

# Gottschalk-Hedlund theorem revisited

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Ergodic optimization and discrete weak KAM theory are two parallel theories with several results in common. For instance, the Mather set is the locus of orbits which minimize the ergodic averages of a given observable. In the favorable cases, the observable is cohomologous to its ergodic minimizing value on the Mather set, and the discrete weak KAM solution plays the role of the transfer function. One possibility of construction of such a coboundary is by using the non linear Lax-Oleinik operator. The other possibility is by using a discounted cohomological equation. It is known that the discounted discrete weak KAM solution converges to some selected weak KAM solution. We show that, in the ergodic optimization case for a coboundary observable over a minimal system, the discounted transfer function converges if and only if the observable is balanced.

## 1. Notations and main statements

We consider a *topological dynamical system*  $(\Omega, \sigma)$  where  $\Omega$  is a compact metric space and  $\sigma : \Omega \rightarrow \Omega$  is a continuous map. We denote by  $\mathcal{P}(\Omega, \sigma)$  the set of probability  $\sigma$ -invariant measures, and for every continuous function  $f \in C^0(\Omega)$ , by  $\bar{f}$ , the *ergodic minimizing value* of  $f$

$$(1) \quad \bar{f} := \min_{\mu \in \mathcal{P}(\Omega, \sigma)} \int f d\mu.$$

A *minimizing measure* is a probability invariant measure realizing the minimum in (1). We denote by  $\mathcal{P}_{min}(\Omega, \sigma, f)$  the set of all the minimizing measures.

Given a continuous function  $f : X \rightarrow \mathbb{R}$ , we want to solve the following *cohomological equation* where  $(M, u)$  are the two unknowns,

$$(CE) \quad \begin{cases} M \text{ is a Borel invariant set, and } \mu(M) = 1 \text{ for some } \mu \in \mathcal{P}(\Omega, \sigma), \\ u : \Omega \rightarrow \mathbb{R} \text{ is a non-negative Borel function,} \\ \forall \omega \in \Omega, f(\omega) - \bar{f} \geq u \circ \sigma(\omega) - u(\omega), \\ \forall \omega \in M, f(\omega) - \bar{f} = u \circ \sigma(\omega) - u(\omega). \end{cases}$$

A function of the form  $u \circ \sigma - u$  for some Borel function is called a *coboundary*, and  $u$  is called a *transfer function*.

Notice that  $\bar{f}$  could also be seen as an unknown of the cohomological equation. We shall see that, if  $(M, u, \bar{f})$  is a solution of (CE), then  $\bar{f}$  is necessarily unique. Notice also that such an invariant measure  $\mu$  giving a unit mass to  $M$  is necessarily a minimizing measure and satisfies  $\text{supp}(\mu) \subseteq \bar{M}$ . As we are interested in the “largest” set  $M$  for which such a transfer function  $u$  exists, it is hence natural to consider the following set, called *Mather set* and defined by

$$(2) \quad \mathcal{M}(f) := \bigcup \left\{ \text{supp}(\mu) : \mu \in \mathcal{P}_{min}(\Omega, \sigma, f) \right\}.$$

It is easy to see that the Mather set is closed, invariant, and is equal to the support of some minimizing measure. The terminology “Mather set”, following Mather [11] (where it is denoted by  $\text{supp}\mathcal{M}_c$  before proposition 3), comes from the weak KAM theory initiated by Mañé [10] (Theorem B, cohomological equation on each  $\text{supp}(\mu)$ ), then extended by Fathi [4] (theorem 1, sub-cohomological equation on the whole set  $\Omega$ ) and later thoroughly studied by Fathi in [5] (the final terminology in section 4.12).

For “hyperbolic systems”, if the dynamical system  $(\Omega, \sigma)$  is a Smale space [13] (for example a sub-shift of finite type) and the function  $f$  is Walters [15] (for example Hölder), then the cohomological equation (CE) admits a solution  $(M, u)$  where  $M = \mathcal{M}(f)$  and  $u$  is Walters, see Bousch [1]. In an opposite direction, if  $(\Omega, \sigma)$  is a topological dynamical system admitting invariant measures with different supports, for  $C^0$  generic function  $f$ , every minimizing measure  $\mu$  has full support,  $\text{supp}(\mu) = \Omega$ , and there is no solution  $(M, u)$  of (CE) with a continuous  $u$ , see Bousch [1]. There also exists  $C^\infty$  lacunary functions on the circle  $f : \mathbb{T} \rightarrow \mathbb{R}$  and Liouville numbers  $\alpha$  such that on the minimal and uniquely ergodic dynamical system  $(\mathbb{T}, R_\alpha)$ , ( $R_\alpha$  denotes the rotation by  $\alpha$ ), there is no solution  $(M, u)$  of (CE) with a Borel  $u$ , see Katok-Robinson [8] (remarks 1 after theorem 3.5) and Herman [7].

Our main goal in this paper is to use the weak KAM approach to provide a new way of solving the cohomological equations under weaker conditions. As an application, we slightly improve the classical Gottshalk-Hedlund theorem by proving that condition (4) below is equivalent to the existence of the continuous coboundary.

We also investigate a discounted weak KAM approach to obtain an approximate (or numerical) solution of (CE) in the Gottschalk-Hedlund setting. The numerical scheme may not converge and we identify those coboundaries that can be obtained as a limit. Unless the coboundary is balanced (that we are going to define) the scheme always oscillates.

Our first result is the following.

**Theorem 1.** *Let  $(\Omega, \sigma)$  be a topological dynamical system and  $f : \Omega \rightarrow \mathbb{R}$  be a continuous function. Assume*

$$(3) \quad \forall \omega \in \Omega, \quad u(\omega) := - \inf_{n \geq 1} \sum_{k=0}^{n-1} (f - \bar{f}) \circ \sigma^k(\omega) < +\infty.$$

Let  $u^+ := \max(u, 0)$ . Define a Borel set

$$M := \left\{ \omega \in \mathcal{M}(f) : \forall k \geq 0, (f - \bar{f} - u^+ \circ \sigma + u^+) \circ \sigma^k(\omega) = 0 \right\}.$$

Then  $(M, u^+)$  is a solution of the cohomological equation (CE):

- i)  $u^+$  is lower semi-continuous,
- ii)  $\forall \omega \in \Omega, f(\omega) - \bar{f} \geq u^+ \circ \sigma(\omega) - u^+(\omega),$
- iii)  $\forall \omega \in M, f(\omega) - \bar{f} = u^+ \circ \sigma(\omega) - u^+(\omega),$
- iv)  $\forall \mu \in \mathcal{P}_{min}(\Omega, \sigma, f), \mu(M) = 1,$
- v)  $M$  is an invariant residual subset of  $\mathcal{M}(f)$ .

A residual set  $M$  in a compact space  $\mathcal{M}$  is a set containing a countable intersection of open and dense subsets in  $\mathcal{M}$ . Unless  $u$  is bounded, condition (3) is not a necessary condition for solving (CE).

The following corollary is an extension of Gottschalk-Hedlund theorem [6] for every minimal subsets of the Mather set.

**Corollary 2.** *Let  $(\Omega, \sigma)$  be a topological dynamical system, and  $f : \Omega \rightarrow \mathbb{R}$  be a continuous function. Assume*

$$\exists C \geq 0, \forall \omega \in \Omega, \forall n \geq 1, \sum_{k=0}^{n-1} (f - \bar{f}) \circ \sigma^k(\omega) \geq -C.$$

Then

- i)  $\forall \omega \in \mathcal{M}(f), \forall n \geq 1, \sum_{k=0}^{n-1} (f - \bar{f}) \circ \sigma^k(\omega) \leq C,$
- ii) *if  $\mu$  is invariant and  $\text{supp}(\mu) \subset \mathcal{M}(f)$  then  $\mu$  is minimizing (the subordination principle),*
- iii) *there exists a lower semi-continuous function  $u : \Omega \rightarrow \mathbb{R}$  such that*
  - a)  $0 \leq u \leq C,$
  - b)  $\forall \omega \in \Omega, f(\omega) - \bar{f} \geq u \circ \sigma(\omega) - u(\omega),$
  - c) *for every minimal subset  $X \subseteq \mathcal{M}(f), u$  is continuous on  $X$  and*

$$\forall \omega \in X, \quad f(\omega) - \bar{f} = u \circ \sigma(\omega) - u(\omega).$$

If  $(\Omega, \sigma)$  is minimal, the Mather set must be equal to  $\Omega$  and we recover the classical Gottschalk-Hedlund theorem. The following statement is a slightly improved extension.

**Theorem 3 (Gottschalk-Hedlund [6]).** *If  $(\Omega, \sigma)$  is minimal,  $f \in C^0(\Omega),$  then the following two properties are equivalent:*

- $\forall \omega \in \Omega,$

$$(4) \quad \inf_{n \geq 1} \sum_{k=0}^{n-1} (f - \bar{f}) \circ \sigma^k(\omega) > -\infty,$$

- $\exists u \in C^0(\Omega), \forall \omega \in \Omega, f(\omega) - \bar{f} = u \circ \sigma(\omega) - u(\omega).$

Notice that if  $(\Omega, \sigma)$  is uniquely ergodic,  $\mathcal{M}(f) = \text{supp}(\mu)$  and  $\bar{f} = \int f d\mu$  for a unique ergodic measure  $\mu$ .

We now consider a weaker form of the cohomological equation that we call *discounted cohomological equation*:

$$(DCE) \quad \begin{cases} \forall \epsilon > 0, u_\epsilon : \Omega \rightarrow \mathbb{R} \text{ is a } C^0 \text{ function,} \\ \forall \epsilon > 0, \forall \omega \in \Omega, f(\omega) = (1 - \epsilon)u_\epsilon \circ \sigma(\omega) - u_\epsilon(\omega). \end{cases}$$

Iterating the previous equality,

$$\begin{aligned} u_\epsilon(\omega) &= (1 - \epsilon)u_\epsilon \circ \sigma(\omega) - f(\omega) \\ &= (1 - \epsilon)^2 u_\epsilon \circ \sigma^2(\omega) - f(\omega) - (1 - \epsilon)f \circ \sigma(\omega) = \dots \end{aligned}$$

we obtain that (DCE) has a unique solution, called *discounted transfer function*, which is given by the formula,

$$(5) \quad U_\epsilon[f](\omega) := - \sum_{k \geq 0} (1 - \epsilon)^k f \circ \sigma^k(\omega).$$

We question whether the discounted solution  $U_\epsilon[f]$  converges to some solution of (CE) as  $\epsilon \rightarrow 0$ . We give a complete answer when  $f$  is a coboundary over a minimal system.

**Definition 4.** Let  $(\Omega, \sigma)$  be a topological dynamical system, and  $f : \Omega \rightarrow \mathbb{R}$  be a continuous function.

- i) We say that  $f$  is a *regular coboundary* if there exists a continuous function  $u : \Omega \rightarrow \mathbb{R}$  such that  $f = u \circ \sigma - u$ .
- ii) We say that  $f$  is a *balanced coboundary* if there exists a continuous function  $u : \Omega \rightarrow \mathbb{R}$  such that  $f = u \circ \sigma - u$  and  $\int u \, d\mu$  is independent of  $\mu \in \mathcal{P}(\Omega, \sigma)$ .

A balanced coboundary is by definition regular. Our second result is the following.

**Theorem 5.** Let  $(\Omega, \sigma)$  be a topological dynamical system, and  $f : \Omega \rightarrow \mathbb{R}$  be a regular coboundary.

- i) If  $f$  is balanced, then there exists a unique  $u \in C^0(\Omega)$  such that  $f = u \circ \sigma - u$  and  $\int u \, d\mu = 0, \forall \mu \in \mathcal{P}(\Omega, \sigma)$ . In that case  $U_\epsilon[f] \rightarrow u$  uniformly in  $\Omega$ .
- ii) If  $(\Omega, \sigma)$  is minimal and  $f$  is regular but not balanced, then there exist  $u \in C^0(\Omega)$  satisfying  $f = u \circ \sigma - u$ , two ergodic invariant measures  $\mu_0, \mu_1$  satisfying  $\int u \, d\mu_0 \neq \int u \, d\mu_1$ , and a residual set  $M \subseteq \Omega$  such that, for every  $\omega \in M$ , there exists a decreasing sequence  $(\epsilon_n)_{n \geq 0}$  converging to 0 such that

$$U_{\epsilon_{2p}}[f](\omega) \rightarrow u - \int u \, d\mu_0, \quad U_{\epsilon_{2p+1}}[f](\omega) \rightarrow u - \int u \, d\mu_1.$$

The notion of discounted cohomological equation is reminiscent of the notion of discounted weak KAM solution discussed in [3] in the continuous setting and in [2, 14] in the discrete setting. Contrary to the phenomenon observed in theorem 5, the discounted weak KAM solution converges to some selected weak KAM solution, called balanced weak KAM solution, see [14] proposition 18 in the discrete setting.

## 2. Proofs for the cohomological equation

The fact that  $\bar{f}$  is the unique solution of the cohomological equation (CE) follows readily from the following standard lemma in ergodic theory. We were not able to find a reference of that lemma and as suggested by the referee we give a short proof in the Appendix.

**Lemma 6.** *Let  $(\Omega, \sigma, \mu)$  be a measurable dynamical system ( $\Omega$  is a Polish space,  $\sigma : \Omega \rightarrow \Omega$  is a Borel map, and  $\mu$  is a  $\sigma$ -invariant probability). Let  $u : \Omega \rightarrow \mathbb{R}$  be a Borel function. If  $(u \circ \sigma - u)^+ \in L^1(\mu)$  then  $u \circ \sigma - u \in L^1(\mu)$  and  $\int (u \circ \sigma - u) d\mu = 0$ .*

**Corollary 7.** *If  $(M, u, \bar{f})$  is a solution of (CE) then  $\bar{f}$  must be equal to the formula given by equation (1).*

*Proof.* As  $f - \bar{f} \geq u \circ \sigma - u$  on  $\Omega$ ,  $(u \circ \sigma - u)^+ \in L^1(\mu)$  for every  $\mu \in \mathcal{P}(\Omega, \sigma)$ . Lemma 6 implies that

$$\forall \mu \in \mathcal{P}(\Omega, \sigma), \quad \int (f - \bar{f}) d\mu \geq 0.$$

We obtain on the one hand

$$\bar{f} \leq \inf_{\mu \in \mathcal{P}(\Omega, \sigma)} \int f d\mu.$$

On the other hand  $f - \bar{f} = u \circ \sigma - u$  on  $M$  and  $\bar{\mu}(M) = 1$  for some  $\bar{\mu} \in \mathcal{P}(\Omega, \sigma)$ . Applying again lemma 6 to  $(M, \sigma, \bar{\mu})$ , we obtain  $\int (f - \bar{f}) d\bar{\mu} = 0$ , and

$$\bar{f} = \int f d\bar{\mu} \geq \inf_{\mu \in \mathcal{P}(\Omega, \sigma)} \int f d\mu.$$

□

*Proof of theorem 1.* Item (i) is a consequence of the fact that the supremum of continuous functions is lower semi-continuous.

Item (ii) is an immediate consequence of the following identity:

$$(6) \quad \forall \omega \in \Omega, \quad f(\omega) - \bar{f} = u^+ \circ \sigma(\omega) - u(\omega).$$

Indeed let  $\omega \in \Omega$ . One could analyze the infimum in the definition of  $u$  in (3) for both cases: the infimum is reached at  $n = 1$  or not, that is, we have either

$$\begin{aligned} (f - \bar{f})(\omega) &= -u(\omega) < \sum_{k=0}^{n-1} (f - \bar{f}) \circ \sigma^k(\omega), \quad \forall n \geq 2, \\ &\leq (f - \bar{f})(\omega) - u \circ \sigma(\omega), \\ u \circ \sigma(\omega) &\leq 0, \quad (f - \bar{f})(\omega) = u^+ \circ \sigma(\omega) - u(\omega), \end{aligned}$$

or

$$\begin{aligned} (f - \bar{f})(\omega) &\geq -u(\omega) = \inf_{n \geq 2} \sum_{k=0}^{n-1} (f - \bar{f}) \circ \sigma^k(\omega), \\ &= (f - \bar{f})(\omega) - u \circ \sigma(\omega), \\ u \circ \sigma(\omega) &\geq 0, \quad (f - \bar{f})(\omega) = u^+ \circ \sigma(\omega) - u(\omega). \end{aligned}$$

We have proved in particular,

$$\forall \omega \in \Omega, \quad (f - \bar{f})(\omega) \geq u^+ \circ \sigma(\omega) - u^+(\omega).$$

Lemma 6 implies that  $u^+ \circ \sigma - u^+ \in L^1(\mu)$  and  $\int (u^+ \circ \sigma - u^+) d\mu = 0$ ,  $\forall \mu \in \mathcal{P}(\Omega, \sigma)$ .

The proof of item (iv) will follow from the fact that  $u \geq 0$ ,  $\mu(d\omega)$  a.e. for every  $\mu \in \mathcal{P}_{min}(\Omega, \sigma, f)$ . Let  $u^- := (-u)^+$  and  $\mu$  be a minimizing measure. We have

$$\begin{aligned} 0 &= \int (f - \bar{f}) d\mu = \int (u^+ \circ \sigma - u) d\mu = \int (u^+ \circ \sigma - u^+) + u^- d\mu, \\ \int u^- d\mu &= 0 \quad \Rightarrow \quad u \circ \sigma^k(\omega) \geq 0, \quad \mu(d\omega), \quad \forall k \geq 0, \quad \text{a.e.} \end{aligned}$$

which implies  $\mu(M) = 1$ .

The proof of item (v) will follow from the following two facts.

First fact: Let  $R$  be the set of points of continuity of  $u$  belonging to the Mather set. As  $u$  is lower semi-continuous,  $R$  is a residual set of  $\mathcal{M}(f)$ . See

[9] for a detailed proof: a lower semi-continuous function is a Baire function of first category (Theorem 3.3.4), and the set of points of discontinuity of a Baire function of first category is a  $F_\sigma$  of empty interior (Theorem 3.5.3).

Second fact:  $u \geq 0$  on  $R$ . Indeed let  $\omega \in R$ . Then  $\omega \in \text{supp}(\mu)$  for some minimizing measure  $\mu$ . By contradiction, if  $u(\omega) < 0$ , we would have  $u < 0$  on a neighborhood  $U$  containing  $\omega$ . Since  $U \cap \text{supp}(\mu) \neq \emptyset$ , we would have  $\mu(U) > 0$ , contradicting  $u \geq 0$ ,  $\mu$  a.e. Therefore,  $u \geq 0$  for any  $\omega \in R$ , which implies (6) holds with  $u^+(\omega)$  instead of  $u(\omega)$ .

Hence,  $\bigcap_{k \geq 0} \sigma^{-k}(R)$  is again residual, invariant, contained in  $M$ , which completes the proof of (v). □

*Proof of corollary 2.* Theorem 1 implies the existence of a lower semi-continuous function  $u : \Omega \rightarrow \mathbb{R}$  and a residual subset  $M \subseteq \mathcal{M}(f)$  such that

- $\forall \omega \in \Omega, f(\omega) - \bar{f} \geq u \circ \sigma(\omega) - u(\omega),$
- $\forall \omega \in M, f(\omega) - \bar{f} = u \circ \sigma(\omega) - u(\omega),$
- $\forall \omega \in \Omega, 0 \leq u(\omega) \leq C.$

The proof of item (i) follows from,

$$\begin{aligned} \forall \omega \in M, \forall n \geq 1, \sum_{k=0}^{n-1} (f - \bar{f}) \circ \sigma^k(\omega) &= u^+ \circ \sigma^n(\omega) - u^+(\omega) \\ &\leq u^+ \circ \sigma^n(\omega) \leq C, \end{aligned}$$

and from the fact that  $M$  is residual and in particular dense in the Mather set.

The proof of item (ii) follows from item (i). If  $\text{supp}(\mu) \subseteq \mathcal{M}(f)$ , then

$$\forall n \geq 0, n \int f d\mu \leq n\bar{f} + C \Rightarrow \int f d\mu = \bar{f}.$$

The proof of item (iii) follows from theorem 1 applied to  $-f$  on any  $(X, \sigma)$ . Indeed, thanks to item (i), we have

$$\sup_{\mu \in \mathcal{P}(X, \sigma)} \int f d\mu = \bar{f} \quad \text{and} \quad \forall \omega \in X, \sup_{n \geq 1} \sum_{k=0}^{n-1} (f - \bar{f}) \circ \sigma^k(\omega) < +\infty.$$

There exists a non-positive upper semi-continuous function  $v : X \rightarrow \mathbb{R}$  such that

$$\forall \omega \in X, f(\omega) - \bar{f} \leq v \circ \sigma(\omega) - v(\omega).$$



Then

$$\forall \omega \in X, \quad u \circ \sigma(\omega) - u(\omega) \leq f\omega - \bar{f} \leq v \circ \sigma(\omega) - v(\omega).$$

Since  $u - v$  is lower semi-continuous on  $X$ ,  $u - v$  attains its infimum on  $X$ . Define

$$D := \min \{(u - v)(\omega) : \omega \in X\}, \quad \tilde{X} := \{\omega \in X : (u - v)(\omega) \leq D\}.$$

Since  $(u - v) \circ \sigma \leq (u - v)$ ,  $\tilde{X}$  is compact,  $\sigma$ -invariant, therefore by minimality is equal to  $X$ :  $u - v = D$  on  $X$ ,  $u$  and  $v$  restricted to the  $X$  are continuous and  $f - \bar{f} = u \circ \sigma - u = v \circ \sigma - v$  on  $X$ .  $\square$

We will need the following lemma for the proof of theorem 3. See proposition A.7 in Morris [12] for a proof.

**Lemma 8.** *Let  $(\Omega, \sigma)$  be a topological dynamical system and  $f \in C^0(\Omega)$ . Then*

$$\exists \omega_* \in \Omega, \forall n \geq 0, \frac{1}{n} \sum_{k=0}^{n-1} f \circ \sigma^k(\omega_*) \leq \min_{\mu \in \mathcal{P}(\Omega, \sigma)} \int f d\mu.$$

*Proof of theorem 3.* It follows from lemma 8 and by assumption of the theorem, there exists  $\omega_* \in \Omega$  and a constant  $C \geq 0$  such that

$$\forall n \geq 0, \quad -C \leq \sum_{k=0}^{n-1} (f - \bar{f}) \circ \sigma^k(\omega_*) \leq 0.$$

Then

$$\forall m, n \geq 0, \quad \sum_{k=m}^{m+n-1} (f - \bar{f}) \circ \sigma^k(\omega_*) \geq -C.$$

By minimality of  $(\Omega, \sigma)$ , the orbit of  $(\sigma^k(\omega_*))_{k \geq 0}$  is dense,

$$\forall \omega \in \Omega, \forall n \geq 1, \sum_{k=0}^{n-1} (f - \bar{f}) \circ \sigma^k(\omega) \geq -C.$$

We conclude the proof by using corollary 2.  $\square$

### 3. Proofs for the discounted cohomological equation

Notice that the unique solution of (DCE), equation (5), can be written as

$$U_\epsilon[f](\omega) = -\frac{1}{\epsilon} \int f d\mu_{\epsilon,\omega}, \quad \forall \omega \in \Omega,$$

where  $\mu_{\epsilon,\omega} := \sum_{k \geq 0} \epsilon(1 - \epsilon)^k \delta_{\sigma^k(\omega)}$  is a probability measure not necessarily invariant.

The proof of item (i) of theorem 5 follows from the following lemma.

**Lemma 9.** *Let  $f \in C^0(\Omega)$ .*

- i) If  $\forall \mu \in \mathcal{P}(\Omega, \sigma)$ ,  $\int f d\mu = 0$ , then  $\int f d\mu_{\epsilon,\omega} \rightarrow 0$  uniformly in  $\omega \in \Omega$ .*
- ii) If  $f = u \circ \sigma - u$ , then  $U_\epsilon[f](\omega) = u(\omega) - \int u \circ \sigma d\mu_{\epsilon,\omega}$ ,  $\forall \omega \in \Omega$ .*

*Proof of item (i).* We first prove that

$$\limsup_{\epsilon \rightarrow 0} \sup_{\omega \in \Omega} \int f d\mu_{\epsilon,\omega} = 0.$$

Let  $(\epsilon_n)_{n \geq 0}$  be a sequence tending to 0 and realizing the above lim sup. Let  $(\omega_n)_{n \geq 0}$  be a sequence of points of  $\Omega$  realizing the supremum of  $\int f d\mu_{\epsilon_n,\omega}$  for each  $\epsilon_n$ . Choose a sub-sequence of  $(\epsilon_n)_{n \geq 0}$ , that we denote in the same way, such that  $(\mu_{\epsilon_n,\omega_n})_{n \geq 0}$  converges to some probability measure  $\mu$ . Notice that

$$\forall n \geq 0, \forall g \in C^0(\Omega), \quad (1 - \epsilon_n)\mu_{\epsilon_n,\omega_n}(g \circ \sigma) = \mu_{\epsilon_n,\omega_n}(g) - \epsilon_n g(\omega_n).$$

Taking  $n \rightarrow +\infty$ , we obtain  $\mu \in \mathcal{P}(\Omega, \sigma)$  and

$$\limsup_{\epsilon \rightarrow 0} \sup_{\omega \in \Omega} \int f d\mu_{\epsilon,\omega} = \lim_{\epsilon \rightarrow 0} \int f d\mu_{\epsilon_n,\omega_n} \rightarrow \int f d\mu = 0.$$

Similarly we show  $\liminf_{\epsilon \rightarrow 0} \inf_{\omega \in \Omega} \int f d\mu_{\epsilon,\omega} = 0$ . Item (i) is proved. □

*Proof of item (ii).* We observe

$$\begin{aligned} u &= u \circ \sigma - f = (1 - \epsilon)u \circ \sigma - f + \epsilon u \circ \sigma \\ &= \sum_{k \geq 0} (1 - \epsilon)^k (-f + \epsilon u \circ \sigma) \circ \sigma^k \\ u(\omega) &= U_\epsilon[f](\omega) + \int u \circ \sigma d\mu_{\epsilon,\omega}. \end{aligned}$$

□

*Proof of item (i) of theorem 5.* If  $f$  is a balanced coboundary,  $f = u \circ \sigma - u$  for some  $u$  satisfying  $\int u \, d\mu = 0$ ,  $\forall \mu \in \mathcal{M}(\Omega, \sigma)$ . Then, thanks to lemma 9,

$$U_\epsilon[f](\omega) = u(\omega) - \int u \circ \sigma \, d\mu_{\epsilon, \omega} \rightarrow u(\omega), \quad \text{uniformly in } \omega \in \Omega.$$

In particular, such a transfer function  $u$  is unique. □

The proof of the second item of theorem 5 will be given after the two following lemmas.

**Lemma 10.** *Let  $(\Omega, \sigma)$  be a minimal dynamical system, and  $\mu_0, \mu_1$  be two ergodic measures. Then there exists a residual subset  $M \subseteq \Omega$  such that for every  $\omega \in M$  there exists a sequence of integers  $(N_p)_{p \geq 1}$  such that*

$$\begin{aligned} &\forall p \geq 1, \quad N_{2p} < \ln(N_{2p+1}), \quad N_{2p-1} < \ln(N_{2p}), \\ &\forall p \geq 1, \quad \forall \ln(N_p) < n < N_p, \quad \left| \frac{1}{n} \sum_{k=0}^{n-1} u \circ \sigma^k(\omega) - \mu_{[p]}(u) \right| < \frac{1}{p}, \end{aligned}$$

where  $[p] = p \pmod 2$ .

*Proof.* Let  $i = 0, 1$ . As  $\mu_i$  is ergodic, thanks to Birkhoff's ergodic theorem, for every  $p \geq 1, q \geq 1$ ,

$$U_{p,q}^{(i)} := \left\{ \omega \in \text{supp}(\mu_i) : \exists N \geq q, \forall \ln(N) < n < N, \left| \frac{1}{n} \sum_{k=0}^{n-1} u \circ \sigma^{k+1}(\omega) - \mu_i(u) \right| < \frac{1}{p} \right\}$$

is an open and dense set of  $\text{supp}(\mu_i)$ . As  $(\Omega, \sigma)$  is minimal,  $\text{supp}(\mu_i) = \Omega$ . The set  $M_i := \bigcap_{p,q \geq 1} U_{p,q}^{(i)}$  is thus a residual set of  $\Omega$ . Define  $M := M_0 \cap M_1$ . Then  $M$  is a residual set. If  $\omega \in M$ , we construct by induction a sequence

of integers  $(N_p)_{p \geq 1}$  satisfying the properties of the above lemma:

$$p = 1, q = 1, \exists N_1, \forall \ln(N_1) < n < N_1, \left| \frac{1}{n} \sum_{k=0}^{n-1} u \circ \sigma^{k+1}(\omega) - \mu_1(u) \right| < 1,$$

$$p = 2, q = \ln(N_1), \exists N_2 > q, \forall \ln(N_2) < n < N_2,$$

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} u \circ \sigma^{k+1}(\omega) - \mu_0(u) \right| < \frac{1}{2},$$

$$p = 3, q = \ln(N_2), \exists N_3 > q, \forall \ln(N_3) < n < N_3,$$

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} u \circ \sigma^{k+1}(\omega) - \mu_1(u) \right| < \frac{1}{3},$$

and so on. □

Denote by  $A_{n,\omega} := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\sigma^k(\omega)}$  the empirical measure.

**Lemma 11.** *For every  $\epsilon > 0$ ,  $\omega \in \Omega$ ,  $n \geq 2$*

$$\begin{aligned} \mu_{\epsilon,\omega} &= \sum_{k=\lfloor \ln(n) \rfloor}^{n-2} (k+1)\epsilon^2(1-\epsilon)^k A_{k+1,\omega} + R_{n,\epsilon,\omega}, \\ R_{n,\epsilon,\omega} &= \sum_{k=0}^{\lfloor \ln(n) \rfloor - 1} (k+1)\epsilon^2(1-\epsilon)^k A_{k+1,\omega} \\ &\quad + n\epsilon(1-\epsilon)^{n-1} A_{n,\omega} + (1-\epsilon)^n \mu_{\epsilon,\sigma^n(\omega)}, \\ \sup_{\omega} R_{n,\epsilon,\omega}(\mathbb{1}) &\leq (\epsilon \ln(n))^2 + (1 + \epsilon n e) e^{-\epsilon n}. \end{aligned}$$

*Proof.* We have

$$\begin{aligned} \mu_{\epsilon,\omega} &= \sum_{k=0}^{n-1} \epsilon(1-\epsilon)^k \delta_{\sigma^k(\omega)} + (1-\epsilon)^n \mu_{\epsilon,\sigma^n(\omega)}, \\ \sum_{k=0}^{n-1} \epsilon(1-\epsilon)^k \delta_{\sigma^k(\omega)} &= \sum_{k=0}^{n-1} \epsilon(1-\epsilon)^k ((k+1)A_{k+1,\omega} - kA_{k,\omega}) \\ &= \sum_{k=0}^{n-2} (k+1)\epsilon^2(1-\epsilon)^k A_{k+1,\omega} + n\epsilon(1-\epsilon)^{n-1} A_{n,\omega}. \end{aligned}$$

□

*Proof of item (ii) of theorem 5.* Let  $(\Omega, \sigma)$  be a minimal dynamical system and  $f$  be a non-balanced coboundary:  $f = u \circ \sigma - u$ ,  $\int u d\mu_0 \neq \int u d\mu_1$  for some ergodic measures  $\mu_0, \mu_1$ . Let  $M$  be the residual set given by lemma 10. Let  $\omega \in M$  and  $(N_p)_{p \geq 1}$  be the sequence of integers given by lemma 10. Let  $\epsilon_p := \frac{\ln(N_p)}{N_p}$ . Define

$$\alpha_p := \sum_{k=\lfloor \ln(N_p) \rfloor}^{N_p-2} (k+1)\epsilon_p^2(1-\epsilon_p)^k.$$

Then, using lemma 11,

$$\begin{aligned} 0 \leq 1 - \alpha_p &= \sup_{\omega} R_{N_p, \epsilon_p, \omega} \\ &\leq (\epsilon_p \ln(N_p))^2 + (1 + \epsilon_p N_p e) e^{-\epsilon_p N_p} \rightarrow 0, \end{aligned}$$

$$\begin{aligned} \mu_{\epsilon_p, \omega}(u \circ \sigma) - \mu_{[p]}(u) &= \sum_{k=\lfloor \ln(N_p) \rfloor}^{N_p-2} (k+1)\epsilon^2(1-\epsilon)^k (A_{k+1, \omega}(u \circ \sigma) - \mu_{[p]}(u)) \\ &\quad + R_{N_p, \epsilon_p, \omega}(u \circ \sigma) - (1 - \alpha_p)\mu_{[p]}(u), \end{aligned}$$

$$|\mu_{\epsilon_p, \omega}(u \circ \sigma) - \mu_{[p]}(u)| \leq \frac{\alpha_p}{p} + 2(1 - \alpha_p)\|u\|_{\infty} \rightarrow 0.$$

We conclude the proof of the theorem using item (ii) of lemma 9. □

### Appendix

*Proof of lemma 6.* We define for every integer  $N \geq 0$  the truncated function

$$u_N = -N\mathbf{1}_{(u \leq -N)} + u\mathbf{1}_{(-N < u < N)} + N\mathbf{1}_{(u \geq N)}.$$

As  $u_N \in L^1(\mu)$ , by invariance of  $\mu$  with respect to  $\sigma$ , we obtain

$$(7) \quad \int (u_N \circ \sigma - u_N)^- d\mu = \int (u_N \circ \sigma - u_N)^+ d\mu.$$

By Fatou's theorem,

$$\int (u \circ \sigma - u)^- d\mu \leq \liminf_{N \rightarrow +\infty} \int (u_N \circ \sigma - u_N)^+ d\mu.$$

The heart of the proof is to show the following a priori estimate (everywhere)

$$(u_N \circ \sigma - u_N)^+ \leq (u \circ \sigma - u)^+.$$

Notice first that it is enough to assume  $u_N \circ \sigma - u_N > 0$ . In particular, this implies  $u < N$  (otherwise we would have  $u_N \circ \sigma > N$  which is not allowed), and for the same reasons  $u \circ \sigma > -N$ . We thus obtain on the set  $\{u_N \circ \sigma - u_N > 0\}$ ,

$$u_N \geq u, \quad u \circ \sigma \geq u_N \circ \sigma, \quad 0 < u_N \circ \sigma - u_N \leq u \circ \sigma - u, \\ (u_N \circ \sigma - u_N)^+ \leq (u \circ \sigma - u)^+.$$

We just have proved  $(u \circ \sigma - u)^- \in L^1(\mu)$ , and  $u \circ \sigma - u \in L^1(\mu)$ . By applying the same a priori estimate to  $v = -u$  and by noticing  $v_N = -u_N$ , we obtain

$$(u_N \circ \sigma - u_N)^- \leq (u \circ \sigma - u)^-.$$

By the dominated convergence theorem applied to (7), we get

$$\int (u \circ \sigma - u) d\mu = 0. \quad \square$$

### Acknowledgements

We would like to thank the anonymous referee for his/her careful reading and for several suggestions that have improved the paper. X. Su is supported by the National Natural Science Foundation of China (Grant No. 11971060, 11871242).

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RECEIVED JANUARY 22, 2019

ACCEPTED JULY, 31, 2019