endowed with the topology induced by the product topology on $\mathrm{sp}(X) \times \mathrm{sp}(Y)$. We are going to study some properties concerning the relation between $\mathrm{sp}(X \times_S Y)$ and $\mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$.

(a) Show that we have a canonical continuous map $f : \mathrm{sp}(X \times_S Y) \to \mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$.

(b) Show that f is surjective.

(c) Let us consider the example $X = Y = \operatorname{Spec} \mathbb{C}$ and $S = \operatorname{Spec} \mathbb{R}$. Show that $X \times_S Y \simeq \operatorname{Spec}(\mathbb{C} \oplus \mathbb{C})$ and that f is not injective.

(d) Show that in the case of Exercise 1.9, with $X = \operatorname{Spec} k(u)$, $Y = \operatorname{Spec} k(v)$, and $S = \operatorname{Spec} k$, the map f has infinite fibers.

(e) Let $S = \operatorname{Spec} k$ be the spectrum of an arbitrary field. By studying the example $X = Y = \mathbb{A}^1_k$, show that the image of an open subset under f is not necessarily an open subset.

1.11. Let k be a field and $z \in \mathbb{P}^n_k(k)$. We choose a homogeneous coordinate system such that $z = (1, 0, \dots, 0)$.

(a) Show that there exists a morphism $p : \mathbb{P}^n_k \setminus \{z\} \to \mathbb{P}^{n-1}_k$ such that over $\overline{k}$, where $\overline{k}$ is the algebraic closure of k, we have

$$p_{\overline{k}}(a_0, a_1, \dots, a_n) = (a_1, \dots, a_n)$$

for every point of $\mathbb{P}^n_{\overline{k}}(\overline{k})$. Such a morphism is called a *projection with center z*.

(b) Let X be a closed subset of $\mathbb{P}^n_k$ not containing z. Show that X cannot contain $p^{-1}(y)$ for any $y \in \mathbb{P}^{n-1}_k$.

3.2 Applications to algebraic varieties

3.2.1 Morphisms of finite type

Most interesting morphisms in algebraic geometry are of finite type and localizations of morphisms of finite type.

Definition 2.1. A morphism $f : X \to Y$ is said to be of *finite type* if f is quasi-compact (Exercise 2.3.17), and if for every affine open subset V of Y, and for every affine open subset U of $f^{-1}(V)$, the canonical homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ makes $\mathcal{O}_X(U)$ into a finitely generated $\mathcal{O}_Y(V)$-algebra. A Y-scheme is said to be of finite type if the structural morphism is of finite type.

Proposition 2.2. *Let $f : X \to Y$ be a morphism of schemes. Let us suppose that there exists a covering $\{V_i\}_i$ of Y by affine open subsets such that for every i, $f^{-1}(V_i)$ is a finite union of affine open subsets U_{ij}, and that $\mathcal{O}_X(U_{ij})$ is a finitely generated algebra over $\mathcal{O}_Y(V_i)$ for every j. Then f is of finite type.*

Proof We first notice that given two affine open subsets U, W in any scheme, we can cover $U \cap W$ by open subsets which are principal in both U and W. Indeed,

$U \cap W$ is covered by principal open subsets W_k of W. Each W_k is covered by principal open subsets W_{kq} of U. The W_{kq}'s cover $U \cap W$ when both k, q vary. Each W_{kq} is principal in U, hence principal in any affine open subset of U. In particular W_{kq} is principal in W_q which is principal in W, it is thus principal in W.

We start to prove the proposition when $Y = \operatorname{Spec} A$ is affine and X is a finite union of affine open subsets U_j with $\mathcal{O}_X(U_j)$ finitely generated over A. Let U be an affine open subset of X. We have to show that $\mathcal{O}_X(U)$ is finitely generated over A. By the above remark, we can cover $U \cap U_j$ by open subsets U_{jq} which are principal in U and in U_j. Each U_{jq} is equal to $D_{U_j}(f_{jq})$ for some $f_{jq} \in \mathcal{O}_X(U_j)$, and $\mathcal{O}_X(U_{jq}) = \mathcal{O}_X(U_j)_{f_{jq}} \simeq \mathcal{O}_X(U_j)[T]/(f_{jq}T - 1)$ is finitely generated over $\mathcal{O}_X(U_j)$, hence finitely generated over A. When j, q vary, the U_{jq}'s cover U. As U is quas-compact, it is covered by a finite number of them. To summarize, U is a finite union of prinicpal open subsets $D_U(f_\alpha)$, $f_\alpha \in B = \mathcal{O}_X(U)$ such that B_{f_α} is finitely generated over A. It follows that there exists a finitely generated sub-A-algebra C of B, containing the b_α's and such that $C_{b_\alpha} = B_{b_\alpha}$ for every α. As the $D_U(f_\alpha)$'s cover U, we have an identity $1 = \sum_\alpha b_\alpha c_\alpha$ with $c_\alpha \in B$. It is easy to conclude (as in the proof of Proposition 2.3.1(a)) that as an A-algebra, B is generated by C and the c_α's. Consequently, B is indeed finitely generated over A.

Now we come to the general case. Let V be an affine open subset of Y. As above, we can cover V by a finite number of open subsets V_{ik} which are principal in V and in V_i. We have $f^{-1}(V_{ik}) = \cup_j U_{ijk}$, where $U_{ijk} := U_{ij} \cap f^{-1}(V_{ik})$ is a principal open subset of U_{ij}, hence affine. So $f^{-1}(V)$ is quasi-compact. As $\mathcal{O}_X(U_{ij})$ is finitely generated over $\mathcal{O}_Y(V_i)$, $\mathcal{O}_X(U_{ijk})$ is finitely generated over $\mathcal{O}_Y(V_{ik})$ because V_{ik} is prinicpal in V_i. As V_{ik} is principal in V, $\mathcal{O}_Y(V_{ik})$ is finitely generated over $\mathcal{O}_Y(V)$. Hence each $\mathcal{O}_X(U_{ijk})$ is finitely generated over $\mathcal{O}_Y(V)$. By the previous case, this implies that for every affine open subset U of $f^{-1}(V)$, $\mathcal{O}_X(U)$ is finitely generated over $\mathcal{O}_Y(V)$. □

Example 2.3. Let k be a field. Then the schemes of finite type over $\operatorname{Spec} k$ are exactly the algebraic varieties over k (Definition 2.3.47).

Proposition 2.4. *We have the following properties:*

(a) *Closed immersions are of finite type.*

(b) *The composition of two morphisms $f : X \to Y$, $g : Y \to Z$ of finite type is of finite type.*

(c) *Morphisms of finite type are stable under base change.*

(d) *If $X \to Z$ and $Y \to Z$ are of finite type, then so is $X \times_Z Y \to Z$.*

(e) *If the composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ is of finite type, and if f is quasi-compact, then f is of finite type.*

Proof (a) results from Proposition 2.3.20; (b) let $h = g \circ f$. Let V be an affine open subset of Z. Then $g^{-1}(V)$ is a finite union of affine open subsets U_i of Y, and each $f^{-1}(U_i)$ is a finite union of affine open subsets W_{ij} of X. It is clear that the composition $W_{ij} \to U_i \to V$ is of finite type. Since $h^{-1}(V) = \cup_{i,j} W_{ij}$, it