C^*-algebras of anisotropic Schrödinger operators on trees

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Abstract

We study a C^*-algebra generated by differential operators on a tree. We give a complete description of its quotient with respect to the compact operators. This allows us to compute the essential spectrum of self-adjoint operators affiliated to this algebra. The results cover Schrödinger operators with highly anisotropic, possibly unbounded potentials.

1 Introduction

Given a ν-fold tree Γ of origin e with its canonical metric d, we write $x \sim y$ when x and y are connected by an edge and we set $|x| = d(x, e)$. For each $x \in \Gamma \setminus \{e\}$, we denote by $x' \equiv x^{(1)}$ the unique element $y \sim x$ such that $|y| = |x| - 1$ and we set $x^{(p)} = (x^{(p-1)})'$ for $1 \leq p \leq |x|$. Let $x\Gamma = \{y \in \Gamma \mid |y| \geq |x| \text{ and } y^{(|y|-|x|)} = x\}$, where the convention $x^{(0)} = x$ has been used.

On $\ell^2(\Gamma)$ we define the bounded operator ∂ given by $(\partial f)(x) = \sum_{y'=x} f(y)$. Its adjoint is given by $(\partial^* f)(e) = 0$ and $(\partial^* f)(x) = f(x')$ for $|x| \geq 1$. Let $\mathscr{D}$ be the C^*-algebra generated by ∂.

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In order to obtain our algebra of potentials, we consider the "hyperbolic" compactification $\widehat{\Gamma} = \Gamma \cup \partial\Gamma$ of Γ constructed as follows. An element x of the boundary at infinity $\partial\Gamma$ is a Γ-valued sequence $x = (x_n)_{n\in\mathbb{N}}$ such that $|x_n| = n$ and $x_{n+1} \sim x_n$ for all $n \in \mathbb{N}$. We set $|x| = \infty$ for $x \in \partial\Gamma$. The space $\widehat{\Gamma}$ is equipped with a natural ultrametric space structure. For $x \in \partial\Gamma$ and $(y_n)_{n\in\mathbb{N}}$ a sequence in Γ we have $\lim_{n\to\infty} y_n = x$ if for each $m \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that for each $n \geq N$ we have $y_n \in x_m\Gamma$. We denote by $C(\widehat{\Gamma})$ the set of complex-valued continuous functions defined on $\widehat{\Gamma}$. Since Γ is dense in $\widehat{\Gamma}$, we can view $C(\widehat{\Gamma})$ as a C^*-subalgebra of $C_b(\Gamma)$, the algebra of bounded complex-valued functions defined on Γ. For $V \in C(\widehat{\Gamma})$, we denote by $V(Q)$ the operator of multiplication by V in $\ell^2(\Gamma)$.

Let us now denote by $\mathscr{C}(\widehat{\Gamma})$ the C^*-algebra generated by $\mathscr{D}$ and $C(\widehat{\Gamma})$. It contains the compact operators of $\ell^2(\Gamma)$. Following the strategy exposed in [6], we shall first compute its quotient with respect to the ideal of compact operators. We stress that the crossed product technique introduced in [6] in order to compute quotients cannot be used in our case. Instead, we shall use the Theorem 4.5 in order to calculate the essential spectrum of self-adjoint operators related to $\mathscr{C}(\widehat{\Gamma})$. In this introduction we consider only the most important case, when $\nu > 1$.

Theorem 1.1 *Let $\nu > 1$. There is a unique morphism $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \mathscr{D} \otimes C(\partial\Gamma)$ such that $\Phi(D) = D \otimes 1$ for all $D \in \mathscr{D}$ and $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$. This morphism is surjective and its kernel is $\mathbb{K}(\Gamma)$.*

The rest of this introduction is devoted to some applications of this theorem to spectral analysis. Let $\nu > 1$ and $H = \sum_{\alpha,\beta} a_{\alpha,\beta}(Q)\partial^{*\alpha}\partial^{\beta} + K$, where K is a compact operator, $a_{\alpha,\beta} \in C(\widehat{\Gamma})$ and $a_{\alpha,\beta} = 0$ for all $(\alpha,\beta) \in \mathbb{N}^2$ but a finite number of pairs. Clearly $H \in \mathscr{C}(\widehat{\Gamma})$. As a consequence of the Theorem 1.1, there is Φ such that $\Phi(H) = \sum_{\alpha,\beta} \partial^{*\alpha}\partial^{\beta} \otimes (a_{\alpha,\beta})|_{\partial\Gamma}$, and, if H self-adjoint, its essential spectrum is:

$$\sigma_{\text{ess}}(H) = \bigcup_{\gamma\in\partial\Gamma} \sigma\Big(\sum_{\alpha,\beta} a_{\alpha,\beta}(\gamma)\partial^{*\alpha}\partial^{\beta}\Big).$$

This result can be made quite explicit in the particular case of a Schrödinger operator

$H = \Delta + V(Q)$ with potential V in $C(\widehat{\Gamma})$. Since Δ is a bounded operator on $\ell^2(\Gamma)$ defined by $(\Delta f)(x) = \sum_{y\sim x}(f(y) - f(x))$, it belongs to $\mathscr{C}(\widehat{\Gamma})$. We then set $\Delta_0 = \partial + \partial^* - \nu\mathrm{Id}$ (which belongs to $\mathscr{D}$) and notice that $\Delta - \Delta_0$ is compact. One then gets (see [1] for instance):

$$\sigma_{ess}(\partial + \partial^*) = \sigma_{ac}(\partial + \partial^*) = \sigma(\partial + \partial^*) = [-2\sqrt{\nu}, 2\sqrt{\nu}],$$

where $\sigma_{ac}(T)$ denotes the absolute continuous part of the spectrum of a given self-adjoint operator T. On the other hand, Theorem 1.1 gives us directly $\sigma_{\mathrm{ess}}(\partial^* + \partial) = \sigma(\partial^* + \partial)$. We thus get

$$\sigma_{ess}(\Delta + V(Q)) = \sigma(\Delta_0) + V(\partial\Gamma) = [-\nu - 2\sqrt{\nu}, -\nu + 2\sqrt{\nu}] + V(\partial\Gamma).$$

In fact this result holds (and is trivial) in the case of $\nu = 1$, i.e. when $\Gamma = \mathbb{N}$.

Given a continuous function on $\partial\Gamma$, the Tietze theorem allows us to extend it to a continuous function on $\widehat{\Gamma}$, so one may construct a large class of Hamiltonians with given essential spectra. Nevertheless, we are able to point out a concrete class of non-trivial potentials $V \in C(\widehat{\Gamma})$ with uniform behaviour at infinity which form a dense family of $C(\widehat{\Gamma})$. Namely, for each bounded function $f : \Gamma \to \mathbb{R}$ and each real $\alpha > 1$ let

$$V(x) = \sum_{k=1}^{|x|} \frac{f(x_k)}{k^\alpha}, \tag{1.1}$$

where $x_k = x^{|x|-k}$ for $x \in \Gamma$ (V belongs to $C(\widehat{\Gamma})$ because of Proposition 2.3).

Concerning finer spectral features, based mainly on the Mourre estimate, we mention that in the case $H = \Delta + V(Q)$, with V as in (1.1) where $\alpha \geq 3$ and such that $V(\partial\Gamma) = 0$, the results of [1] can be applied (the hypotheses of the Lemmas 6 and 7 from [1] are verified since $V(x) = O(|x|^{-\alpha+1})$ when $|x| \to \infty$). The aim of our work in preparation [8] is to prove that the Mourre estimate holds for more general classes of Hamiltonians affiliated to $\mathscr{C}(\widehat{\Gamma})$ and to develop a scattering theory for them. Theorem 1.1 remains the key technical point for these purposes.

The preceding results on trees allow us to treat more general graphs. We recall that a graph is said to be *connected* if two of its elements can be joined by a sequence of neighbours. Let $G = \bigcup_{i=1}^{n} \Gamma_i \cup G_0$ be a finite disjoint union of Γ_i, each Γ_i being a ν_i-fold branching tree with $\nu_i \geq 1$ and of G_0, a compact connected graph. We endow G with a connected graph structure that respects the graph structure of each Γ_i and the one of G_0, such that Γ_i is connected to Γ_j ($i \neq j$) only through G_0 and such that Γ_i is connected to G_0 only through e_i, the origin of Γ_i. The graph G is hyperbolic and its boundary at infinity ∂G is the disjoint union $\cup_{i=1}^{n} \partial\Gamma_i$. We now choose $V \in C(G \cup \partial G)$. One has $V|_{\widehat{\Gamma}_i} \in C(\widehat{\Gamma}_i)$ for all $i = 1, \dots, n$ and we easily obtain:

$$\sigma_{ess}(\Delta + V(Q)) = \bigcup_{i=1}^{n} \big([-\nu_i - 2\sqrt{\nu_i}, -\nu_i + 2\sqrt{\nu_i}\,] + V(\partial\Gamma_i) \big).$$

This covers in particular the case of the Cayley graph of a free group with finite system of generators. We recall that the Cayley graph of a group G with a system of generators S is the graph defined on the set G with the relation $x \sim y$ if $xy^{-1} \in S$ or $yx^{-1} \in S$. Let G be a free group with a system of generators S such that $S = S^{-1}$. We denote by e its neutral element and we set $|S| = \nu + 1$. One may associate the restriction of the Cayley graph to the set of words starting with a given generator with a ν-fold branching tree having as origin the generator. Hence, the Cayley graph of G will be $\cup_{i=1}^{\nu} \Gamma_i \cup \{e\}$ where Γ_i is a ν-fold branching tree with the above graph structure.

We now go further by taking $V \in C(\widehat{\Gamma}, \overline{\mathbb{R}})$ such that $V(\Gamma) \subset \mathbb{R}$ (here $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the Alexandrov compactification of $\mathbb{R}$). More precisely, $V \in C(\widehat{\Gamma}, \overline{\mathbb{R}})$ if and only if for each $\gamma \in \partial\Gamma$ we have either $\lim_{x \to \gamma} V(x) = l$ where $l \in \mathbb{R}$ or for each $M \geq 0$ there is $N \in \mathbb{N}$ such that $|V(x)| \geq M$ for all $n \geq N$ and $x \in \gamma_n \Gamma$ (see Proposition 2.3). We set

$$D(V) = \{ f \in \ell^2(\Gamma) \mid \|V(Q)f\|^2 < \infty \}.$$

Let $T \in \mathscr{D}$ and $T_0 = \Phi(T)$. Since T is bounded, the operator $H = T + V(Q)$ with domain $D(V)$ is self-adjoint and it is affiliated to $\mathscr{C}(\widehat{\Gamma})$ (i.e. its resolvent belongs to $\mathscr{C}(\widehat{\Gamma})$). Indeed, we have $(V(Q) + z)^{-1} \in C(\widehat{\Gamma})$ for each $z \in \mathbb{C} \setminus \mathbb{R}$, and for large such z,

$$(H + z)^{-1} = (V(Q) + z)^{-1} \sum_{n \geq 0} (T(V(Q) + z)^{-1})^n,$$

where the series is norm convergent. Now, with the same z, we use the Theorem 1.1 and the fact that $\mathscr{D} \otimes C(\partial\Gamma) \simeq C(\partial\Gamma, \mathscr{D})$ to obtain

$$\Phi_\gamma\big((H+z)^{-1}\big) \equiv \Phi\big((H+z)^{-1}\big)(\gamma) = (V(\gamma)+z)^{-1} \sum_{n\geq 0} (T_0(V(\gamma)+z)^{-1})^n.$$

Note that $(V(\gamma)+z)^{-1} = 0$ if $V(\gamma) = \infty$. By analytic continuation we get $\Phi_\gamma((T + V(Q) + z)^{-1}) = (T_0 + V(\gamma) + z)^{-1}$, for all $z \in \mathbb{C} \setminus \mathbb{R}$. We used the convention $(T_0 + V(\gamma) + z)^{-1} = 0$ if $V(\gamma) = \infty$.

We now compute the essential spectrum of H. If $V(\gamma) = \infty$ then $\sigma(\Phi_\gamma(H)) = \emptyset$. Otherwise, one has $\sigma(\Phi_\gamma(H)) = \sigma(T_0+V(\gamma)) = \sigma(T_0)+V(\gamma)$. Hence we obtain:

$$\sigma_{\text{ess}}(T + V(Q)) = \sigma(T_0) + V(\partial\Gamma_0),$$

where $\partial\Gamma_0$ is the set of $\gamma \in \partial\Gamma$ such that $V(\gamma) \in \mathbb{R}$.

Remark: We mention an interesting question which has not been studied in this paper. In fact, one could replace the algebra $\mathscr{D}$ by the (much bigger) C^*-algebra generated by all the right translations ρ_a (see Subsection 3.4 for notations) and consider the corresponding algebra $\mathscr{C}(\widehat{\Gamma})$. This is a natural object, since it contains all the "right-differential" operators acting on the tree (not only polynomials in ∂ and ∂^*). A combination of the techniques that we use and that of [9, 10] could allow one to compute the quotient in this case too. We also note that in [9, 10] a certain connection with the notion of crossed-product is pointed out, and this could be useful in further investigations. I would like to thank the referee for bringing to my attention the two papers of A. Nica quoted above.

2 Trees and related objects

2.1 The free monoïd Γ

Let $\mathscr{A}$ be a finite set consisting of ν objects. Let Γ be the free monoïd over $\mathscr{A}$; its elements are *words* and those of $\mathscr{A}$ *letters*. We refer to [3, Chapter I, §7] for a detailed discussion of these notions, but we recall that a word x is an $\mathscr{A}$-valued map defined on a set of the form[1] $[\![1, n]\!]$ with $n \in \mathbb{N}$, $x(i)$

[1] We use the notation $[\![1, n]\!] = [1, n] \cap \mathbb{N}$ where $\mathbb{N}$ is the set of integers ≥ 0 and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

being the i-th letter of the word x. The integer n (the number of letters of x) is the length of the word and will be denoted $|x|$. There is a unique word e of length 0, its domain being the empty set. This is the neutral element of Γ. We will also identify $\mathscr{A}$ with the set of words of length 1.

The monoïd Γ will be endowed with the discrete topology. If $x \in \Gamma$, we denote $x\Gamma$ and Γx the right and left ideals generated by x. We have on Γ a canonical order relation which is by definition:

$$x \leq y \Leftrightarrow y \in x\Gamma.$$

We recall some terminology from the theory of ordered sets. If Γ is an arbitrary ordered set and $x, y \in \Gamma$, then one says that y *covers* x if $x < y$ and if $x \leq z \leq y \Rightarrow z = x$ or $z = y$. If $x \in \Gamma$, we denote $\widetilde{x} = \{y \in \Gamma \mid y$ covers $x\}$

In our case, y covers x if $x \leq y$ and $|y| = |x| + 1$. Notice that each element $x \in \Gamma \backslash \{e\}$ covers a unique element x', its *father*, and each element $x \in \Gamma$ is covered by ν elements, its *sons*. The set of sons of x clearly is $\widetilde{x} = \{x\varepsilon \mid \varepsilon \in \mathscr{A}\}$. Hence:

$$y \text{ covers } x \Leftrightarrow y' = x \Leftrightarrow y \in \widetilde{x}.$$

For $|x| \geq n$, we define $x^{(n)}$ inductively by setting $x^{(0)} = x$ and $x^{(m+1)} = (x^{(m)})'$ for $m \leq n - 1$. One may also notice that: $|x^{(\alpha)}| = |x| - \alpha$, if $\alpha \leq |x|$, and for $\alpha \leq |ab|$:

$$(ab)^{(\alpha)} = \begin{cases} ab^{(\alpha)}, & \text{if } \alpha \leq |b| \\ a^{(\alpha - |b|)}, & \text{if } \alpha \geq |b|. \end{cases}$$

We remark that if $\nu = 1$ then $\Gamma = \mathbb{N}$ and if $\nu > 1$ then Γ is the set of monoms of ν non-commutative variables.

2.2 The tree Γ and the extended tree associated to $\mathscr{A}$

Recall that a graph is a couple $G = (V, E)$, where V is a set (of *vertices*) and E is a set of pairs of elements of V (the *edges*). If x and y are joined by an edge, one says that they are *neighbours* and one abbreviates $x \sim y$. The graph structure allows one to endow V with a canonical metric d, where $d(x, y)$ is the length of the shortest path in G joining x to y.

The graph G_Γ associated to the free monoïd Γ is defined as follows: $V = \Gamma$ and $x \sim y$ if x covers y or y covers x. It is usual to identify Γ and G_Γ, the so-called ν-fold branching tree. For all $x \in \Gamma$, we have $|x| = d(e, x)$. We set $B(x, r) = \{y \in \Gamma \mid d(x, y) < r\}$ and $S^n = \{x \in \Gamma \mid |x| = n\}$.

We shall now define an extended tree by mimicking the definition of a free monoïd over $\mathscr{A}$. We choose $o \in \mathscr{A}$; this element will be fixed from now on. For each integer r, we set $\mathbb{Z}_r = \{i \in \mathbb{Z} \mid i \leq r\}$. The *extended tree* $\widetilde{\Gamma}$ *associated to* $\mathscr{A}$ is the set of $\mathscr{A}$-valued maps x defined on sets of the form $\mathbb{Z}_r$ such that $\{i \mid x(i) \neq o\}$ is finite. For $x \in \widetilde{\Gamma}$, the unique $r \in \mathbb{Z}$ such that x is a map $\mathbb{Z}_r \to \mathscr{A}$ will be denoted $|x|$ and will be called *length* of x.

We shall identify Γ with the set $\{x \mid |x| \geq 0 \text{ and } x(i) = o \text{ if } i \leq 0\}$ as follows: if $x \in \Gamma$ then we associate to it the element of $\widetilde{\Gamma}$ defined on $\mathbb{Z}_{|x|}$ by extending x with $x(i) = o$ if $i \leq 0$. The element e will be identified with the map $e \in \widetilde{\Gamma}$ such that $|e| = 0$ and $e(i) = o, \forall i \leq 0$. Notice that the two notions of length are consistent on Γ.

There is a natural right action of Γ on $\widetilde{\Gamma}$ by concatenation, i.e. for $x \in \widetilde{\Gamma}$ and $y \in \Gamma$, xy will be the function z defined on $\mathbb{Z}_{|x|+|y|}$ such that $z(i) = x(i)$, for $i \in \mathbb{Z}_{|x|}$ and $z(|x| + i) = y(i)$ for $i \in [\![1, |y|\,]\!]$. Then we equip $\widetilde{\Gamma}$ with an order relation by setting:

$$x \leq y \Leftrightarrow y \in x\Gamma.$$

As before, y covers x if and only if $x \leq y$ and $|y| = |x| + 1$. Now, each $x \in \widetilde{\Gamma}$ covers a unique $x' \in \widetilde{\Gamma}$ and each $x \in \widetilde{\Gamma}$ is covered by ν elements, namely those of $\widetilde{x} = \{x\varepsilon \mid \varepsilon \in \mathscr{A}\}$. We still have: $y \text{ covers } x \Leftrightarrow y' = x \Leftrightarrow y \in \widetilde{x}$. Observe that $x' = x|_{\mathbb{Z}_{|x|-1}}$. We will set $x^{(\alpha)} = x|_{\mathbb{Z}_{|x|-\alpha}}$ for all $\alpha \in \mathbb{Z}$. As we did it for Γ, we shall indentify the graph $G_{\widetilde{\Gamma}}$ with $\widetilde{\Gamma}$. This justifies the notion of extended *tree* used for $\widetilde{\Gamma}$.

2.3 The boundary at infinity of Γ

We shall see in the ending remark of this subsection that the boundary at infinity of Γ can be thought as the boundary of a 0-hyperbolic space in the sense of Gromov. We prefer, however, to give a simpler presentation that is closer to the theory of p-adic numbers (see [11] for instance). In fact, if ν is prime the boundary will be the set of ν-adic integers.

Definition 2.1 *The boundary at infinity of* Γ *is the set* $\partial\Gamma = \{x : \mathbb{N}^* \to \mathscr{A}\}$. *For* $x \in \partial\Gamma$, *we set* $|x| = \infty$.

Let $\widehat{\Gamma}$ be $\Gamma \cup \partial\Gamma$. For $x \in \widehat{\Gamma}$, we define the sequence $(x_n)_{n \in [\![0,|x|]\!]}$ with values in Γ by setting $x_0 = e$ and $x_n = x|_{[\![1,n]\!]}$ for $n \geq 1$. Observe that the map $x \mapsto (x_n)_{n \in [\![0,|x|]\!]}$ is injective. There is a natural left action of Γ on $\widehat{\Gamma}$. For $x \in \Gamma$ and $y \in \widehat{\Gamma}$, xy will be defined on the set[2] $[\![1, |x| + |y|]\!]$ by $x(i)$ for $i \leq |x|$ and by $y(i - |x|)$ for $i > |x|$.

We will now equip $\widehat{\Gamma}$ with a structure of ultrametric space. We define a kind of valuation v on $\widehat{\Gamma} \times \widehat{\Gamma}$ by

$$v(x,y) = \begin{cases} \max\{n \mid x_n = y_n\} & \text{if} \quad x \neq y \\ \infty & \text{if} \quad x = y. \end{cases} \tag{2.1}$$

If $x, y, z \in \widehat{\Gamma}$ it is easy to see that:

$$v(x,y) \geq \min(v(x,z), v(z,y)). \tag{2.2}$$

Let us set on $\widehat{\Gamma}$:

$$\widehat{d}(x,y) = \exp(-v(x,y)).$$

The relation (2.2) clearly implies that $(\widehat{\Gamma}, \widehat{d})$ is an ultrametric space, i.e. a metric space such that $\widehat{d}(x,y) \leq \max(\widehat{d}(x,z), \widehat{d}(z,y))$, for $x, y, z \in \widehat{\Gamma}$. We will denote, for $r > 0$, $\widehat{B}(x,r) = \{y \in \widehat{\Gamma} \mid \widehat{d}(x,y) < r\}$. Notice that ultrametricity implies that $\widehat{B}(x,r)$ is closed for all $x \in \widehat{\Gamma}$ and $r > 0$.

The topology induced by $\widehat{\Gamma}$ on Γ coincides with the initial topology of Γ, the discrete one. For $x \in \partial\Gamma$ and $n \in \mathbb{N}$,

$$x_n\widehat{\Gamma} \;=\; \{y \in \widehat{\Gamma} \mid v(x,y) \geq n\} = \widehat{B}(x, \exp(-n+1))$$

which is the closure of $x_n\Gamma$ in $\widehat{\Gamma}$. Hence for each $x \in \partial\Gamma$, $\{x_n\widehat{\Gamma}\}_{n \in \mathbb{N}}$ is a basis of neighbourhoods of x in $\widehat{\Gamma}$. Observe that if $x \in \Gamma$ then $x\partial\Gamma = x\widehat{\Gamma} \cap \partial\Gamma$.

[2]We use the convention $[\![1, \infty]\!] = \mathbb{N}^* \cup \{\infty\}$.

Proposition 2.2 *$\widehat{\Gamma}$ and $\partial\Gamma$ are compact spaces. $\widehat{\Gamma}$ is a compactification of Γ.*

Proof: $\partial\Gamma = \mathscr{A}^{\mathbb{N}^*}$, thus the set $\partial\Gamma$ endowed with the product topology is compact. This topology coincides with the one induced by the restriction of $\widehat{d}$ on $\partial\Gamma$ (for $x \in \partial\Gamma$, the product topology gives us the same basis of neighbourhoods $\{x_n\partial\Gamma\}_{n\in\mathbb{N}}$ as $\widehat{d}|_{\partial\Gamma}$).

Since $\partial\Gamma$ is compact, in order to show that $\widehat{\Gamma}$ is compact, it suffices to remark that $\cup_{x\in\partial\Gamma}\widehat{B}(x, \exp(-k)) = \{y\widehat{\Gamma} \mid |y| = k+1\}$ has a finite complementary in $\widehat{\Gamma}$, for all $k \in \mathbb{N}$. Since Γ is dense in $\widehat{\Gamma}$, $\widehat{\Gamma}$ is a compactification of Γ. $\square$

Notice also that if $\nu > 1$, the topological space $\partial\Gamma$ is perfect.

The C^*-algebra $C(\widehat{\Gamma})$ of continuous complex-valued functions on $\widehat{\Gamma}$ plays an important rôle. The dense embedding $\Gamma \subset \widehat{\Gamma}$ gives a canonical inclusion $C(\widehat{\Gamma}) \subset C_b(\Gamma)$ ($C_b(\Gamma)$ is the space of bounded complex-valued functions on Γ). Moreover, we have

$$C_0(\Gamma) = \{f \in C(\widehat{\Gamma}) \mid f|_{\partial\Gamma} = 0\}, \tag{2.3}$$

where $C_0(\Gamma) = \{f : \Gamma \to \mathbb{C} \mid \forall\varepsilon > 0, \exists M > 0 \mid |x| > M \Rightarrow |f(x)| < \varepsilon\}$. We shall often abbreviate $C_0(\Gamma)$ by C_0.

The following proposition gives us a better understanding of the functions in $C(\widehat{\Gamma})$.

Proposition 2.3 *Let E be a metrisable topological space. A function $V : \Gamma \to E$ extends to a continuous function $\widehat{V} : \widehat{\Gamma} \to E$ if and only if for each $x \in \partial\Gamma$ the limit of $V(y)$, when $y \in \Gamma$ converges to x, exists.*

Proof: Let $x \in \partial\Gamma$ and $\widehat{V}(x)$ be the above limit. Let F be a closed neighbourhood of $\widehat{V}(x)$ in E; there is k such that $V(x_k\Gamma) \subset F$. Then $x_k\widehat{\Gamma}$ is a neighbourhood of x in $\widehat{\Gamma}$ and, since F is closed, we have $\widehat{V}(x_k\widehat{\Gamma}) \subset F$. $\square$

Later on, we will need the next ultrametricity result. We will say that $\mathscr{U} = \{x_i\Gamma\}$ is a *covering of* $\partial\Gamma$ if $\widehat{\mathscr{U}} = \{x_i\widehat{\Gamma}\}$ is a covering of $\partial\Gamma$.

Proposition 2.4 *For each open covering $\{\mathscr{O}_i\}_{i\in I}$ of $\partial\Gamma$, there is a disjoint and finite covering $\{x_j\Gamma\}_{j\in J}$ of $\partial\Gamma$ such that for each $j \in J$ there is $i \in I$ such that $x_j\widehat{\Gamma} \subset \mathscr{O}_i$.*

Proof: For each $x \in \partial\Gamma$ there is i such that x belongs to the open set $\mathscr{O}_i$ and there is $n = n(x,i)$ such that $x_n\widehat{\Gamma} \subset \mathscr{O}_i$. Since $\partial\Gamma$ is compact, there is a finite sub-covering of $\partial\Gamma$ made by sets $\{y_j\widehat{\Gamma}\}_{j\in[\![1,m]\!]}$ such that each of its elements is a subset of some $\mathscr{O}_i$. But in ultrametric spaces two balls are either disjoint or one of them is included in the other one. Since $\{y_j\widehat{\Gamma}\}$ are balls, we get the result. One may also choose $\{y\widehat{\Gamma} \mid |y| = \max_{j\in[\![1,m]\!]} |y_j|\}$ as the required covering. □

Remark: As we said previously, this section could be presented from the perspective of hyperbolicity in the sense of Gromov, see [2, Chapter V] (a deeper investigation can be found in [4] and [7]). Let (M,d) be a metric space. For $x, y \in M$ and a given $O \in M$, we define the *Gromov product* as:

$$(x,y)_O = \frac{1}{2}(d(O,x) + d(O,y) - d(x,y)). \tag{2.4}$$

The space (M,d) is called δ*-hyperbolic* if there is δ such that for all x, y, z, $O \in M$,

$$(x,y)_O \geq \min((x,z)_O, (z,y)_O) - \delta. \tag{2.5}$$

A metric space is *hyperbolic* if it is δ-hyperbolic for a certain δ. In fact, if there is δ such that (2.5) holds for all $x, y, z \in M$ and a given O then (M,d) is 2δ-hyperbolic. Classical examples of 0-hyperbolic spaces are trees (connected graphs with no cycle) and real trees (see [7] for this notion). Cartan-Hadamard manifolds, the Poincaré half-plane and, more generally, complete simply connected manifolds with sectional curvature bounded by $\kappa < 0$ are δ-hyperbolic spaces with $\delta > 0$.

We equip the set of sequences with values in M with an equivalence relation between (u_n) and (v_n) defined by the condition $\lim_{(n,m)\to\infty}(u_n, v_m)_O = \infty$. The boundary at infinity ∂M is the set of equivalence classes. A basis of open sets of ∂M is given by

$$\widetilde{\mathscr{O}} = \{\gamma \in \partial M \mid \gamma \text{ is not associated to any sequence of } M \setminus \mathscr{O}\},$$

where $\mathscr{O}$ is an open set of M. The boundary of a 0-hyperbolic space is ultrametric.

In our context, if we drop the convention $v(x,x) = \infty$, our valuation (2.1) is exactly (2.4). Hence (2.2) implies that Γ is 0-hyperbolic. We define a *geodesic ray* as being $\gamma : \mathbb{N} \to \Gamma$ such that $|\gamma(n)| = n$ and $\gamma(n+1) \sim$

$\gamma(n)$. Geodesic rays are representative elements of the above equivalence classes. The two notions of boundary at infinity are identified by setting $x_n = \gamma(n)$.

3 Operators in $\ell^2(\Gamma)$

3.1 Bounded and compact operators

We are interested in operators acting on the Hilbert space $\ell^2(\Gamma) = \{f : \Gamma \to \mathbb{C} \mid \sum_{x\in\Gamma} |f(x)|^2 < \infty\}$ endowed with the inner product: $\langle f, g\rangle = \sum_{x\in\Gamma} \overline{f(x)}g(x)$. We embed $\Gamma \subset \ell^2(\Gamma)$ by identifying x with $\chi_{\{x\}}$, where χ_A is the characteristic function of the set A. Observe that Γ is the canonical orthonormal basis in $\ell^2(\Gamma)$ and each $f \in \ell^2(\Gamma)$ writes as $f = \sum_{x\in\Gamma} f(x)x$.

We denote by $\mathbb{B}(\Gamma)$, $\mathbb{K}(\Gamma)$ the sets of bounded, respectively compact operators in $\ell^2(\Gamma)$. For $T \in \mathbb{B}(\Gamma)$, we will denote by T^* its adjoint. Given $A \subset \Gamma$ we denote by $\mathbf{1}_A$ the operator of multiplication by χ_A in $\ell^2(\Gamma)$. The orthogonal projection associated to $\{x \in \Gamma \mid |x| \geq r\}$ is denoted by $\mathbf{1}_{\geq r}$. For $T \in \Gamma$, we have the following compacity criterion for bounded operators T in $\ell^2(\Gamma)$:

Proposition 3.1 $T \in \mathbb{K}(\Gamma) \iff \|\mathbf{1}_{\geq r}T\| \underset{r\to\infty}{\longrightarrow} 0 \iff \|T\mathbf{1}_{\geq r}\| \underset{r\to\infty}{\longrightarrow} 0.$

Proof: If one has for example $\|\mathbf{1}_{\geq r}T\| \to 0$, then T is the norm limit of the sequence of finite rank operators $\mathbf{1}_{B(e,r)}T$, hence is compact. $\square$

3.2 The operator ∂

We now extend $x \mapsto x'$ to a map $\ell^2(\Gamma) \to \ell^2(\Gamma)$. We set $e' = 0$ and define the derivative of any $f \in \ell^2(\Gamma)$ as:

$$(\partial f)(x) \equiv f'(x) = \sum_{y\in\Gamma} f(y)y'(x) = \sum_{y'=x} f(y) = \sum_{y\in\widetilde{x}} f(y).$$

Thus $\partial \in \mathbb{B}(\Gamma)$. Indeed, $\|f'\|^2 = \sum_{x\in\Gamma} |f'(x)|^2 \leq \nu \sum_{x\in\Gamma} \sum_{y\in\widetilde{x}} |f(y)|^2 \leq \nu\|f\|^2$. The adjoint ∂^* acts on each $f \in \ell^2(\Gamma)$ as follows:

$$\partial^* f(x) = \chi_{\Gamma\setminus\{e\}}(x)f(x').$$

Indeed, $\langle \partial f, f\rangle = \sum_{x\in\Gamma}\sum_{y\in\widetilde{x}} \overline{f(y)} f(x) = \sum_{x\in\Gamma} \overline{f(x)}\chi_{\Gamma\setminus\{e\}}(x) f(x') = \langle f, \partial^* f\rangle$. Moreover, $\|\partial^* f\|^2 = \sum_{x\in\Gamma\setminus\{e\}} |f(x')|^2 = \nu \sum_{x\in\Gamma} |f(x)|^2 = \nu\|f\|^2$ shows that

$$\partial\partial^* = \nu \mathrm{Id}. \tag{3.1}$$

Thus $\partial^*/\sqrt{\nu}$ is isometric on $\ell^2(\Gamma)$ and $\|\partial\| = \|\partial^*\| = \sqrt{\nu}$.

For $\alpha \in \mathbb{N}$ we set $f^{(\alpha)} = \partial^\alpha f$. Thus for each $x \in \Gamma$, $x^{(\alpha)}$ is well defined in $\ell^2(\Gamma)$ and $x^{(\alpha)} = 0 \Leftrightarrow \alpha > |x|$. For $|x| \geq \alpha$ the notation is consistent with our old definition.

3.3 C^*-algebras of energy observables related to Γ

We first summarize the method used in [6] to study the essential spectrum of large families of operators. Let $\mathscr{H}$ be a Hilbert space and H a bounded self-adjoint operator on $\mathscr{H}$. If $C(\mathscr{H}) = B(\mathscr{H})/K(\mathscr{H})$ is the Calkin C^*-algebra, we denote by $S \mapsto \widehat{S}$ the canonical surjection of $B(\mathscr{H})$ onto $C(\mathscr{H})$ and we recall that $\sigma_{ess}(H) = \sigma(\widehat{H})$ (this is a version of Weyl's Theorem). If $\mathfrak{C}$ is a C^*-subalgebra of $B(\mathscr{H})$ which contains the compact operators, then one has a canonical embedding $\mathfrak{C}/K(\mathscr{H}) \subset C(\mathscr{H})$. Thus, in order to determine the essential spectrum of an operator $H \in \mathfrak{C}$ it suffices to give a good description of the quotient $\mathfrak{C}/K(\mathscr{H})$ and to compute $\widehat{H}$ as element of it. As explained in [6], we can actually go further by taking H as an unbounded operator over $\mathscr{H}$ such that $(H+i)^{-1} \in \mathfrak{C}$. We shall apply this strategy in our context.

Let $\mathscr{D}_{\text{alg}}$ be the $*$-algebra of operators in $\ell^2(\Gamma)$ generated by ∂ and $\mathscr{D}$ the C^*-algebra of operators in $\ell^2(\Gamma)$ generated by ∂. Because of (3.1), $\mathscr{D}_{\text{alg}}$ is unital. We denote by $\varphi(Q)$ the operator of multiplication by φ on $\ell^2(\Gamma)$. If C is a C^*-subalgebra of $\ell^\infty(\Gamma)$ then we embed C in $\mathbb{B}(\Gamma)$ by $\varphi \mapsto \varphi(Q)$. Let $\langle \mathscr{D}, C\rangle$ be the C^*-algebra generated by $\mathscr{D} \cup C$. In this paper we shall take $\mathfrak{C} = \langle \mathscr{D}, C\rangle$. This algebra contains many Hamiltonians of physical interest, for instance Schrödinger operators with potentials in C. We recall that given a graph G the Laplace operator acts on $\ell^2(G)$ as follows:

$$(\Delta f)(x) = \sum_{y\sim x} (f(y) - f(x)).$$

With our definitions $\Delta = \partial + \partial^* - \nu \mathrm{Id} + \chi_{\{e\}}$. Notice that if $\nu > 1$ then $\mathscr{D}$ does not contain compact operators (see below), so $\Delta \notin \mathscr{D}$. On the other hand, if $C \supset C_0$ and $V \in C$ then the Schrödinger operator $\Delta + V(Q)$ clearly belongs to $\langle \mathscr{D}, C \rangle$.

We now give a new description of $\mathbb{K}(\Gamma)$.

Proposition 3.2 *If $\mathscr{C}_0$ be the C^*-algebra generated by $\mathscr{D} \cdot C_0$ then $\mathscr{C}_0 = \mathbb{K}(\Gamma)$.*

Proof: For each $\varphi \in C_0$, Proposition 3.1 shows $\varphi(Q) \in \mathbb{K}(\Gamma)$. Hence $\mathscr{C}_0 \subset \mathbb{K}(\Gamma)$. For the opposite inclusion, let $T \in \mathbb{K}(\Gamma)$ and fix $\varepsilon > 0$. Proposition 3.1, shows that there is an operator T' with compactly supported kernel such that $\|T - T'\| \leq \varepsilon$. Define $\delta_{x,y} \in \mathbb{K}(\Gamma)$ by $(\delta_{x,y} f)(z) = f(y)$ if $z = x$ and 0 elsewhere. We have $\delta_{x,x} = \chi_{\{x\}}(Q) \in C_0$. As T' is a linear combination of $\delta_{x,y}$, it suffices to show that $\delta_{x,y}$ is in $\mathscr{C}_0$. But this follows from $\delta_{x,y} = \delta_{x,x}(\partial^*)^{|x|}\partial^{|y|}\delta_{y,y}$. $\square$

If C is a C^*-subalgebra of $\ell^\infty(\Gamma)$ that contains C_0, then $\mathbb{K}(\Gamma) \subset \langle \mathscr{D}, C \rangle$. Hence, in order to apply the technique described above, we have to give a sufficiently explicit description of the quotient $\langle \mathscr{D}, C \rangle / \mathbb{K}(\Gamma)$. In this paper we concentrate on the case $C \equiv C(\widehat{\Gamma})$ which is, geometrically speaking, the most interesting one (see the last Remark in §2.3). *The C^*-algebra generated by ∂ and $C(\widehat{\Gamma})$ will be denoted by $\mathscr{C}(\widehat{\Gamma})$ and the $*$-subalgebra generated by ∂ and $C(\widehat{\Gamma})$ will be denoted by $\mathscr{C}(\widehat{\Gamma})_{\mathrm{alg}}$.* We will need the next fundamental property.

Proposition 3.3 $[\partial, C(\widehat{\Gamma})] \subset \mathbb{K}(\Gamma)$.

Proof: For each $\varphi \in C(\widehat{\Gamma})$ one has $([\partial, \varphi(Q)]f)(x) = \sum_{y'=x}(\varphi(y) - \varphi(x))f(y) = (\partial \circ \psi(Q)f)(x)$, where ψ belongs to $C(\widehat{\Gamma})$ and is defined by $\psi(y) = \varphi(y) - \varphi(y')$ when $|y| \geq 1$ and $\psi(e) = 0$. Observe that for $\gamma \in \partial\Gamma$ we have $\psi(\gamma) = \varphi(\gamma) - \varphi(\gamma) = 0$. Hence by (2.3), $\psi \in C_0$. Proposition 3.2 implies $\psi(Q) \in \mathbb{K}(\Gamma)$. $\square$

Remark: The algebra $\mathscr{D}$ is the tree analogous of the algebra generated by the momentum operator on the real line. However, these algebras are rather different: $\mathscr{D}$ is not commutative and the spectrum and the essential spectrum of the operators from $\mathscr{D}$ are not connected sets in general. For instance, one has $\sigma(\partial^*\partial) = \sigma_{\mathrm{ess}}(\partial^*\partial) = \{0, \nu\}$ if $\nu > 1$. Indeed, we remind that if A, B are elements of a Banach algebra we have $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$ and, as noticed below, $\dim \operatorname{Ker} \partial$ is infinite for $\nu > 1$.

3.4 Translations in $\ell^2(\Gamma)$

Γ acts on itself to the left and to the right: for each $a \in \Gamma$ we may define $\lambda_a, \rho_a : \Gamma \to \Gamma$ by $\lambda_a(x) = ax$ and $\rho_a(x) = xa$ respectively. Clearly, for $a, b \in \Gamma$, $\lambda_a \rho_b = \rho_b \lambda_a$ and for any $x \in a\Gamma$ we define $a^{-1}x$ as being the y for which $x = ay$. For each $x \in \Gamma a = \{y \in \Gamma \mid \exists z \in \Gamma \text{ s.t. } y = za\}$, we define $y = xa^{-1}$ by $x = ya$. We extend now these translations to $\ell^2(\Gamma)$. The translation λ_a acts on each $f \in \ell^2(\Gamma)$ as $\sum_{x\in\Gamma} f(x)ax$, i.e. $(\lambda_a f)(x) = \chi_{a\Gamma}(x) f(a^{-1}x)$. In the same manner, we define $(\rho_a f)(x) = \chi_{\Gamma a}(x) f(xa^{-1})$. The operators λ_a and ρ_a are isometries:

$$\lambda_a^* \lambda_a = \text{Id and } \rho_a^* \rho_a = \text{Id}. \tag{3.2}$$

It is easy to check that the adjoints act on any $f \in \ell^2(\Gamma)$ as $(\lambda_a^* f)(x) = f(ax)$ and $(\rho_a^* f)(x) = f(xa)$. Moreover,

$$\lambda_a \lambda_a^* = \mathbf{1}_{a\Gamma} \text{ and } \rho_a \rho_a^* = \mathbf{1}_{\Gamma a}. \tag{3.3}$$

Note also that $\partial^* = \sum_{|a|=1} \rho_a$ and $\partial = \sum_{|a|=1} \rho_a^*$.

3.5 Localizations at infinity

In order to study $\mathscr{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma)$ we have to define the localizations at infinity of $T \in \mathscr{C}(\widehat{\Gamma})$ by looking at the behavior of the translated operator $\lambda_a^* T \lambda_a$ as a converges to γ in $\widehat{\Gamma}$ (abbreviated $a \to \gamma$), for each $\gamma \in \partial\Gamma$.

If $T \in \mathbb{K}(\Gamma)$ then $\text{u-}\lim_{a\to\gamma} \lambda_a^* T \lambda_a = 0$, where u-lim means convergence in norm. Indeed, by (3.2), (3.3) and Proposition 3.1 we get $\|\lambda_a^* T \lambda_a\| = \|\mathbf{1}_{a\Gamma} T \mathbf{1}_{a\Gamma}\| \to 0$, as $a \to \gamma$. Now, we compute the uniform limit of $\lambda_a^* T \lambda_a$ when $T \in \mathscr{C}(\widehat{\Gamma})_{\text{alg}}$. There is P, a non-commutative complex polynomial in $m+2$ variables, and functions $\varphi_i \in C(\widehat{\Gamma})$ for $i = [\![1, m]\!]$, such that $T = P(\varphi_1, \varphi_2, \ldots, \varphi_m, \partial, \partial^*)$. We set $T(\gamma) = P(\varphi_1(\gamma), \varphi_2(\gamma), \ldots, \varphi_m(\gamma), \partial, \partial^*)$.

Lemma 3.4 *There is* $a_0 \in \Gamma$ *such that* u-$\lim_{a\to\gamma} \lambda_a^* T \lambda_a = \lambda_{a_0}^* T(\gamma)\lambda_{a_0}$.

Proof: The Proposition 3.3 and (3.1) give some $\phi_k \in C(\widehat{\Gamma})$, $K \in \mathbb{K}(\Gamma)$ and $\alpha_k, \beta_k \in \mathbb{N}$ such that $T = \sum_{k=1}^n \phi_k(Q)\partial^{*\alpha_k}\partial^{\beta_k} + K$ and $T(\gamma) = \sum_{k=1}^n \phi_k(\gamma)\partial^{*\alpha_k}\partial^{\beta_k}$. Thus, it suffices to compute a limit of the form u-$\lim_{a\to\gamma} \lambda_a^*\varphi(Q)\partial^{*\alpha}\partial^\beta\lambda_a$ with $\varphi \in C(\widehat{\Gamma})$. We suppose $|a| \geq \alpha$ and take $f \in \ell^2(\Gamma)$. We first show the result for $\varphi = 1$. Since

$$(\lambda_a^*\partial^{*\alpha}\partial^\beta\lambda_a f)(x) = \sum_{\{y|y^{(\beta)}=(ax)^{(\alpha)}\}} (\lambda_a f)(y) = \sum_{\{y|(ay)^{(\beta)}=(ax)^{(\alpha)}\}} f(y), \tag{3.4}$$

it suffices to show that the set $\{y \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\}$ is independent of a if $|a| \geq \alpha$. But this is precisely what asserts the Lemma 3.5 below.

We now treat the general case $\varphi \in \mathscr{C}(\widehat{\Gamma})$. The identity $(\lambda_a^*\varphi(Q)\partial^{*\alpha}\partial^\beta\lambda_a f)(x) = \varphi(ax)(\lambda_a^*\partial^{*\alpha}\partial^\beta\lambda_a f)(x)$ gives us that $\|\lambda_a^*\varphi(Q)\partial^{*\alpha}\partial^\beta\lambda_a - \varphi(\gamma)\lambda_a^*\partial^{*\alpha}\partial^\beta\lambda_a\| \leq \|\varphi(aQ)-\varphi(\gamma)\|\cdot\|\partial^{*\alpha}\partial^\beta\| \to 0$ as $a \to \gamma$. On the other hand, by the Lemma 3.5, $\varphi(\gamma)\lambda_a^*\partial^{*\alpha}\partial^\beta\lambda_a$ is constant for $|a| \geq \alpha$. Thus, it suffices to choose $|a_0| \geq \max\{\alpha_k \mid k = 1, \ldots, n\}$ in the statement of the lemma to end the proof. $\square$

Lemma 3.5 *For* $|a| \geq \alpha$ *we have:*

$$\{y \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\} = \begin{cases} \emptyset & \text{for } |x| + \beta - \alpha < 0, \\ S^{|x|+\beta-\alpha} & \text{for } |x| < \alpha \text{ and } |x| + \beta - \alpha \geq 0, \\ x^{(\alpha)}S^\beta & \text{for } |x| \geq \alpha \text{ and } |x| + \beta - \alpha \geq 0. \end{cases} \tag{3.5}$$

Proof: Let $J_x = \{y \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\}$. Then

$$\begin{aligned} aJ_x &= \{ay \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\} = \{y \mid y^{(\beta)} = (ax)^{(\alpha)}\} \cap a\Gamma \\ &= ((ax)^{(\alpha)}S^\beta(\Gamma)) \cap a\Gamma. \end{aligned}$$

We first notice that $(ax)^{(\alpha)}S^\beta \subset S^{|a|+|x|-\alpha+\beta}$. If $|x| - \alpha + \beta < 0$ then $((ax)^{(\alpha)}S^\beta) \cap a\Gamma = \emptyset$, so $aJ_x = \emptyset$. This implies $J_x = \emptyset$. If $|x|-\alpha+\beta \geq 0$ then $((ax)^{(\alpha)}S^\beta) \cap a\Gamma \neq \emptyset$. If we suppose that $|x| < \alpha$, i.e. $|(ax)^{(\alpha)}| < |a|$, we have $a \in (ax)^{(\alpha)}\Gamma$. Let b such that $a = (ax)^{(\alpha)}b$. Thus

$$\begin{aligned} ((ax)^{(\alpha)}S^\beta) \cap a\Gamma &= ((ax)^{(\alpha)}S^\beta) \cap (ax)^{(\alpha)}b\Gamma = (ax)^{(\alpha)}(S^\beta \cap b\Gamma) \\ &= (ax)^{(\alpha)}bS^{\beta-|b|} = aS^{\beta-|b|} = aS^{\beta+|x|-\alpha}, \end{aligned}$$

so we have $aJ_x = aS^{\beta+|x|-\alpha}$, hence $J_x = S^{\beta+|x|-\alpha}$.

Finally, if $|x| \geq \alpha$, i.e. $|(ax)^{(\alpha)}| \geq |a|$, one has $(ax)^{(\alpha)} \in a\Gamma$. Thus we obtain $aJ_x = (ax)^{(\alpha)}S^\beta = ax^{(\alpha)}S^\beta$, hence $J_x = x^{(\alpha)}S^\beta$. $\square$

Remark: As seen in the proof of lemma 3.4, one may choose any a_0 such that $|a_0| \geq \deg(P)$. On the other hand, we stress that the limit is not a multiplicative function of T. Indeed,

$$\text{u-}\lim_{a\to\gamma} \lambda_a^*\partial^*\partial\lambda_a \neq (\text{u-}\lim_{a\to\gamma} \lambda_a^*\partial^*\lambda_a) \cdot (\text{u-}\lim_{a\to\gamma} \lambda_a^*\partial\lambda_a).$$

Therefore, in order to describe the morphism of the algebra $\mathscr{C}(\widehat{\Gamma})$ onto its quotient $\mathscr{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma)$ we have to improve our definition of the localizations at infinity.

3.6 Extensions to $\widetilde{\Gamma}$

The space $\ell^2(\widetilde{\Gamma})$ is defined similarly to $\ell^2(\Gamma)$. Since $\Gamma \subset \widetilde{\Gamma}$, we have $\ell^2(\Gamma) \hookrightarrow \ell^2(\widetilde{\Gamma})$. As before, we embed $\widetilde{\Gamma}$ in $\ell^2(\widetilde{\Gamma})$ by sending x on $\chi_{\{x\}}$ and we notice that $\widetilde{\Gamma}$ is an orthonormal basis of $\ell^2(\widetilde{\Gamma})$. We define $\widetilde{\partial} : \ell^2(\widetilde{\Gamma}) \to \ell^2(\widetilde{\Gamma})$ by

$$(\widetilde{\partial} f)(x) = f'(x) = \sum_{y'=x} f(y).$$

For $\alpha \in \mathbb{N}$, we set $f^{(\alpha)} = \widetilde{\partial}^\alpha f$, notation which is consistent with our old definition of $x^{(\alpha)}$ as the restriction of x to $\mathbb{Z}_{|x|-\alpha}$. Obviously $\widetilde{\partial} \in \mathbb{B}(\Gamma)$, its adjoint $\widetilde{\partial}^*$ acts as $(\widetilde{\partial}^* f)(x) = f(x')$, $\widetilde{\partial}^*/\sqrt{\nu}$ is an isometry on $\ell^2(\widetilde{\Gamma})$:

$$\widetilde{\partial}\widetilde{\partial}^* = \nu\text{Id}, \tag{3.6}$$

thus $\|\widetilde{\partial}\| = \|\widetilde{\partial}^*\| = \nu$. *We denote by $\widetilde{\mathscr{D}}$ the C^*-algebra generated by $\widetilde{\partial}$ and by $\widetilde{\mathscr{D}}_{\text{alg}}$ the $*$-algebra generated by $\widetilde{\partial}$.* Both of them are unital.

We now make the connection between $\mathscr{D}_{alg}$ and $\widetilde{\mathscr{D}}_{alg}$.

Lemma 3.6 *For $|a| \geq \alpha$, one has: $\lambda_a^*\partial^{*\alpha}\partial^\beta\lambda_a = \mathbf{1}_\Gamma\widetilde{\partial}^{*\alpha}\widetilde{\partial}^\beta\mathbf{1}_\Gamma$.*

Proof: For any $f \in \ell^2(\widetilde{\Gamma})$, one has $(\mathbf{1}_\Gamma\widetilde{\partial}^{*\alpha}\widetilde{\partial}^\beta\mathbf{1}_\Gamma f)(x) = \mathbf{1}_\Gamma(x)\sum_{\{y|y^{(\beta)}=x^{(\alpha)}\}} \mathbf{1}_\Gamma(y)f(y)$. Using the same arguments as in the proof of the Lemma 3.5, one shows that for each $x \in \Gamma$ the set $\{y \in \Gamma \mid y^{(\beta)} = x^{(\alpha)}\}$ equals the r.h.s. of (3.5). Thus the above sum is the same as that of the r.h.s. of (3.4).□

We will also need a result concerning the localization of the norm on $\widetilde{\mathscr{D}}_{\mathrm{alg}}$.

Lemma 3.7 *If $\widetilde{T} \in \widetilde{\mathscr{D}}_{alg}$, then $\|\widetilde{T}\| = \|\mathbf{1}_\Gamma \widetilde{T} \mathbf{1}_\Gamma\|$.*

Proof: Because of (3.6), we can suppose that $\widetilde{T} = \sum_{k=1}^{n} c_k \widetilde{\partial}^{*\alpha_k} \widetilde{\partial}^{\beta_k}$. We denote by β the integer $\max\{\beta_k \mid k \in [\![1, n]\!]\}$. For each $\varepsilon > 0$, there is some $g \in \ell^2(\widetilde{\Gamma})$ with compact support such that $\|g\| = 1$ and $\|\widetilde{T}g\| \geq \|\widetilde{T}\| - \varepsilon$. Note that if $y_1, y_2, \ldots, y_m$ are distinct points of Γ, $a_1, a_2, \ldots, a_m$ are complex numbers and $x_1, x_2 \in \widetilde{\Gamma}$, we have

$$\| \sum_{i=1}^{m} a_i x_1 y_i \|^2 = \sum_{i=1}^{m} |a_i|^2 = \| \sum_{i=1}^{m} a_i x_2 y_i \|^2. \tag{3.7}$$

Thus, since g has compact support, there are $x \in \widetilde{\Gamma}$, $m \in \mathbb{N}^*$ and $y_i \in \Gamma$, $|y_i| \geq \beta$, $a_i \in \mathbb{C}$, for all $i \in [\![1, m]\!]$ such that $g = \sum_{k=1}^{m} a_i x y_i$. We set $f = \sum_{k=1}^{m} a_i e y_i$. Then (3.7) gives us $\|f\| = \|g\| = 1$. Using $|y_i| \geq \beta$, we get $f \in \ell^2(\Gamma)$ and $\widetilde{T}f \in \ell^2(\Gamma)$. Also with (3.7) we obtain for $z \in \Gamma$,

$$\begin{aligned}
\|\widetilde{T}g\| &= \|\sum_{k=1}^{n} \sum_{i=1}^{m} c_k a_i \widetilde{\partial}^{*\alpha_k} \widetilde{\partial}^{\beta_k} x y_i\| = \|\sum_{k=1}^{n} \sum_{i=1}^{m} \sum_{|z|=\alpha_k} c_k a_i (x y_i)^{(\beta_k)} z\| \\
&= \|\sum_{k=1}^{n} \sum_{i=1}^{m} \sum_{|z|=\alpha_k} c_k a_i x (y_i)^{(\beta_k)} z\| = \|\sum_{k=1}^{n} \sum_{i=1}^{m} \sum_{|z|=\alpha_k} c_k a_i e (y_i)^{(\beta_k)} z\| \\
&= \|\sum_{k=1}^{n} \sum_{i=1}^{m} \sum_{|z|=\alpha_k} c_k a_i (e y_i)^{(\beta_k)} z\| = \|\sum_{k=1}^{n} \sum_{i=1}^{m} c_k a_i \widetilde{\partial}^{*\alpha_k} \widetilde{\partial}^{\beta_k} e y_i\| = \|\widetilde{T}f\|.
\end{aligned}$$

Hence, there is $f \in \ell^2(\widetilde{\Gamma})$ such that $\|\mathbf{1}_\Gamma \widetilde{T} \mathbf{1}_\Gamma f\| = \|\widetilde{T}f\| = \|\widetilde{T}g\| \geq \|\widetilde{T}\| - \varepsilon$. □

4 The main results

4.1 The morphism

In the sequel, a morphism will be understood as a morphism of C^*-algebras. To describe the quotient $\mathscr{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma)$, we need to find an adapted morphism.

Theorem 4.1 *For each* $\gamma \in \partial\Gamma$ *there is a unique morphism* $\Phi_\gamma : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}}$ *such that* $\Phi_\gamma(\partial) = \widetilde{\partial}$ *and* $\Phi_\gamma(\varphi(Q)) = \varphi(\gamma)$, *for all* $\varphi \in C(\widehat{\Gamma})$. *One has* $\mathbb{K}(\Gamma) \subset \operatorname{Ker}\Phi_\gamma$.

Proof: We use the notations from §3.5. If $T \in \mathscr{C}(\widehat{\Gamma})_{\rm alg}$ then by Lemma 3.4 we have u-$\lim_{a\to\gamma} \lambda_a^* T \lambda_a = \lambda_{a_0}^* T(\gamma)\lambda_{a_0}$. Let $\widetilde{T}(\gamma)$ be $P(\varphi_1(\gamma), \varphi_2(\gamma), \ldots, \varphi_m(\gamma), \widetilde{\partial}, \widetilde{\partial}^*)$. By Lemma 3.6 and (3.6) one can choose a_0 such that $\lambda_{a_0}^* T(\gamma)\lambda_{a_0} = \mathbf{1}_\Gamma \widetilde{T}(\gamma)\mathbf{1}_\Gamma$. Lemma 3.7 implies

$$\|\widetilde{T}(\gamma)\| = \|\mathbf{1}_\Gamma \widetilde{T}(\gamma)\mathbf{1}_\Gamma\| = \|\lambda_{a_0}^* T(\gamma)\lambda_{a_0}\| = \|\text{u-}\lim_{a\to\gamma} \lambda_a^* T \lambda_a\| \leq \|T\|.$$

Thus there is a linear multiplicative contraction $\Phi_\gamma^0 : \mathscr{C}(\widehat{\Gamma})_{\rm alg} \to \widetilde{\mathscr{D}}$, $\Phi_\gamma^0(T) = T(\gamma)$. The density of $\mathscr{C}(\widehat{\Gamma})_{\rm alg}$ in $\mathscr{C}(\widehat{\Gamma})$ allows us to extend Φ_γ^0 to a morphism $\Phi_\gamma : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}}$ which clearly satisfies the conditions of the theorem. The uniqueness of Φ_γ is obvious and the last assertion of the theorem follows from the Proposition 3.2. □

4.2 The case $\nu > 1$

In this case, we are able to improve the Theorem 4.1. We recall first that an isometry is said to be *proper* if it is not unitary. The operators ∂^* and $\widetilde{\partial}^*$ are proper isometries and the dimensions of the kernels of ∂ and $\widetilde{\partial}$ are infinite: in the case of ∂, if one lets a, b be two different letters of $\mathscr{A}$, and one chooses $g \in \ell^2(\Gamma a)$ and $h \in \ell^2(\Gamma b)$ such that $h(xb) = g(xa)$ for all $x \in \Gamma$, then $g - h$ is in $\operatorname{Ker}\partial$.

Let $\mathbb{T}$ be the unit circle of $\mathbb{R}^2$ and H^2 the closure of the subspace spanned by $\{e^{inQ}, n \in \mathbb{N}\}$ in $\ell^2(\mathbb{T})$. For $g \in L^\infty(\mathbb{T})$, we define the *Toeplitz operator* T_g on H^2 by $T_g h = P_{H^2} gh,$ where P_{H^2} is the projection on H^2.

For each $z \in \mathbb{C} \setminus \{0\}$, we denote by $\mathscr{T}$ the C^*-algebra generated by T_z. The next theorem is due to Coburn (see [5] for a proof).

Theorem 4.2 *If S is a proper isometry, then there is a unique isomorphism $\mathscr{J}$ of $\mathscr{T}$ onto $\mathscr{S}$, the C^*-algebra generated by S, such that $\mathscr{J}(T_z) = S$.*

Thus there is a unique isomorphism $\mathscr{J}$ of $\mathscr{D}$ onto $\widetilde{\mathscr{D}}$ such that $\mathscr{J}(\partial) = \mathscr{J}(\widetilde{\partial})$, so in the case $\nu > 1$ we can rewrite our Theorem 4.1 as follows.

Theorem 4.3 *Let $\gamma \in \partial\Gamma$. There is a unique morphism $\Phi_\gamma : \mathscr{C}(\widehat{\Gamma}) \to \mathscr{D}$ such that $\Phi_\gamma(\varphi(Q)) = \varphi(\gamma)$ for all $\varphi \in C(\widehat{\Gamma})$ and $\Phi_\gamma(D) = D$ for all $D \in \mathscr{D}$.*

Remark: When $\nu = 1$, there is no isomorphism $\mathscr{J} : \mathscr{D} \to \widetilde{\mathscr{D}}$ such that $\mathscr{J}(\partial) = \widetilde{\partial}$ because $\widetilde{\mathscr{D}}$ is commutative. Thus, in this case, one cannot hope in a result as above. There is an other way of proving Theorem 4.3 which uses the next proposition.

Proposition 4.4 *If $\nu \geq 1$ then $\{\partial^{*\alpha}\partial^\beta\}_{\{\alpha,\beta\in\mathbb{N}\}}$ is a basis of the vector space $\mathscr{D}_{\mathrm{alg}}$. One has $\nu > 1$ if and only if $\{\widetilde{\partial}^{*\alpha}\widetilde{\partial}^\beta\}_{\{\alpha,\beta\in\mathbb{N}\}}$ is a basis of space $\widetilde{\mathscr{D}}_{\mathrm{alg}}$.*

Proof: Let $\lambda_i \neq 0$ for all $i \in [\![1, n]\!]$. Assume that $\sum_{i=1}^n \lambda_i \partial^{*\alpha_i}\partial^{\beta_i} = 0$, where (α_i, β_i) are distinct couples. We set $\underline{\alpha} = \min\{\alpha_i \mid i \in [\![1, n]\!]\}$ and $I = \{i \mid \alpha_i = \underline{\alpha}\}$. We take $x \in \Gamma$ such that $|x| = \underline{\alpha}$ and we obtain $\sum_{i\in I} \lambda_i(\partial^{\beta_i} f)(e) = 0$. Notice that $\{\beta_i\}_{i\in I}$ are pairwise distinct by hypothesis. Now, by taking $i_0 \in I$ and f the characteristic function of $S_{\beta_{i_0}}$, we get that $\lambda_{i_0} = 0$ which is a contradiction. Hence $\sum_{i=1}^n \lambda_i \partial^{*\alpha_i}\partial^{\beta_i} \neq 0$, i.e. the familly is free. Let now $\nu > 1$ and $\lambda_i \neq 0$ for all $i \in [\![1, n]\!]$. We suppose $\sum_{i=1}^n \lambda_i \widetilde{\partial}^{*\alpha_i}\widetilde{\partial}^{\beta_i} = 0$, with (α_i, β_i) pairwise distinct. We fix $x \in \widetilde{\Gamma}$ and set $\bar{\alpha} = \max\{\alpha_i, i \in [\![1, n]\!]\}$. One has $(\sum_{i=1}^n \lambda_i \widetilde{\partial}^{*\alpha_i}\widetilde{\partial}^{\beta_i} f)(x) = \sum_{i=1}^n \lambda_i \sum_{y\in x^{(\alpha_i)}S^{\beta_i}} f(y) = 0$. Notice that $x^{(\alpha)}S^\beta \cap x^{(\alpha')}S^{\beta'} = \emptyset$ if and only if $\alpha' - \alpha \neq \beta' - \beta$. Taking $f \in \ell^2(S^{|x|-\alpha_1+\beta_1})$, we see that one can reduce oneself o the case when there is some k such that $\alpha_i - \beta_i = k$ for all $i \in [\![1, n]\!]$. Since $x^{(\bar{\alpha}-l)}S^{\bar{\alpha}-k-l} \subset x^{(\bar{\alpha}-1)}S^{\bar{\alpha}-k-1} \subsetneq x^{(\bar{\alpha})}S^{\bar{\alpha}-k}$ for all $l \in [\![1, (\bar{\alpha} - k)]\!]$, there is some $y_0 \in x^{(\bar{\alpha})}S^{\bar{\alpha}-k} \setminus \cup_{\alpha_i\neq\bar{\alpha}} x^{(\alpha_i)}S^{\beta_i}$. Then, taking $f = \chi_{\{y_0\}}$ we get some i_0 such that $\lambda_{i_0} = 0$, which is a contradiction. Hence $\sum_{i=1}^n \lambda_i \widetilde{\partial}^{*\alpha_i}\widetilde{\partial}^{*\beta_i} \neq 0$. Finally, since when $\nu = 1$ one has $\widetilde{\partial}\widetilde{\partial}^* = \widetilde{\partial}^*\widetilde{\partial} = \mathrm{Id}$, $\{\widetilde{\partial}^{*\alpha}\widetilde{\partial}^\beta\}_{\alpha,\beta\in\mathbb{N}}$ is obviously not a basis. □

4.3 Description of $\mathscr{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma)$

Theorem 4.5 i) *For any* $\nu \geq 1$*, there is a unique morphism* $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}} \otimes C(\partial\Gamma)$ *such that* $\Phi(\partial) = \widetilde{\partial} \otimes 1$ *and* $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$*. This morphism is surjective and its kernel is* $\mathbb{K}(\Gamma)$*.*
ii) *For* $\nu > 1$*, there is a unique surjective morphism* $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \mathscr{D} \otimes C(\partial\Gamma)$ *such that* $\Phi(\partial) = \partial \otimes 1$*,* $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$ *and* $\operatorname{Ker}\Phi = \mathbb{K}(\Gamma)$*.*

Once again, as in Remark 4.2, the statement (ii) of the theorem is false if $\nu = 1$. As a corollary of Theorem 4.5 we obtain the following result.

Proposition 4.6 *If* $\nu > 1$ *then* $\mathscr{D} \cap \mathbb{K}(\Gamma) = \{0\}$ *and if* $\nu = 1$ *one has* $\mathbb{K}(\Gamma) \subset \mathscr{D}$*.*

Proof: Let $\nu > 1$ and $T \in \mathscr{D} \cap \mathbb{K}(\Gamma)$. Theorem 4.5 gives us both $\Phi(T) = T \otimes 1$ and $\Phi(T) = 0$ (since T is compact). For $\nu = 1$, as in the proof of Proposition 3.2, it suffices to prove that $\delta_{x,x}$ is in $\mathscr{D}$. But this is clear since $\delta_{x,x} = \partial^{*|x+1|}\partial^{|x+1|} - \partial^{*|x|}\partial^{|x|}$. $\square$

We devote the rest of the section to the proof of the Theorem 4.5.

Proof: By Theorem 4.1 there is a morphism $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}}^{\partial\Gamma}$ such that $(\Phi(\partial))(\gamma) = \widetilde{\partial}$ and $(\Phi(\varphi(Q)))(\gamma) = \varphi(\gamma)$, for all $\gamma \in \partial\Gamma$, $\varphi \in C(\widehat{\Gamma})$. Since the images of ∂ and $\varphi(Q)$ through Φ belong to the C^*-subalgebra $C(\partial\Gamma, \widetilde{\mathscr{D}})$, and since $\mathscr{C}(\widehat{\Gamma})$ is generated by ∂ and such $\varphi(Q)$, it follows that the range of Φ is included in $C(\partial\Gamma, \widetilde{\mathscr{D}})$. We have $C(\partial\Gamma, \widetilde{\mathscr{D}}) \cong \widetilde{\mathscr{D}} \otimes C(\partial\Gamma)$, so we get the required morphism $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}} \otimes C(\partial\Gamma)$. Now since $\Phi(\partial) = \widetilde{\partial} \otimes 1$ and $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$, and since any function in $C(\partial\Gamma)$ is the restriction of some function from $C(\widehat{\Gamma})$, it follows that Φ is surjective. Its uniqueness is clear. It remains to compute the kernel.

As seen in the Theorem 4.1, $\mathbb{K}(\Gamma) \subset \operatorname{Ker}\Phi$. In the remainder of this section we shall prove the reverse inclusion. For this we need some preliminary lemmas.

Lemma 4.7 *Let* $R = \varphi(Q)\partial^{*\alpha}\partial^{\beta}$ *and* $\mathscr{U} = \{a_i\Gamma\}_{i\in[\![1,n]\!]}$ *be a disjoint covering of* $\partial\Gamma$*. For each* $\varepsilon > 0$ *there are* $c_1, c_2, \ldots, c_m \in \operatorname{Ran}(\varphi)$ *and there is a disjoint covering* $\mathscr{U}' = \{b_j\Gamma\}_{j\in[\![1,m]\!]}$ *of* $\partial\Gamma$ *finer than* $\mathscr{U}$ *such that* $\|\mathbf{1}_{U'}R - R'\| \leq \varepsilon$*, where* $R' = \sum_{j=1}^{m} \mathbf{1}_{b_j\Gamma}c_j\partial^{*\alpha}\partial^{\beta}$ *and* $U' = \cup_{j=1}^{m} b_j\Gamma$*.*

Proof: Let $\varepsilon > 0$ and denote $\varepsilon/\|\partial^{*\alpha}\partial^{\beta}\|$ by ε'. Since $\varphi(\partial\Gamma)$ is compact, there are $\gamma_1, \gamma_2, \ldots, \gamma_N \subset \partial\Gamma$ such that $\varphi(\partial\Gamma) \subset \cup_{k=1}^{N} D(\varphi(\gamma_k), \varepsilon')$, where $D(z, r)$ is the complex open disk of center z and ray r. The open sets $\mathscr{O}_{i,k} = a_i\widehat{\Gamma} \cap \varphi^{-1}(D(\varphi(\gamma_k), \varepsilon'))$ cover $\partial\Gamma$. The Proposition 2.4 gives us a disjoint covering $\{b_j\Gamma\}_{j\in[\![1,m]\!]}$ of $\partial\Gamma$ such that for each $j \in [\![1, m]\!]$ there are i and k such that $b_j\widehat{\Gamma} \subset \mathscr{O}_{i,k}$. To simplify the notations, we will denote by γ_j those γ_k associated to $b_j\Gamma$. We set $\mathscr{U}' = \{b_j\Gamma\}_{j\in[\![1,m]\!]}$ and $R' = \sum_{j=1}^{n} \mathbf{1}_{b_j\Gamma}\varphi(\gamma_j)\partial^{*\alpha}\partial^{\beta}$. Recall that $\sup_{x\in b_j\Gamma} |\varphi(\gamma_j) - \varphi(x)| \leq \varepsilon'$, so

$$
\begin{aligned}
\|(R' - \mathbf{1}_{U'}R)f\|^2 &= \sum_{x\in\Gamma} \Big|\sum_{j=1}^{m} \mathbf{1}_{b_j\Gamma}(x)(\varphi(\gamma_j) - \varphi(x))(\partial^{*\alpha}\partial^{\beta} f)(x)\Big|^2 \\
&= \sum_{j=1}^{m} \sum_{x\in b_j\Gamma} |(\varphi(\gamma_j) - \varphi(x))(\partial^{*\alpha}\partial^{\beta} f)(x)|^2 \\
&\leq \sum_{j=1}^{m} \sup_{x\in b_j\Gamma} |\varphi(\gamma_j) - \varphi(x)|^2 \sum_{x\in b_j\Gamma} |(\partial^{*\alpha}\partial^{\beta} f)(x)|^2 \\
&\leq \varepsilon'^2 \sum_{j=1}^{m} \sum_{x\in b_j\Gamma} |(\partial^{*\alpha}\partial^{\beta} f)(x)|^2 \\
&\leq \varepsilon^2 \|\partial^{*\alpha}\partial^{\beta}\|^{-2} \cdot \|\partial^{*\alpha}\partial^{\beta}\|^2 \cdot \|f\|^2 = \varepsilon^2\|f\|^2.
\end{aligned}
$$

Denoting $\varphi(\gamma_j)$ by c_j we obtain the result. □

Lemma 4.8 *Let $T = \sum_{k=1}^{n} \varphi_k(Q)\partial^{*\alpha_k}\partial^{\beta_k}$ with $\varphi_k \in C(\widehat{\Gamma})$ and let $\varepsilon > 0$. There are a compact operator K, a disjoint covering $\{a_j\Gamma\}_{j\in[\![1,m]\!]}$ of $\partial\Gamma$ and $S = \sum_{k=1}^{n} \sum_{j=1}^{m} \mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}$, with $\min_{j\in[\![1,m]\!]} |a_j| \geq \max_{k\in[\![1,n]\!]} \alpha_k$ and $\gamma_{j,k} \in \partial\Gamma$ such that $\|T - S - K\| \leq \varepsilon$.*

Proof: We denote by $\alpha = \max\{\alpha_k \mid k \in [\![1, n]\!]\}$. Let T_k be $\varphi_k(Q)\partial^{*\alpha_k}\partial^{\beta_k}$. Setting $\mathscr{U}_0 = \cup_{\{a||a|=\alpha\}}\{a\Gamma\}$, we apply the Lemma 4.7 inductively for $k \in [\![1, n]\!]$ with ε/n instead of ε, $\mathscr{U} = \mathscr{U}_{k-1}$ and $R = T_k$, denoting $\mathscr{U}'$ by $\mathscr{U}_k$ and R' by S_k. Then, for $k \in [\![1, n]\!]$ we get $\|\mathbf{1}_{U_k}T_k - S_k\| \leq \varepsilon/k$. Since $\mathscr{U}_{k+1}$ is finer than $\mathscr{U}_k$ for $k \in [\![1, n-1]\!]$, we obtain $\|\mathbf{1}_{U_n}\sum_{k=1}^{n}(T_k - S_k)\| \leq \varepsilon$, hence $\|T - \mathbf{1}_{U_n^c}T - \mathbf{1}_{U_n}\sum_{k=1}^{n} S_k\| \leq \varepsilon$. To finish the proof, we denote the compact operator $\mathbf{1}_{U_n^c}T$ by K, $\mathbf{1}_{U_n}\sum_{k=1}^{n} S_k$ by S and $\mathscr{U}_n$ by $\{a_j\Gamma\}_{j\in[\![1,m]\!]}$. □

We now go back to the proof of Theorem 4.5. Let $T \in$ Ker Φ. For each $\varepsilon > 0$ there is $T' \in \mathscr{C}(\widehat{\Gamma})_{\mathrm{alg}}$ such that $\|T - T'\| \leq \varepsilon/4$. By relation (3.1) and Proposition 3.3, we can write $T' = \sum_{k=1}^{n} \varphi_k(Q)\partial^{*\alpha_k}\partial^{\beta_k} + K$, where $K \in \mathbb{K}(\Gamma)$ and $\varphi_k \in C(\widehat{\Gamma})$. Thus $\|\Phi(T')\| \leq \varepsilon/4$. Using Lemma 4.8, we get an operator S and a compact operator K_1 such that $\|T' - S - K_1\| \leq \varepsilon/4$. This implies that $\|\Phi(S)\| \leq \varepsilon/2$.

Lemma 4.9 *There is* $K_2 \in \mathbb{K}(\Gamma)$ *such that* $\|S - K_2\| \leq \|\Phi(S)\|$.

Before proving the lemma, let us remark that it implies

$$\|T - K_1 - K_2\| \quad \leq \quad \|T - T'\| + \|T' - S - K_1\| + \|S - K_2\| \leq \varepsilon.$$

Hence $T \in \mathbb{K}(\Gamma)$. Thus Theorem 4.5 is proved.□

Proof of Lemma 4.9. First, we remark that for each $a \in \Gamma$ and $\alpha, \beta \geq 0$, the Proposition 3.3 gives us that $\mathbf{1}_{a\Gamma}\partial^{*\alpha}\partial^{\beta} - \mathbf{1}_{a\Gamma}\partial^{*\alpha}\partial^{\beta}\mathbf{1}_{a\Gamma}$ is a compact operator. We define $S' = \sum_{k=1}^{n}\sum_{j=1}^{m} \mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\mathbf{1}_{a_j\Gamma}$ and we set $K_2 = S - S'$, which is a compact operator. Since $\{a_j\Gamma\}_{j\in[\![1,m]\!]}$ is a disjoint covering of $\partial\Gamma$, for any $f \in \ell^2(\Gamma)$:

$$\begin{aligned}
\|S'f\|^2 &= \sum_{x\in\Gamma} \Big|\sum_{k=1}^{n}\sum_{j=1}^{m} (\mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\mathbf{1}_{a_j\Gamma}f)(x)\Big|^2 \\
&= \sum_{j=1}^{m}\sum_{x\in\Gamma} \Big|\sum_{k=1}^{n} (\mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\mathbf{1}_{a_j\Gamma}f)(x)\Big|^2 \\
&\leq \sum_{j=1}^{m} \Big\|\sum_{k=1}^{n} \mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\mathbf{1}_{a_j\Gamma}\Big\|^2 \cdot \|\mathbf{1}_{a_j\Gamma}f\|^2.
\end{aligned}$$

Now we use (3.2) and (3.3) and get:

$$\|\mathbf{1}_{a_j\Gamma}\big(\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\big)\mathbf{1}_{a_j\Gamma}\| = \|\lambda_{a_j}^{*}\big(\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\big)\lambda_{a_j}\|.$$

Since $|a_j| \geq \max\{\alpha_k \mid k \in [\![1,n]\!]\}$, the Lemmas 3.6 and 3.7 give us:

$$\begin{aligned}
\|\lambda_{a_j}^{*}\big(\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\big)\lambda_{a_j}\| &= \|\mathbf{1}_{\Gamma}\big(\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\widetilde{\partial}^{*\alpha_k}\widetilde{\partial}^{\beta_k}\big)\mathbf{1}_{\Gamma}\| \\
&= \|\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\widetilde{\partial}^{*\alpha_k}\widetilde{\partial}^{\beta_k}\|.
\end{aligned}$$

For each j we choose $\gamma_j \in a_j\partial\Gamma$. The family $\{a_j\Gamma\}_{j\in[\![1,m]\!]}$ is a disjoint covering of $\partial\Gamma$, so we have $\lim_{x\to\gamma_j}\chi_{a_j\Gamma}(x)=1$ and $\lim_{x\to\gamma_j}\chi_{a_i\Gamma}(x)=0$ for $i\neq j$. Hence $\Phi_{\gamma_j}(S')=\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\widetilde{\partial}^{*\alpha_k}\widetilde{\partial}^{\beta_k}$. We obtain

$$\|S'f\|^2 \;\leq\; \sum_{j=1}^{m}\|\Phi_{\gamma_j}(S')\|^2\cdot\|\mathbf{1}_{a_j\Gamma}f\|^2 \leq \sup_{\gamma\in\partial\Gamma}\|\Phi_\gamma(S')\|^2\cdot\|f\|^2.$$

Finally, since $\mathbb{K}(\Gamma)\subset \operatorname{Ker}\Phi$, $\|\Phi(S)\|=\|\Phi(S')\|=\sup_{\gamma\in\partial\Gamma}\|\Phi_\gamma(S')\|$. □

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