Isometries, Fock Spaces, and Spectral Analysis of Schrödinger Operators on Trees

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Abstract

We construct conjugate operators for the real part of a completely non unitary isometry and we give applications to the spectral and scattering theory of a class of operators on (complete) Fock spaces, natural generalizations of the Schrödinger operators on trees. We consider C^*-algebras generated by such Hamiltonians with certain types of anisotropy at infinity, we compute their quotient with respect to the ideal of compact operators, and give formulas for the essential spectrum of these Hamiltonians.

1 Introduction

The Laplace operator on a graph Γ acts on functions $f : \Gamma \to \mathbb{C}$ according to the relation

$$(\Delta f)(x) = \sum_{y \leftrightarrow x} (f(y) - f(x)), \tag{1.1}$$

where $y \leftrightarrow x$ means that x and y are connected by an edge. The spectral analysis and the scattering theory of the operators on $\ell^2(\Gamma)$ associated to expressions of the form $L = \Delta + V$, where V is a real function on Γ, is an interesting question which does not seem to have been much studied (we have in mind here only situations involving non trivial essential spectrum).

Our interest on these questions has been aroused by the work of C. Allard and R. Froese [All, AlF] devoted to the case when Γ is a binary tree: their main results are the construction of a conjugate operator for L under suitable conditions on the potential V and the proof of the Mourre estimate. As it is well known, this allows one to deduce various non trivial spectral properties of L, for example the absence of the singularly continuous spectrum.

The starting point of this paper is the observation that if Γ is a tree then $\ell^2(\Gamma)$ can be naturally viewed as a Fock space[1] over a finite dimensional Hilbert space and that the operator L has a very simple interpretation in this framework. This suggests the consideration of a general class of operators, abstractly defined only in terms of the Fock space structure. Our purpose then is twofold: first, to construct conjugate operators for this class of operators, hence to point out some of their basic spectral properties, and second to reconsider the kind of anisotropy studied in [Gol] in the present framework.

It seems interesting to emphasize the non technical character of our approach: once the correct objects are isolated (the general framework, the notion of number operator associated to an isometry, the C^*-algebras of anisotropic potentials), the proofs are very easy, of a purely algebraic nature, the arguments needed to justify some formally obvious computations being very simple.

We recall the definition of a ν-fold tree with origin e, where ν is a positive integer and $\nu = 2$ corresponds to a binary tree (see [Gol]). Let A be a set consisting of ν elements and let

$$\Gamma = \bigcup_{n\geq 0} A^n \tag{1.2}$$

where A^n is the n-th Cartesian power of A. If $n = 0$ then A^0 consists of a single element that we denote e. An element $x = (a_1, a_2, \ldots, a_n) \in A^n$ is written $x = a_1a_2\ldots a_n$ and if $y = b_1b_2\ldots b_m \in A^m$ then $xy = a_1a_2\ldots a_nb_1b_2\ldots b_n \in A^{n+m}$ with the convention $xe = ex = x$. This provides Γ with a monoïd structure. The graph structure on Γ is defined as follows: $x \leftrightarrow y$ if and only if there is $a \in A$ such that $y = xa$ or $x = ya$.

We embed Γ in $\ell^2(\Gamma)$ by identifying $x \in \Gamma$ with the characteristic function of the set $\{x\}$. Thus Γ becomes the canonical orthonormal basis of $\ell^2(\Gamma)$. In particular, linear combinations of elements of Γ are well defined elements of $\ell^2(\Gamma)$, for example $\sum_{a\in A} a$ belongs to $\ell^2(\Gamma)$ and has norm equal to $\sqrt{\nu}$.

Due to the monoïd structure of Γ, each element v of the linear subspace generated by Γ in $\ell^2(\Gamma)$ defines two bounded operators λ_v and ρ_v on $\ell^2(\Gamma)$, namely the operators of left and right multiplication by v. It is then easy to see that if

[1] Note that we use the notion of Fock space in a slightly unusual sense, since no symmetrization or anti-symmetrization is involved in its defi nition. Maybe we should say "Boltzmann-Fock space".

$v = \sum_{a \in A} a$ then the adjoint operator ρ_v^* acts as follows: if $x \in \Gamma$ then $\rho_v^* x = x'$, where $x' = 0$ if $x = e$ and x' is the unique element in Γ such that $x = x'a$ for some $a \in A$ otherwise. Thus the Laplace operator defined by (1.1) can be expressed as follows:

$$\Delta = \rho_v + \rho_v^* + e - (\nu + 1)$$

In the rest of this paper we shall not include in Δ the terms $e - (\nu + 1)$ because e is a function on Γ with support equal to $\{e\}$, hence can be considered as part of the potential, and $\nu + 1$ is a number, so has a trivial contribution to the spectrum. It will also be convenient to renormalize Δ by replacing v by a vector of norm $1/2$, hence by $v/(2\sqrt{\nu})$ if $v = \sum_{a \in A} a$.

We shall explain now how to pass from trees to Fock spaces. We use the following equality (or, rather, canonical isomorphism): if A, B are sets, then

$$\ell^2(A \times B) = \ell^2(A) \otimes \ell^2(B).$$

Thus $\ell^2(A^n) = \ell^2(A)^{\otimes n}$ if $n \geq 1$ and clearly $\ell^2(A^0) = \mathbb{C}$. Then, since the union in (1.2) is disjoint, we have

$$\ell^2(\Gamma) = \bigoplus_{n=0}^{\infty} \ell^2(A^n) = \bigoplus_{n=0}^{\infty} \ell^2(A)^{\otimes n}$$

which is the Fock space constructed over the "one particle" Hilbert space $H = \ell^2(A)$. Thus we are naturally led to the following abstract framework. Let H be a complex Hilbert space and let $\mathscr{H}$ be the Fock space associated to it:

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} H^{\otimes n}. \tag{1.3}$$

Note that H could be infinite dimensional, but this is not an important point here and in the main applications we assume it finite dimensional. We choose an arbitrary vector $u \in H$ with $\|u\| = 1$ and consider the operator $U \equiv \rho_u : \mathscr{H} \to \mathscr{H}$ defined by $Uf = f \otimes u$ if $f \in H^{\otimes n}$. It is clear that U is an isometry on $\mathscr{H}$ and the self-adjoint operator of interest for us is

$$\Delta = \text{Re}\, U = \frac{1}{2}(U + U^*), \tag{1.4}$$

our purpose being to study perturbations $L = \Delta + V$ where the conditions on V are suggested by the Fock space structure of $\mathscr{H}$. In the second part of the paper we shall replace Δ by an arbitrary self-adjoint operator in the C^*-algebra generated by U.

Translating the problem into a Fock space language does not solve it. The main point of the first part of our paper is that we treat a more general problem. The question is: given an arbitrary isometry on a Hilbert space $\mathscr{H}$ and defining Δ by (1.4), can one construct a conjugate operator for it? We also would like that this conjugate operator be relatively explicit and simple, because we should be able to use it also for perturbations L of Δ.

If U is unitary, there is no much hope to have an elegant solution to this problem. Indeed, for most unitary U the spectrum of Δ will be purely singular. On the other hand, we show that in the opposite case of completely non unitary U, there is a very simple prescription for the construction of a ”canonical” conjugate operator. Sections 2 and 3 are devoted to this question in all generality and in Section 4 we give applications in the Fock space framework.

The construction is easy and elementary. Let U be an isometry on a Hilbert space $\mathscr{H}$. We call *number operator associated to* U a self-adjoint operator N on $\mathscr{H}$ such that $UNU^* = N - 1$. The simplest examples of such operators are described in Examples 2.5 and 2.6. It is trivial then to check that, if S is the imaginary part of U, the operator $A := (SN + NS)/2$, satisfies $[\Delta, iA] = 1 - \Delta^2$, hence we have a (strict) Mourre estimate on $[-a, a]$ for each $a \in]0, 1[$.

The intuition behind this construction should be immediate for people using the positive commutator method: in Examples 2.5 and 2.6 the operator Δ is the Laplacian on $\mathbb{Z}$ or $\mathbb{N}$ respectively and S is the operator of derivation, the analog of $P = -i\frac{d}{dx}$ on $\mathbb{R}$, so it is natural to look after something similar to the position operator Q and then to consider the analog of $(PQ + QP)/2$. Note that we got such a simple prescription because we did *not* make a Fourier transform in order to realize Δ as a multiplication operator, as it is usually done when studying discrete Laplacians (e.g. in [AlF]). Note also that the relation $UNU^* = N - 1$ is a discrete version of the canonical commutation relations, cf. (2) of Lemma 2.4.

In the unitary case the existence of N is a very restrictive condition, see Example 2.5. The nice thing is that in the completely non unitary case N exists and is uniquely defined. This is an obvious fact: the formal solution of the equation $N = 1 + UNU^*$ obtained by iteration $N = 1 + UU^* + U^2U^{*2} + \dots$ exists as a densely defined self-adjoint operator if and only if $U^{*n} \to 0$ strongly on $\mathscr{H}$, which means that U is completely non unitary. Finally, observe that the operators ρ_u on the Fock space are completely non unitary, so we can apply them this construction.

Our notation N should not be confused with that used in [AlF]: our N is proportional to their $R - N + 1$, in our notations R being the particle number operator $\boldsymbol{N}$ (see below). We could have used the notation Q for our N, in view of the intuition mentioned above. We have preferred not to do so, because the number operator associated to U in the tree case has no geometric interpretation, as we explain below.

There is no essential difference between the tree model and the Fock space model, besides the fact that we tend to emphasize the geometric aspects in the first representation and the algebraic aspects in the second one. In fact, if H is a finite dimensional Hilbert space equipped with an orthonormal basis $A \subset H$ then the tree Γ associated to A can be identified with the orthonormal basis of $\mathscr{H}$ canonically associated to A, namely the set of vectors of the form $a_1 \otimes a_2 \cdots \otimes a_n$ with $a_k \in A$. In other terms, giving a tree is equivalent with giving a Fock space over a finite dimensional Hilbert space equipped with a certain orthonormal basis. However, this gives more structure than usual on a Fock space: the notions of positivity and locality inherent to the space $\ell^2(\Gamma)$ are missing in the pure Fock space situation, there is no analog of the spaces $\ell^p(\Gamma)$, etc. But our results show that this structure specific to the tree is irrelevant for the spectral and scattering properties of L.

We stress, however, that an important operator in the Fock space setting has a simple geometric interpretation in any tree version. More precisely, let $\boldsymbol{N}$ be the *particle number operator* defined on $\mathscr{H}$ by the condition $\boldsymbol{N}f = nf$ if f belongs to $H^{\otimes n}$. Clearly, if $\mathscr{H}$ is represented as $\ell^2(\Gamma)$, then $\boldsymbol{N}$ becomes the operator of multiplication by the function d, where $d(x) \equiv d(x, e)$ is the distance from the point x to the origin e (see [Gol]).

On the other hand, the number operator N associated to an isometry of the form $U = \rho_u$ is quite different from $\boldsymbol{N}$, it has not a simple geometrical meaning and is not a local operator in the tree case, unless we are in rather trivial situations like the case $\nu = 1$ (see Example 2.6). For this reason we make an effort in Section 4 to eliminate the conditions from Section 3 involving the operator N and to replace them by conditions involving $\boldsymbol{N}$. This gives us statements like that of the Theorem 1.1 below, a particular case of our main result concerning the spectral and scattering theory of the operators L.

We first have to introduce some notations. Let $\mathbf{1}_n$ and $\mathbf{1}_{\geq n}$ be the orthogonal projections of $\mathscr{H}$ onto the subspaces $H^{\otimes n}$ and $\bigoplus_{k \geq n} H^{\otimes k}$ respectively. For real s let $\mathscr{H}_{(s)}$ be the Hilbert space defined by the norm

$$\|f\|^2 = \|\mathbf{1}_0 f\|^2 + \sum_{n \geq 1} n^{2s} \|\mathbf{1}_n f\|^2.$$

If T is an operator on a finite dimensional space E then $\langle T \rangle$ is its *normalized trace*: $\langle T \rangle = \mathrm{Tr}(T)/\dim E$. We denote by $\sigma_{\text{ess}}(L)$ and $\sigma_{\text{p}}(L)$ the essential spectrum and the set of eigenvalues of L. As a consequence of Theorem 4.6, we have:

Theorem 1.1 *Assume that H is finite dimensional, choose $u \in H$ with $\|u\| = 1$, and let us set $\Delta = (\rho_u + \rho_u^*)/2$. Let V be a self-adjoint operator of the form $V = \sum_{n \geq 0} V_n \mathbf{1}_n$, with $V_n \in \mathcal{B}(H^{\otimes n})$, $\lim_{n \to \infty} \|V_n\| = 0$, and such that $\|V_n - \langle V_n \rangle\| + \|V_{n+1} - V_n \otimes \mathbf{1}_H\| \leq \delta(n)$ where δ is a decreasing function*

such that $\sum_n \delta(n) < \infty$. *Let* W *be a bounded self-adjoint operator satisfying* $\sum_n \|W\mathbf{1}_{\geq n}\| < \infty$. *We set* $L_0 = \Delta + V$ *and* $L = L_0 + W$. *Then:*
(1) $\sigma_{\rm ess}(L) = [-1, +1]$*;*
(2) *the eigenvalues of* L *distinct from* ± 1 *are of finite multiplicity and can accumulate only toward* ± 1*;*
(3) *if* $s > 1/2$ *and* $\lambda \notin \kappa(L) := \sigma_{\rm p}(L) \cup \{\pm 1\}$, *then* $\lim_{\mu \to 0}(L - \lambda - i\mu)^{-1}$ *exists in norm in* $\mathcal{B}(\mathscr{H}_{(s)}, \mathscr{H}_{(-s)})$, *locally uniformly in* $\lambda \in \mathbb{R} \setminus \kappa(L)$*;*
(4) *the wave operators for the pair* (L, L_0) *exist and are complete.*

These results show a complete analogy with the standard two body problem on an Euclidean space, the particle number operator $\boldsymbol{N}$ playing the rôle of the position operator. Note that V, W are the analogs of the long range and short range components of the potential. See Proposition 4.4 for a result of a slightly different nature, covering those from [AlF]. Our most general results in the Fock space setting are contained in Theorem 4.6.

The second part of the paper (Section 5) is devoted to a problem of a completely different nature. Our purpose is to compute the essential spectrum of a general class of operators on a Fock space in terms of their "localizations at infinity", as it was done in [GeI] for the case when Γ is an abelian locally compact group.

The basic idea of [GeI] is very general and we shall use it here too: the first step is to isolate the class of operators we want to study by considering the C^*-algebra $\mathscr{C}$ generated by some elementary Hamiltonians and the second one is to compute the quotient of $\mathscr{C}$ with respect to the ideal $\mathscr{C}_0 = \mathscr{C} \cap \mathcal{K}(\mathscr{H})$ of compact operators belonging to $\mathscr{C}$. Then, if $L \in \mathscr{C}$ the projection $\widehat{L}$ of L in the quotient $\mathscr{C}/\mathscr{C}_0$ is the localization of L at infinity we need (or the set of such localizations, depending on the way the quotient is represented). The interest of $\widehat{L}$ comes from the relation $\sigma_{\rm ess}(L) = \sigma(\widehat{L})$. In all the situations studied in [GeI] these localizations at infinity correspond effectively with what we would intuitively expect.

We stress that both steps of this approach are non trivial in general. The algebra $\mathscr{C}$ must be chosen with care, if it is too small or too large then the quotient will either be too complicated to provide interesting information, or the information we get will be less precise than expected. Moreover, there does not seem to be many techniques for the effective computation of the quotient. One of the main observations in [GeI] is that in many situations of interest in quantum mechanics the configuration space of the system is an abelian locally compact group and then the algebras of interest can be constructed as crossed products; in such a case there is a systematic procedure for computing the quotient.

The techniques from [GeI] cannot be used in the situations of interest here, because the monoïd structure of the tree is not rich enough and in the Fock space version the situation is even worse. However, a natural C^*-algebra of anisotropic operators associated to the hyperbolic compactification of a tree has been pointed out in [Gol]. This algebra contains the compact operators on $\ell^2(\Gamma)$ and an embedding of the quotient algebra into a tensor product, which allows the computation of the essential spectrum, has also been described in [Gol]. In Section 5 and in the Appendix we shall improve these results in two directions: we consider more general types of anisotropy and we develop new abstract techniques for the computation of the quotient algebra. To clarify this, we give an example below.

We place ourselves in the Fock space setting with H finite dimensional and we fix a vector $u \in H$ and the isometry U associated to it. We are interested in self-adjoint operators of the form $L = D + V$ where D is a "continuous function" of U and U^*, i.e. it belongs to the C^*-algebra $\mathscr{D}$ generated by U, and V is of the form $\sum V_n \mathbf{1}_n$ where V_n are bounded operators on $H^{\otimes n}$ and are asymptotically constant in some sense (when $n \to \infty$). In order to get more precise results, we make more specific assumptions on the operators V_n.

Let $\mathcal{A} \subset \mathcal{B}(H)$ be a C^*-algebra with $\mathbf{1}_H \in \mathcal{A}$. Let $\mathscr{A}_{\rm vo}$ be the set of operators V as above such that $V_n \in \mathcal{A}^{\otimes n}$, $\sup \|V_n\| < \infty$ and $\|V_n - V_{n-1} \otimes \mathbf{1}_H\| \to 0$ as $n \to \infty$. If $\nu = 1$, i.e. in the setting of Example 2.6, $\mathscr{A}_{\rm vo}$ is the algebra of bounded sequences of vanishing oscillation at infinity. We mention that the C^*-algebra of bounded continuous functions with vanishing oscillation at infinity on a group has first been considered in the context of [GeI] in [Man] (cf. also references therein).

Observe that the algebras $\mathcal{A}^{\otimes n}$ are embedded in the infinite tensor product C^*-algebra $\mathcal{A}^{\otimes\infty}$. Thus we may also introduce the C^*-subalgebra $\mathscr{A}_\infty$ of $\mathscr{A}_{\rm vo}$ consisting of the operators V such that $V_\infty := \lim V_n$ exists in norm in $\mathcal{A}^{\otimes\infty}$. Note that the subset $\mathscr{A}_0$ of operators V such that $\lim V_n = 0$ is an ideal of $\mathscr{A}_{\rm vo}$.

The algebras of Hamiltonians of interest for us can now be defined as the C^*-algebras $\mathscr{C}_{\rm vo}$ and $\mathscr{C}_\infty$ generated by the operators of the form $L = D + V$ where D is a polynomial in U, U^* and $V \in \mathscr{A}_{\rm vo}$ or $V \in \mathscr{A}_\infty$ respectively. Let us denote $\mathscr{C}_0 = \mathscr{C}_{\rm vo} \cap \mathcal{K}(\mathscr{H})$. Below we assume H of dimension at least 2, see Proposition A.5 for the one dimensional case.

Theorem 1.2 *There are canonical isomorphisms*

$$\mathscr{C}_{\rm vo}/\mathscr{C}_0 \simeq (\mathscr{A}_{\rm vo}/\mathscr{A}_0) \otimes \mathscr{D}, \quad \mathscr{C}_\infty/\mathscr{C}_0 \simeq \mathcal{A}^{\otimes\infty} \otimes \mathscr{D}. \tag{1.5}$$

For applications in the computation of the essential spectrum, see Propositions 5.15 and 5.16. For example, if $D \in \mathscr{D}$ and $V \in \mathscr{A}_\infty$ are self-adjoint operators and $L = D + V$, then

$$\sigma_{\rm ess}(L) = \sigma(D) + \sigma(V_\infty). \tag{1.6}$$

The localization of L at infinity in this case is $\widehat{L} = 1 \otimes D + V_\infty \otimes 1$.

To cover perturbations of the Laplacian on a tree by functions V, it suffices to consider an abelian algebra $\mathcal{A}$, see Example 5.13. In this case, if A is the spectrum of $\mathcal{A}$, then $\mathcal{A}^{\otimes\infty} = C(A^\infty)$ where $A^\infty = A^{\mathbb{N}}$ is a compact topological space with the product topology, and then we can speak of the set of localizations at infinity of L. Indeed, we have then

$$\mathcal{A}^{\otimes\infty} \otimes \mathscr{D} \simeq C(A^\infty, \mathscr{D}),$$

hence $\widehat{L}$ is a continuous map $\widehat{L} : A^\infty \to \mathscr{D}$ and we can say that $\widehat{L}(x)$ is the localization of L at the point $x \in A^\infty$ on the boundary at infinity of the tree (or in the direction x). More explicitly, if $L = D+V$ as above, then $\widehat{L}(x) = D+V_\infty(x)$.

Plan of the paper: The notion of number operator associated to an isometry is introduced and studied in Section 2. The spectral theory of the operators L is studied via the Mourre estimate in Section 3: after some technicalities in the first two subsections, our main abstract results concerning these matters can be found in Subsection 3.3 and the applications in the Fock space setting in Subsection 4.2. Section 5 is devoted to the study of several C^*-algebras generated by more general classes of anisotropic Hamiltonians on a Fock space. Subsections 5.1 and 5.2 contain some preparatory material which is used in Subsection 5.3 in order to prove our main result in this direction, Theorem 5.10. The Appendix, concerned with the representability of some C^*-algebras as tensor products, is devoted to an important ingredient of this proof. The case $\nu = 1$, which is simpler but not covered by the techniques of Section 5, is treated at the end of the Appendix.

Notations: $\mathcal{B}(\mathscr{H})$, $\mathcal{K}(\mathscr{H})$ are the spaces of bounded or compact operators on a Hilbert space $\mathscr{H}$. If S, T are operators such that $S - T \in \mathcal{K}(\mathscr{H})$, we write $S \approx T$. If S, T are quadratic forms with the same domain and $S - T$ is continuous for the topology of $\mathscr{H}$, we write $S \sim T$. $\mathcal{D}(T)$ is the domain of the operator T. We denote by 1 the identity of a unital algebra, but for the clarity of the argument we sometimes adopt a special notation, e.g. the identity operator on $\mathscr{H}$ could be denoted $\mathbf{1}_{\mathscr{H}}$. A *morphism* between two C^*-algebras is a $*$-homomorphism and an *ideal* of a C^*-algebra is a closed bilateral ideal.

Acknowledgments: We are grateful to George Skandalis for a very helpful conversation related to the questions we treat in the Appendix (see the comments before Proposition A.2 and in its proof).

2 Number operator associated to an isometry

2.1 Definition and first examples

Let U be an isometry on a Hilbert space $\mathscr{H}$. Thus $U^*U = 1$ and UU^* is the (orthogonal) projection onto the closed subspace $\operatorname{ran} U = U\mathscr{H}$, hence $P_0 := [U^*, U] = 1 - UU^*$ is the projection onto $(\operatorname{ran} U)^\perp = \ker U^*$.

Definition 2.1 *A* number operator associated to U *is a self-adjoint operator* N *satisfying* $UNU^* = N - 1$.

In fact, N *is a number operator for* U *if and only if* $U^*\mathcal{D}(N) \subset \mathcal{D}(N)$ *and* $UNU^* = N - 1$ *holds on* $\mathcal{D}(N)$. Indeed, this means $N - 1 \subset UNU^*$ and $N - 1$ is a self-adjoint operator, so it cannot have a strict symmetric extension.

In this section we discuss several aspects of this definition. If the operator U is unitary (situation of no interest in this paper), then U^kNU^{-k} is a well defined self-adjoint operator for each $k \in \mathbb{Z}$ and the equality $UNU^* = N-1$ is equivalent to $U^kNU^{-k} = N - k$ for all $k \in \mathbb{Z}$. In particular, a number operator associated to a unitary operator cannot be semibounded. Example 2.5 allows one to easily understand the structure of a unitary operator which has an associated number operator.

Note that if U is unitary, than N does not exist in general and if it exists, then it is not unique, since $N + \lambda$ is also a number operator for each real λ. On the other hand, we will see in the Subsection 2.2 that N exists, is positive and is uniquely defined if U is a completely non unitary isometry.

In order to express Definition 2.1 in other, sometimes more convenient, forms, we recall some elementary facts. If A, B are linear operators on $\mathscr{H}$ then the domain of AB is the set of $f \in \mathcal{D}(B)$ such that $Bf \in \mathcal{D}(A)$. It is then clear that if A is closed and B is bounded, then AB is closed, but in general BA is not. However, if B is isometric, then BA is closed. Thus, if N is self-adjoint and U is isometric, then UNU^* is a closed symmetric operator.

Lemma 2.2 *Let* N *be a number operator associated to* U. *Then* $\mathcal{D}(N)$ *is stable under* U *and* U^* *and we have* $NU = U(N+1)$ *and* $NU^* = U^*(N-1)$. *Moreover,* $\operatorname{ran} P_0 \subset \ker(N-1)$ *and* $NP_0 = P_0N = P_0$.

Proof: From $UNU^* = N - 1$ and $U^*U = 1$ we get $U^*\mathcal{D}(N) \subset \mathcal{D}(N)$ and $NU^* = U^*(N-1)$ on the domain on N. Moreover, since $U^*P_0 = 0$, we have $P_0\mathscr{H} \subset \mathcal{D}(UNU^*) = \mathcal{D}(N)$ and $(N-1)P_0 = 0$, so $NP_0 = P_0$, which clearly implies $P_0N = P_0$. If $f, g \in \mathcal{D}(N)$ then

$$\langle (N-1)f, Ug\rangle = \langle U^*(N-1)f, g\rangle = \langle NU^*f, g\rangle = \langle f, UNg\rangle$$

hence $Ug \in \mathcal{D}(N^*) = \mathcal{D}(N)$ and $UNg = (N-1)Ug$. Thus $U\mathcal{D}(N) \subset \mathcal{D}(N)$ and $NU = U(N+1)$ on the domain on $\mathcal{D}(N)$. If $f \in \mathscr{H}$ and $Uf \in \mathcal{D}(N)$ then $f = U^*Uf \in \mathcal{D}(N)$, so we have $NU = U(N+1)$ as operators. If $f \in \mathscr{H}$ and $U^*f \in \mathcal{D}(N)$ then $UU^*f \in \mathcal{D}(N)$ and $P_0 f \in \mathcal{D}(N)$, so $f = UU^*f + P_0 f$ belongs to $\mathcal{D}(N)$, hence $NU^* = U^*(N-1)$ as operators. ■

Note that the relation $NU = U(N+1)$ can also be written $[N, U] = U$. Reciprocally, we have:

Lemma 2.3 *If a self-adjoint operator N satisfies $[N, U] = U$ in the sense of forms on $\mathcal{D}(N)$ and $P_0 N = P_0$ on $\mathcal{D}(N)$, then N is a number operator associated to U.*

Proof: The first hypothesis means $\langle Nf, Ug\rangle - \langle U^*f, Ng\rangle = \langle f, Ug\rangle$ for all f, g in $\mathcal{D}(N)$. But this clearly implies $U^*f \in \mathcal{D}(N)$ and $NU^*f = U^*(N-1)f$ for all $f \in \mathcal{D}(N)$. Then we get

$$UNU^*f = UU^*(N-1)f = (N-1)f - P_0(N-1)f = (N-1)f$$

for all such f, so N is a number operator by the comment after Definition 2.1. ■

Observe that by induction we get $[N, U^n] = nU^n$, hence $\|[N, U^n]\| = n$ if $U \neq 0$. In particular, N is not a bounded operator.

Lemma 2.4 *If N is a self-adjoint operator, then the condition $[N, U] = U$ in the sense of forms on $\mathcal{D}(N)$ is equivalent to each of the following ones:*

(1) $U\mathcal{D}(N) \subset \mathcal{D}(N)$*and* $[N, U] = U$ *as operators on* $\mathcal{D}(N)$*;*

(2) $e^{itN}Ue^{-itN} = e^{it}U$ *for all* $t \in \mathbb{R}$*;*

(3) $\varphi(N)U = U\varphi(N+1)$ *for all* $\varphi : \mathbb{R} \to \mathbb{C}$ *bounded and Borel.*

Proof: The implications (3) $\Rightarrow$ (2) and (1) $\Rightarrow$ (0) are immediate, condition (0) being that $[N, U] = U$ in the sense of forms on $\mathcal{D}(N)$. If (0) holds, then for all $f, g \in \mathcal{D}(N)$ one has $\langle Nf, Ug\rangle - \langle f, UNg\rangle = \langle f, Ug\rangle$. This gives us $Ug \in \mathcal{D}(N^*) = \mathcal{D}(N)$, hence we get (1). If (2) is satisfied then $\langle e^{-itN}f, Ue^{-itN}g\rangle = e^{it}\langle f, Ug\rangle$ for all $f, g \in \mathcal{D}(N)$, so by taking the derivatives at $t = 0$, we get (0). If (1) holds then by using $NU = U(N+1)$ we get $(N+z)^{-1}U = U(1+N-z)^{-1}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$, hence by standard approximation procedures we obtain (3). ■

It is easy to check that the map $\mathscr{U}$ defined by $S \mapsto USU^*$ is a morphism of $\mathcal{B}(\mathscr{H})$ onto $\mathcal{B}(U\mathscr{H})$. We identify $\mathcal{B}(U\mathscr{H})$ with the C^*-subalgebra of $\mathcal{B}(\mathscr{H})$ consisting of the operators T such that $TP_0 = P_0T = 0$; note that $P_0^\perp$ is the identity of the algebra $\mathcal{B}(U\mathscr{H})$ and that the linear positive map $T \mapsto U^*TU$ is a right-inverse for $\mathscr{U}$. Clearly

$$U\varphi(N)U^* = \varphi(N-1)P_0^\perp \quad \text{for all bounded Borel functions} \quad \varphi : \mathbb{R} \to \mathbb{C}. \tag{2.1}$$

By standard approximation procedures we now see that each of the following conditions is necessary and sufficient in order that N be a number operator associated to U: (i) $Ue^{itN}U^* = e^{-it}e^{itN}P_0^\perp$ for all $t \in \mathbb{R}$; (ii) $U(N-z)^{-1}U^* = (N-1-z)^{-1}P_0^\perp$ for some $z \in \mathbb{C} \setminus \mathbb{R}$.

We now give the simplest examples of number operators.

Example 2.5 Let $\mathscr{H} = \ell^2(\mathbb{Z})$ and $(Uf)(x) = f(x-1)$. If $\{e_n\}$ is the canonical orthonormal basis of $\mathscr{H}$ then $Ue_n = e_{n+1}$. It suffices to define N by the condition $Ne_n = ne_n$. Any other number operator is of the form $N + \lambda$ for some real λ. It is an easy exercise to show that if (U, N) is an abstract irreducible couple consisting of a unitary operator U and a self-adjoint operator N such that $[N, U] = U$ in the sense of forms on $\mathcal{D}(N)$, then there is a unique real λ such that this couple is unitarily equivalent to the couple $(U, N + \lambda)$ constructed above.

Example 2.6 Let $\mathscr{H} = \ell^2(\mathbb{N})$ and U as above. Then $U^*e_n = e_{n-1}$ with $e_{-1} = 0$, so $P_0 = |e_0\rangle\langle e_0|$. We obtain a number operator by defining $Ne_n = (n+1)e_n$ and it is easy to see that this is the only possibility. We shall prove this in a more general context below.

2.2 Completely non unitary isometries

An isometry U is called *completely non unitary* if $\text{s-}\lim_{k\to\infty} U^{*k} = 0$. This is equivalent to the fact that the only closed subspace $\mathscr{K}$ such that $U\mathscr{K} = \mathscr{K}$ is $\mathscr{K} = \{0\}$. We introduce below several objects naturally associated to such an isometry, see [Bea].

Consider the decreasing sequence $\mathscr{H} = U^0\mathscr{H} \supset U^1\mathscr{H} \supset U^2\mathscr{H} \supset \dots$ of closed subspaces of $\mathscr{H}$. Since U^k is an isometric operator with range $U^k\mathscr{H}$, the operator $P^k := U^kU^{*k}$ is the orthogonal projection of $\mathscr{H}$ onto $U^k\mathscr{H}$ and we have $1 = P^0 \geq P^1 \geq P^2 \dots$ and $\text{s-}\lim_{k\to\infty} P^k = 0$, because $\|P^kf\| = \|U^{*k}f\| \to 0$.

Recall that $P_0 = 1 - UU^* = 1 - P^1$ is the projection onto $\ker U^*$. More generally, let $\mathscr{H}_k$ be the closed subspace

$$\mathscr{H}_k = \ker U^{*k+1} \ominus \ker U^{*k} = \operatorname{ran} U^k \ominus \operatorname{ran} U^{k+1} = U^k(\ker U^*)$$

and let P_k be the projection onto it, so

$$P_k = P^k - P^{k+1} = U^kU^{*k} - U^{k+1}U^{*k+1} = U^kP_0U^{*k}$$

Notice that $P_{k+1} = UP_kU^*$, hence $UP_k = P_{k+1}U$, and

$$P_kP_m = 0 \text{ if } k \neq m \text{ and } \sum_{k=0}^{\infty} P_k = 1. \tag{2.2}$$

We have $\dim \mathscr{H}_k = \dim \mathscr{H}_0 \neq 0$ for all $k \in \mathbb{N}$. Indeed, it suffices to show that $U_k := U|_{\mathscr{H}_k} : \mathscr{H}_k \to \mathscr{H}_{k+1}$ is a bijective isometry with inverse equal to $U^*|_{\mathscr{H}_{k+1}}$.

In fact, from $UP_k = P_{k+1}U$ we get $U\mathscr{H}_k \subset \mathscr{H}_{k+1}$ so U_k is isometric from $\mathscr{H}_k$ to $\mathscr{H}_{k+1}$. To prove surjectivity, note that $U^*P_{k+1} = P_kU^*$, hence $U^*\mathscr{H}_{k+1} \subset \mathscr{H}_k$ and $UU^*P_{k+1} = UP_kU^* = P_{k+1}$. Thus $U_k : \mathscr{H}_k \to \mathscr{H}_{k+1}$ is bijective and its inverse is $U^*|_{\mathscr{H}_{k+1}}$.

Proposition 2.7 *If U is a completely non unitary isometry then there is a unique number operator associated to it, and we have*

$$N \equiv N_U = \sum_{k=0}^{\infty} P^k = \sum_{k=0}^{\infty} (k+1)P_k, \tag{2.3}$$

the sums being interpreted in form sense. Thus each $k+1$, with $k \in \mathbb{N}$, is an eigenvalue of N_U of multiplicity equal to $\dim \ker U^*$ *and $\mathscr{H}_k$ is the corresponding eigenspace.*

Proof: Since $P_k = P^k - P^{k+1}$, the two sums from (2.3) are equal and define a self-adjoint operator N_U with $\mathbb{N} + 1$ as spectrum and $\mathscr{H}_k$ as eigenspace of the eigenvalue $k+1$. Since $UP_k = P_{k+1}U$, condition (3) of Lemma 2.4 is clearly verified, hence N_U is a number operator for U by Lemma 2.3. Of course, one can also check directly that the conditions of the Definition 2.1 are satisfied. It remains to show uniqueness.

It is clear that an operator N is a number operator if and only if it is of the form $N = M + 1$ where M is a self-adjoint operator such that $M = UU^* + UMU^*$. With a notation introduced above, this can be written $M = UU^* + \mathscr{U}(M)$ hence we get a unique formal solution by iteration: $M = \sum_{k \geq 0} \mathscr{U}^k(UU^*) = \sum_{k\geq 1} P^k$ which gives (2.3). In order to make this rigorous, we argue as follows.

Recall that, by Lemma 2.2, U and U^* leave invariant the domain of M. Hence by iteration we have on $\mathcal{D}(M)$:

$$M = P^1 + UMU^* = P^1 + UP^1U^* + U^2MU^{*2} = P^1 + P^2 + \ldots + P^n + U^nMU^{*n}$$

for all $n \in \mathbb{N}$. It is clear that $P^m\mathcal{D}(M) \subset \mathcal{D}(M)$ for all m and $(1 - P^n)U^n = U^{*n}(1 - P^n) = 0$, hence

$$M(1 - P^n) = (1 - P^n)M = \sum_{1 \leq k \leq n-1} P^k(1 - P^n) = \sum_{1 \leq k \leq n-1} kP^k$$

Then $MP_k = P_kM = kP_k$ for all $k \in \mathbb{N}$, hence $M = \sum_k kP_k$. ■

3 The Mourre estimate

3.1 The free case

Our purpose in this section is to construct a conjugate operator A and to establish a Mourre estimate for the “free” operator

$$\Delta := \text{Re }(U) = \frac{1}{2}(U + U^*) \tag{3.1}$$

where U is an isometry which admits a number operator N on a Hilbert space $\mathscr{H}$. The operator A will be constructed in terms of N and of the imaginary part of U:

$$S := \text{Im }(U) = \frac{1}{2}(U - U^*). \tag{3.2}$$

More precisely, we define A as the closure of the operator

$$A_0 = \frac{1}{2}(SN + NS), \quad \mathcal{D}(A_0) = \mathcal{D}(N). \tag{3.3}$$

We shall prove below that A_0 is essentially self-adjoint and we shall determine the domain of A. That A_0 is not self-adjoint is clear in the situations considered in Examples 2.5 and 2.6. Note that in these examples S is an analog of the derivation operator. Before, we make some comments concerning the operators introduced above.

We have $U = \Delta + iS$ and $\|\Delta\| = \|S\| = 1$. In fact, by using [Mur, Theorem 3.5.17] in case U is not unitary and (2) of Lemma 2.4 if U is unitary, we see that $\sigma(\Delta) = \sigma(S) = [-1, 1]$. By Lemma 2.2 the polynomials in U, U^* (hence in Δ, S) leave invariant the domain of N. If not otherwise mentioned, the computations which follow are done on $\mathcal{D}(N)$ and the equalities are understood to hold on $\mathcal{D}(N)$. The main relations

$$NU = U(N + 1) \text{ and } NU^* = U^*(N - 1) \tag{3.4}$$

will be frequently used without comment. In particular, this gives us

$$[N, S] = -i\Delta \text{ and } [N, \Delta] = iS \tag{3.5}$$

These relations imply that Δ and S are of class $C^\infty(N)$ (we use the terminology of [ABG]). We also have

$$[U, \Delta] = -P_0/2, \quad [U^*, \Delta] = P_0/2, \quad [S, \Delta] = iP_0/2. \tag{3.6}$$

A simple computation gives then:

$$\Delta^2 + S^2 = 1 - P_0/2. \tag{3.7}$$

It follows that we have on the domain of N:

$$A_0 = NS + \frac{i}{2}\Delta = SN - \frac{i}{2}\Delta = \frac{1}{2i}\big((N-\frac{1}{2})U - U^*(N-\frac{1}{2})\big). \tag{3.8}$$

Remark: If we denote $a = iU^*(N - 1/2)$ then on the domain of N we have $A = (a + a^*)/2$. Note that a looks like a bosonic annihilation operator (the normalization with respect to N being, however, different) and that

$$aa^* = (N+1/2)^2,\ a^*a = (N-1/2)^2 P_0^\perp,\ [a,a^*] = 2N + P_0/4,\ [N,a] = a.$$

Lemma 3.1 *A is self-adjoint with $\mathcal{D}(A) = \mathcal{D}(NS) = \{f \in \mathscr{H} \mid Sf \in \mathcal{D}(N)\}$.*

Proof: Note that NS is closed on the specified domain and that $\mathcal{D}(N) \subset \mathcal{D}(NS)$, because $S\mathcal{D}(N) \subset \mathcal{D}(N)$. Let us show that $\mathcal{D}(N)$ is dense in $\mathcal{D}(NS)$ (i.e. NS is the closure of $NS|\mathcal{D}(N)$). Let $f \in \mathcal{D}(NS)$, then $f_\varepsilon = (1 + i\varepsilon N)^{-1} f \in \mathcal{D}(N)$ and $\|f_\varepsilon - f\| \to 0$ when $\varepsilon \to 0$. Then, since $S \in C^1(N)$:

$$\begin{aligned}
NSf_\varepsilon &= NS(1+i\varepsilon N)^{-1}f \\
&= N(1+i\varepsilon N)^{-1}[i\varepsilon N, S](1+i\varepsilon N)^{-1}f + N(1+i\varepsilon N)^{-1}Sf \\
&= \varepsilon N(1+i\varepsilon N)^{-1}\Delta(1+i\varepsilon N)^{-1}f + (1+i\varepsilon N)^{-1}NSf.
\end{aligned}$$

The last term converges to NSf as ε tends to 0. So it suffices to observe that $\varepsilon N(1 + i\varepsilon N)^{-1} \to 0$ strongly as $\varepsilon \to 0$.

Let $A_0 = SN - i\Delta/2$, $\mathcal{D}(A_0) = \mathcal{D}(N)$. It is trivial to prove that $A_0^* = NS + i\Delta/2$, $\mathcal{D}(A_0^*) = \mathcal{D}(NS)$. By what we proved and the fact that $A_0^*|_{\mathcal{D}(N)} = A_0$, we see that A_0^* is the closure of A_0. So A_0 is essentially self-adjoint. ■

The next proposition clearly implies the Mourre estimate for Δ outside ± 1.

Proposition 3.2 *$\Delta \in C^\infty(A)$ and $[\Delta, iA] = 1 - \Delta^2 = S^2 + P_0/2$.*

Proof: On $\mathcal{D}(N)$ we have

$$\begin{aligned}
[\Delta, iA] &= [\Delta, iNS] = [\Delta, iN]S + N[\Delta, iS] \\
&= S^2 + NP_0/2 = S^2 + P_0/2 = 1 - \Delta^2,
\end{aligned}$$

which implies $\Delta \in C^\infty(A)$ by an obvious induction argument. ■

We mention two other useful commutation relations:

$$[iA, S] = \text{Re}\,(S\Delta) \quad \text{and} \quad [iA, N] = -\text{Re}\,(N\Delta). \tag{3.9}$$

Indeed:

$$[iA, S] = [iSN + \frac{1}{2}\Delta, S] = iS[N, S] + \frac{1}{2}[\Delta, S] = S\Delta + \frac{1}{2}[\Delta, S]$$

and

$$[iA, N] = [iSN + \frac{1}{2}\Delta, N] = [iS, N]N + \frac{1}{2}[\Delta, N] = -\Delta N + \frac{1}{2}[\Delta, N].$$

3.2 Commutator bounds

The following abbreviations will be convenient. For $T \in \mathcal{B}(\mathscr{H})$ we set $\dot{T} \equiv T^{\cdot} = [iN, T]$, interpreted as a form on $\mathcal{D}(N)$, and $T' = [S, T]$, $T_\Delta = [\Delta, T]$, which are bounded operators on $\mathscr{H}$. Iterated operations like $\ddot{T} \equiv T^{\cdot\cdot}$, $T'^{\cdot}$ or $\dot{T}' \equiv T^{\cdot\prime}$ are obviously defined. Note that

$$\dot{T}' - T'^{\cdot} = [S, [iN, T]] - [iN, [S, T]] = [T, [iN, S]]] = -T_\Delta \tag{3.10}$$

because of the Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ and (3.5).

If T is a bounded operator then both NT and TN are well defined quadratic forms with domain $\mathcal{D}(N)$. We write $\|NT\| = \infty$, for example, if NT is not continuous for the topology of $\mathscr{H}$. If NT is continuous, then $T\mathcal{D}(N) \subset \mathcal{D}(N)$ and the operator NT with domain $\mathcal{D}(N)$ extends to a unique bounded operator on $\mathscr{H}$ which will also be denoted NT and whose adjoint is the continuous extension of T^*N to $\mathscr{H}$. If $T^* = \pm T$ then the continuity of NT is equivalent to that of TN. Such arguments will be used without comment below.

Proposition 3.3 *For each $V \in \mathcal{B}(\mathscr{H})$ we have, in the sense of forms on $\mathcal{D}(N)$,*

$$[iA, V] = \dot{V}S + iNV' - \frac{1}{2}V_\Delta. \tag{3.11}$$

In particular

$$\|[iA, V]\| \le \|\dot{V}\| + \|NV'\| + \frac{1}{2}\|V\|. \tag{3.12}$$

Moreover, for the form $[iA, [iA, V]]$ with domain $\mathcal{D}(N^2)$, we have

$$\begin{aligned} \frac{1}{4}\|\,[iA, [iA, V]]\,\| &\le \|V\| + \|\dot{V}\| + \|\ddot{V}\| + \|V'\| \\ &+ \|NV'\| + \|NV_\Delta\| + \|N\dot{V}'\| + \|N^2V''\|. \end{aligned} \tag{3.13}$$

Proof: The relation (3.11) follows immediately from $A = iNS - \frac{1}{2}\Delta$. For the second commutator, note that $A\mathcal{D}(N^2) \subset \mathcal{D}(N)$, hence in the sense of forms on $\mathcal{D}(N^2)$ we have:

$$\begin{aligned} [iA, [iA, V]] &= [iA, \dot{V}S] + [iA, iNV'] - \frac{1}{2}[iA, V_\Delta] \\ &= [iA, \dot{V}]S + \dot{V}[iA, S] + [iA, iN]V' + iN[iA, V'] - \frac{1}{2}[iA, V_\Delta]. \end{aligned}$$

By (3.9) we have $\|\dot{V}[iA, S]\| \le \|\dot{V}\|$ and then (3.5) gives

$$\begin{aligned} [iA, iN]V' &= -i\mathrm{Re}\,(N\Delta)V' = -\frac{i}{2}(N\Delta V' + \Delta NV') \\ &= -\frac{i}{2}[N, \Delta]V' - i\Delta NV' = \frac{1}{2}SV' - i\Delta NV' \end{aligned}$$

Thus, we have

$$\|[iA,[iA,V]]-[iA,\dot{V}]S-iN[iA,V']+\frac{1}{2}[iA,V_\Delta]\| \le \|\dot{V}\|+\|V'\|/2+\|NV'\|.$$

We now apply (3.11) three times with V replaced successively by $\dot{V}$, V' and V_Δ. First, we get

$$\|[iA,\dot{V}]S\| = \|\ddot{V}S^2 + iN\dot{V}'S - \dot{V}_\Delta S/2\| \le \|\ddot{V}\| + \|N\dot{V}'\| + \|\dot{V}\|.$$

Then, by using also (3.10) and the notation $V'_\Delta = (V')_\Delta$, we get

$$N[iA,V'] = NV'^{\cdot}S + iN^2V'' - NV'_\Delta/2 = N(\dot{V}'+V_\Delta)S + iN^2V'' - NV'_\Delta/2.$$

Now (3.5) gives

$$NV'_\Delta = N\Delta V' - NV'\Delta = [N,\Delta]V' + [\Delta, NV'] = iSV' + [\Delta, NV']$$

hence

$$\|N[iA,V']\| \le \|N\dot{V}'\| + \|NV_\Delta\| + \|N^2V''\| + \|V'\|/2 + \|NV'\|.$$

Then

$$[iA,V_\Delta] = (V_\Delta)^{\cdot} + iN(V_\Delta)' - (1/2)V_{\Delta\Delta}.$$

The first two terms on the right hand side are estimated as follows:

$$(V_\Delta)^{\cdot} = [iN,[\Delta,V]] = -[\Delta,[V,iN]] - [V,[iN,\Delta]] = [\Delta,\dot{V}] + [V,S]$$

and

$$\begin{aligned} N(V_\Delta)' &= N[S,[\Delta,V]] = -N[\Delta,[V,S]] - N[V,[S,\Delta]] = N[\Delta,V'] \\ &- \frac{i}{2}N[V,P_0] = [N,\Delta]V' + \Delta NV' - NV'\Delta - \frac{i}{2}N[V,P_0] \\ &= iSV' + [\Delta,NV'] - \frac{i}{2}N[V,P_0]. \end{aligned}$$

Since $NP_0 = P_0$ we have

$$N[V,P_0] = NVP_0 - NP_0V = [N,V]P_0 + VNP_0 - NP_0V = -i\dot{V} + [V,P_0].$$

hence we get

$$\|[iA,V_\Delta]\| \le 5\|V\| + (5/2)\|\dot{V}\| + \|V'\| + \|NV'\|.$$

Adding all these estimates we get a more precise form of the inequality (3.13). ■

The following result simplifies later computations. The notation $X \sim Y$ means that X,Y are quadratic forms on the domain of N or N^2 and $X-Y$ extends to a bounded operator. *From now on we suppose* $0 \notin \sigma(N)$. In fact, in the case of interest for us we have $N \ge 1$.

Lemma 3.4 *Let V be a bounded self-adjoint operator. If $[U,V]N$ is bounded, then $[U^*,V]N$ is bounded, so $\|NV'\| + \|NV_\Delta\| < \infty$. If $[U,V]N$ is compact, then $[U^*,V]N$ is compact, so NV' is compact. If $\dot{V}$ and $[U,\dot{V}]N$ are bounded, then $\|N\dot{V}'\| < \infty$. If $[U,[U,V]]N^2$ is bounded, then $\|N^2V''\| < \infty$.*

Proof: We have

$$N = UU^*N + P_0N = U(N+1)U^* + P_0 \tag{3.14}$$

hence

$$[U^*,V]N = U^*[V,U](N+1)U^* + [U^*,V]P_0, \tag{3.15}$$

which proves the first two assertions. The assertion involving $\dot{V}$ is a particular case, because $\dot{V}$ is self-adjoint if it is bounded.

For the rest of the proof we need the following relation:

$$N = P_0 + 2P_1 + U^2(N+2)U^{*2}. \tag{3.16}$$

This follows easily directly from the definition of N:

$$\begin{aligned} N &= 1 + UNU^* = 1 + U(1+UNU^*)U^* = 1 + UU^* + U^2NU^{*2} \\ &= (1-UU^*) + 2(UU^* - U^2U^{*2}) + U^2(N+2)U^{*2}. \end{aligned}$$

Since $P_kU^2 = U^{*2}P_k = 0$ for $k = 0,1$, we get from (3.17):

$$N^2 = P_0 + 4P_1 + U^2(N+2)^2U^{*2}. \tag{3.17}$$

We clearly have:

$$-4N^2V'' = N^2[U^*,[U^*,V]] + N^2[U,[U,V]] - N^2([U^*,[U,V]] + [U,[U^*,V]]$$

We shall prove that the three terms from the right hand side are bounded. Since $N^2[U^*,[U^*,V]] = ([U,[U,V]]N^2)^*$, this is trivial for the first one. The second term is the adjoint of $[U^*,[U^*,V]]N^2$ and due to (3.17) we have

$$\begin{aligned} [U^*,[U^*,V]]N^2 &= (U^{*2}V - 2U^*VU^* + VU^{*2})N^2 \\ &\sim (U^{*2}V - 2U^*VU^* + VU^{*2})U^2(N+2)^2U^{*2} \\ &= U^{*2}[U,[U,V]](N+2)^2U^{*2}, \end{aligned}$$

hence we have the required boundedness. Finally, the third term is the adjoint of $([U,[U^*,V]] + [U^*,[U,V]])N^2$ and by a simple computation this is equal to

$$2(V - UVU^* - U^*VU + VUU^*)N^2 \sim -2U^*[U,[U,V]](N+1)^2U^*$$

where we used $N^2 = UU^*N^2 + P_0N^2 = U(N+1)^2U^* + P_0$. ■

If the right hand side of the relation (3.12) or (3.13) is finite, then the operator V is of class $C^1(A)$ or $C^2(A)$ respectively. We shall now point out criteria which are less general than (3.12), (3.13) but are easier to check.

Proposition 3.5 *Let $\Lambda \in \mathcal{B}(\mathscr{H})$ be a self-adjoint operator such that $[\Lambda, N] = 0$ and $[U, \Lambda]N \in \mathcal{B}(\mathscr{H})$. Let V be a bounded self-adjoint operator.*
(1) *If $(V - \Lambda)N$ is bounded, then $V \in C^1(A)$.*
(2) *If $[U, [U, \Lambda]]N^2$ and $(V - \Lambda)N^2$ are bounded, then $V \in C^2(A)$.*
(3) *If $[U, \Lambda]N,\ [\Delta, V]$ and $(V - \Lambda)N$ are compact, then $[iA, V]$ is compact.*

Proof: We have $-i\dot{V} = [N, V] = [N, V - \Lambda] = N(V - \Lambda) - (V - \Lambda)N$ so this is a bounded (or even compact) operator under the conditions of the proposition. Then by using (3.5) we get

$$\begin{aligned} NV' &= N[S, \Lambda] + N[S, V - \Lambda] = N[S, \Lambda] + NS(V - \Lambda) - N(V - \Lambda)S \\ &= N[S, \Lambda] - i\Delta(V - \Lambda) + [S, N(V - \Lambda)] \end{aligned}$$

hence NV' is bounded (or compact). Now in order to get (1) and (3) it suffices to use (3.11) and (3.12) and Lemma 3.4 with V replaced by Λ.

Now we prove (2). We have $V \in C^1(A)$ by what we have shown above. The assumption $\|(V - \Lambda)N^2\| < \infty$ implies $\|N^2(V - \Lambda)\| < \infty$ and then by interpolation $\|N(V - \Lambda)N\| < \infty$. Thus

$$\begin{aligned} -\ddot{V} &= [N, [N, V]] = [N, [N, V - \Lambda]] \\ &= N^2(V - \Lambda) - 2N(V - \Lambda)N + (V - \Lambda)N^2 \end{aligned}$$

is bounded. Moreover,

$$\begin{aligned} -iN\dot{V}' &= N[S, [N, V]] = N[S, [N, V - \Lambda]] = NSN(V - \Lambda) \\ &- NS(V - \Lambda)N - N^2(V - \Lambda)S + N(V - \Lambda)NS, \end{aligned}$$

is bounded by (3.5). Lemma 3.4 shows that $[U^*, \Lambda]N$ is a bounded operator. Hence, by using again (3.5),

$$\begin{aligned} NV_\Delta &= N[\Delta, V - \Lambda] + N[\Delta, \Lambda] \sim N[\Delta, V - \Lambda] \\ &= N\Delta(V - \Lambda) - N(V - \Lambda) \sim \Delta N(V - \Lambda) + iS(V - \Lambda). \end{aligned}$$

So NV_Δ is bounded. At last $N^2V'' = N^2[S, [S, V]] \sim N^2[S, [S, V - \Lambda]]$ by Lemma 3.4 applied to Λ, and this is a bounded operator. ■

3.3 Spectral and scattering theory

We shall now study the spectral theory of abstract self-adjoint operators of the form $L = \Delta + V$ with the help of the theory of conjugate operators initiated in [Mou] and the estimates. We first give conditions which ensure that a Mourre estimate holds. Recall that U is an arbitrary isometry on a Hilbert space $\mathscr{H}$ which admits a number operator N such that $0 \notin \sigma(N)$ and $\Delta = \operatorname{Re} U$. In this subsection the operator V is assumed to be at least self-adjoint and compact. We recall the notation: $S \approx 0$ if $S \in \mathcal{K}(\mathscr{H})$.

Definition 3.6 *We say that the self-adjoint operator L has* normal spectrum *if $\sigma_{\mathrm{ess}}(L) = [-1, +1]$ and the eigenvalues of L different from ± 1 are of finite multiplicity and can accumulate only toward ± 1. Let $\sigma_{\mathrm{p}}(L)$ be the set of eigenvalues of L; then $\kappa(L) = \{-1, +1\} \cup \sigma_{\mathrm{p}}(L)$ is the set of* critical values *of L.*

Theorem 3.7 *Let V be a compact self-adjoint operator on $\mathscr{H}$ such that $[N, V]$ and $[U, V]N$ are compact operators. Then L has normal spectrum and if J is a compact subset of $]-1, +1[$, then there are a real number $a > 0$ and a compact operator K such that $E(J)[L, iA]E(J) \geq aE(J) + K$, where E is the spectral measure of L.*

Proof: We have $\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(\Delta) = [-1, +1]$ because V is compact. This also implies that $\varphi(L) - \varphi(\Delta)$ is compact if φ is a continuous function. From (3.11) and Lemma 3.4 it follows that $[V, iA]$ is a compact operator, so V is of class $C^1(A)$ in the sense of [ABG]. Then, if $\operatorname{supp} \varphi$ is a compact subset of $]-1, +1[$ we have

$$\varphi(L)^*[L, iA]\varphi(L) \approx \varphi(\Delta)^*[\Delta, iA]\varphi(\Delta) \geq a|\varphi(\Delta)|^2 \approx a|\varphi(L)|^2$$

because $[\Delta, iA] = 1 - \Delta^2 \geq a$ on $\varphi(\Delta)\mathscr{H}$. This clearly implies the Mourre estimate, which in turn implies the the assertions concerning the eigenvalues, see [Mou] or [ABG, Corollary 7.2.11]. ■

The next result summarizes the consequences of the Mourre theorem [Mou], with an improvement concerning the regularity of the boundary values of the resolvent, cf. [GGM] and references there. If s is a positive real number we denote by $\mathcal{N}_s$ the domain of $|N|^s$ equipped with the graph topology and we set $\mathcal{N}_{-s} := (\mathcal{N}_s)^*$, where the adjoint spaces are defined such as to have $\mathcal{N}_s \subset \mathscr{H} \subset \mathcal{N}_{-s}$. If J is a real set then $J_\pm$ is the set of complex numbers of the form $\lambda \pm i\mu$ with $\lambda \in J$ and $\mu > 0$.

Theorem 3.8 *Let V be a compact self-adjoint operator on $\mathscr{H}$ such that $[N, V]$ and $[U, V]N$ are compact operators. Assume also that $[N, [N, V]]$, $[U, [N, V]]N$*

and $[U,[U,V]]N^2$ are bounded operators. Then L has no singularly continuous spectrum. Moreover, if J is a compact real set such that $J \cap \kappa(L) = \emptyset$, then for each real $s \in]1/2, 3/2[$ there is a constant C such that for all $z_1, z_2 \in J_\pm$

$$\|(L-z_1)^{-1} - (L-z_2)^{-1}\|_{B(\mathcal{N}_s,\mathcal{N}_{-s})} \le C|z_1 - z_2|^{s-1/2}. \tag{3.18}$$

We have used the obvious fact that $\mathcal{N}_s \subset \mathcal{D}(|A|^s)$ for all real $s > 0$ (for our purposes, it suffices to check this for $s = 2$). The theorem can be improved by using [ABG, Theorem 7.4.1], in the sense that one can eliminate the conditions on the second order commutators, replacing them with the optimal Besov type condition $V \in \mathscr{C}^{1,1}(A)$, but we shall consider this question only in particular cases below.

With the terminology of [ABG], the rôle of the conditions on the second order commutators imposed in Theorem 3.8 is to ensure that V (hence L) is of class $C^2(A)$. We shall now consider more general operators, which admit short and long range type components which are less regular. We also make a statement concerning scattering theory under short range perturbations.

Definition 3.9 *Let W be a bounded self-adjoint operator. We say that W is* short range with respect to N*, or* N-short range, *if*

$$\int_1^\infty \|W\chi_0(|N|/r)\| dr < \infty, \tag{3.19}$$

where χ_0 is the characteristic function of the interval $[1,2]$ in $\mathbb{R}$. We say that W is long range with respect to N*, or* N-long range*, if $[N,W]$ and $[U,W]N$ are bounded operators and*

$$\int_1^\infty \Big(\|[N,W]\chi(|N|/r)\| + \|[U,W]N\chi(|N|/r)\|\Big)\frac{dr}{r} < \infty, \tag{3.20}$$

where χ is the characteristic function of the interval $[1,\infty[$ in $\mathbb{R}$.

The condition (3.19) is obviously satisfied if there is $\varepsilon > 0$ such that

$$\|W|N|^{1+\varepsilon}\| < \infty. \tag{3.21}$$

Similarly, (3.20) is a consequence of

$$\|[N,W]\,|N|^{\varepsilon}\| + \|[U,W]\,|N|^{1+\varepsilon}\| < \infty. \tag{3.22}$$

Lemma 3.10 *If W is compact and N-short range, then WN is a compact operator. If W is N-long range, then $\int_1^\infty \|[U^*,W]N\chi(|N|/r)\|dr/r < \infty$.*

Proof: Let φ be a smooth function on $\mathbb{R}$ such that $\varphi(x) = 0$ if $x < 1$ and $\varphi(x) = 1$ if $x > 2$ and let $\theta(x) = x\varphi(x)$. Then $\int_0^\infty \theta(x)dx/x = 1$ hence $\int_0^\infty \theta(|N|/r)dr/r = 1$ in the strong topology. If $\theta_1(x) = x\theta(x)$ then we get $\int_0^\infty W\theta_0(|N|/r)dr = W|N|$ on the domain of N, which clearly proves the first part of the lemma. The second part follows from (3.15) and (3) of Lemma 2.4. ■

Theorem 3.11 *Let V be a compact self-adjoint operator such that $[N,V]$ and $[U,V]N$ are compact. Assume that we can decompose $V = V_s + V_\ell + V_m$ where V_s is compact and N-short range, V_ℓ is N-long range, and V_m is such that*

$$[N,[N,V_m]], \quad [U,[N,V_m]]N \quad \textit{and} \quad [U,[U,V_m]]N^2$$

are bounded operators. Then $L = \Delta + V$ has normal spectrum and no singularly continuous spectrum. Moreover, $\lim_{\mu\to 0}(L-\lambda-i\mu)^{-1}$ exists in norm in $\mathcal{B}(\mathcal{N}_s, \mathcal{N}_{-s})$ if $s > 1/2$ and $\lambda \notin \kappa(L)$, and the convergence is locally uniform in λ outside $\kappa(L)$. Let $L_0 = \Delta + V_\ell + V_m$ and let Π_0, Π be the projections onto the subspaces orthogonal to the set of eigenvectors of L_0, L respectively. Then the wave operators

$$\Omega_\pm := \text{s-}\lim_{t\to\pm\infty} e^{itL}e^{-itL_0}\Pi_0$$

exist and are complete, i.e. $\Omega_\pm \mathscr{H} = \Pi\mathscr{H}$.

Proof: From the Lemma 3.10 it follows easily that $[N,V_s]$ and $[U,V_s]N$ are compact operators, hence the potentials V and $V_\ell + V_m$ satisfy the hypotheses of Theorem 3.7, so the Mourre estimate holds for L and L_0 on each compact subset of $]-1,+1[$. From [ABG, Theorem 7.5.8] it follows that the operator V_s is of class $\mathscr{C}^{1,1}(A)$. By using (3.11), the second part of Lemma 3.10 and [ABG, Proposition 7.5.7] we see that $[iA, V_\ell]$ is of class $\mathscr{C}^{0,1}(A)$, hence V_ℓ is of class $\mathscr{C}^{1,1}(A)$. Finally, V_m is of class $C^2(A)$ by Proposition 3.3 and Lemma 3.4. Thus, L_0 and L are of class $\mathscr{C}^{1,1}(A)$. Then an application of [ABG, Theorem 7.4.1] gives the spectral properties of L and the existence of the boundary values of the resolvent. Finally, the existence and completeness of the wave operators is a consequence of [ABG, Proposition 7.5.6] and [GeM, Theorem 2.14]. ■

4 A Fock space model

4.1 The Fock space

Let H be a complex Hilbert space and let $\mathscr{H} = \bigoplus_{n=0}^\infty H^{\otimes n}$ be the (complete) Fock space associated to it. We make the conventions $H^{\otimes 0} = \mathbb{C}$ and $H^{\otimes n} = \{0\}$

if $n < 0$. We fix $u \in H$ with $\|u\| = 1$. Let $U = \rho_u$ be the right multiplication by u. More precisely:

$$\begin{array}{rcl} \rho_u h_1 \otimes \ldots \otimes h_n & = & h_1 \otimes \ldots \otimes h_n \otimes u \\ \rho_u^* h_1 \otimes \ldots \otimes h_n & = & \left\{ \begin{array}{l} h_1 \otimes \ldots \otimes h_{n-1} \langle u, h_n \rangle \quad \text{if} \quad n \geq 1 \\ 0 \quad \text{if} \quad n = 0. \end{array} \right. \end{array}$$

Clearly $\rho_u^* \rho_u = 1$, so U is an isometric operator. Then $\Delta = (U + U^*)/2$ acts as follows:

$$\Delta h_1 \otimes \ldots \otimes h_n = h_1 \otimes \ldots \otimes h_{n-1} \otimes (h_n \otimes u + \langle u, h_n \rangle)$$

if $n \geq 1$ and $\Delta h = hu$ if $h \in \mathbb{C} = H^{\otimes 0}$. We have

$$UH^{\otimes n} \subset H^{\otimes n+1}, \quad U^* H^{\otimes n} \subset H^{\otimes n-1}. \tag{4.1}$$

In particular $U^{*n} H^{\otimes m} = 0$ if $n > m$, hence we have s–$\lim_{n \to \infty} U^{*n} = 0$.

Thus U is a completely non unitary isometry, hence there is a unique number operator $N_U \equiv N$ associated to it. We shall keep the notations $P^k = \rho_u^k \rho_u^{*k}$ and $P_k = \rho_u^k [\rho_u^*, \rho_u] \rho_u^{*k}$ introduced in the general setting of Subsection 2.2.

Let us denote by $p_u = |u\rangle\langle u|$ the orthogonal projection in H onto the subspace $\mathbb{C}u$. Then it is easy to check that

$$P^k | H^{\otimes n} = \left\{ \begin{array}{l} 0 \quad \text{if} \quad 0 \leq n < k \\ \mathbf{1}_{n-k} \otimes p_u^{\otimes k} \quad \text{if} \quad n \geq k. \end{array} \right. \tag{4.2}$$

Here $\mathbf{1}_n$ is the identity operator in $H^{\otimes n}$ and the tensor product refers to the natural factorization $H^{\otimes n} = H^{\otimes n-k} \otimes H^{\otimes k}$. In particular, we get $P^k H^{\otimes n} \subset H^{\otimes n}$ or $[P^k, \mathbf{1}_n] = 0$ for all $k, n \in \mathbb{N}$ and similarly for the P_k.

Lemma 4.1 *N leaves stable each $H^{\otimes n}$. We have*

$$N_n := N | H^{\otimes n} = \sum_{k=0}^{n} (k+1) P_k | H^{\otimes n} \tag{4.3}$$

and $\sigma(N_n) = \{1, 2, \ldots n+1\}$, hence $1 \leq N_n \leq n+1$ and $\|N_n\| = n+1$.

Proof: The first assertion is clear because each spectral projection P_k of N leaves $H^{\otimes n}$ invariant. We obtain (4.3) from $P_k = P^k - P^{k+1}$ and the relations (2.3) and (4.2). To see that each $k+1$ is effectively an eigenvalue, one may check that

$$N_n w \otimes v \otimes u^{\otimes k} = (k+1) w \otimes v \otimes u^{\otimes k}$$

if $k < n$, $w \in H^{n-k-1}$ and $v \in H$ with $v \perp u$, and $N_n u^{\otimes n} = (n+1)u^{\otimes n}$. ■

The following more explicit representations of N_n can be proved without difficulty. Let $p_u^\perp$ be the projection in H onto the subspace K orthogonal to u. Then:

$$\begin{aligned} N_n &= \mathbf{1}_n + \mathbf{1}_{n-1} \otimes p_u + \mathbf{1}_{n-2} \otimes p_u^{\otimes 2} + \cdots + p_u^{\otimes n} \\ &= \mathbf{1}_{n-1} \otimes p_u^\perp + 2\mathbf{1}_{n-2} \otimes p_u^\perp \otimes p_u + 3\mathbf{1}_{n-3} \otimes p_u^\perp \otimes p_u^{\otimes 2} + \ldots \\ &+ (n+1)p_u^{\otimes n}. \end{aligned}$$

The last representation corresponds to the following orthogonal decomposition:

$$H^{\otimes n} = \oplus_{k=0}^{n} (H^{\otimes n-k-1} \otimes K \otimes u^{\otimes k})$$

where the term corresponding to $k = n$ must be interpreted as $\mathbb{C}u^{\otimes n}$.

The number operator N associated to U should not be confused with the *particle number operator* $\boldsymbol{N}$ acting on the Fock space according to the rule $\boldsymbol{N} f = nf$ if $f \in H^{\otimes n}$. In fact, while $\boldsymbol{N}$ counts the total number of particles, $N - 1$ counts (in some sense, i.e. after a symmetrization) the number of particles in the state u. From (4.3) we get a simple estimate of N in terms of $\boldsymbol{N}$:

$$N \leq \boldsymbol{N} + 1. \tag{4.4}$$

It is clear that an operator $V \in \mathcal{B}(\mathscr{H})$ commutes with $\boldsymbol{N}$ if and only if it is of the form

$$V = \sum_{n \geq 0} V_n \mathbf{1}_n, \quad \text{with} \quad V_n \in \mathcal{B}(H^{\otimes n}) \quad \text{and} \quad \sup_n \|V_n\| < \infty. \tag{4.5}$$

Note that we use the same notation $\mathbf{1}_n$ for the identity operator in $H^{\otimes n}$ and for the orthogonal projection of $\mathscr{H}$ onto $H^{\otimes n}$. For each operator V of this form we set $V_{-1} = 0$ and then we define

$$\delta(V) = \sum_{n \geq 0} (V_{n-1} \otimes \mathbf{1}_H - V_n)\mathbf{1}_n, \tag{4.6}$$

which is again a bounded operator which commutes with $\boldsymbol{N}$. We have:

$$[U, V] = \delta(V)U. \tag{4.7}$$

Indeed, if $f \in H^{\otimes n}$ then

$$UVf = UV_n f = (V_n f) \otimes u = (V_n \otimes \mathbf{1}_H)(f \otimes u) = (V_n \otimes \mathbf{1}_H)Uf.$$

On the other hand, since $Uf \in H^{\otimes n+1}$, we have $VUf = V_{n+1}Uf$ and $\delta(V)Uf = (V_n \otimes \mathbf{1}_H - V_{n+1})Uf$, which proves the relation (4.7).

Lemma 4.2 *If V is a bounded self-adjoint operator which commutes with $\boldsymbol{N}$ then the quadratic forms $\dot{V}$ and $\ddot{V}$ are essentially self-adjoint operators. With the notations from (4.5), the closures of these operators are given by the direct sums*

$$\begin{aligned} \dot{V} &= \sum_{n\geq 0}[iN_n, V_n]\mathbf{1}_n \equiv \sum_{n\geq 0}\dot{V}_n\mathbf{1}_n, &(4.8)\\ \ddot{V} &= \sum_{n\geq 0}[iN_n[iN_n, V_n]]\mathbf{1}_n \equiv \sum_{n\geq 0}\ddot{V}_n\mathbf{1}_n. &(4.9)\end{aligned}$$

The proof is easy and will not be given. In particular: *$\dot{V}$ is bounded if and only if $\sup_n \|[N_n, V_n]\| < \infty$ and $\ddot{V}$ is bounded if and only if $\sup_n \|[N_n[N_n, V_n]]\| < \infty$.*

4.2 The Hamiltonian

In this subsection *we assume that H is finite dimensional* and we apply the general theory of Section 3 to the Hamiltonian of the form $L = \Delta + V$ where V is a compact self-adjoint operator on $\mathscr{H}$ such that $[V, \boldsymbol{N}] = 0$, so V preserves the number of particles (but V *does not commute with* N in the cases of interest for us). Equivalently, this means that V has the form

$$V = \sum_{n\geq 0} V_n\mathbf{1}_n, \quad \text{with} \quad V_n \in \mathcal{B}(H^{\otimes n}) \quad \text{and} \quad \lim_{n\to\infty}\|V_n\| = 0. \tag{4.10}$$

We shall also consider perturbations of such an L by potentials which do not commute with $\boldsymbol{N}$ but satisfy stronger decay conditions.

The following results are straightforward consequences of the theorems proved in Subsection 3.3, of the remarks at the end of Subsection 4.1, and of the relation (4.7). For example, in order to check the compactness of $[U, V]N$, we argue as follows: we have $[U, V]N = \delta(V)UN = \delta(V)(N-1)U$ and $(\boldsymbol{N}+1)^{-1}N$ is bounded, hence the compactness of $\delta(V)\boldsymbol{N}$ suffices. Note also the relations

$$\begin{aligned} [U,[U,V]] &= [U, \delta(V)U] = [U, \delta(V)]U = \delta^2(V)U^2 &(4.11)\\ \delta^2(V) &= \sum_{n\geq 0}(V_{n-2}\otimes \mathbf{1}_{H^{\otimes 2}} - 2V_{n-1}\otimes\mathbf{1}_H + V_n)\mathbf{1}_n. &(4.12)\end{aligned}$$

Proposition 4.3 *Assume that H is finite dimensional and let V be a self-adjoint operator of the form (4.10) and such that $\|\dot{V}_n\| + n\|V_{n-1}\otimes\mathbf{1}_H - V_n\| \to 0$ when $n\to\infty$. Then the spectrum of L is normal and the Mourre estimate holds on each compact subset of $]-1, +1[$.*

Proposition 4.4 *Assume that H is finite dimensional and let V be a self-adjoint operator of the form (4.10) and such that*
(1) $\|\dot{V}_n\| + n\|V_{n-1}\otimes \mathbf{1}_H - V_n\| \to 0$ *when* $n\to\infty$
(2) $\|\ddot{V}_n\| + n\|\dot{V}_{n-1}\otimes \mathbf{1}_H - \dot{V}_n\| + \|(V_{n-2}\otimes \mathbf{1}_{H^{\otimes 2}} - 2V_{n-1}\otimes \mathbf{1}_H + V_n\| \leq C < \infty$
Then L has normal spectrum and no singularly continuous spectrum.

This result is of the same nature as those of C. Allard and R. Froese. To see this, we state a corollary with simpler and explicit conditions on the potential. If T is a linear operator on a finite dimensional Hilbert space E, we denote by $\langle T\rangle$ its normalized trace:

$$\langle T\rangle = \frac{1}{\dim E}\operatorname{Tr} T \tag{4.13}$$

Observe that $|\langle T\rangle| \leq \|T\|$.

Corollary 4.5 *Let H be finite dimensional and let V be as in (4.10) and such that:*
(1) $\|V_n - \langle V_n\rangle\| = O(1/n^2)$,
(2) $\langle V_{n+1}\rangle - \langle V_n\rangle = o(1/n)$,
(3) $\langle V_{n+1}\rangle - 2\langle V_n\rangle + \langle V_{n-1}\rangle = O(1/n^2)$.
Then L has normal spectrum and no singularly continuous spectrum, the Mourre estimate holds on each compact subset of $]-1,+1[$, and estimates of the form (3.18) are valid.

This follows easily from Proposition 3.5 with $\Lambda = \sum_{n\geq 0}\langle V_n\rangle \mathbf{1}_n$. In the case when V is a function on a tree, the conditions (1)-(3) of the corollary are equivalent to those of Lemma 7 and Theorem 8 in [AlF]. Note, however, that even in the tree case we do not assume that the V_n are functions. Now we improve these results.

Let $\mathbf{1}_{\geq n} = \sum_{k\geq n}\mathbf{1}_k$ be the orthogonal projection of $\mathscr{H}$ onto $\bigoplus_{k\geq n} H^{\otimes k}$.

Theorem 4.6 *Let H be finite dimensional and let V be a self-adjoint operator of the form (4.10) and such that*

$$\sum_{k\geq 0}\sup_{n\geq k}\|V_n - \langle V_n\rangle\| < \infty \quad \textit{and} \quad \langle V_{n+1}\rangle - \langle V_n\rangle = o(1/n). \tag{4.14}$$

Furthermore, assume that $\langle V_n\rangle = \lambda_n + \mu_n$ where $\{\lambda_n\},\{\mu_n\}$ are sequences of real numbers which converge to zero and such that:
(1) $\lambda_{n+1} - \lambda_n = o(1/n)$ *and* $\lambda_{n+1} - 2\lambda_n + \lambda_{n-1} = O(1/n^2)$,
(2) $\sum_{n\geq 0}\sup_{m\geq n}|\mu_{m+1} - \mu_m| < \infty$.
Finally, let W be a bounded self-adjoint operator satisfying $\sum_n \|W\mathbf{1}_{\geq n}\| < \infty$. Then the operators $L_0 = \Delta + V$ and $L = L_0 + W$ have normal spectrum and no singularly continuous spectrum, and the wave operators for the pair (L, L_0) exist and are complete.

Proof: Let $\Lambda = \sum \lambda_n \mathbf{1}_n$ and $M = \sum \mu_n \mathbf{1}_n$. We shall apply Theorem 3.11 to L with the following identifications: $V_s = V + W - (\Lambda + M)$, $V_\ell = M$ and $V_m = \Lambda$. Note that the condition imposed on W implies that W is a compact N-short range operator (in fact, the condition says that W is $\boldsymbol{N}$-short range). Moreover, the first condition in (4.14) is of the same nature, so it implies that $V - (\Lambda + M)$ is N-short range. Hence V_s is compact and N-short range. The fact that M is N-long range is an easy consequence of $[M, N] = 0$ and of the condition (2) (which says, in fact, that M is $\boldsymbol{N}$-long range). Finally, the fact that V_m satisfies the conditions required in Theorem 3.11 is obvious, by (1) and by what we have seen before. The compactness of $[N, V]$ and $[U, V]N$ is proved as follows. Since $V - (\Lambda + M)$ is N-short range and due to Lemma 3.10, it suffices to show the compactness of the operators $[N, \Lambda + M]$ and $[U, \Lambda + M]N$. But the first one is zero and for the second one we use the first part of condition (1) and condition (2). In the case of $V + W$ one must use again Lemma 3.10 ■

Under the conditions of the preceding theorem, we also have the following version of the "limiting absorption principle", cf. Theorem 3.11. For real s let $\mathscr{H}_{(s)}$ be the Hilbert space defined by the norm

$$\|f\|^2 = \|\mathbf{1}_0 f\|^2 + \sum_{n \geq 1} n^{2s} \|\mathbf{1}_n f\|^2.$$

Then, if $s > 1/2$ and $\lambda \notin \kappa(L)$, the limit $\lim_{\mu \to 0}(L - \lambda - i\mu)^{-1}$ exists in norm in the space $\mathcal{B}(\mathscr{H}_{(s)}, \mathscr{H}_{(-s)})$, the convergence being locally uniform on $\mathbb{R} \setminus \kappa(L)$.

5 The anisotropic tree algebra

5.1 The free algebra

Our purpose now is to study more general operators of the form $L = D + V$, where D is a function of U and U^* (in the sense that it belongs to the C^*-algebra generated by U) and V has the same structure as in Subsection 4.2, i.e. is a direct sum of operators V_n acting in $H^{\otimes n}$, but V_n does not vanish as $n \to \infty$, so V is anisotropic in a sense which will be specified later on.

In this section we keep the assumptions and notations of Subsection 4.1 but *assume that H is of dimension $\nu \geq 2$* (possibly infinite). Then both the range of U and the kernel of U^* are infinite dimensional. It follows easily that each P_k is a projection of infinite rank.

The free algebra $\mathscr{D}$ is the C^*-algebra of operators on $\mathscr{H}$ generated by the isometry U. Since $U^*U = 1$ on $\mathscr{H}$, the set $\mathscr{D}_0$ of operator of the form

$$D = \sum_{n,m\geq 0} \alpha_{nm} U^n U^{*m} \tag{5.1}$$

with $\alpha_{nm} \in \mathbb{C}$ and $\alpha_{nm} \neq 0$ only for a finite number of n, m, is a $*$-subalgebra of $\mathscr{D}$, dense in $\mathscr{D}$. Observe that the projections $P^k = U^k U^{*k}$ and $P_k = P^k - P^{k+1}$ belong to $\mathscr{D}_0$. In the tree case the elements of $\mathscr{D}$ are interpreted as "differential" operators on the tree, which justifies our notation.

We introduce now a formalism needed for the proof of Lemma 5.4, a result important for what follows. For each operator $S \in \mathcal{B}(\mathscr{H})$ we define

$$S^\circ = \sum_{n=0}^{\infty} \mathbf{1}_n S \mathbf{1}_n. \tag{5.2}$$

It is clear that the series is strongly convergent and that $\|S^\circ\| \leq \|S\|$. Thus $S \mapsto S^\circ$ is a linear contraction of $\mathcal{B}(\mathscr{H})$ into itself such that $1^\circ = 1$. This map is also positive and faithful in the following sense:

$$S \geq 0 \text{ and } S \neq 0 \Rightarrow S^\circ \geq 0 \text{ and } S^\circ \neq 0 \tag{5.3}$$

Indeed, $S^\circ \geq 0$ is obvious and if $S^\circ = 0$ then $(\sqrt{S}\mathbf{1}_n)^*(\sqrt{S}\mathbf{1}_n) = \mathbf{1}_n S \mathbf{1}_n = 0$ hence $\sqrt{S}\mathbf{1}_n = 0$ for all n, so $\sqrt{S} = 0$ and then $S = 0$.

We need one more property of the map $S \mapsto S^\circ$:

$$S \in \mathcal{K}(\mathscr{H}) \Rightarrow S^\circ \in \mathcal{K}(\mathscr{H}). \tag{5.4}$$

In fact, this follows from

$$\|S^\circ - \sum_{0\leq m\leq n} \mathbf{1}_m S \mathbf{1}_m\| \leq \sup_{m>n} \|\mathbf{1}_m S \mathbf{1}_m\|$$

because $\|\mathbf{1}_n S \mathbf{1}_n\| \to 0$ as $n \to 0$ if S is compact.

Lemma 5.1 *The restriction to $\mathscr{D}$ of the map $S \mapsto S^\circ$ is a map $\theta : \mathscr{D} \to \mathscr{D}$ whose range is equal to the (abelian, unital) C^*-algebra $\mathscr{P}$ generated by the projections P^k, $k \geq 0$. Moreover, θ is a norm one projection of $\mathscr{D}$ onto its linear subspace $\mathscr{P}$, i.e. $\theta(D) = D$ if and only if $D \in \mathscr{P}$.*

Proof: Since $U^n U^{*m} H^{\otimes k} \subset H^{\otimes(k-m+n)}$, we have $\mathbf{1}_k U^n U^{*m} \mathbf{1}_k \neq 0$ only if $n = m$. Thus, if $D \in \mathscr{D}_0$ is as in (5.1), then

$$\mathbf{1}_k D \mathbf{1}_k = \sum_n \alpha_{n,n} \mathbf{1}_k U^n U^{*n} \mathbf{1}_k = \sum_n \alpha_{n,n} P^n \mathbf{1}_k,$$

because $[P^n, \mathbf{1}_k] = 0$. Thus we get $D^\circ = \sum_n \alpha_{n,n} P^n \in \mathscr{P}$. Since $D \mapsto D^\circ$ is a linear contraction and $\mathscr{D}_0$ is dense in $\mathscr{D}$, we get that $D^\circ \in \mathscr{P}$ for all $D \in \mathscr{D}$. To finish the proof, note that $(P^n)^\circ = P^n$ for all n and $\mathscr{P}$ is the closed linear subspace of $\mathscr{D}$ generated by the operators P^n, hence $D^\circ = D$ for all $D \in \mathscr{P}$. ■

The pairwise orthogonal projections P_n belong to $\mathscr{P}$ but the C^*-algebra (equal to the norm closed subspace) generated by them is strictly smaller than $\mathscr{P}$. On the other hand, the Von Neumann algebra $\mathscr{P}_w$ generated by $\mathscr{P}$ (i.e. the strong closure of $\mathscr{P}$) coincides with that generated by $\{P_n\}_{n\geq 0}$. Indeed, for each $n \geq 0$ we have $P^n = \sum_{m\geq n} P_m$ the series being strongly convergent.

Lemma 5.2 *For each $D \in \mathscr{D}$ there is a unique bounded sequence $\{\alpha_n\}_{n\geq 0}$ of complex numbers such that $D^\circ = \sum_{n\geq 0} \alpha_n P_n$. If $D \geq 0$ then $\alpha_n \geq 0$ for all n. If $D \in \mathscr{D}$, $D \geq 0$ and $D \neq 0$, one has $D^\circ \geq \alpha P_n$ for some real $\alpha > 0$ and some $n \in \mathbb{N}$.*

Proof: Since $P_n P_m = 0$ if $n \neq m$ and $\sum_{k\geq 0} P_k = 1$, each element of the Von Neumann algebra generated by $\{P_n\}_{n\geq 0}$ can be written as $\sum_{n\geq 0} \alpha_n P_n$ for some unique bounded sequence of comples numbers α_n. If $D \geq 0$, then $D^\circ \geq 0$ and this is equivalent to $\alpha_n \geq 0$ for all n. If $D \geq 0$ and $D \neq 0$, then $D^\circ \neq 0$ by (5.3) hence $\alpha_n > 0$ for some n. ■

Corollary 5.3 $\mathscr{D} \cap \mathcal{K}(\mathscr{H}) = \{0\}$.

Proof: $\mathscr{D} \cap \mathcal{K}(\mathscr{H})$ is a C^*-algebra, so that if the intersection is not zero, then it contains some D with $D \geq 0$ and $D \neq 0$. But then D° is a compact operator by (5.4) and we have $D^\circ \geq \alpha P_n$ for some $\alpha > 0$ and $n \in \mathbb{N}$.

We note that if $0 \leq S \leq K$ and $K \approx 0$ then $S \approx 0$. Indeed, for each $\varepsilon > 0$ there is a finite range projection F such that $\|F'KF'\| \leq \varepsilon$, where $F' = 1 - F$. Thus $0 \leq F'SF' \leq \varepsilon$ and so $S = FS + F'SF + F'SF'$ is the sum of a finite range operator and of an operator of norm $\leq \varepsilon$. Hence $S \approx 0$.

Thus P_n is compact, or P_n is an infinite dimension projection. ■

Finally, we are able to prove the result we need.

Lemma 5.4 *Let $V \in \mathcal{B}(\mathscr{H})$ such that $V = V^\circ$ and $[V, U] \in \mathcal{K}(\mathscr{H})$. If there is $D \in \mathscr{D}$, $D \neq 0$, such that $VD \in \mathcal{K}(\mathscr{H})$, then $VP_0 \in \mathcal{K}(\mathscr{H})$.*

Proof: From $VD \approx 0$ it follows that $VDD^*V^* \approx 0$. Then (5.4) gives

$$V(DD^*)^\circ V^* = (VDD^*V^*)^\circ \approx 0.$$

By Lemma 5.2, since $DD^* \in \mathscr{D}$ is positive and not zero, we have $DD^* \geq \alpha P_n$ for some $n \geq 0$, with $\alpha > 0$. Thus $0 \leq VP_nV^* \leq \alpha^{-1}VDD^*V^*$. Or $VDD^*V^* \approx 0$ so $VP_nV^* \approx 0$ and since $VP_n = \sqrt{VP_nV^*}J$ for some partial isometry J we see that $VP_n \approx 0$. But $P_n = U^nP_0U^{*n}$ and $U^*U = 1$ so $VU^nP_0 \approx 0$. If $n \geq 1$ then $UVU^{n-1}P_0 = [U,V]U^{n-1}P_0 + VU^nP_0 \approx 0$ and since $U^*U = 1$ we get $VU^{n-1}P_0 \approx 0$. Repeating, if necessary, the argument, we obtain that $VP_0 \approx 0$. ■

5.2 The interaction algebra

The classes of interaction operators $V \in \mathcal{B}(\mathscr{H})$ we isolate below must be such that $V = V^\circ$ and $VP_0 \approx 0 \Rightarrow V \approx 0$. We shall use the embedding ($n \geq 0$)

$$\mathcal{B}(H^{\otimes n}) \hookrightarrow \mathcal{B}(H^{\otimes n+1}) \text{ defined by } S \mapsto S \otimes \mathbf{1}_H. \tag{5.5}$$

Let us set $\mathcal{A}_0 = \mathbb{C}$ and for each $n \geq 1$ let $\mathcal{A}_n$ be a C^*-algebra of operators on $H^{\otimes n}$ such that

$$\mathcal{A}_n \otimes \mathbf{1}_H \subset \mathcal{A}_{n+1}. \tag{5.6}$$

Note that this implies $\mathbf{1}_n \in \mathcal{A}_n$. The convention (5.5) gives us natural embeddings

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \subset \mathcal{A}_n \subset \ldots \tag{5.7}$$

and we can define $\mathcal{A}_\infty$ as the completion of the $*$-algebra $\cup_{n=0}^{\infty}\mathcal{A}_n$ under the unique C^*-norm we have on it (note that $\mathcal{A}_{n+1}$ induces on $\mathcal{A}_n$ the initial norm of $\mathcal{A}_n$). Thus $\mathcal{A}_\infty$ is a unital C^*-algebra, each $\mathcal{A}_n$ is a unital subalgebra of $\mathcal{A}_\infty$ and we can write:

$$\mathcal{A}_\infty = \overline{\bigcup_{n\geq 0} \mathcal{A}_n} \quad \text{(norm closure)}. \tag{5.8}$$

We emphasize that $\mathcal{A}_\infty$ *has not* a natural realization as algebra of operators on $\mathscr{H}$. On the other hand, the following is a unital C^*-algebra of operators on $\mathscr{H}$:

$$\mathscr{A} = \prod_{n\geq 0} \mathcal{A}_n = \big\{ V = (V_n)_{n\geq 0} \mid V_n \in \mathcal{A}_n \text{ and } \|V\| := \sup_{n\geq 0} \|V_n\| < \infty \big\}. \tag{5.9}$$

Indeed, if $f = (f_n)_{n\geq 0} \in \mathscr{H}$ and V is as above, we set $Vf = (V_nf_n)_{n\geq 0}$. In other terms, we identify

$$V = \sum_{n=0}^{\infty} V_n \mathbf{1}_n \tag{5.10}$$

the right hand side being strongly convergent on $\mathscr{H}$. Observe that

$$\mathscr{A}_0 = \bigoplus_{n\geq 0} \mathcal{A}_n = \big\{ V \in \mathscr{A} \mid \lim_{n\to\infty} \|V_n\| = 0 \big\}. \tag{5.11}$$

is an ideal in $\mathscr{A}$.

Lemma 5.5 *We have $\mathscr{A} \cap \mathbb{K}(\mathscr{H}) \subset \mathscr{A}_0$ and the inclusion becomes an equality if H is finite dimensional.*

Proof: We have $\mathbf{1}_n \to 0$ strongly on $\mathscr{H}$ if $n \to \infty$, hence if V is compact then $\|V\mathbf{1}_n\| \to 0$. In the finite dimensional case, note that $\sum_{m=0}^{n} V_m \mathbf{1}_m$ is compact for all n and converges in norm to V if $V \in \mathscr{A}_0$. ■

Let $\tau : \mathscr{A} \to \mathscr{A}$ be the morphism defined by:

$$\tau(V_0, V_1, V_2, \ldots) = (0, V_0\mathbf{1}_H, V_1 \otimes \mathbf{1}_H, V_2 \otimes \mathbf{1}_H, \ldots),$$

or $\tau(V)_n = V_{n-1} \otimes \mathbf{1}_H$, where $V_{-1} = 0$. Clearly $\tau^n(V) \to 0$ as $n \to \infty$ strongly on $\mathscr{H}$, for each $V \in \mathscr{A}$. Observe that the map $\delta = \tau - \mathrm{Id}$ coincides with that defined in (4.6), because

$$\delta(V)_n = V_{n-1} \otimes \mathbf{1}_H - V_n.$$

Since $\delta(V'V'') = \delta(V')\tau(V'') + V'\delta(V'')$ and since $\mathscr{A}_0$ is an ideal of $\mathscr{A}$, the space

$$\mathscr{A}_{\mathrm{vo}} = \{V \in \mathscr{A} \mid \delta(V) \in \mathscr{A}_0\} \tag{5.12}$$

is a C^*-subalgebra of $\mathscr{A}$ which contains $\mathscr{A}_0$. This algebra is an analog of the algebra of bounded continuous functions with vanishing oscillation at infinity on $\mathbb{R}$, or that of bounded functions with vanishing at infinity derivative on $\mathbb{Z}$ or $\mathbb{N}$.

Proposition 5.6 *Assume that H is finite dimensional and let $V \in \mathscr{A}_{\mathrm{vo}}$. If $D \in \mathscr{D}$, $D \neq 0$, and $VD \in \mathbb{K}(\mathscr{H})$, then $V \in \mathbb{K}(\mathscr{H})$.*

Proof: We have $\delta(V) \approx 0$ and $[U, V] \approx 0$ by (4.7) and Lemma 5.5. Now according to Lemma 5.4, it remains to prove that $V \approx 0$ follows from $VP_0 \approx 0$. Since $\mathbf{1}_n \to 0$ strongly as $n \to \infty$ and since $[\mathbf{1}_n, P_0] = 0$ and $V\mathbf{1}_n = V_n\mathbf{1}_n$, we get $\|V_nP_0\mathbf{1}_n\| \to 0$ as $n \to \infty$. By using $P_0 = 1 - P^1$ we get

$$P_0\mathbf{1}_n = \mathbf{1}_n - \mathbf{1}_{n-1} \otimes p_u = \mathbf{1}_{n-1} \otimes p'_u,$$

where $p'_u = \mathbf{1}_H - p_u$ is the projection of H onto the subspace orthogonal to u, hence $\|p'_u\| = 1$ (recall that $\dim H = \nu \geq 2$). Thus we have $\|V_n \cdot \mathbf{1}_{n-1} \otimes p'_u\| \to 0$. But $\delta(V) \in \mathscr{A}_0$ means $\|V_n - V_{n-1} \otimes \mathbf{1}_H\| \to 0$. So

$$\|V_{n-1}\| = \|V_{n-1} \otimes p'_u\| \leq \|(V_n - V_{n-1} \otimes \mathbf{1}_H) \cdot \mathbf{1}_{n-1} \otimes p'_u\| + \|V_n \cdot \mathbf{1}_{n-1} \otimes p'_u\|$$

converges to 0 as $n \to \infty$. ■

We are mainly interested in the particular class of algebras $\mathcal{A}_n$ constructed as follows. Let $\mathcal{A}$ be a C^*-algebra of operators on H such that $\mathbf{1}_H \in \mathcal{A}$ and let us set:

$$\mathcal{A}_0 = \mathcal{A}^{\otimes 0} = \mathbb{C} \quad \text{and} \quad \mathcal{A}_n = \mathcal{A}^{\otimes n} \quad \text{if} \quad n \geq 1. \tag{5.13}$$

Then $\mathcal{A}_\infty$ *is just the infinite tensor product* $\mathcal{A}^{\otimes\infty}$. Note that the embedding $\mathcal{A}^{\otimes n} \subset \mathcal{A}^{\otimes\infty}$ amounts now to identify $V_n \in \mathcal{A}^{\otimes n}$ with $V_n \otimes \mathbf{1}_H \otimes \mathbf{1}_H \otimes \ldots \in \mathcal{A}^{\otimes\infty}$. We summarize the preceeding notations and introduce new ones specific to this situation:

$$\begin{aligned}
\mathscr{A} &= \prod_{n\geq 0} \mathcal{A}^{\otimes n} = \{V = (V_n)_{n\geq 0} \mid V_n \in \mathcal{A}^{\otimes n}, \|V\| = \sup_{n\geq 0} \|V_n\| < \infty\} \\
\mathscr{A}_0 &= \bigoplus_{n\geq 0} \mathcal{A}^{\otimes n} = \{V \in \mathscr{A} \mid \lim_{n\to\infty} \|V_n\| = 0\} \\
\mathscr{A}_{\mathrm{vo}} &= \{V \in \mathscr{A} \mid \delta(V) \in \mathscr{A}_0\} \\
\mathscr{A}_\infty &= \{V \in \mathscr{A} \mid V_\infty := \lim_{n\to\infty} V_n \text{ exists in } \mathcal{A}^{\otimes\infty}\} \\
\mathscr{A}_{\mathrm{f}} &= \{V \in \mathscr{A} \mid \exists N \text{ such that } V_n = V_N \text{ if } n \geq N\}.
\end{aligned}$$

Note that $V_n = V_N$ means $V_n = V_N \otimes \mathbf{1}_{n-N}$ if $n > N$. The space of main interest for us is the C^*-algebra $\mathscr{A}_\infty$. Clearly, $\mathscr{A}_0$ is a closed self-adjoint ideal in $\mathscr{A}_\infty$ and

$$V \in \mathscr{A}_\infty \Rightarrow \delta(V) \in \mathscr{A}_0, \tag{5.14}$$

in other terms $\mathscr{A}_\infty \subset \mathscr{A}_{\mathrm{vo}}$.

Proposition 5.7 *The map $V \mapsto V_\infty$ is a surjective morphism of the C^*-algebra $\mathscr{A}_\infty$ onto $\mathcal{A}^{\otimes\infty}$ whose kernel is $\mathscr{A}_0$. Thus, we have a canonical isomorphism*

$$\mathscr{A}_\infty / \mathscr{A}_0 \simeq \mathcal{A}^{\otimes\infty}. \tag{5.15}$$

Moreover, $\mathscr{A}_{\mathrm{f}}$ is a dense $*$-subalgebra of $\mathscr{A}_\infty$ and we have

$$\mathscr{A}_{\mathrm{f}} = \Big\{V \in \mathscr{A}_\infty \mid V_\infty \in \bigcup_{n\geq 0} \mathcal{A}^{\otimes n}\Big\}. \tag{5.16}$$

Proof: That $V \mapsto V_\infty$ is a morphism and is obvious. $\mathscr{A}_{\mathrm{f}}$ is clearly a $*$-subalgebra. If $V \in \mathscr{A}_\infty$ and if we set $V_n^N = V_n$ for $n \leq N$, $V_n^N = V_N$ for $n > N$, then $V^N \in \mathscr{A}_{\mathrm{f}}$ and $\|V - V^N\| = \sup_{n>N} \|V_n - V_N\| \to 0$ as $N \to \infty$. Thus $\mathscr{A}_{\mathrm{f}}$ is dense in $\mathcal{A}_\infty$.

If $W \in \mathcal{A}^{\otimes N}$ and if we define $V \in \mathscr{A}$ by $V_n = 0$ for $n < N$, $V_n = W$ if $n \geq N$, then $V \in \mathscr{A}_{\mathrm{f}}$ and $V_\infty = W$. Thus the range of the morphism $V \mapsto V_\infty$

contains the dense subset $\cup_{n\geq 0}\mathcal{A}^{\otimes n}$ of $\mathcal{A}^{\otimes\infty}$. Since the range of a morphism is closed, the morphism is surjective. ■

The following remarks concerning the linear map $\mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ defined by $S \mapsto U^*SU$ will be needed below (see also the comments after Lemma 2.4). If we use the natural embedding $\mathcal{B}(H^{\otimes n}) \hookrightarrow \mathcal{B}(\mathscr{H})$ then we clearly have

$$U^*\mathcal{B}(H^{\otimes n+1})U \subset \mathcal{B}(H^{\otimes n})$$

and if $S' \in \mathcal{B}(H^{\otimes n})$ and $S'' \in \mathcal{B}(H)$ then

$$U^*(S' \otimes S'')U = S'\langle u, S''u\rangle.$$

Of course, $U^*SU = 0$ if $S \in \mathcal{B}(H^{\otimes 0})$. It is clear then that $\omega(V) := U^*VU$ defines a linear positive contraction $\omega : \mathscr{A} \to \mathscr{A}$ which leaves invariant the subalgebras $\mathscr{A}_0$ and $\mathscr{A}_{\mathrm{f}}$, hence $\mathscr{A}_\infty$ too. From (4.7) we then get for all $V \in \mathscr{A}$:

$$UV = [V + \delta(V)]U \quad \text{and} \quad U^*V = [V - \omega\circ\delta(V)]U^*. \tag{5.17}$$

We make two final remarks which are not needed in what follows. First, note that the map ω could be defined with the help of [Tak, Corollary 4.4.25]. Then, observe that for $S \in \mathcal{B}(H^{\otimes n})$ we have $USU^* = S \otimes p_u$. Thus in general the morphism $S \mapsto USU^*$ does not leave invariant the algebras we are interested in.

5.3 The anisotropic tree algebra

In this subsection we study C^*-algebras of operators on the Fock space $\mathscr{H}$ generated by self-adjoint Hamiltonians of the form $L = D+V$, where D is a polynomial in U and U^* and V belongs to a C^*-subalgebra of $\mathscr{A}$. We are interested in computing the quotient of such an algebra with respect to the ideal of compact operators. The largest algebra for which this quotient has a rather simple form is obtained starting with $\mathscr{A}_{\mathrm{vo}}$ and the quotient becomes quite explicit if we start with $\mathscr{A}_\infty$.

More precisely, we fix a vector $u \in H$ with $\|u\| = 1$ and a C^*-algebra $\mathcal{A}$ of operators on H containing $\mathbf{1}_H$. Recall that H is a Hilbert space of dimension $\nu \geq 2$. Throughout this subsection *we assume that H is finite dimensional*, although part of the results hold in general. Then we define $U = \rho_u$ as in Section 4 and we consider the C^*-algebras on $\mathscr{H}$

$$\mathscr{A}_0 \subset \mathscr{A}_\infty \subset \mathscr{A}_{\mathrm{vo}} \subset \mathscr{A}$$

associated to $\mathcal{A}$ as in Subsection 5.2. Then we define

$$\begin{aligned}
\mathscr{C}_{\mathrm{vo}} &= \text{norm closure of } \mathscr{A}_{\mathrm{vo}}\cdot\mathscr{D},\\
\mathscr{C}_\infty &= \text{norm closure of } \mathscr{A}_\infty\cdot\mathscr{D},\\
\mathscr{C}_0 &= \text{norm closure of } \mathscr{A}_0\cdot\mathscr{D}.
\end{aligned}$$

We recall the notation: if A, B are subspaces of an algebra C, then $A \cdot B$ is the linear subspace of C generated by the products ab with $a \in A$ and $b \in B$. Observe that, $\mathscr{D}$ and $\mathscr{A}_{\rm vo}$ being unital algebras, we have and $\mathscr{D} \cup \mathscr{A}_{\rm vo} \subset \mathscr{C}_{\rm vo}$ and, similarly, $\mathscr{D} \cup \mathscr{A}_\infty \subset \mathscr{C}_\infty$. Clearly $\mathscr{C}_0 \subset \mathscr{C}_\infty \subset \mathscr{C}_{\rm vo}$.

Lemma 5.8 *$\mathscr{C}_{\rm vo}$ and $\mathscr{C}_\infty$ are C^*-algebras and $\mathscr{C}_0$ is an ideal in each of them.*

Proof: Indeed, from (5.17) it follows easily that for each $V \in \mathscr{A}_\infty$ there are $V', V'' \in \mathscr{A}_\infty$ such that $UV = V'U$ and $U^*V = V''U^*$ and similarly in the case of $\mathscr{A}_{\rm vo}$. This proves the first part of the lemma. Then note that $V', V'' \in \mathscr{A}_0$ if $V \in \mathscr{A}_0$ and use (5.14). ■

It is not difficult to prove that $\mathscr{C}_{\rm vo}$ is the C^*-algebra generated by the operators $L = D + V$, where D and V are self-adjoint elements of $\mathscr{D}$ and $\mathscr{A}_{\rm vo}$ respectively, and similarly for $\mathscr{C}_\infty$ (see the proof of Proposition 4.1 from [GeI]). Since only the obvious fact that such operators belong to the indicated algebras matters here, we do not give the details.

Lemma 5.9 *If H finite dimensional, then $\mathscr{C}_0 = \mathcal{K}(\mathscr{H}) \cap \mathscr{C}_\infty = \mathcal{K}(\mathscr{H}) \cap \mathscr{C}_{\rm vo}$. If, moreover, u is a cyclic vector for $\mathcal{A}$ in H, then we have $\mathscr{C}_0 = \mathcal{K}(\mathscr{H})$.*

Proof: Since H is finite dimensional, we have $\mathscr{A}_0 \subset \mathcal{K}(\mathscr{H})$, hence $\mathscr{C}_0 \subset \mathcal{K}(\mathscr{H})$. Reciprocally, let $S \in \mathscr{C}_{\rm vo}$ be a compact operator. Let π_n be the projection of $\mathscr{H}$ onto $\bigoplus_{0 \le m \le n} H^{\otimes m}$. Then $\pi_n = \sum_{0 \le m \le n} \mathbf{1}_m \in \mathscr{A}_0$ and $\pi_n \to \mathbf{1}_{\mathscr{H}}$ strongly when $n \to \infty$. Since S is compact, we get $\pi_n S \to S$ in norm, so it suffices to show that $\pi_n S \in \mathscr{C}_0$ for each n. We prove that this holds for any $S \in \mathscr{C} =$ norm closure of $\mathscr{A} \cdot \mathscr{D}$: it suffices to consider the case $S = VD$ with $V \in \mathscr{A}$ and $D \in \mathscr{D}$, and then the assertion is obvious.

Since H is finite dimensional, u is cyclic for $\mathcal{A}$ if and only if $\mathcal{A}u = H$. If this is the case, then $u^{\otimes n}$ is cyclic for $\mathcal{A}^{\otimes n}$ on $H^{\otimes n}$ for each n. Let $n, m \in \mathbb{N}$ and $f \in H^{\otimes n}$, $g \in H^{\otimes m}$. Then there are $V \in \mathcal{A}^{\otimes n}$ and $W \in \mathcal{A}^{\otimes m}$ such that $f = Vu^{\otimes n} = VU^n e$ and $g = Wu^{\otimes m} = WU^m e$, where $e = 1 \in \mathbb{C} = H^{\otimes 0}$. So we have $|f\rangle\langle g| = VU^n|e\rangle\langle e|U^*W^*$. Clearly V, W and $|e\rangle\langle e|$ belong to $\mathscr{A}_0$, so $|f\rangle\langle g| \in \mathscr{C}_0$. An easy approximation argument gives then $\mathcal{K}(\mathscr{H}) \subset \mathscr{C}_0$. ■

We can now describe the quotient $\mathscr{C}_{\rm vo}/\mathscr{C}_0$ of the algebra $\mathscr{C}_{\rm vo}$ with respect to the ideal of compact operators which belong to it.

Theorem 5.10 *Assume that H is finite dimensional. Then there is a unique morphism $\Phi : \mathscr{C}_{\rm vo} \to (\mathscr{A}_{\rm vo}/\mathscr{A}_0) \otimes \mathscr{D}$ such that $\Phi(VD) = \widehat{V} \otimes D$ for all $V \in \mathscr{A}_{\rm vo}$ and $D \in \mathscr{D}$, where $V \mapsto \widehat{V}$ is the canonical map $\mathscr{A}_{\rm vo} \to \mathscr{A}_{\rm vo}/\mathscr{A}_0$. This morphism is surjective and $\ker \Phi = \mathscr{C}_0$, hence we get a canonical isomorphism*

$$\mathscr{C}_{\rm vo}/\mathscr{C}_0 \simeq (\mathscr{A}_{\rm vo}/\mathscr{A}_0) \otimes \mathscr{D}. \tag{5.18}$$

Proof: We shall check the hypotheses of Corollary A.4 with the choices:

$$u \equiv U,\ B = \mathscr{A}_{\rm vo},\ C = \mathscr{C}_{\rm vo},\ C_0 = \mathscr{C}_0 = \mathscr{C}_{\rm vo} \cap \mathcal{K}(\mathscr{H}).$$

Thus $A = \mathscr{D}$. From Corollary 5.3 we get $A_0 = \{0\}$ and then

$$B_0 = \mathscr{A}_{\rm vo} \cap \mathscr{C}_0 = \mathscr{A}_{\rm vo} \cap \mathscr{C}_{\rm vo} \cap \mathcal{K}(\mathscr{H}) = \mathscr{A}_{\rm vo} \cap \mathcal{K}(\mathscr{H}) = \mathscr{A}_0$$

by Lemma 5.5. Then we use Proposition 5.6 and the fact that $[V, U] \in \mathcal{K}(\mathscr{H})$ if $V \in \mathscr{A}_{\rm vo}$ (see (4.7) and note that $\delta(V) \in \mathscr{A}_0 \in \mathcal{K}(\mathscr{H})$). ∎

The quotient $\mathscr{C}_\infty/\mathscr{C}_0$ has a more explicit form. This follows immediately from Theorem 5.10 and Proposition 5.7.

Corollary 5.11 *If H is finite dimensional, then there is a unique morphism $\Phi : \mathscr{C}_\infty \to \mathcal{A}^{\otimes\infty} \otimes \mathscr{D}$ such that $\Phi(VD) = V_\infty \otimes D$ for all $V \in \mathscr{A}_\infty$ and $D \in \mathscr{D}$. This morphism is surjective and* $\ker \Phi = \mathscr{C}_0$*, hence we have a canonical isomorphism*

$$\mathscr{C}_\infty/\mathscr{C}_0 \simeq \mathcal{A}^{\otimes\infty} \otimes \mathscr{D}. \tag{5.19}$$

Example 5.12 The simplest choice is $\mathcal{A} = \mathbb{C}\mathbf{1}_H$. Then $\mathcal{A}^{\otimes n} = \mathbb{C}\mathbf{1}_n$ and $\mathscr{A}_\infty$ is the set of operators $V \in \mathcal{B}(\mathscr{H})$ of the form $V = \sum_{n\geq 0} V_n \mathbf{1}_n$, where $\{V_n\}$ is a convergent sequence of complex numbers, and $V_\infty = \lim_{n\to\infty} V_n$. In this case, Theorem 5.10 gives us a canonical isomorphism $\mathscr{C}_\infty/\mathscr{C}_0 \simeq \mathscr{D}$. On the other hand, $\mathscr{A}_{\rm vo}$ corresponds to the bounded sequences $\{V_n\}$ such that $\lim |V_{n+1} - V_n| = 0$, and the quotient $\mathscr{A}_{\rm vo}/\mathscr{A}_0$ is quite complicated (it can be described in terms of the Stone-Cech compactification of $\mathbb{N}$).

Example 5.13 In order to cover the tree case considered in [Gol] (see the Introduction) it suffices to choose $\mathcal{A}$ an abelian algebra. Since H is finite dimensional, the spectrum of $\mathcal{A}$ is a finite set A and we have $\mathcal{A} \simeq C(A)$ hence $\mathcal{A}^{\otimes n} \simeq C(A^n)$ canonically. If $A^\infty \equiv A^{\mathbb{N}^*}$ equipped with the product topology, then we get a natural identification $\mathcal{A}^{\otimes\infty} \simeq C(A^\infty)$. Let $\Gamma := \bigcup_{n\geq 0} A^n$, then $\mathscr{A}$ can be identified with the set of bounded functions $V : \Gamma \to \mathbb{C}$ and $\mathscr{A}_0$ is the subset of functions which tend to zero at infinity. The embedding (5.6) is obtained by extending a function $\varphi : A^n \to \mathbb{C}$ to a function on A^{n+1} by setting $\varphi(a_1, \dots, a_n, a_{n+1}) = \varphi(a_1, \dots, a_n)$. Thus $V \in \mathscr{A}_{\rm vo}$ if and only if

$$\lim_{n\to\infty} \sup_{a\in A^n,\, b\in A} |V(a, b) - V(a)| = 0.$$

Let $\pi_n : A^\infty \to A^n$ be the projection onto the n first factors. Then $V \in \mathscr{A}_\infty$ if and only if there is $V_\infty \in C(A^\infty)$ such that

$$\lim_{n\to\infty} \sup_{a\in A^\infty} |V \circ \pi_n(a) - V_\infty(a)| = 0.$$

This means that the function $\widetilde{V}$ defined on the space $\widetilde{\Gamma} = \Gamma \cup A^\infty$ equipped with the natural hyperbolic topology (see [Gol]) by the conditions $\widetilde{V}|\Gamma = V$ and $\widetilde{V}|A^\infty = V_\infty$ is continuous. And reciprocally, each continuous function $\widetilde{V} : \widetilde{\Gamma} \to \mathbb{C}$ defines by $\widetilde{V}|\Gamma = V$ an element of $\mathscr{A}_\infty$. This shows that our results cover those of [Gol].

We mention that in order to have a complete equivalence with the tree model as considered in [Gol] the vector u must be a cyclic vector of $\mathcal{A}$, in particular $\mathcal{A}$ must be maximal abelian. Indeed, in this case A can be identified with an orthonormal basis of H diagonalizing $\mathcal{A}$ (the vectors a are uniquely determined modulo a factor of modulus 1 and the associated character of $\mathcal{A}$ is $V \mapsto \langle a, Va \rangle$). Then $u = \sum_{a\in A} c_a a$ is cyclic for $\mathcal{A}$ if and only if $c_a \neq 0$ for all a. If $c_a = |A|^{-1/2}$ with $|A|$ the number of elements of A, we get the standard tree case.

Example 5.14 Another natural choice is $\mathcal{A} = \mathcal{B}(H)$. Then u is a cyclic vector for $\mathcal{A}$ because $u \neq 0$, so $\mathscr{C}_0 = \mathcal{K}(\mathscr{H})$. In this case we have

$$\mathscr{C}_\infty / \mathcal{K}(\mathscr{H}) \simeq \mathcal{B}(H)^{\otimes\infty} \otimes \mathscr{D}$$

and $\mathcal{B}(H)^{\otimes\infty}$ is a simple C^*-algebra.

We give an application to the computation of the essential spectrum. Note that if $L = \sum_{k=1}^n V^k D_k$, with $V^k \in \mathscr{A}_{\mathrm{vo}}$ and $D_k \in \mathscr{D}$, then $\Phi(L) = \sum_{k=1}^n \widehat{V^k} \otimes D_k$. In particular, we get

Proposition 5.15 *Let $L = D + V$ with $D \in \mathscr{D}$ and $V \in \mathscr{A}_{\mathrm{vo}}$ self-adjoint. Then*

$$\sigma_{\mathrm{ess}}(L) = \sigma(D) + \sigma(\widehat{V}). \tag{5.20}$$

If $V \in \mathscr{A}_\infty$, then

$$\sigma_{\mathrm{ess}}(L) = \sigma(D) + \sigma(V_\infty). \tag{5.21}$$

Proof: It suffices to note that $\Phi(L) = 1 \otimes D + \widehat{V} \otimes 1$ and to use the general relation: if A, B are self-adjoint then $\sigma(A \otimes 1 + 1 \otimes B) = \sigma(A) + \sigma(B)$. ■

In the abelian case the result is more general and more explicit.

Proposition 5.16 *Assume that we are in the framework of Example 5.13 and let $L = \sum_{k=1}^n V^k D_k$ be a self-adjoint operator with $V^k \in \mathscr{A}_\infty$ and $D_k \in \mathscr{D}$. Then*

$$\sigma_{\mathrm{ess}}(L) = \bigcup_{a \in A^\infty} \sigma\Big(\sum_k V_\infty^k(a) D_k \Big). \tag{5.22}$$

For the proof, observe that $a \mapsto \sum_k V_\infty^k(a) D_k$ is a norm continuous map on the compact space A^∞, which explains why the right hand side above is a closed set. A formula similar to (5.22) holds if $\mathscr{A}_\infty$ is replaced by $\mathscr{A}_{\rm vo}$, the only difference being that A^∞ must be replaced with the spectrum of the abelian algebra $\mathscr{A}_{\rm vo}/\mathscr{A}_0$.

Remarks: We shall make some final comments concerning various natural generalizations of the algebras considered above. Assume that $\mathcal{A}_n$ are C^*-algebras as at the beginning of Subsection 5.2 and let $\mathscr{A}$ be given by (5.9). Then

$$\mathscr{A}_{\rm rc} = \big\{ V = (V_n)_{n\geq 0} \mid V_n \in \mathcal{A}_n \text{ and } \{V_n \mid n \geq 0\} \text{ is relatively compact in } \mathcal{A}_\infty \big\}$$

is a C^*-subalgebra of $\mathscr{A}$ which contains $\mathscr{A}_{\rm vo}$. Interesting subalgebras of $\mathscr{A}_{\rm rc}$ can be defined as follows (this is the analog of a construction from [GeI]): let α be a filter on $\mathbb{N}$ finner than the Fr´echet filter and let $\mathscr{A}_\alpha$ be the set of $V = (V_n) \in \mathscr{A}$ such that $\lim_\alpha V_n$ exists in $\mathcal{A}_\infty$, where $\lim_\alpha$ means norm limit along the filter α. Note that $\mathscr{A}_\alpha = \mathscr{A}_{\rm rc}$ if α is an ultrafilter. Now it is natural to consider the C^*-algebra $\mathscr{C}_{\rm rc}$ generated by the Hamiltonians with potentials $V \in \mathscr{A}_{\rm rc}$, so the C^*-algebra generated by $\mathscr{A}_{\rm rc} \cup \mathscr{D}$, and the similarly defined algebras $\mathscr{C}_\alpha$. It would be interesting to describe the quotient $\mathscr{C}_\alpha/\mathscr{C}_0$, but neither the techniques of the Appendix nor those from [GeI] do not seem to be of any use for this. Indeed, the main ingredients of our proof where Proposition 5.6 and the fact that the commutator of a potential with U is compact, or these properties will not hold in general. Moreover, the examples treated in [GeI], more precisely the Klaus (or bumps) algebra, which has an obvious analog here, show that we cannot expect a simple embedding of the quotient into a tensor product. Note that "localizations at infinity" in the sense of [GeI] can be defined for the elements of $\mathscr{C}_{\rm rc}$ by using iterations of the operators λ_v of *left* multiplication by elements $v \in H$ in the Fock space $\mathscr{H}$, a technique already used in [GeI, Gol], and this could be used in order to define the canonical morphism which describes the quotient.

A Appendix

Let us consider two C^*-subalgebras A and B of a C^*-algebra C satisfying the following conditions:

- A or B is nuclear,
- $ab = ba$ if $a \in A$ and $b \in B$.

We denote by $A \otimes B$ the minimal C^*-algebra tensor product of the two algebras A and B. Since, by the nuclearity assumption, $A \otimes B$ is also the maximal tensor product of A and B, there is a unique morphism $\phi : A \otimes B \to C$ such that $\phi(a \otimes b) = ab$, see [Mur, Theorem 6.3.7].

Our purpose is to find conditions which ensure that ϕ is injective. Then ϕ is isometric and so it gives a canonical identification of the tensor product $A \otimes B$ with the C^*-subalgebra of C generated by A and B. The following simple observation is useful.

Lemma A.1 *The morphism ϕ is injective if and only if the following condition is satisfied: if $b_1, \ldots, b_n$ is a linearly independent family of elements of B, then*

$$a_1, \ldots, a_n \in A \text{ and } a_1 b_1 + \cdots + a_n b_n = 0 \Rightarrow a_1 = \cdots = a_n = 0. \tag{A.1}$$

Proof: This condition is clearly necessary. Reciprocally, let $A \odot B$ be the algebraic tensor product of A and B, identified with a dense subspace of $A \otimes B$. Then each $x \in A \odot B$ can be written $x = \sum a_i \otimes b_i$ for some linearly independent family $b_1, \ldots, b_n$ of elements of B and then $\phi(x) = \sum a_i b_i$. It follows immediately that $x \mapsto \|\phi(x)\|$ is a C^*-norm on $A \odot B$. But the nuclearity of A or B ensures that there is only one such norm, hence $\|\phi(x)\| = \|x\|$, so that ϕ extends to an isometry on $A \otimes B$. ■

The condition (A.1) is not easy to check in general, so it would be convenient to replace it with the simpler:

$$a \in A, b \in B, b \neq 0 \text{ and } ab = 0 \Rightarrow a = 0. \tag{A.2}$$

Exercise 2 in [Tak, Sec. 4.4] treats the case when A is abelian. The following result, which was suggested to us by a discussion with Georges Scandalis, is more suited to our purposes.

Let us say that a self-adjoint projection p in a C^*-algebra K is *minimal* if $p \neq 0$ and if the only projections $q \in K$ such that $q \leq p$ are 0 and p. We say that the algebra is *generated by minimal projections* if for each positive non zero element $a \in K$ there is a minimal projection p and a real $\alpha > 0$ such that $a \geq \alpha p$.

We also recall that an ideal K of A is called *essential* if for $a \in A$ the relation $aK = 0$ implies $a = 0$.

Proposition A.2 *If (A.2) is fulfilled and if A contains an essential ideal K which is generated by its minimal projections, then ϕ is injective.*

Proof: The following proof of the proposition in the case $A = \mathscr{D}$, which is the only case of interest in this paper, is due to Georges Scandalis: since $\mathscr{D}$ is isomorphic to the Toeplitz algebra, $\mathscr{D}$ contains a copy K of the algebra of compact operators on $\ell^2(\mathbb{N})$ as an essential ideal. Then it is clear that it suffices to assume that $A = K$

and in this case the assertion is essentially obvious, because $\ker(\varphi \otimes \psi)$ is an ideal of $K \otimes B$. These ideas are certainly sufficient to convince an expert in C^*-algebras, but since we have in mind a rather different audience, we shall develop and give the details of the preceding argument. We also follow a different idea in the last part of the proof.

(i) We first explain why it suffices to consider the case $A = K$. Note that one can identify $K \otimes B$ with the closed subspace of $A \otimes B$ generated by the elements of the form $a \otimes b$ with $a \in K, b \in B$ (see [Mur, Theorem 6.5.1]) and so $K \otimes B$ is an ideal in $A \otimes B$. Let us show that this is an essential ideal.

We can assume that K and B are faithfully and non-degenerately represented on Hilbert spaces $\mathscr{E}, \mathscr{F}$. Since K is essential in A, the representation of K extends to a faithful and non-degenerate representation of A on $\mathscr{E}$ (this is an easy exercise). Thus we are in the situation $K \subset A \subset \mathcal{B}(\mathscr{E})$, $B \subset \mathcal{B}(\mathscr{F})$, the action of K on $\mathscr{E}$ being non-degenerate. Let $\{k_\alpha\}$ be an approximate unit of K. Then $\text{s-}\lim k_\alpha = 1$ on $\mathscr{E}$, because $\|k_\alpha\| \leq 1$ and the linear subspace generated by the vectors ke, with $k \in K$ and $e \in \mathscr{E}$, is dense in $\mathscr{E}$ (in fact $K\mathscr{E} = \mathscr{E}$). Similarly, if $\{b_\beta\}$ is an approximate unit for B then $\text{s-}\lim b_\beta = 1$ on $\mathscr{F}$ and then clearly $\text{s-}\lim_{\alpha,\beta} k_\alpha \otimes b_\beta = 1$ on $\mathscr{E} \otimes \mathscr{F}$. From our assumptions (the tensor products are equal to the minimal ones) we get $K \otimes B \subset A \otimes B \subset \mathcal{B}(\mathscr{E} \otimes \mathscr{F})$. Let $x \in A \otimes B$ such that $x \cdot K \otimes B = 0$. Then $x \cdot k_\alpha \otimes b_\beta = 0$ for all α, β, hence $x = \text{s-}\lim_{\alpha,\beta} x \cdot k_\alpha \otimes b_\beta = 0$. Thus $K \otimes B$ is an essential ideal in $A \otimes B$.

Now it is obvious that a morphism $A \otimes B \to C$ whose restriction to $K \otimes B$ is injective, is injective. Thus it suffices to show that the restriction of ϕ to $K \otimes B$ is injective, so from now on we may, and we shall, assume that $A = K$.

(ii) We make a preliminary remark: let P be the set of minimal projections in A; then for each $p \in P$ we have $pAp = \mathbb{C}p$. Note that this is equivalent to the fact that for each $p \in P$ there is a state τ_p of A such that $pap = \tau_p(a)p$ for all $a \in A$.

Since pAp is the C^*-subalgebra of A consisting of the elements a such that $ap = pa = a$, it suffices to show that each $a \in pAp$ with $a \geq 0$, $a \neq 0$, is of the form λp for some real λ. Let $q \in P$ such that $a \geq \varepsilon q$ for some real $\varepsilon > 0$. Then $\varepsilon q \leq a = pap \leq \|a\| p$ from which it is easy to deduce that $q \leq p$, hence $q = p$ (p and q being minimal). Let λ be the largest positive number such that $a \geq \lambda p$. If $a - \lambda p \neq 0$, then there is $r \in P$ and a real $\nu > 0$ such that $a - \lambda p \geq \nu r$. In particular $a \geq \nu r$ and so $r = p$ by the preceding argument. Hence $a \geq (\lambda + \nu)p$, which contradicts the maximality of λ. Thus $a = \lambda p$.

(iii) Finally, we check (A.1). Let $b_1, \dots, b_n$ be a linearly independent family of elements of B and $a_1, \dots, a_n \in A$ such that $\sum a_i b_i = 0$. Then for all $a \in A$ and $p \in P$ we have

$$p\left(\sum \tau_p(aa_i)b_i\right) = \sum paa_ipb_i = pa\left(\sum a_ib_i\right)p = 0.$$

Since $p \in A$, $p \neq 0$, and $\sum \tau_p(aa_i)b_i \in B$, we must have $\sum \tau_p(aa_i)b_i = 0$. But $\tau_p(aa_i)$ are complex numbers, so $\tau_p(aa_i) = 0$ for each i and all $a \in A$. In particular, we have $\tau_p(a_i^* a_i) = 0$, which is equivalent to $pa_i^* a_i p = 0$ for all $p \in P$. If $a_i^* a_i \neq 0$, then there are $\alpha > 0$ and $q \in P$ such that $a_i^* a_i \geq \alpha q$. By taking $p = q$, we get $0 = qa_i^* a_i q \geq \alpha q$, which is absurd. Thus $a_i^* a_i = 0$, i.e. $a_i = 0$. ■

The next proposition is a simple extension of the preceding one. We recall that a C^*-algebra is called *elementary* if it is isomorphic with the C^*-algebra of all compact operators on some Hilbert space.

Proposition A.3 *Let A, B be C^*-subalgebras of a C^*-algebra C, let C_0 be an ideal of C, and let $A_0 = A \cap C_0$ and $B_0 = B \cap C_0$ be the corresponding ideals of A and B respectively. Denote by $\widehat{A} = A/A_0$, $\widehat{B} = B/B_0$ and $\widehat{C} = C/C_0$ the associated quotient algebras and assume that:*

- *$\widehat{A}$ contains an essential ideal K which is an elementary algebra and such that $\widehat{A}/K$ is nuclear (e.g. abelian)*
- *if $a \in A, b \in B$ then $[a, b] \in C_0$*
- *if $a \in A, b \in B$ and $ab \in C_0$ then either $a \in C_0$ or $b \in C_0$.*
- *C is the C^*-algebra generated by $A \cup B$*

Then there is a unique morphism $\Phi : C \to \widehat{A} \otimes \widehat{B}$ such that $\Phi(ab) = \widehat{a} \otimes \widehat{b}$ for all $a \in A, b \in B$. This morphism is surjective and has C_0 as kernel. In other terms, we have a canonical isomorphism

$$C/C_0 \simeq (A/A_0) \otimes (B/B_0). \tag{A.3}$$

Proof: It is clear that an elementary algebra is generated by minimal projections and is nuclear hence, by [Mur, Theorem 6.5.3], the conditions we impose on A imply the nuclearity of $\widehat{A}$. Note that $\widehat{A}$ and $\widehat{B}$ are C^*-subalgebras of $\widehat{C}$ and that they generate $\widehat{C}$. Moreover, we have $\widehat{a}\widehat{b} = \widehat{b}\widehat{a}$ for all $a \in A, b \in B$ and if $\widehat{a}\widehat{b} = 0$ then $\widehat{a} = 0$ or $\widehat{b} = 0$. By Proposition A.2 the natural morphism $\widehat{A} \otimes \widehat{B} \to \widehat{C}$ is an isomorphism. Denote ψ its inverse, let $\pi : C \to \widehat{C}$ be the canonical map, and let $\Phi = \psi \circ \pi$. This proves the existence of a morphism with the required properties. Its uniqueness is obvious. ■

Now we summarize the facts needed in this paper.

Corollary A.4 *Let C be a C^*-algebra, C_0 an ideal of C, B a C^*-subalgebra of C, $B_0 = B \cap C_0$, and $u \in C$ a non unitary isometry such that $B \cup \{u\}$ generates C.*

Let A be the C^*-subalgebra generated by u and let us assume that $A \cap C_0 = \{0\}$ and that $[u, b] \in C_0$ for all $b \in B$. Finally, assume that:

$$a \in A,\ b \in B \ \text{ and } \ ab \in C_0 \Rightarrow a \in C_0 \ \text{ or } \ b \in C_0.$$

Then there is a unique morphism $\Phi : C \to A \otimes (B/B_0)$ such that $\Phi(ab) = a \otimes \widehat{b}$ for all $a \in A, b \in B$ (where $\widehat{b}$ is the image of b in B/B_0). This morphism is surjective and has C_0 as kernel. In other terms, we have a canonical isomorphism

$$C/C_0 \simeq A \otimes (B/B_0). \tag{A.4}$$

Proof: The assumption $[u, b] \in C_0$ for all $b \in B$ clearly implies $[a, b] \in C_0$ for all $a \in A, b \in B$. Moreover, the algebra $A = \widehat{A}$ is isomorphic with the Toeplitz algebra, see [Mur, Theorem 3.5.18], and so all the conditions imposed on it in Proposition A.3 are satisfied, see [Mur, Example 6.5.1]. ■

We shall now study a more elementary situation which is relevant in the context of Section 5. Our purpose is to treat the case when the Hilbert space H is of dimension 1 (this situation, although much simpler, is not covered by the arguments from Section 5).

This is in fact the case considered in Example 2.6, namely we take $\mathscr{H} = \ell^2(\mathbb{N})$ and define the isometry U by $Ue_n = e_{n+1}$. Then the C^*-algebra $\mathscr{D}(\mathbb{N})$ generated by U is just the Toeplitz algebra [Mur, Section 3.5]. We also consider the situation of Example 2.5, where $\mathscr{H} = \ell^2(\mathbb{Z})$ and U acts in the same way, but now it is a unitary operator and the C^*-algebra $\mathscr{D}(\mathbb{Z})$ generated by it is isomorphic to the algebra $C(T)$ of continuous functions on the unit circle T (make a Fourier transformation). Let $\mathscr{K}(\mathbb{N}) := \mathcal{K}(\ell^2(\mathbb{N}))$ and $\mathscr{K}(\mathbb{Z}) := \mathcal{K}(\ell^2(\mathbb{Z}))$ be the ideals of compact operators on $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ respectively.

It is clear that $\mathscr{D}(\mathbb{Z}) \cap \mathscr{K}(\mathbb{Z}) = \{0\}$ and it is easily shown that $\mathscr{K}(\mathbb{N}) \subset \mathscr{D}(\mathbb{N})$. From [Mur, Theorem 3.5.11] it follows that we have a canonical isomorphism $\mathscr{D}(\mathbb{N})/\mathscr{K}(\mathbb{N}) \simeq \mathscr{D}(\mathbb{Z})$. This isomorphism is uniquely defined by the fact that it sends the shift operator U on $\mathbb{N}$ into the the shift operator U on $\mathbb{Z}$, cf. the Coburn theorem [Mur, Theorem 3.5.18]).

We identify $\ell^\infty(\mathbb{N})$ with the set of bounded multiplication operators on $\ell^2(\mathbb{N})$.

Proposition A.5 *Let $\mathscr{A}$ be a unital C^*-subalgebra of $\ell^\infty(\mathbb{N})$ such that for each $V \in \mathscr{A}$ the operator $[U, V]$ is compact. Let $\mathscr{C}$ be the C^*-algebra generated by $\mathscr{A} \cup \{U\}$ and let us denote $\mathscr{A}_0 = \mathscr{A} \cap \mathscr{K}(\mathbb{N})$ and $\mathscr{C}_0 = \mathscr{C} \cap \mathscr{K}(\mathbb{N})$. Then*

$$\mathscr{C}/\mathscr{C}_0 \simeq (\mathscr{A}/\mathscr{A}_0) \otimes \mathscr{D}(\mathbb{Z}). \tag{A.5}$$

This relation holds also if $\mathbb{N}$ is replaced with $\mathbb{Z}$.

Proof: Clearly $[D, V] \in \mathscr{K}(\mathbb{N})$ for all $D \in \mathscr{D}(\mathbb{N})$ and $V \in \mathscr{A}$, hence we have a natural surjective morphism $(\mathscr{A}/\mathscr{A}_0) \otimes \mathscr{D}(\mathbb{Z}) \to \mathscr{C}/\mathscr{C}_0$. It remains to show that this is an injective map. According to [Tak, Sec. 4.4, Exercice 2], it suffices to prove the following: if $D \in \mathscr{D}(\mathbb{N})$ is not compact and if $V \in \ell^\infty(\mathbb{N})$ has the property $VD \in \mathscr{K}(\mathbb{N})$, then V is compact. We may assume that $D \geq 0$, otherwise we replace it by DD^*.

To each $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ we associate a unitary operator S_α on $\ell^2(\mathbb{N})$ by the rule $S_\alpha e_n = \alpha^n e_n$. We clearly have $S_\alpha U S_\alpha^* = \alpha U$, thus $A \mapsto A_\alpha := S_\alpha A S_\alpha^*$ is an automorphism of $\mathcal{B}(\ell^2(\mathbb{N}))$ which leaves invariant the algebra $\mathscr{D}(\mathbb{N})$ and the ideal $\mathscr{K}(\mathbb{N})$ and reduces to the identity on $\ell^\infty(\mathbb{N})$. Thus $VD_\alpha \in \mathscr{K}(\mathbb{N})$ for each such α. We shall prove the following: there are $\alpha_1, \ldots, \alpha_n$ such that $\sum D_{\alpha_i} = A + K$, where A is an invertible operator and K is compact. Then VA is compact and $V = VAA^{-1}$ too, which finishes the proof of the proposition.

We shall denote by $\widehat{S}$ the image of an operator $S \in \mathcal{B}(\ell^2(\mathbb{N}))$ in the Calkin algebra $\mathcal{B}(\ell^2(\mathbb{N}))/\mathcal{K}(\ell^2(\mathbb{N}))$. Thus we have $\widehat{D} \geq 0$, $\widehat{D} \neq 0$. As explained before the proof, we have $\mathscr{D}(\mathbb{N})/\mathscr{K}(\mathbb{N}) \simeq \mathscr{D}(\mathbb{Z}) \simeq C(T)$. Let θ_α be the automorphism of $C(T)$ defined by $\theta_\alpha(\varphi)(z) = \varphi(z\alpha)$. Then we have $\widehat{D_\alpha} = \theta_\alpha(\widehat{D})$ (because this holds for U, hence for all the elements of the C^*-algebra generated by U). But $\widehat{D}$ is a positive continuous function on T which is strictly positive at some point, hence the sum of a finite number of translates of the function is strictly positive, thus invertible in $C(T)$. So there are $\alpha_1, \ldots, \alpha_n$ such that the image of $\sum D_{\alpha_i}$ be invertible in the Calkin algebra and this is exactly what we need. ■

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