{\Sigma}_{\gamma,\kappa}$. Suppose by contradiction $X_* \in \partial\bar{\Sigma}_{\gamma,\kappa}$. Let

$$\delta_{i,j}^*(t) := x_i^*(t) - x_j^*(t), \quad \delta_*(t) := \max_{i,j} \delta_{i,j}^*(t) \text{ and } \mu_*(t) := \frac{1}{N} \sum_{i=1}^N x_i^*(t).$$

Then, there exist $1 \leq i_0, j_0 \leq N$, such that $\delta_{i_0,j_0}^*(0) = \Delta_{\gamma,\kappa}(0)$. As in the proof of item 1 of **Theorem 1.1**, there exists $t_0 > 0$ small enough such that, for every $0 < t < t_0$, $\delta_{i_0,j_0}^*(t) < \Delta_{\gamma,\kappa}(\mu_*(t))$. By repeating this argument for every $1 \leq i_1, j_1 \leq N$ satisfying the equality $\delta_{i_1,j_1}^*(t) = \Delta_{\gamma,\kappa}(\mu_*(t))$, we obtain for some $t' > 0$, $\delta_*(t') < \Delta_{\gamma,\kappa}(\mu_*(t'))$. The invariance of $C_{\gamma,\kappa}^N$ implies $\delta_*(t) < \Delta_{\gamma,\kappa}(\mu_*(t))$ for every $t > t'$. We have obtained a contradiction with the fact that

$$\delta_*(\theta(X_*)) = \delta_*(0) = \Delta_{\gamma,\kappa}(0) = \Delta_{\gamma,\kappa}(2\pi) = \Delta_{\gamma,\kappa}(\mu_*(\theta(X_*))).$$

■

Proof of Theorem 1.1 - Item 2: **Corollary 2.1** implies the existence of a point $X_* \in C_{\gamma,\kappa}^N$ and a time $\theta_* > 0$ such that $\Phi^{\theta_*}(X_*) = X_* + 2\pi\mathbb{1}$. By the uniqueness property of the solutions of an ODE, we have

$$\Phi^{\theta_*+t}(X_*) = \Phi^t(X_*) + 2\pi\mathbb{1}, \quad \forall t \geq 0.$$

Let $\Psi(s) := \Phi^s(X_*) - \frac{2\pi s}{\theta_*}\mathbb{1} = (\Psi_1(s), \dots, \Psi_N(s))$, $\forall s \geq 0$. Then, Ψ is periodic of period θ_* . We have indeed

$$\Psi(s + \theta_*) = \Phi^{s+\theta_*}(X_*) - 2\pi \frac{s + \theta_*}{\theta_*}\mathbb{1} = \Phi^s(X_*) + 2\pi\mathbb{1} - 2\pi \frac{s + \theta_*}{\theta_*}\mathbb{1} = \Psi(s).$$

Moreover, the return time θ_* is uniformly bounded from above with respect to N as in **Lemma 2.6**. ■

3. Numerical results

The study of organized populations has been modelled by Winfree in [1] in 1967. He observed that a weak coupling between independent individuals tends to synchronize them. An example of biological synchronization is given by flashes of tropical fireflies. The synchronization appears when the individuals emit their flashes simultaneously with the same frequency. In this section, we analyze a simplified Winfree model given by Equation (4) with $P_\beta(x) = 1 + \cos(x + \beta)$ and $R(x) = \sin(x)$.

When $\beta = 0$, the synchronization hypothesis H3 is satisfied. Indeed,

$$\frac{d}{ds}(\ln(1 - \kappa P_0 R)) = \frac{-\kappa(P'_0 R + P_0 R')}{1 - \kappa P_0 R},$$

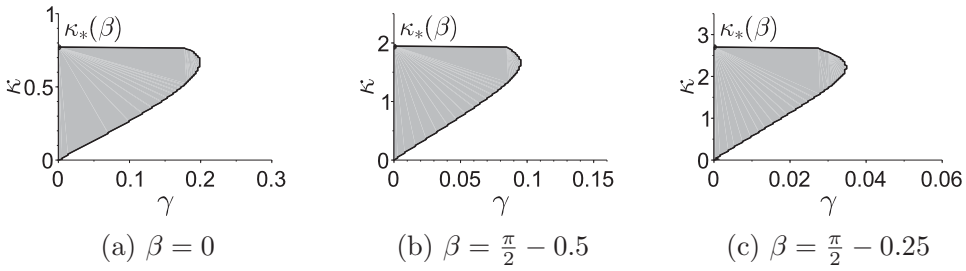


Figure 1. Numerical domain of synchronization U_β of model (4). In all cases, we choose initial conditions randomly distributed in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $N = 100$ oscillators with ordered and equidistant natural frequencies ω_i in $[1 - \gamma, 1 + \gamma]$. U_β is in grey and obtained by testing the largest γ satisfying $d_X(T) < 3\pi$. In Figure 1(a), $T = 1500$, $\beta = 0$ and $\kappa_*(\beta) \approx 0.769$. In Figure 1(b), $T = 1800$, $\beta = \frac{\pi}{2} - 0.5$ and $\kappa_*(\beta) \approx 1.936$. In Figure 1(c), $T = 3500$, $\beta = \frac{\pi}{2} - 0.25$ and $\kappa_*(\beta) \approx 2.694$.

and

$$\begin{aligned} \int_0^{2\pi} \frac{P_0 R'}{1 - \kappa P_0 R} ds &= - \int_0^{2\pi} \frac{P_0' R}{1 - \kappa P_0 R} ds \\ &= \int_0^{2\pi} \frac{\sin^2(s)}{1 - \kappa(1 + \cos(s)) \sin(s)} ds > \frac{\pi}{3}. \end{aligned}$$

The numerical value of κ_* is $\frac{4}{3\sqrt{3}} \sim 0.769$ and the two constants C and \tilde{C} of Lemmas 2.2 and 2.3 are bounded from above by 3. The theoretical domain of parameters U for which the Winfree model is synchronized is built in Lemma 2.5 using $D(\kappa)$ and $L(\kappa)$. Its size is very small, 10^{-3} smaller than the size of the numerical domain one can expect as in Figure 1(a).

In order to analyze numerically the hypothesis H3, we discuss model (4) for different values of $\beta \in [0, \pi]$. We use two different order parameters to measure the synchronization

$$r_X(\beta) := \frac{1}{N} \sum_{j=1}^N e^{ix_j(T)}, \quad d_X(T) := \max_i |x_i(T) - \mu(T)|.$$

Figure 1 shows the numerical domains U_β of synchronization for three different values of β . The domain decreases along the γ -direction as β increases to $\frac{\pi}{2}$ as we shall see. The critical parameter giving the transition to the death state is defined as in Equation (3) by $\kappa_*(\beta) := \max\{P_\beta(x)R(x) : x \in \mathbb{R}\}^{-1}$.

We show in Figure 2(a) the variation of the critical parameter $\kappa_*(\beta)$ for $\beta \in [0, \pi]$; we notice that the minimum is obtained at $\kappa_*(0) \approx 0.769$. Let

$$H_\kappa(\beta) := \int_0^{2\pi} \frac{P_\beta(s)R'(s)}{1 - \kappa P_\beta(s)R(s)} ds.$$

We also show numerically in Figure 2(a), that the grey region corresponds to the domain of (β, κ) satisfying $\beta \in [0, \pi]$, $\kappa \in [0, \kappa_*(\beta)]$, and $H_\kappa(\beta) > 0$, that is to the domain $\beta < \frac{\pi}{2}$. As noticed by the referee, the symmetry $H_\kappa(\beta) = -H_\kappa(\pi - \beta)$ implies $H_\kappa(\frac{\pi}{2}) = 0$ for every

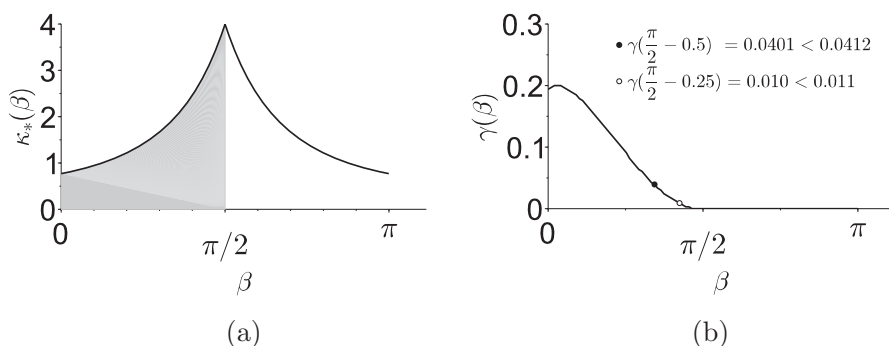


Figure 2. We plot in Figure 2(a) the critical parameter of the death state transition $\kappa_*(\beta)$. We choose in Figure 2(b) $\kappa = 0.6, T = 3 \times 10^4$ and draw the *desynchronization curve*, the largest γ for which $d_X(T) < 3\pi$. We use the same experiment parameters as in Figure 1.

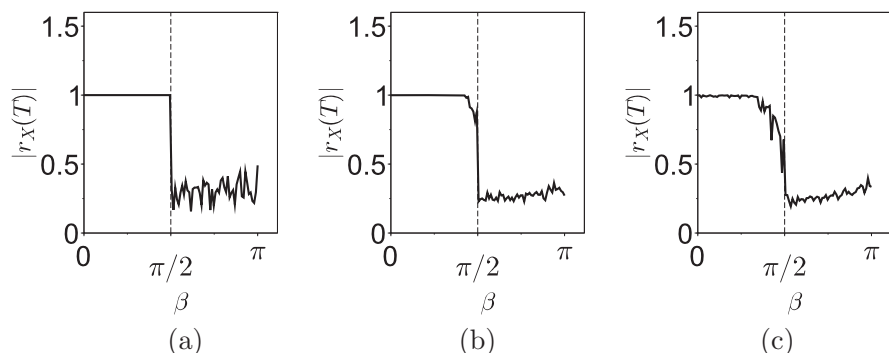


Figure 3. The plot the order parameter $|r_X(\beta)|$ at $T = 3000$ and $\kappa = 0.6$. The other parameters are chosen as in Figure 1. (a) $\gamma_0 = 0$, (b) $\gamma_1 = 0.011$, (c) $\gamma_2 = 0.0412$.

$\kappa \in [0, \kappa_*(\frac{\pi}{2})]$. We choose $\kappa = 0.6$ in Figure 2(b) and compute the largest γ satisfying $d_X(T) < 3\pi$, that is, the boundary of U_β at $\kappa = 0.6$. We notice that the numerical domain of synchronization is negligible if and only if $\beta \in [\frac{\pi}{2}, \pi]$.

We study in Figure 3 the variation of the order parameter $|r_X(\beta)|$ as β increases in $[0, \pi]$ at a given time T and coupling strength κ . A numerical value $|r_X(\beta)| \approx 1$ suggests a very tight cluster of almost all oscillators on the circle; a value $|r_X(\beta)| \approx 0$ suggests on the contrary symmetrically distributed oscillators. We observe a sharp decrease of $|r_X(\beta)|$ at $\beta = \frac{\pi}{2}$, that is, when $H_\kappa(\beta)$ becomes negative.

Figure 4 shows the graph of $d_X(t)$ as a function of $t \in [0, 3 \times 10^4]$ for different values of γ and β at a given $\kappa = 0.6$. It also shows the location of the oscillators on the circle at the final time $T = 3 \times 10^4$. Synchronization has been defined by the condition $\sup_{t>0} d_X(t) < +\infty$. For $\beta > \frac{\pi}{2}$, except at $\gamma \approx 0$, $t \mapsto d_X(t)$ does not seem to be bounded: the oscillators are desynchronized. This observation corroborates the disappearance of the synchronization domain in Figure 2 when $\beta > \frac{\pi}{2}$. For $\gamma = 0$, the oscillators have identical natural frequencies; the comparison principle of ODEs in the periodic case implies, if $\max_{1 \leq i, j \leq N} |x_i(0) - x_j(0)| \leq 2\pi$, that $d_X(t) \leq 2\pi$ for every $t > 0$, the oscillators are always synchronized. For $\beta < \frac{\pi}{2}$, we choose two values of γ which are close, but above, the desynchronization curve:

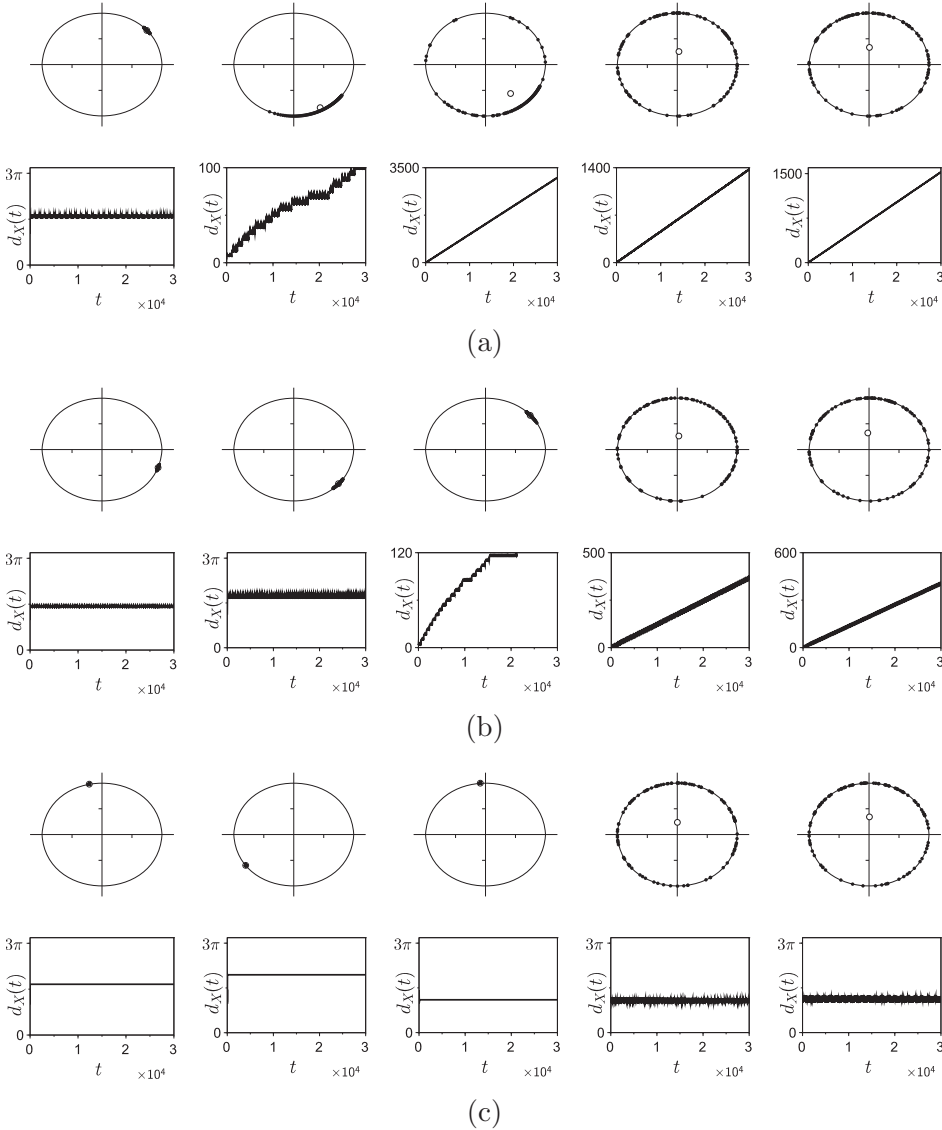


Figure 4. We choose $\kappa = 0.6$, random initial conditions in $[-\pi, \pi]$, and the other parameters as in Figure 1. We choose vertically from bottom to top, $\gamma_0 = 0, \gamma_1 = 0.011$ and $\gamma_2 = 0.0412$, horizontally from left to right, $\beta = 0, \beta = \frac{\pi}{2} - 0.5, \beta = \frac{\pi}{2} - 0.25, \beta = \frac{\pi}{2} + 0.25$ and $\beta = \frac{\pi}{2} + 0.5$. We plot the order parameter $d_X(t)$ for $t \in [0, 3 \times 10^4]$, the oscillators on the circle (bold dots) and the order parameter $r_X(\beta)$ (circle dot) at time $T = 3 \times 10^4$. (a) $\gamma_2 = 0.0412$, (b) $\gamma_1 = 0.011$, (c) $\gamma_0 = 0$.

$\gamma_1 > \gamma(\frac{\pi}{2} - 0.25)$ and $\gamma_2 > \gamma(\frac{\pi}{2} - 0.5)$. The plots suggest a non-decreasing function $t \mapsto d_X(t)$ going to infinity with flat parts.

4. Conclusion

We defined synchronization as a state where the dispersion of the oscillators is uniformly bounded in time. For every given coupling strength κ , corresponding to the non-death

state, we showed analytically the existence of synchronized oscillators when hypothesis H3 is satisfied and when the spectrum width γ is sufficiently smaller than some value depending on κ . In addition, under the same hypothesis, we showed the existence of N periodic synchronized oscillators with the same strong rotation number. We also showed numerically, for a parametrized simplified Winfree model at a given coupling strength, that hypothesis H3 seems to be necessary for the existence of synchronization.

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