

An improved Branch-Cut-and-Price Algorithm for Heterogeneous Vehicle Routing Problems

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Heterogeneous Vehicle Routing

Set I of n customers, each $i \in I$ with demand d_i .

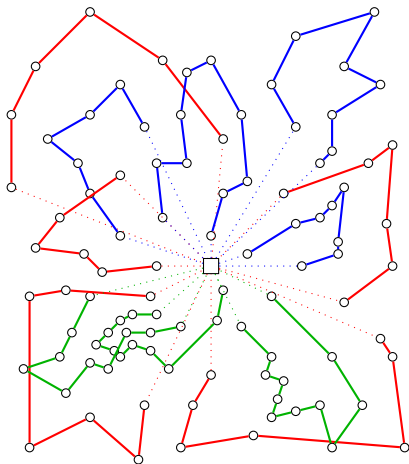
Set U of vehicle types, each $u \in U$ has a depot with K_u vehicles of capacity Q_u , with fixed cost f_u and travel costs c_a^u for each edge a .

Objective: minimize the total fixed and travel cost.

Variants

- ▶ Multi-depot VRP
- ▶ Site-dependent VRP

Instance HVRPFV-20-100



6 , 4 , 3 

Optimum **4760.68** (BKS 4761.26)

Set partitioning (master) formulation

- ▶ R_u — set of q -routes feasible for a vehicle of type u
- ▶ a_i^r — number of times that customer i appears in route r .
- ▶ c_r — cost of route r .
- ▶ **Binary variable** $\lambda_u^r = 1$ if and only if a vehicle of type u uses route r

$$\begin{aligned} \min \quad & \sum_{u \in U} \sum_{r \in R_u} c_r \lambda_r \\ & \sum_{u \in U} \sum_{r \in R_u} a_i^r \lambda_r = 1, \quad \forall i \in I, \\ & \sum_{r \in R_u} \lambda_r \leq K_u, \quad \forall u \in U, \\ & \lambda_r \in \{0, 1\}, \quad \forall r \in R_u, \forall u \in U. \end{aligned}$$

Pricing subproblem for a vehicle type

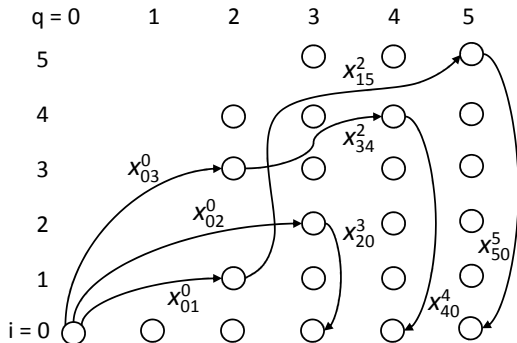
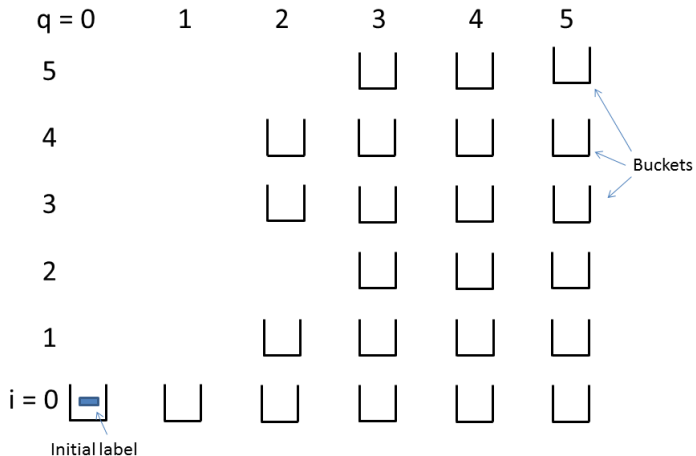
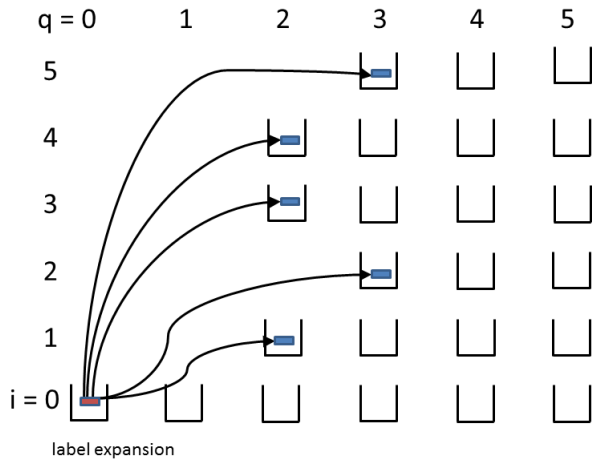


Figure: $|I| = Q = 5$, $d_1 = d_3 = d_4 = 2$, $d_2 = d_5 = 3$; routes 0-1-5-0, 0-2-0, 0-3-4-0 are shown

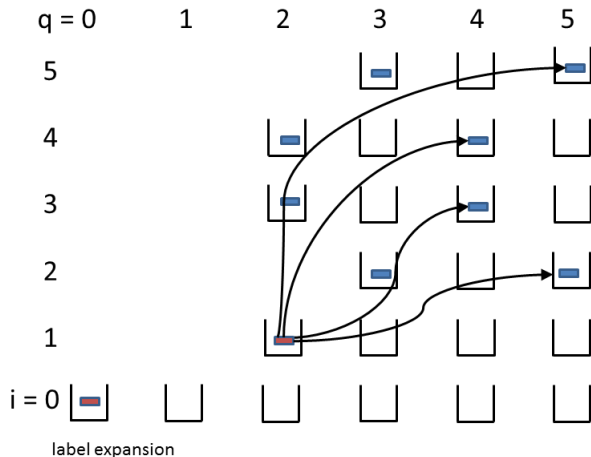
The labeling algorithm



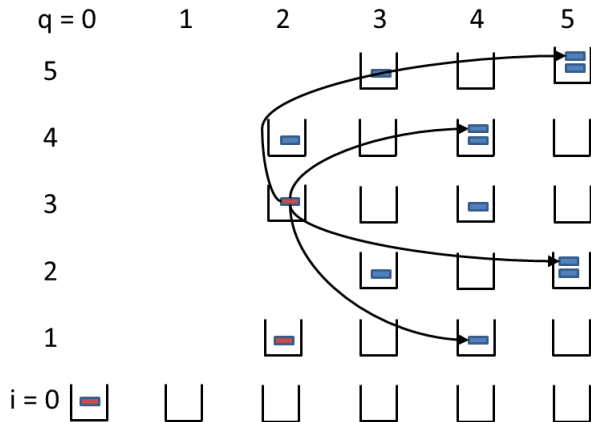
The labeling algorithm



The labeling algorithm

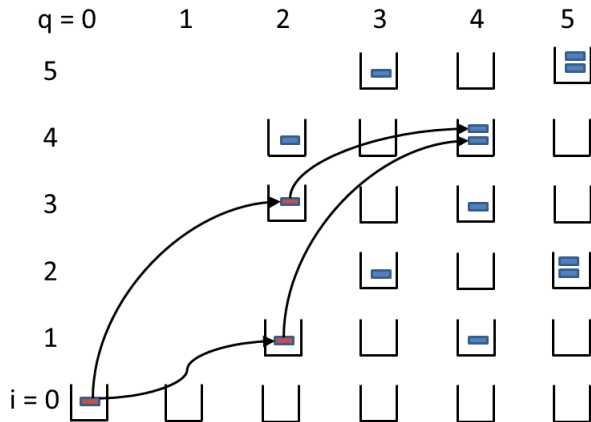


The labeling algorithm



Do both labels need to be kept in bucket (4,4)?

The labeling algorithm



The labels represent partial paths 0-1-4 and 0-3-4

Subset Row Cuts (SRCs)

Given $C \subseteq V_+$ and a multiplier p , the (C, p) -Subset Row Cut is:

$$\sum_{u \in U} \sum_{r \in R_u} \left\lfloor p \sum_{i \in C} a_i^r \right\rfloor \lambda_r \leq \lfloor p|C| \rfloor$$

Special case of **Chvátal-Gomory rank-1 cuts** obtained by rounding of $|C|$ constraints in the master

Each cut adds **an additional resource** in the shortest path pricing problem

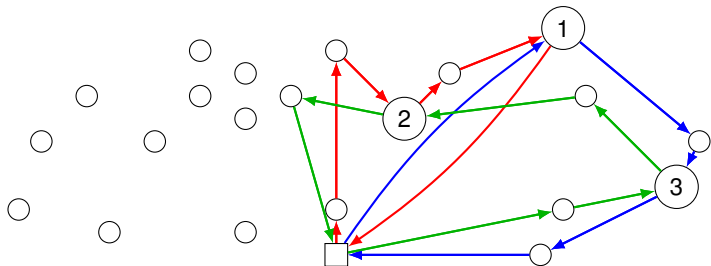


Mads Jepsen and Bjorn Petersen and Simon Spoorendonk and David Pisinger (2008).

Subset-Row Inequalities Applied to the Vehicle-Routing Problem with Time Windows.

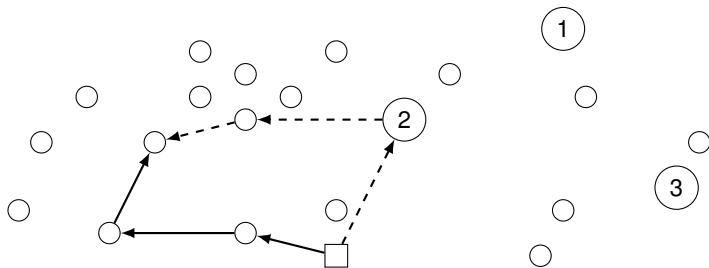
Operations Research, 56(2):497–511.

Example of 3-Subset Row Cut, $|C| = 3$, $\rho = \frac{1}{2}$



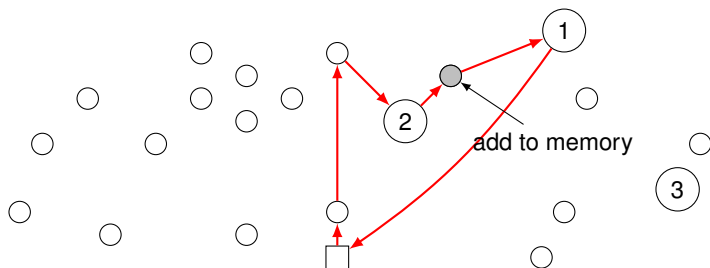
If $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, and $\lambda_3 = 0.5$, cut $C = \{1, 2, 3\}$ is violated.

Example of 3-Subset Row Cut, $|C| = 3$, $\rho = \frac{1}{2}$



Less possibilities for domination after adding the cut.

Example of 3-Subset Row Cut, $|C| = 3$, $p = \frac{1}{2}$



Concept of **limited memory cuts** [Pecin et al., 2017b].

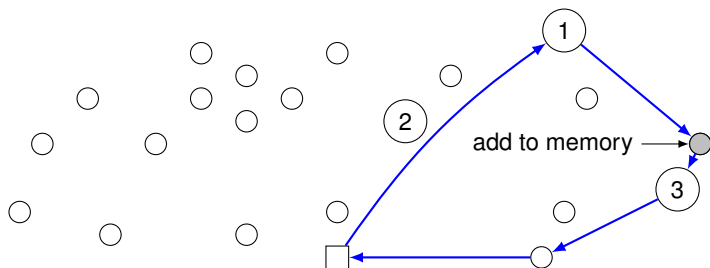


Pecin, D., Pessoa, A., Poggi, M., and Uchoa, E. (2017a).

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Mathematical Programming Computation, 9(1):61–100.

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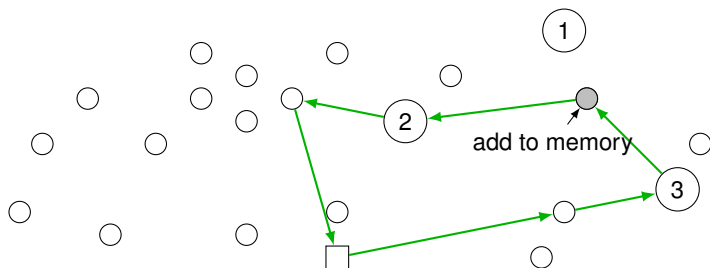


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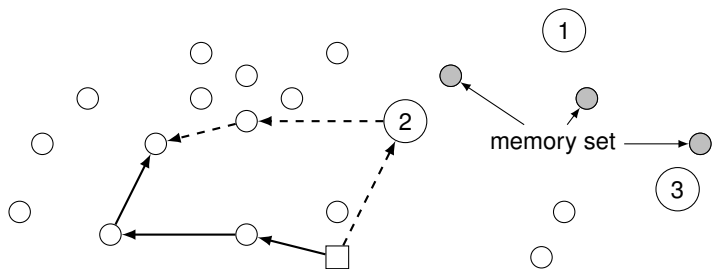


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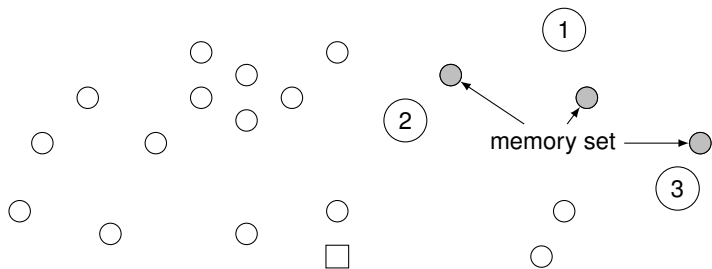
Mathematical Programming Computation, 9(1):61–100.

Example of 3-Subset Row Cut, $|C| = 3$, $p = \frac{1}{2}$



Dashed partial path “forgot” the cut (cut state in the label is 0)
 \Rightarrow larger domination probability.

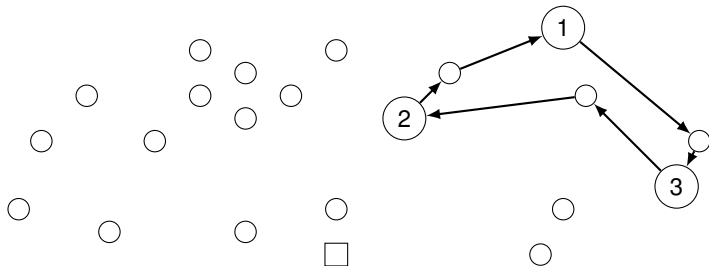
Example of 3-Subset Row Cut, $|C| = 3$, $p = \frac{1}{2}$



Different memory sets

- ▶ Node memory

Example of 3-Subset Row Cut, $|C| = 3$, $p = \frac{1}{2}$



Different memory sets

- ▶ Node memory
- ▶ Arc memory [Pecin et al., 2017a]

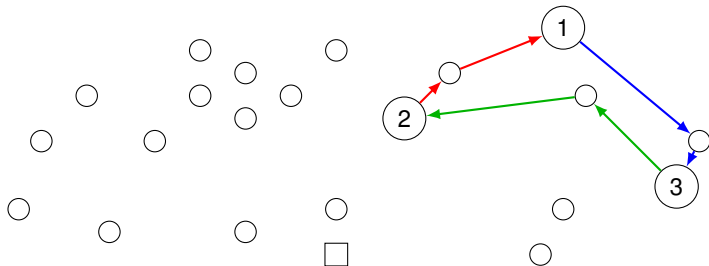


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New enhancements for the exact solution of the vehicle routing problem with time windows.

INFORMS Journal on Computing, 29(3):489–502.

Example of 3-Subset Row Cut, $|C| = 3$, $p = \frac{1}{2}$



Different memory sets

- ▶ Node memory
- ▶ Arc memory [Pecin et al., 2017a]
- ▶ **Subproblem dependent memory** (this work)



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Arbitrary cuts of Rank 1

Rounding using a **vector** p instead of single value:

$$\sum_{u \in U} \sum_{r \in R_u} \left\lfloor \sum_{i \in C} p_i a_i^r \right\rfloor \lambda_r \leq \left\lfloor \sum_{i \in C} p_i \right\rfloor$$

All facet-defining vectors p for cuts up to 5 rows
[Pecin et al., 2017c]:

- ▶ $|C| = 1, p = (\frac{1}{2})$
- ▶ $|C| = 3, p = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
- ▶ $|C| = 4, p = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
- ▶ $|C| = 5, p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
- ▶ $|C| = 5, p = (\frac{2}{4}, \frac{2}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
- ▶ $|C| = 5, p = (\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
- ▶ $|C| = 5, p = (\frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5})$
- ▶ $|C| = 5, p = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
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- ▶ $|C| = 5, p = (\frac{3}{4}, \frac{3}{4}, \frac{2}{4}, \frac{2}{4}, \frac{1}{4})$



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Limited memory rank-1 cuts for vehicle routing problems.

Operations Research Letters, 45(3):206 – 209.

Extended Capacity Cuts

Definition

An Extended Capacity Cut (ECC) [Pessoa et al., 2009] over subset C of customers is any inequality valid for $P(C)$, the polyhedron given by the convex hull of the 0 – 1 solutions of

$$\sum_{u \in U} \left(\sum_{a \in \delta_u^-(C)} \sum_{q=1}^Q dx_a^q - \sum_{a \in \delta_u^+(C)} \sum_{q=0}^{Q-1} dx_a^q \right) = d(C)$$



Pessoa, Artur and Uchoa, Eduardo and Poggi, Marcus (2009).

A robust branch-cut-and-price algorithm for the heterogeneous fleet vehicle routing problem.

Networks, 54(4):167–177.

Homogeneous Extended Capacity Cuts

$$y^{m,q} = \sum_{a^q \in \delta_u^-(C)} x_a^q, \quad z^q = \sum_{a^q \in \delta_u^+(C)} x_a^{m,q}, \quad (q = 0, \dots, Q).$$

Definition

A Homogeneous Extended Capacity Cut (HECC) over set C of customers is any inequality valid for the polyhedron given by the convex hull of the integral solutions of

$$\sum_{u \in U} \left(\sum_{q=1}^Q dy^{m,q} - \sum_{q=0}^{Q-1} dz^{m,q} \right) = d(C). \quad (1)$$

Separation

- ▶ Cuts obtained by applying integer rounding of (1).
- ▶ Heuristic separation of [Pessoa et al., 2009] is used.

Labeling algorithm enhancements

- ▶ **ng-routes** to impose partial elementarity [Baldacci et al., 2011].
- ▶ **Bi-directional** labelling [Righini and Salani, 2006]
- ▶ **Reduced cost fixing** of subproblem arc variables x [Irnich et al., 2010]



Baldacci, R., Mingozzi, A., and Roberti, R. (2011).

New route relaxation and pricing strategies for the vehicle routing problem.
Operations Research, 59(5):1269–1283.



Righini, G. and Salani, M. (2006).

Symmetry helps: Bounded bi-directional dynamic programming for the elementary shortest path problem with resource constraints.
Discrete Optimization, 3(3):255 – 273.



Irnich, S., Desaulniers, G., Desrosiers, J., and Hadjar, A. (2010).

Path-reduced costs for eliminating arcs in routing and scheduling.
INFORMS Journal on Computing, 22(2):297–313.

Elementary routes enumeration

We try to enumerate all elementary routes whose reduced cost is smaller than the current gap [Baldacci et al., 2008], possibly to a pool [Contardo and Martinelli, 2014].

This work contributions

- ▶ **Subproblem dependent** enumeration
- ▶ If succeeded, a subproblem passes to the **enumerated state**:
 - ▶ Pricing is performed by **inspection**
 - ▶ **Cut coefficients** of columns in the master are **lifted**



Baldacci, R., Christofides, N., and Mingozzi, A. (2008).

An exact algorithm for the vehicle routing problem based on the set partitioning formulation with additional cuts.

Mathematical Programming, 115:351–385.

Lifting of cuts in enumerated state

Rank-1 cuts

Limited memory is extended to **full memory**

Homogeneous Extended Capacity Cuts

Integer rounding of

$$\sum_{u \in EU} \sum_{r \in R_u} d_r(C) \lambda_r + \sum_{u \in U \setminus EU} \left(\sum_{q=1}^Q dy^{m,q} - \sum_{q=0}^{Q-1} dz^{m,q} \right) = d(C).$$

For a particular rounding multiplier and $EU = U$ (all subproblems are in the enumerated state), equivalent to Strong Capacity Cuts [Baldacci et al., 2008].

Other enhancements

- ▶ Heuristic pricing (keeping one label per bucket)
- ▶ **Automatic** dual price smoothing **stabilization** [Pessoa et al., 2017].
- ▶ **Rollback** mechanism [Pecin et al., 2017b]



Pessoa, A., Sadykov, R., Uchoa, E., and Vanderbeck, F. (2017).

Automation and combination of linear-programming based stabilization techniques in column generation.

INFORMS Journal on Computing, (Forthcoming).

Branching

Strong branching [Pecin et al., 2017b]

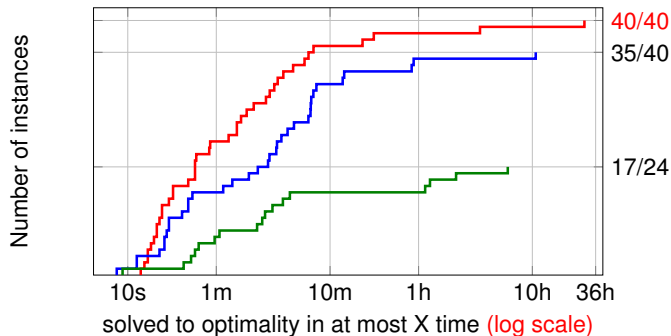
- ▶ Multi-strategy
- ▶ Branching history (**pseudo-costs**)
- ▶ Multi-phase

Branching strategies

- ▶ Number of vehicles
- ▶ Assignment of customers to vehicle types
- ▶ Participation of arcs in routes

Results for classic Heterogeneous VRP instances

40 instances with 50-100 customers by [Taillard, E. D., 1999]



— Our algorithm — [Baldacci and Mingozzi, 2009] — [Pessoa et al., 2009]



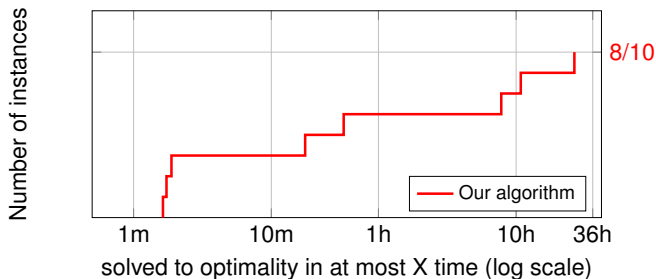
Baldacci, R. and Mingozzi, A. (2009).

A unified exact method for solving different classes of vehicle routing problems.

Mathematical Programming, 120(2):347–380.

Results for larger Heterogeneous VRP instances

10 instances with 100-200 customers [Brandao, 2011]



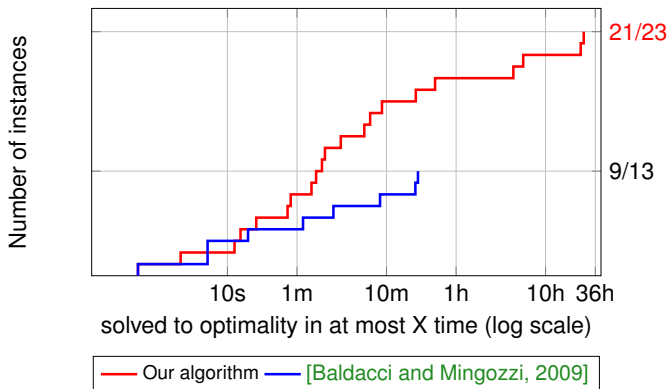
Brandao, J. (2011).

A tabu search algorithm for the heterogeneous fixed fleet vehicle routing problem.

Computers and Operations Research, 38(1):140 – 151.

Results for standard Site-Dependent VRP instances

Instances with 27-324 customers by [Cordeau and Laporte, 2001]

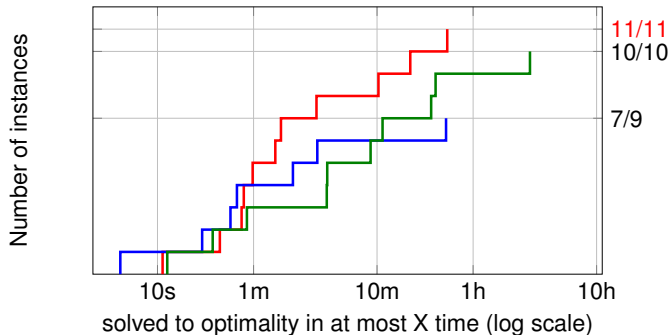


Largest solved instance has **216 customers**

[Baldacci and Mingozzi, 2009] solved only 1 of 5 instances with 100 customers and more

Results for standard Multi-Depot VRP instances

Instances with 50–360 customers [Cordeau et al., 1997]



— Our algorithm — [Baldacci and Mingozzi, 2009] — [Contardo and Martinelli, 2014]



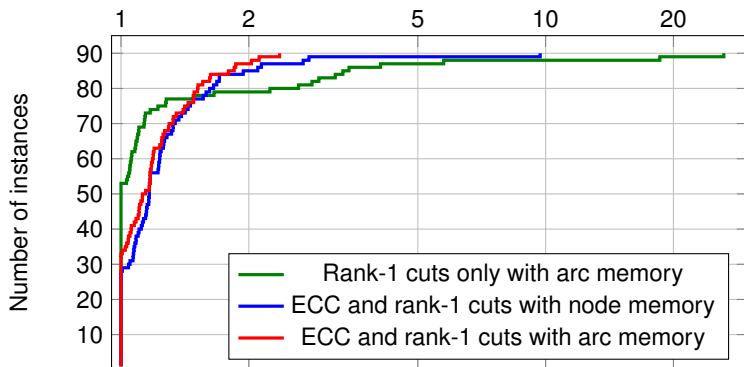
Contardo, C. and Martinelli, R. (2014).

A new exact algorithm for the multi-depot vehicle routing problem under capacity and route length constraints.

Discrete Optimization, 12:129 – 146.

Impact of Extended Capacity Cuts

Cuts	Rank-1 cuts memory	Root gap	Root time	Nodes num.	Solved	Total time
R1C only	arc, sp.dep	0.323%	93	67.8	87/90	181
R1C+ECC	node, sp.dep	0.133%	106	38.7	86/90	178
R1C+ECC	arc, sp.dep	0.105%	113	29.6	88/90	170



for which variant is at most X times slower than the best

Improved Best Known Solutions

Problem	Instance	Size	Previous BKS	Reference	Improved value
HVRP	BrandaoN1fsm	150	2212.77	[SPUS12]	2211.63
	BrandaoN1hd	150	2234.13	[S16]	2233.90
	BrandaoN2fsm	199	2823.75	[SPUS12]	2810.20
	BrandaoN2hd	199	2859.82	[S16]	2851.94
	c100_20fsmf	100	4032.81	[SPUS12]	4029.61
	c100_20hvrp	100	4761.26	[SPUS12]	4760.68
MDVRP	n200-k16-3-80	200	1757.86	[BM09]	1756.48
SDVRP	p16	216	3393.55	[CM12]	3393.31
	p18	324	4751.27	[CM12]	4747.75 ¹
	p21	209	1263.71	[CM12]	1260.01

¹ optimality is not proved, other values are optimal

[SPUS12] [Subramanian et al., 2012]

[S16] [Subramanian, 2016]

[BM09] [Baldacci and Mingozzi, 2009]

[CM12] [Cordeau and Maischberger, 2012]

Contributions

- ▶ **Large computational improvement** over the state-of-the-art algorithms for the problem
- ▶ Showed **importance** of **Extended Capacity Cuts**
- ▶ New concept of **subproblem dependent memory** for rank-1 cuts
- ▶ New concept of **“enumerated state”** for pricing subproblems
- ▶ **New family** of lifted Extended Capacity Cuts

Case of multiple and “non-discretizable” resources

Presentation at IFORS in Quebec next week

**A Bucket Graph Based Labelling Algorithm for the RCSPP
with Applications to Vehicle Routing**

Case of multiple and “non-discretizable” resources

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A Bucket Graph Based Labelling Algorithm for the RCSP with Applications to Vehicle Routing

A glimpse of the results

- ▶ Solved 5/9 open VRPTW instances of [Gehring and Homberger, 2002] with 200 customers
- ▶ Solved 6/7 distance-constrained CVRP instances of [Christofides et al., 1979] (CMT) with up to 200 customers
- ▶ Solved all 22 distance-constrained MDVRP instances of [Cordeau et al., 1997] with up to 288 customers
- ▶ Solved 7/10 distance-constrained SDVRP instances of [Cordeau and Laporte, 2001] with up to 216 customers
- ▶ Solved 56/96 “nightmare” HFVRP instances of [Duhamel et al., 2011] with up to 186 customers.

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Computers and Operations Research, 38(4):723 – 739.



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Journal of Heuristics, 8(3):251–276.



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European Journal of Operational Research, 221(2):285 – 295.



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RAIRO-Oper. Res., 33(1):1–14.