A compact MIP formulation for single machine scheduling to minimize a piecewise linear objective function

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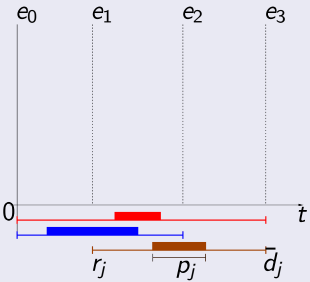
The problem PWL (I)

Data

1 machine, m intervals
$I_u = [e_{u-1}, e_u]$, $u \in M$;
for each job $j \in N = \{1, \dots, n\}$

- r_j - release date,
- p_j - processing time,
- $\bar{d}_j$ - deadline,

such that $\{r_j, \bar{d}_j\}_{j \in M} \subseteq \{e_u\}_{u \in M}$.



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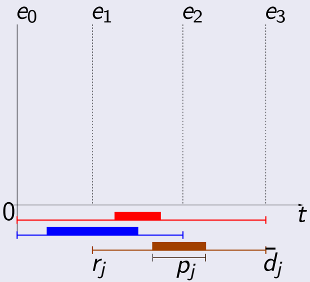


Constraints

- the machine can process only one job at a time,
- preemtion is not allowed,

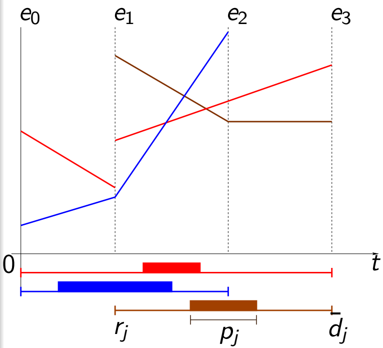
The problem PWL (II)

Objective

$\forall j \in N$, the cost function $F_j(C_j)$ is linear in each interval :

$$\text{if } e_{u-1} < C_j \leq e_u \text{ then}$$

$$F_j(C_j) = f_j^u + w_j^u \cdot (C_j - e_{u-1}).$$



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We minimize the total cost :

$$\sum_{j \in N} F_j(C_j)$$

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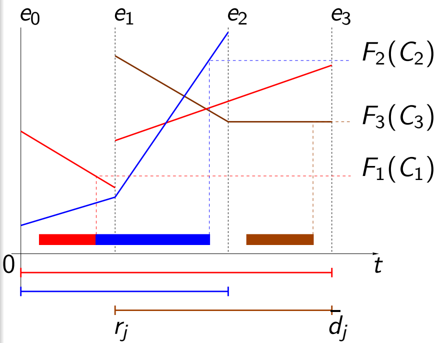

Generality of the problem PWL

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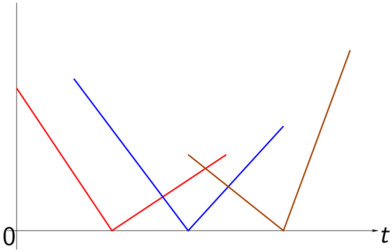
Earliness-tardiness scheduling



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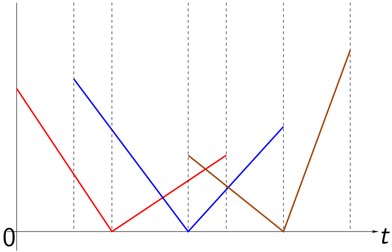

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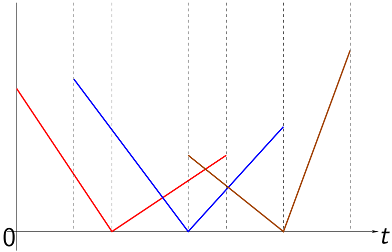
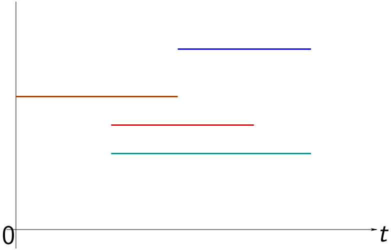


Maximizing weighted number of on-time jobs



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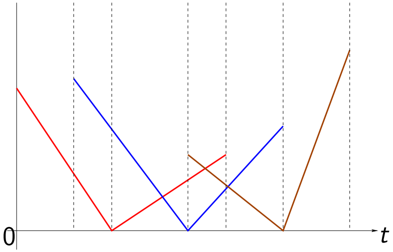

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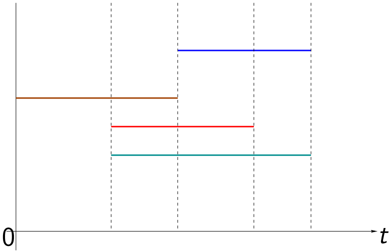

Time-indexed formulation

Data is integer $\rightarrow$ Time-indexed formulation [Sousa, Wolsey, 92]

$X_{jt} \in \{0, 1\}$, $X_{jt} = 1$ iff job j is started at time moment t

$$\min \quad \sum_{t=0}^{T} F_j(t + p_j) X_{jt}$$

$$s.t. \quad \sum_{t=r_j}^{\overline{d}_j - p_j} X_{jt} = 1, \quad j \in N,$$

$$\sum_{j \in N} \sum_{s=t-p_j+1}^{t} X_{js} \leq 1, \quad t \in [0, T),$$

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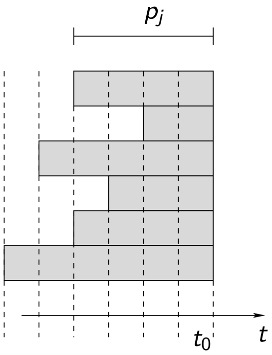

Interval decomposition

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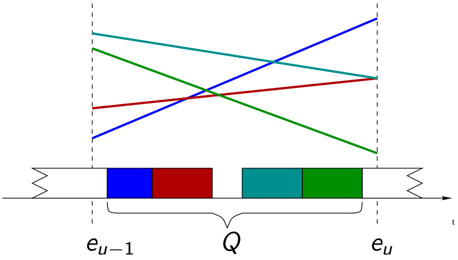
Property. If all jobs in a set Q are started and completed in the same interval $[e_{u-1}, e_u]$, it is optimal to process them according to the Smith rule :

$$i \to j \text{ if } \frac{w_i^u}{p_i} > \frac{w_j^u}{p_j}.$$

Idle time between jobs with positive and negative weights w_j^u



Interval-indexed formulation

Variables

$X_j^u, Y_j^u \in \{0,1\}$ — whether job j is started (completed) before e_u
W_u — length of the idle time in I_u
F_j^u — difference between the completion time and the border of I_u

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Variables constraints

$$\begin{aligned}
\forall u \in M,\ \forall j \in N, \quad & Y_j^{u-1} \le Y_j^u,\ X_j^{u-1} \le X_j^u,\ Y_j^u \le X_j^u \\
\forall u \in M,\ \forall j \in NS_u, \quad & X_j^{u-1} \le Y_j^u \\
\forall u \in M,\ \forall j \in NB_u, \quad & Y_j^u \le X_j^{u-1} \\
\forall j \in N,\ r_j = e_u, \quad & X_j^u = 0 \\
\forall j \in N,\ \bar{d}_j = e_u, \quad & Y_j^u = 1
\end{aligned}$$

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Knapsack constraints

$$\forall u \in M, \qquad \sum_{j \in N} p_j Y_j^u + \sum_{v=1}^{u} W_v \leq e_u$$

$$\forall u \in M, \qquad \sum_{j \in N} p_j X_j^u + \sum_{v=1}^{u} W_v \geq e_u$$

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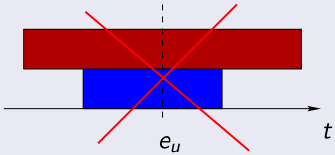
Additional constraints I

$$\forall u \in M, \qquad \sum_{i \in N} (X_i^u - Y_i^u) \leq 1$$



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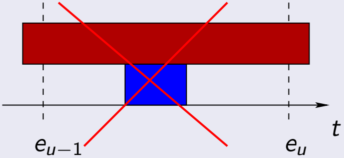
NS_u, NB_u — set of "small" jobs and "big" jobs for I_u

Additional constraints II

$$\forall u \in M,\ \forall j \in NS_u, \qquad \sum_{i \in NB_u} (X_i^{u-1} - Y_i^u) + Y_j^u - X_j^{u-1} \leq 1$$



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Theorem

Vector (X, Y, W) *satisfies the constraints presented*
$\Leftrightarrow$ *the corresponding schedule is feasible*

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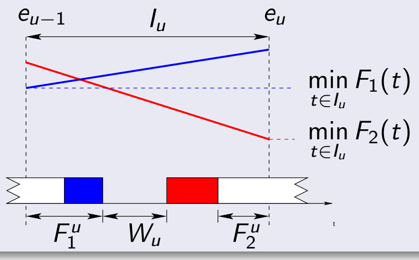
Objective

$$\min \sum_{j\in N}\sum_{u\in M} \mid w_j^u \mid F_j^u + \sum_{j\in N}\sum_{u\in M} \min_{t\in I_u}\{F_j(t)\}\cdot(Y_j^u - Y_j^{u-1})$$



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$$\forall u \in M,\ \forall j \in N, \quad F_j^u \geq \alpha X + \beta Y + \gamma W$$

Tightening interval-indexed formulation

- Redundant constraintes.
  Example : $p_j \leq e_u - e_v$: $C_j \leq e_v \Rightarrow C_j \leq e_u$ $(Y_j^u \geq X_j^u)$
- Appropriate partition of the time horizon
  Known order of jobs completed but not necessarily started in the same interval.

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Numerical experiments

Cplex 10.0, standard settings and only standard cuts,
1000 seconds time limit

- **Minimizing total weighted earliness and tardiness with release dates** $(1 \mid r_j \mid \alpha_j E_j + \beta_j T_j)$ — some 15 jobs instances could not be solved (all are solved if number of intervals is small).
- **Minimizing total weighted earliness and tardiness** $(1 \mid\mid \alpha_j E_j + \beta_j T_j)$ — all 15 jobs instances are solved (majority of 20 jobs instances are not solved).
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Similar performance with the Time-indexed formulation

Conclusions

- A new compact MIP formulation for a general machine scheduling problem (size is $O(nm) \times O(nm)$)
- More efficient in practical situations (few different release/due dates)
- 4 open practical instances from ILOG [Le Pape, Robert, 07] has been solved (up to 25 jobs)
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