The power of non-robust cuts in branch-cut-and-price algorithms

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Outline of the talk

Introduction

Branch-Cut-and-Price

Non-robust cutting planes

- Chvátal-Gomory rounding
- Covering cuts
- Knapsack cuts
- Consistency cuts

Issues

- Impact on the pricing problem difficulty
- Implementation
- Generality/efficiency trade-off

Takeaways

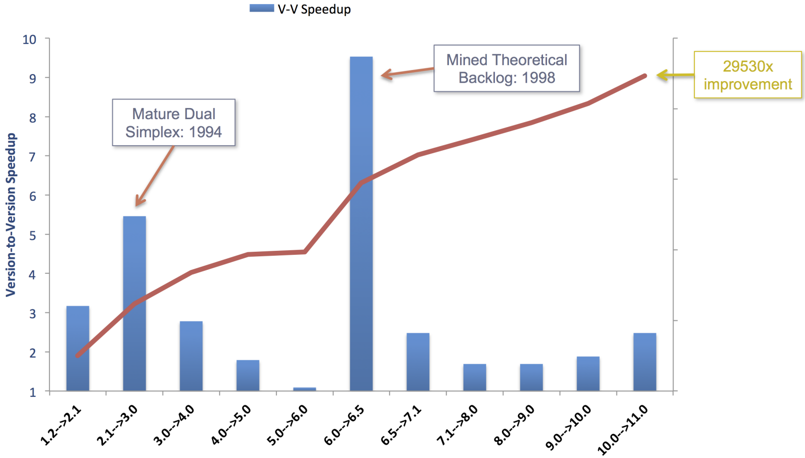
Contents

MIP Solvers: one of the most useful contribution of Operations Research to the society.

- Accounting
- Advertising
- Agriculture
- Airlines
- ATM provisioning
- Compilers
- Defense
- Electrical power
- Energy
- Finance
- Food service
- Forestry
- Gas distribution
- Government
- Internet applications
- Logistics/supply chain
- Medical
- Mining
- National research labs
- Online dating
- Portfolio management
- Railways
- Recycling
- Revenue management
- Semiconductor
- Shipping
- Social networking
- Sourcing
- Sports betting
- Sports scheduling
- Statistics
- Steel Manufacturing
- Telecommunications
- Transportation
- Utilities
- Workforce Management

Progress of MIP solvers

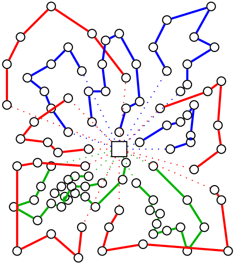
1991-2008 progress



Progress (independently from the computer power) 1991-2015 :
1.1M X – 1.8X/year

Sequential decision making: problems which “resist”

- Vehicle routing problems



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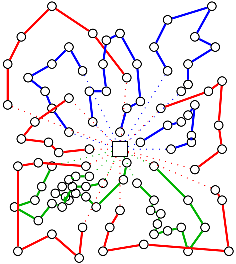
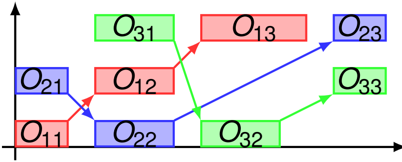


- Scheduling problems



Sequential decision making: problems which “resist”

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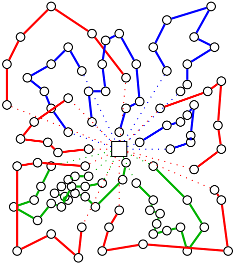

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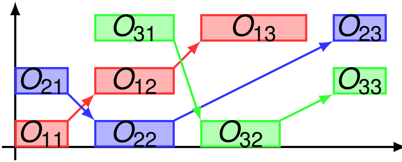
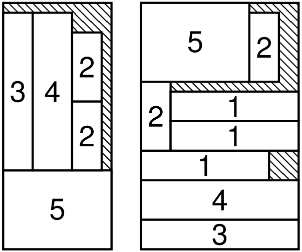
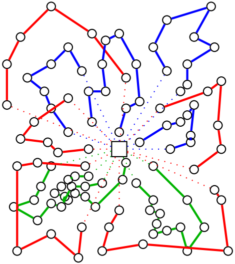


- Packing/cutting problems



Sequential decision making: problems which “resist”

- Vehicle routing problems



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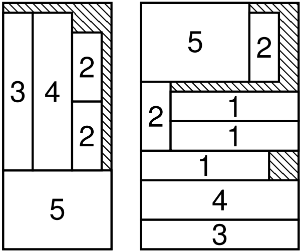

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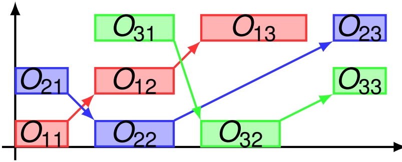
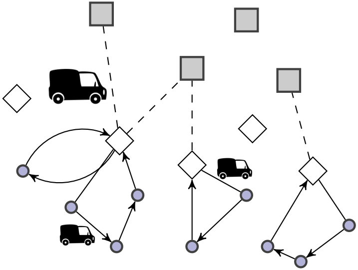


- Network design problems (especially integrated)



Models with a very large number of variables

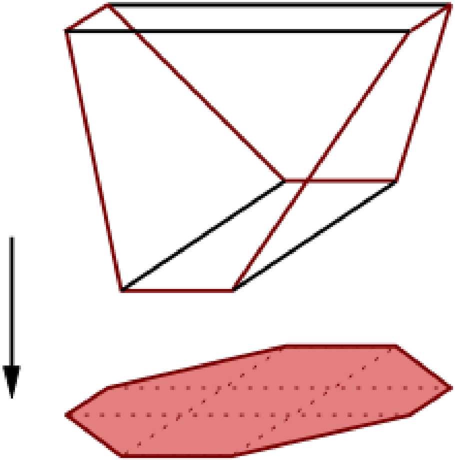


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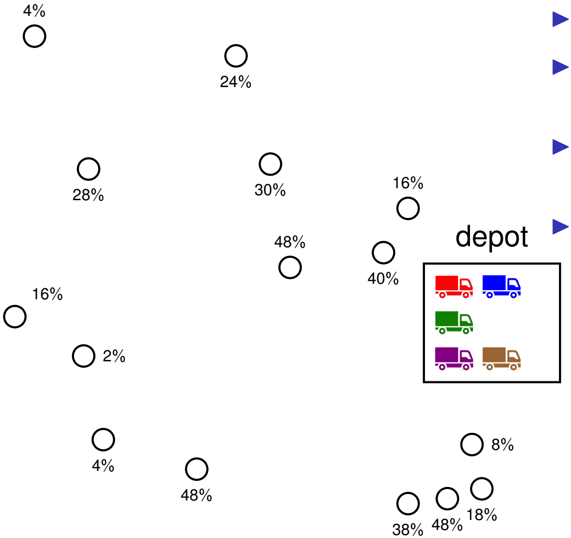
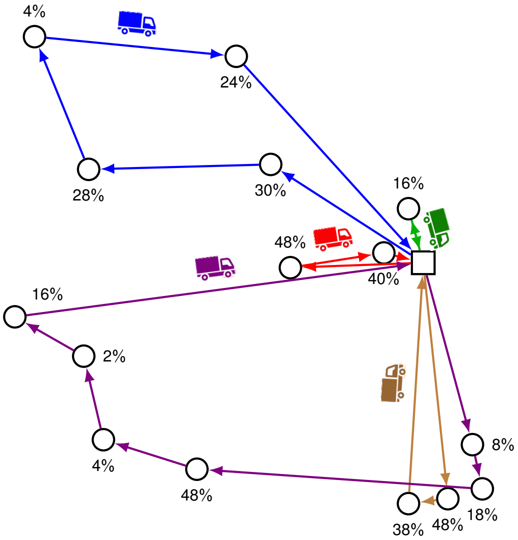
Illustration: Capacitated Vehicle Routing Problem (CVRP) (Dantzig and Ramser 1959)



- Depot
- Identical vehicles of capacity Q
- Clients $i \in V$ with demand d_i
- Cost matrix c

Illustration: Capacitated Vehicle Routing Problem (CVRP) (Dantzig and Ramser 1959)



- Depot
- Identical vehicles of capacity Q
- Clients $i \in V$ with demand d_i
- Cost matrix c

Minimize the total travelling cost

- such that every client is served
- total demand of clients served by the same vehicle does not exceed its capacity

Route-based (extended) formulation

- Variable x_a — arc $a \in A$ is used in the solution or not
- Variable λ_p — feasible route $p \in P$ is used in the solution or not
- $h_a^p = 1$ if and only if path p contains arc a, otherwise 0
- $\delta^-(v)$ — set of arcs in A incoming to $v \in V$

$$
\begin{aligned}
\text{Min} \quad & \sum_{a \in A} c_a x_a \\
\text{S.t.} \quad & \sum_{a \in \delta^-(v)} x_a = 1, && v \in V, \\
& x_a = \sum_{p \in P} h_a^p \lambda_p, && a \in A, \\
& \sum_{p \in P} \lambda_p \leq K, \\
& x_a \in \{0, 1\}, && a \in A, \\
& \lambda_p \in \{0, 1\}, && p \in P.
\end{aligned}
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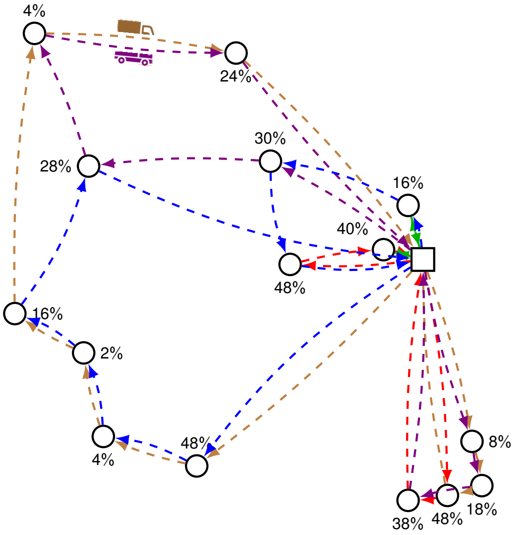
& x_a = \sum_{p\in P} h_a^p \lambda_p, & a \in A, & \quad (\pi_a) \\
& \sum_{p\in P} \lambda_p \le K, & & \quad (\mu) \\
& 0 \le x_a \le 1, & a \in A, & \\
& 0 \le \lambda_p \le 1, & p \in P. &
\end{aligned}
$$

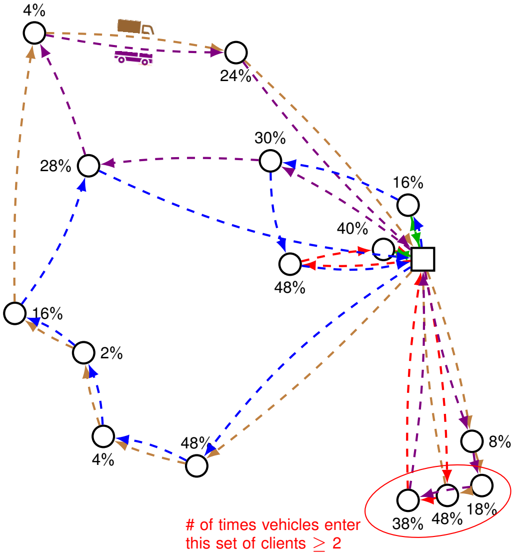
Column and cut generation: illustration



One continuous variable per feasible route.

Pricing problem is the Elementary Resource Constrained Shortest Path problem.

Column and cut generation: illustration



One continuous variable per feasible route.

Pricing problem is the Elementary Resource Constrained Shortest Path problem.

Additional constraints (cuts) are added to reduce the number of feasible non-integer solutions

A generic extended formulation

- x — a vector of original (natural) variables
- λ — a vector of extended variables (feasible sequences of decisions)
- H — mapping between natural and extended variables

$$
\begin{aligned}
\text{Min} \quad & cx \\
\text{S.t.} \quad & Ax \geq a, \\
& x = H\lambda, \\
& L \leq \mathbf{1}\lambda \leq U, \\
& x \in \mathbb{Z}_+^n \times \mathbb{R}_+^m, \\
& \lambda \in \mathbb{Z}_+^k.
\end{aligned}
$$

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$$
\begin{aligned}
\text{Min} \quad & cx \\
\text{S.t.} \quad & Ax \geq a, \\
& Bx \geq b, \qquad \leftarrow \text{ robust}^1 \text{cuts} \\
& x = H\lambda, \\
& L \leq \mathbf{1}\lambda \leq U, \\
& x \in \mathbb{Z}_+^n \times \mathbb{R}_+^m, \\
& \lambda \in \mathbb{Z}_+^k.
\end{aligned}
$$

[1] Marcus Poggi de Aragão and Eduardo Uchoa (2003). "Integer program reformulation for robust branch-and-cut-and-price". In: *Annals of Mathematical Programming in Rio*. Ed. by Laurence A. Wolsey. Búzios, Brazil, pp. 56–61

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\begin{array}{lrll}
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\text{S.t.} & Ax \geq & a, & \\
& Bx \geq & b, & \leftarrow \text{ robust cuts} \\
& x = & H\lambda, & \\
& D\lambda \geq & d, & \leftarrow \text{ non-robust}^{2}\text{cuts} \\
& L \leq \mathbf{1}\lambda \leq & U, & \\
& x \in & \mathbb{Z}_+^n \times \mathbb{R}_+^m, & \\
& \lambda \in & \mathbb{Z}_+^k. &
\end{array}
$$

[2] Mads Jepsen, Bjorn Petersen, Simon Spoorendonk, and David Pisinger (2006). *A Non-Robust Branch-And-Cut-And-Price Algorithm for the Vehicle Routing Problem with Time Windows.* Technical report 06/03. Dept. of Computer Science, University of Copenhagen

Contents

Non-robust cuts: literature review I

(Nemhauser and Park 1991) Odd-circuit cuts for the Edge Colouring Problem

(Belov and Scheithauer 2002, 2006) Gomory cuts for the 1D and 2D Cutting Stock Problems

(Jepsen, Petersen, Spoorendonk, and Pisinger 2006, 2008) Subset-row cuts for Vehicle Routing Problems

(Baldacci, Christofides, and Mingozzi 2008) Strong k-path (rounded capacity) inequalities, clique inequalities for Vehicle Routing Problems

(Petersen, Pisinger, and Spoorendonk 2008)

(Pecin, Pessoa, Poggi, Uchoa, and Santos 2017) General Chvátal-Gomory rank-1 cuts for Vehicle Routing Problems

(Dabia, Ropke, and Woensel 2019) Cover inequalities based on the knapsack constraint for the VRPTW with Shifts

(Dabia, Lai, and Vigo 2019) Generalized subset-row cuts for VRPs with Private Fleet and Common Carrier

Non-robust cuts: literature review II

(Liguori, Mahjoub, Marques, S., and Uchoa 2021) Strong knapsack cuts for the Location-Routing and other problems with the nested knapsack structure

(Rivera Letelier, Clautiaux, and S. 2022) Positive cycle inequalities for the Bin Packing Problem with Time Lags

(Clausen, Lusby, and Ropke 2022) General consistency cuts (work well for the Temporal Knapsack Problem)

This ROADEF

- (Dupont-Bouillard, Fouilhoux, Grappe, and Lacroix 2022) Gomory cuts for the Vertex Colouring Problem
- (Prunet, Absi, Borodin, and Cattaruzza 2022) Operational Storage Location Assignment Problem
- (Balster, Bulhoes, Munari, and S. 2022) Subset-row covering cuts for the Split-Delivery Vehicle Routing

Subset-row cuts[3]

- Replacing arc variables x in the set-partitioning constraints and relaxing to inequality:

$$\sum_{a \in \delta^-(v)} \sum_{p \in P} h_a^p \lambda_p \leq 1, \quad v \in V. \tag{1}$$

- Aggregating (1) for a set $C \subset V$, $|C| = 3$, with multiplier $\frac{1}{2}$:

$$\sum_{p \in P} \frac{1}{2} \sum_{v \in C} \sum_{a \in \delta^-(v)} h_a^p \lambda_p \leq \frac{3}{2}, \tag{2}$$

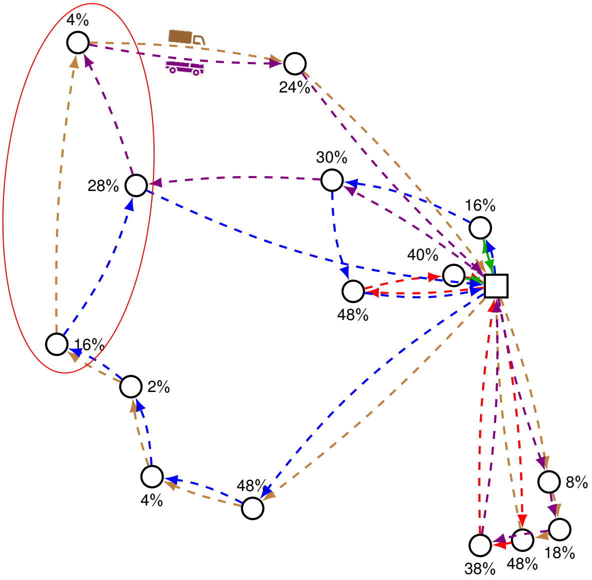
- Performing Chvátal-Gomory rounding of (2):

$$\sum_{p \in P} \left\lfloor \frac{1}{2} \sum_{v \in C} \sum_{a \in \delta^-(v)} h_a^p \right\rfloor \lambda_p \leq 1,$$

[3] Mads Jepsen, Bjorn Petersen, Simon Spoorendonk, and David Pisinger (2008). "Subset-Row Inequalities Applied to the Vehicle-Routing Problem with Time Windows". In: *Operations Research* 56.2, pp. 497–511.

Subset-row cuts: example of violation

of paths serving
at least two of these
three clients ≤ 1



Arbitrary cuts of Chvátal-Gomory rank 1

Chvátal-Gomory rounding using a vector p of multipliers:

$$\sum_{p \in P} \left\lfloor \sum_{v \in C} \sum_{a \in \delta^-(v)} p_v h_a^p \right\rfloor \lambda_p \leq \left\lfloor \sum_{v \in C} p_v \right\rfloor$$

All best possible multiplier vectors p for Chvátal-Gomory rounding of up to 5 constraints were found by (Pecin, Pessoa, Poggi, Uchoa, and Santos 2017):

- $|C| = 1$, $p = \{\frac{1}{2}\}$
- $|C| = 3$, $p = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$
- $|C| = 4$, $p = \{\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$
- $|C| = 5$, $p = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$
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Arbitrary cuts of Chvátal-Gomory rank 1

Chvátal-Gomory rounding using a vector p of multipliers:

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All best possible multiplier vectors p for Chvátal-Gomory rounding of up to 5 constraints were found by (Pecin, Pessoa, Poggi, Uchoa, and Santos 2017):

	Gap(%)
Only CG (elementary routes)	2.63
+ robust cuts	0.98
+ 3SRCs	0.35
+ 4SRCs + 5SRCs	0.24
+ other R1Cs up to 5 rows	0.17

Covering cuts

General idea

Take a subset of elements, and calculate the minimum number of "decision sequences" (columns) that should cover it.

Odd-circuit cuts (Nemhauser and Park 1991)

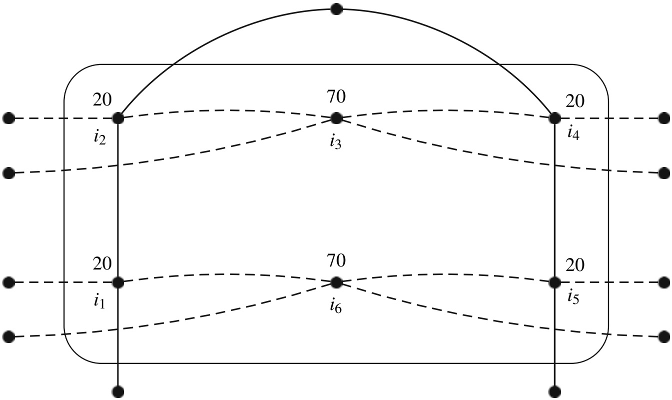
Edge in an odd circuit in a graph should be covered by at least 3 matchings.

Strong *k*-path inequalities (Baldacci, Christofides, and Mingozzi 2008)

A set C of customers should be visited by at least $\lceil \sum_{i \in C} d_i / Q \rceil$ routes. Let $V(p)$ be the set of customers visited by p, then

$$\sum_{\substack{p \in P: \\ V(p) \cap C \neq \emptyset}} \lambda_p \geq \left\lceil \frac{\sum_{i \in C} d_i}{Q} \right\rceil .$$

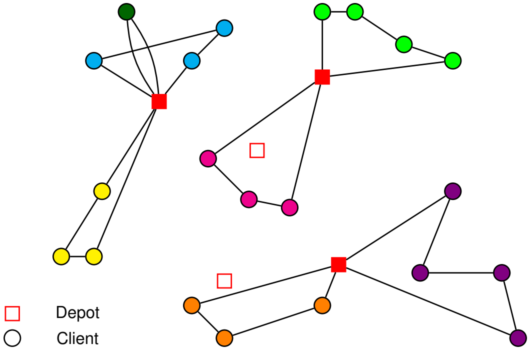
Strong k-path inequalities[4]: illustration



[4] Roberto Baldacci, Nicos Christofides, and Aristide Mingozzi (2008). "An exact algorithm for the vehicle routing problem based on the set partitioning formulation with additional cuts". In: *Mathematical Programming* 115, pp. 351–385.

Capacitated Location-Routing Problem

- Depots $k \in K$ with capacity W_k
- Identical vehicles of capacity Q
- Clients $i \in V$ with demand d_i
- Matrix c of travelling costs



Minimize the total travelling and depot opening costs

- such that every client is served
- each route starts and finished at a same opened depot
- depot and vehicle capacities are satisfied

Depot capacity knapsack constraints

- Variable y_k — depot $k \in K$ is open or not
- $d(p)$ — the load (total delivered demand) of route p.

$$\sum_{q=1}^{Q} \sum_{\substack{p \in P^k: \\ d(p)=q}} q\lambda_p \leq W_k y_k.$$

Valid inequalities can be generated again by Chvátal-Gomory rounding...

Depot capacity knapsack constraints

- Variable y_k — depot $k \in K$ is open or not
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$$\sum_{q=1}^{Q} \sum_{\substack{p \in P^k: \\ d(p)=q}} q\lambda_p \leq W_k y_k.$$

Valid inequalities can be generated again by Chvátal-Gomory rounding... but not only.

$$\sum_{q=1}^{Q} q t_q^k \leq W_k y_k, \text{ where } t_q^k = \sum_{\substack{p \in P^k: \\ d(p)=q}} q\lambda_p.$$

The master knapsack polytope

Theorem ((Aráoz, Evans, Gomory, and Johnson 2003))

The coefficient vectors ξ of the knapsack facets $\xi x \leq 1$ of polytope $\mathrm{conv}\{t \in \mathbb{Z}_+^W : \sum_{q=1}^W q t_q \leq W\}$, are the extreme points of the following system of linear constraints

$$\xi_1 = 0, \quad \xi_W = 1, \tag{3}$$
$$\xi_q + \xi_{W-q} = 1 \qquad \forall 1 \leq i \leq W/2, \tag{4}$$
$$\xi_q + \xi_{q'} \leq \xi_{q+q'} \quad \text{whenever } q + q' \leq W. \tag{5}$$

Road Load Knapsack Cuts (Liguori, Mahjoub, Marques, S., and Uchoa 2021)

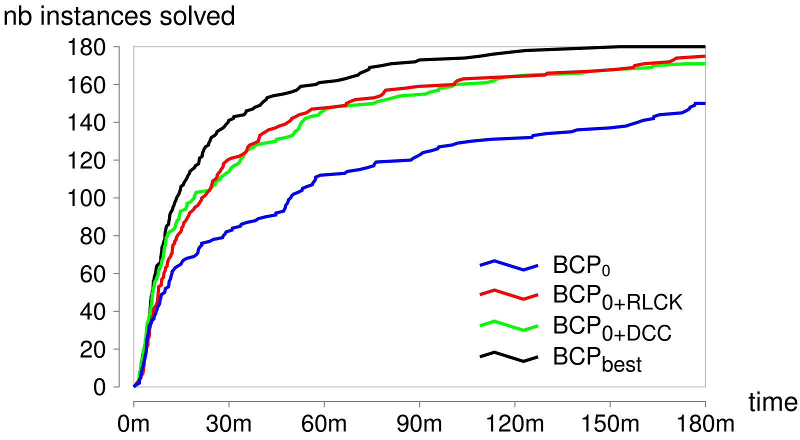
Given a depot $k \in K$ and a vector $\xi \in \mathbb{R}_+^{W_k}$ satisfying (3) and (5), the inequality

$$\sum_{q=1}^{W_k} \sum_{p \in P^k:\, d(p)=q} \xi_q \lambda_p \leq y_k.$$

is valid for the extended formulation of the CLRP.

Computational results for instances with all depots open

From (Liguori, Mahjoub, Marques, S., and Uchoa 2021)



Extended formulation with linking variables

- x^0 — linking natural variables
- x^1, x^2 — natural variables for blocks 1 and 2
- λ^1, λ^2 — extended variables representing all feasible solutions in blocks 1 and 2.

$$
\begin{aligned}
\text{Min} \quad & c^0x^0 + c^1x^1 + c^2x^2 \\
\text{S.t.} \quad & A^0x^0 + A^1x^1 + A^2x^2 \geq a, \\
& (x^0, x^1) = H^1\lambda^1, \\
& (x^0, x^2) = H^2\lambda^2, \\
& \mathbf{1}\lambda^1 = 1, \\
& \mathbf{1}\lambda^2 = 1, \\
& (x^0, x^1, x^2) \in \mathbb{Z}^{n_0+n_1+n_2}, \\
& (\lambda^1, \lambda^2) \in \{0, 1\}^{k_1+k_2}.
\end{aligned}
$$

Consistency cuts[5]

- Let $\bar{x}^0$ be a particular partial solution for linking variables x^0 (a pattern).
- Let $P^1(\bar{x}^0)$ and $P^1(\bar{x}^0)$ be the sets of solutions of block 1 and 2 which "match" pattern $\bar{x}^0$.

Then the following constraint is valid

$$\sum_{p \in P^1(\bar{x}^0)} \lambda_p^1 = \sum_{p \in P^2(\bar{x}^0)} \lambda_p^2.$$

[5] Jens Vinther Clausen, Richard Lusby, and Stefan Ropke (2022). "Consistency Cuts for Dantzig-Wolfe Reformulations". In: *Operations Research* Ahead of Print.

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Then the following constraint is valid

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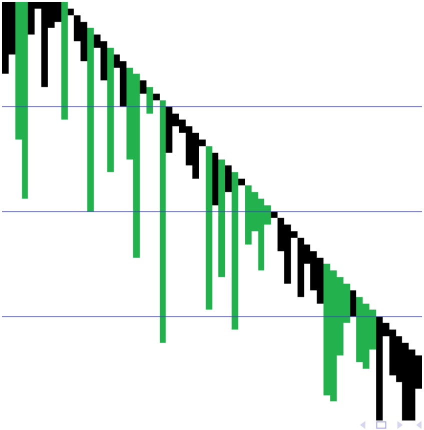
When decomposition forms a staircase structure, there are no linking constraints, and linking variables are binary, then consistency cuts are enough to obtain an optimal integer solution (no branching is necessary).

[5] Jens Vinther Clausen, Richard Lusby, and Stefan Ropke (2022). "Consistency Cuts for Dantzig-Wolfe Reformulations". In: *Operations Research* Ahead of Print.

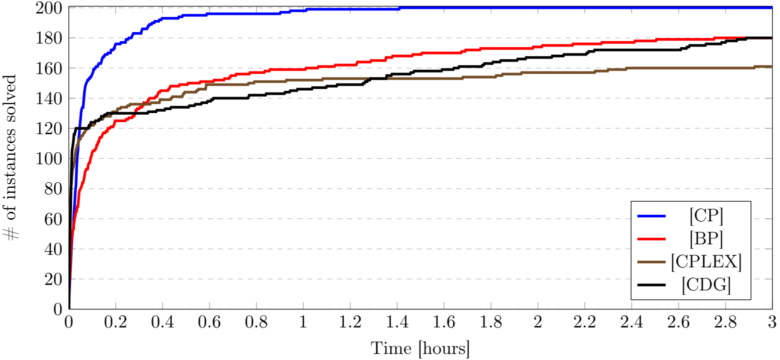
Consistency cuts: application

Temporal knapsack problem

- Every item is active during a certain time interval.
- Knapsack constraints are imposed only for groups of items active at the same time



Consistency cuts: computational results for the temporal knapsack instances



Contents

Non-robust cuts and the pricing problem

- Non-robust cuts change the structure (or dimension) of the pricing problem!
- Usually cannot generate many cuts, as the pricing problem may become intractable.
- Family of non-robust cuts and pricing algorithm are always interdependent.

Non-robust cuts and the pricing problem

- Non-robust cuts change the structure (or dimension) of the pricing problem!
- Usually cannot generate many cuts, as the pricing problem may become intractable.
- Family of non-robust cuts and pricing algorithm are always interdependent.

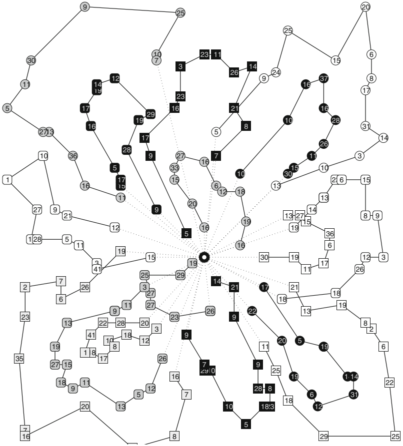
A way to limit impact on the pricing difficulty

- Design a weaker variant of non-robust cuts adapted for the pricing algorithm
- Example: Limited-memory Chvátal-Gomory rank-1 cuts for vehicle routing problems
- Limited-memory cuts are adapted for the labelling pricing algorithm

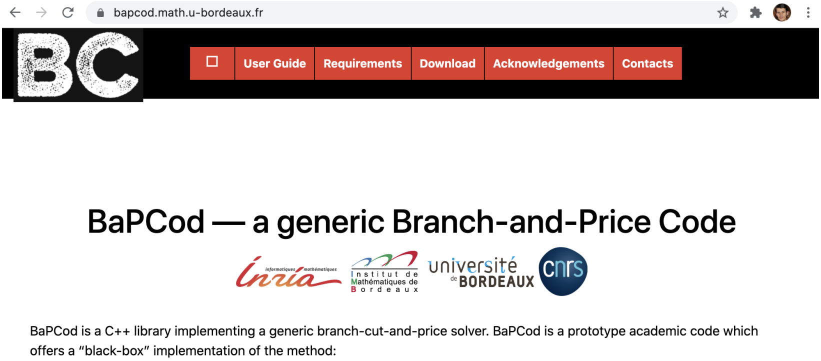
Limited-memory Chvátal-Gomory rank-1 cuts

(Pecin, Pessoa, Poggi, and Uchoa 2017)

Optimal solution of instance M-n200-k16



Implementation: BaPCod C++ library



- Has been developed for ≈ 15 years.
- Source code: `bapcod.math.u-bordeaux.fr` (only for research purposes)
- User guide is available[6]

[6] R. S. and François Vanderbeck (2021). *BaPCod — a generic Branch-And-Price Code*. Technical report HAL-03340548. Inria Bordeaux — Sud-Ouest.

BaPCod: implementation of non-robust cuts

User should define

- A data structure (a class) which characterises a non-robust cut
- A separation algorithm which takes a master solution as input
- Pricing algorithm(s) which retrieve and take into account active non-robust cuts
- A function which calculates the coefficient of a column in a non-robust cut

BaPCod provides

Column generation procedure with automatic stabilization, strong branching, cut and column clean-up, some primal heuristics.

Other libraries

GCG (gcg.or.rwth-aachen.de)

- A subclass of Gomory cuts ($\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts) can be used
- Generation of clique cuts is in plans
- However, only when the pricing is solved by MIP

SCIP (www.scipopt.org)

- Flexible branch-cut-and-price framework
- Non-robust cuts implementation is similar to BaPCod

Coluna.jl (https://github.com/atoptima/Coluna.jl)

- Support of non-robust cuts is in plans

Generality/efficiency trade-off

- Branch-cut-and-price (BCP) approaches for specific problems are rarely used in practice
- Fully generic BCP (pricing solved by MIP) is rarely competitive with MIP solvers

Generality/efficiency trade-off

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VRPSolver[7] (`vrpsolver.math.u-bordeaux.fr`)

- A "semi-generic" BCP algorithm for a wide range of vehicle routing problems
- Accessible through a MIP-and-Graph-based model

julia JUMP

- A state-of-the art performance
- Robust and non-robust cuts are activated by using the collection of packing sets modelling concept.

[7] Artur Pessoa, R. S., Eduardo Uchoa, and François Vanderbeck (2020). "A Generic Exact Solver for Vehicle Routing and Related Problems". In: *Mathematical Programming* 183, pp. 483–523.

Collection of packing sets in VRPSolver

Definition

A packing set is a subset of arcs (vertices) such that, in an optimal solution of the problem, at most one arc (vertex) in the subset appears at most once.

- Definition of packing sets is a part of modeling
- Packing sets generalize customers in CVRP

Collection of packing sets in VRPSolver

Definition

A packing set is a subset of arcs (vertices) such that, in an optimal solution of the problem, at most one arc (vertex) in the subset appears at most once.

- Definition of packing sets is a part of modeling
- Packing sets generalize customers in CVRP
- Generalization examples:
  - Heterogeneous Fleet: customer copies for each vehicle type
  - Multiple time windows: customer copies for each time window
  - Alternative delivery locations: all delivery locations for each client
  - Arc routing: two possible directions for a required edge

Contents

Takeaways

- Generation of non-robust cuts is sometimes necessary to outperform MIP solvers and attain the state-of-the-art performance
- Relatively unexplored area of research
- Coordination with pricing algorithm is important
- Tools which make implementation easier start to appear
- More or less generic approaches are especially welcome

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