D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### Generalized Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

Bordeaux, 10 Nov 2011

Introduction

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# Describe some robust and general method to control spectra for symmetric diffusion operators

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities



Introduction

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# Describe some robust and general method to control spectra for symmetric diffusion operators

L generator of a Markov semigroup,  $P_t = \exp(tL)$  symmetric in  $\mathcal{L}^2(\mu)$ 

$$\int fL(g)d\mu = \int gL(f)d\mu.$$

 $\mu$  finite (probability) measure,  $P_t(1) = 1$ ,  $f \ge 0 \Longrightarrow P_t f \ge 0$ .

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### Introduction

Describe some robust and general method to control spectra for symmetric diffusion operators

L generator of a Markov semigroup,  $P_t = \exp(tL)$  symmetric in  $\mathcal{L}^2(\mu)$ 

$$\int fL(g)d\mu = \int gL(f)d\mu.$$

 $\mu$  finite (probability) measure,  $P_t(1) = 1$ ,  $f \ge 0 \Longrightarrow P_t f \ge 0$ .

Model case L: Laplace operator on a compact or finite measure manifold.

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### Introduction

Describe some robust and general method to control spectra for symmetric diffusion operators

L generator of a Markov semigroup,  $P_t = \exp(tL)$  symmetric in  $\mathcal{L}^2(\mu)$ 

$$\int fL(g)d\mu = \int gL(f)d\mu.$$

 $\mu$  finite (probability) measure,  $P_t(1) = 1$ ,  $f \ge 0 \Longrightarrow P_t f \ge 0$ .

Model case L: Laplace operator on a compact or finite measure manifold.

Carré du Champ : 
$$\Gamma(f, f) = \frac{1}{2}(Lf^2 - 2fLf) \ge 0.$$

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

## Introduction

Describe some robust and general method to control spectra for symmetric diffusion operators

L generator of a Markov semigroup,  $P_t = \exp(tL)$  symmetric in  $\mathcal{L}^2(\mu)$ 

$$\int fL(g)d\mu = \int gL(f)d\mu.$$

 $\mu$  finite (probability) measure,  $P_t(1) = 1$ ,  $f \ge 0 \Longrightarrow P_t f \ge 0$ .

Model case L: Laplace operator on a compact or finite measure manifold.

Carré du Champ :  $\Gamma(f, f) = \frac{1}{2}(Lf^2 - 2fLf) \ge 0$ . Dirichlet form  $\mathcal{E}(f, f) = -\int fL(f)d\mu = \int \Gamma(f, f)d\mu$ ;

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### Introduction

Describe some robust and general method to control spectra for symmetric diffusion operators

L generator of a Markov semigroup,  $P_t = \exp(tL)$  symmetric in  $\mathcal{L}^2(\mu)$ 

$$\int fL(g)d\mu = \int gL(f)d\mu.$$

 $\mu$  finite (probability) measure,  $P_t(1) = 1$ ,  $f \ge 0 \Longrightarrow P_t f \ge 0$ .

Model case L: Laplace operator on a compact or finite measure manifold.

Carré du Champ :  $\Gamma(f, f) = \frac{1}{2}(Lf^2 - 2fLf) \ge 0$ . Dirichlet form  $\mathcal{E}(f, f) = -\int fL(f)d\mu = \int \Gamma(f, f)d\mu$ ; In the model case  $\Gamma(f, f) = g^{ij}\partial_i f\partial_j f$ , where  $g^{ij}$  is the Riemann metric.

# Nash and spectrum Functional inequalities and spectrum

D. Bakry				
Joint works				
with F.				
Bolley, I.				
Gentil, P.				
Maheux				

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities



D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

## Functional inequalities and spectrum

Aim : describe functional inequalities (involving  $\mathcal{L}^p$  norms and the Dirichlet form) which lead to control of the spectrum of L : eg which show that the spectrum is discrete and control  $\sum_n \exp(-t\lambda_n)$ 

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# Functional inequalities and spectrum

Aim : describe functional inequalities (involving  $\mathcal{L}^p$  norms and the Dirichlet form) which lead to control of the spectrum of L : eg which show that the spectrum is discrete and control  $\sum_n \exp(-t\lambda_n)$ 

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

General method : control  $\|P_t f\|_{\infty} \leq K(t) \|f\|_1$  or only  $\|P_t f\|_2 \leq K_1(t) \|f\|_1$ 

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

## Functional inequalities and spectrum

Aim : describe functional inequalities (involving  $\mathcal{L}^p$  norms and the Dirichlet form) which lead to control of the spectrum of L : eg which show that the spectrum is discrete and control  $\sum_n \exp(-t\lambda_n)$ 

General method : control  $\|P_t f\|_{\infty} \leq K(t) \|f\|_1$  or only  $\|P_t f\|_2 \leq K_1(t) \|f\|_1$ 

### then pointwise

$$P_t f(x) = \int f(y) p_t(x, y) d\mu(y)$$

for a *density*  $p_t$ , and

 $\|p_t\|_{\infty} \leqslant K(t) \text{ or } \|p_t\|_{\infty} \leqslant K_1(t/2)^2.$ 

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### Aim : describe functional inequalities (involving $\mathcal{L}^p$ norms and the Dirichlet form) which lead to control of the spectrum of L : eg which show that the spectrum is discrete and control $\sum_n \exp(-t\lambda_n)$

Functional inequalities and spectrum

General method : control  $\|P_t f\|_{\infty} \leq K(t) \|f\|_1$  or only  $\|P_t f\|_2 \leq K_1(t) \|f\|_1$ 

### then pointwise

$$P_t f(x) = \int f(y) p_t(x, y) d\mu(y)$$

for a *density*  $p_t$ , and

I

$$\|p_t\|_{\infty} \leqslant K(t) \text{ or } \|p_t\|_{\infty} \leqslant K_1(t/2)^2.$$

Then

$$\int p_t(x,x)d\mu = \sum_n e^{-t\lambda_n} \leqslant \mathcal{K}(t).$$

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Plan

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

◆□▶ ◆□▶ ◆臣▶ ◆臣▶ 臣 のへぐ

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Plan

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### Nash and Sobolev

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Plan

### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

- Nash and Sobolev
- Ultracontractivity

◆□▶ ◆□▶ ◆臣▶ ◆臣▶ □臣 = のへで

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Plan

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

- Nash and Sobolev
- Ultracontractivity
- Super Poincaré inequalities

▲□▶ ▲□▶ ▲□▶ ▲□▶ □ のQ@

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Plan

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

- Nash and Sobolev
- Ultracontractivity
- Super Poincaré inequalities
- Weighted Nash inequalities

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Plan

### Introduction

- Classical Nash and Sobolev Inequalities
- Ultracontractivity
- Super Poincaré Inequalities
- Weighted Nash Inequalities
- Examples

- Nash and Sobolev
- Ultracontractivity
- Super Poincaré inequalities
- Weighted Nash inequalities

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

Nash	and
spect	rum

## Model inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Model inequalities

### Sobolev inequalities $(Sob_n(C))$ :

$$\|f\|_p^2 \leqslant \|f\|_2^2 + C\mathcal{E}(f),$$

Introduction

### Classical Nash and Sobolev Inequalities

# with p > 2 (p = 2n/(n-2)). *n* is the dimension in the Sobolev inequality.

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Model inequalities

### Sobolev inequalities $(Sob_n(C))$ :

$$\|f\|_p^2 \leqslant \|f\|_2^2 + C\mathcal{E}(f),$$

Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

with p > 2 (p = 2n/(n-2)). *n* is the dimension in the Sobolev inequality.

Logarithmic Sobolev inequality (LS(C)):

$$\operatorname{Ent}_{\mu}(f^2) \leqslant C\mathcal{E}(f),$$

$$\operatorname{Ent}_{\mu}(f) = \int f \ln(f) d\mu - \int f d\mu \ln(\int f d\mu)$$

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Model inequalities

Sobolev inequalities  $(Sob_n(C))$  :

$$\|f\|_p^2 \leqslant \|f\|_2^2 + C\mathcal{E}(f),$$

Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

with 
$$p > 2$$
 ( $p = 2n/(n-2)$ ).  $n$  is the dimension in the Sobolev inequality.

Logarithmic Sobolev inequality (LS(C)):

$$\operatorname{Ent}_{\mu}(f^2) \leqslant C\mathcal{E}(f),$$

Ent<sub>$$\mu$$</sub>( $f$ ) =  $\int f \ln(f) d\mu - \int f d\mu \ln(\int f d\mu)$ .  
Poincaré inequalities ( $P(C)$ ) :

 $\sigma_{\mu}^2(f) \leqslant C\mathcal{E}(f),$ 

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

$$\sigma_{\mu}^{2}(f) = \int f^{2}d\mu - (\int fd\mu)^{2}.$$

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Model inequalities

Sobolev inequalities  $(Sob_n(C))$  :

$$\|f\|_p^2 \leqslant \|f\|_2^2 + C\mathcal{E}(f),$$

Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

with 
$$p > 2$$
 ( $p = 2n/(n-2)$ ).  $n$  is the dimension in the Sobolev inequality.

Logarithmic Sobolev inequality (LS(C)):

$$\operatorname{Ent}_{\mu}(f^2) \leqslant C\mathcal{E}(f),$$

Ent<sub>$$\mu$$</sub>(f) =  $\int f \ln(f) d\mu - \int f d\mu \ln(\int f d\mu)$ 

Poincaré inequalities 
$$(P(C))$$
 :

 $\sigma_{\mu}^{2}(f) \leqslant C\mathcal{E}(f),$ 

 $\sigma_{\mu}^{2}(f) = \int f^{2} d\mu - (\int f d\mu)^{2}.$ Sobolev  $\Longrightarrow$  Logarithmic Sobolev  $\Longrightarrow$  Poincaré

Nash	and
spect	rum

# Poincaré Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities



Poincaré Inequalities

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# $\begin{array}{rcl} P(C) & \Longleftrightarrow & \operatorname{spec}(-L) \subset \{0\} \cup [1/C,\infty) \\ & \Leftrightarrow & \sigma^2(P_t f) \leqslant \exp(-2t/C)\sigma^2(f). \end{array}$

### Introduction

### Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities



# Poincaré Inequalities

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# $\begin{array}{rcl} P(C) & \Longleftrightarrow & \operatorname{spec}(-L) \subset \{0\} \cup [1/C,\infty) \\ & \Leftrightarrow & \sigma^2(P_t f) \leqslant \exp(-2t/C)\sigma^2(f). \end{array}$

▲ロト ▲帰ト ▲ヨト ▲ヨト 三日 - の々ぐ

### Introduction

### Classical Nash and Sobolev Inequalities

Proof :

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

# Nash and spectrum Poincaré Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

$$\begin{array}{rcl} P(\mathcal{C}) & \Longleftrightarrow & \operatorname{spec}(-L) \subset \{0\} \cup [1/\mathcal{C}, \infty) \\ & \Leftrightarrow & \sigma^2(P_t f) \leqslant \exp(-2t/\mathcal{C})\sigma^2(f). \end{array}$$

◆□▶ ◆□▶ ◆臣▶ ◆臣▶ □臣 = のへで

#### Introduction

### Classical Nash and Sobolev Inequalities

Proof : 
$$\partial_t P_t f = L(P_t f)$$
, and  $\partial_t \|P_t f\|_2^2 = -2\mathcal{E}(P_t f)$ .

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

# Nash and spectrum Poincaré Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

$$\begin{array}{rcl} P(\mathcal{C}) & \Longleftrightarrow & \operatorname{spec}(-L) \subset \{0\} \cup [1/\mathcal{C}, \infty) \\ & \Leftrightarrow & \sigma^2(P_t f) \leqslant \exp(-2t/\mathcal{C})\sigma^2(f). \end{array}$$

◆□▶ ◆□▶ ◆臣▶ ◆臣▶ 臣 の�?

### Introduction

Classical Nash and Sobolev Inequalities

Proof : 
$$\partial_t P_t f = L(P_t f)$$
, and  $\partial_t ||P_t f||_2^2 = -2\mathcal{E}(P_t f)$ .  
Apply to  $f - \int f d\mu$  and use  $\int f d\mu = 0 \Longrightarrow \int P_t f d\mu = 0$ .

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

# Poincaré Inequalities

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

$$\begin{array}{rcl} \mathsf{P}(\mathsf{C}) & \Longleftrightarrow & \operatorname{spec}(-\mathsf{L}) \subset \{0\} \cup [1/\mathsf{C},\infty) \\ & \Leftrightarrow & \sigma^2(\mathsf{P}_t f) \leqslant \exp(-2t/\mathsf{C})\sigma^2(f). \end{array}$$

#### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

Proof :  $\partial_t P_t f = L(P_t f)$ , and  $\partial_t ||P_t f||_2^2 = -2\mathcal{E}(P_t f)$ . Apply to  $f - \int f d\mu$  and use  $\int f d\mu = 0 \implies \int P_t f d\mu = 0$ . With  $H(t) = ||P_t f||_2^2$ ,

$$P(C) \Longrightarrow H' \leqslant -(2/C)H.$$

### D. Bakry Joint works with F. Bolley, I. Gentil, P.

$$\begin{array}{rcl} {\cal P}({\cal C}) & \Longleftrightarrow & \operatorname{spec}(-L) \subset \{0\} \cup [1/{\cal C},\infty) \\ & \Leftrightarrow & \sigma^2({\cal P}_t f) \leqslant \exp(-2t/{\cal C})\sigma^2(f). \end{array}$$

### Introduction

Maheux

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

Proof : 
$$\partial_t P_t f = L(P_t f)$$
, and  $\partial_t ||P_t f||_2^2 = -2\mathcal{E}(P_t f)$ .  
Apply to  $f - \int f d\mu$  and use  $\int f d\mu = 0 \Longrightarrow \int P_t f d\mu = 0$ . With  $H(t) = ||P_t f||_2^2$ ,

$$P(C) \Longrightarrow H' \leqslant -(2/C)H.$$

▲□▶ ▲□▶ ▲□▶ ▲□▶ □ のQ@

Converse : use : H(t) is convex.

Poincaré Inequalities

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# $\begin{array}{rcl} P(\mathcal{C}) & \Longleftrightarrow & \operatorname{spec}(-L) \subset \{0\} \cup [1/\mathcal{C}, \infty) \\ & \Leftrightarrow & \sigma^2(\mathcal{P}_t f) \leqslant \exp(-2t/\mathcal{C})\sigma^2(f). \end{array}$

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

Proof : 
$$\partial_t P_t f = L(P_t f)$$
, and  $\partial_t ||P_t f||_2^2 = -2\mathcal{E}(P_t f)$ .  
Apply to  $f - \int f d\mu$  and use  $\int f d\mu = 0 \Longrightarrow \int P_t f d\mu = 0$ . With  $H(t) = ||P_t f||_2^2$ ,

$$P(C) \Longrightarrow H' \leqslant -(2/C)H.$$

Converse : use : H(t) is convex. Indeed stronger : In H is convex

Poincaré Inequalities

$$\begin{split} H'' &= 4 \int (LP_t f)^2 d\mu, \ H' = -2 \int P_t f L(P_t f) d\mu \\ & \Longrightarrow (\mathrm{Cauchy} - \mathrm{Shwartz}) \Longrightarrow H H'' \geqslant H'^2. \end{split}$$

▲ロト ▲帰ト ▲ヨト ▲ヨト 三日 - の々ぐ

Nash and spectrum	From	inequalities	to bounds	
D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux				
Introduction				
Classical Nash and Sobolev Inequalities				
Ultracontractivity				
Super Poincaré Inequalities				
Weighted Nash Inequalities				
Examples				

・ロト・(四)・(日)・(日)・(つ)・

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### From inequalities to bounds

### Introduction Poincaré not enoug Classical Nash and Sobolev Inequalities Ultracontractivity Super Poincaré Inequalities Weighted Nash Inequalities Examples

Poincaré not enough to go beyond first eigenvalue.

▲□▶ ▲□▶ ▲□▶ ▲□▶ □ のQ@

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### From inequalities to bounds

### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

- Poincaré not enough to go beyond first eigenvalue.
- Sobolev + Poincaré ⇒ Discrete spectrum, bounded diameter.

▲ロト ▲帰ト ▲ヨト ▲ヨト 三日 - の々ぐ

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

Poincaré not enough to go beyond first eigenvalue.

From inequalities to bounds

- Sobolev + Poincaré ⇒ Discrete spectrum, bounded diameter.
- Sobolev  $\iff \|P_t f\|_{\infty} \leq K(t) \|f\|_1$ ,  $K(t) = Ct^{-n/2}$ ,  $0 < t \leq 1$ . (Ultracontractivity)

▲ロト ▲帰ト ▲ヨト ▲ヨト 三日 - の々ぐ

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

## From inequalities to bounds

### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

- Poincaré not enough to go beyond first eigenvalue.
- Sobolev + Poincaré ⇒ Discrete spectrum, bounded diameter.
- ► Sobolev  $\iff ||P_t f||_{\infty} \le K(t) ||f||_1$ ,  $K(t) = Ct^{-n/2}$ ,  $0 < t \le 1$ . (Ultracontractivity)
- Logarithmic Sobolev "almost enough" to get discrete spectrum (but not ultracontractivity)

Nash	and
spect	rum

# Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

★□> <圖> < E> < E> E のQ@

# Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

$$\|f\|_2^2 \leqslant \|f\|_1^{2(1- heta)} \Big[\|f\|_2^2 + C\mathcal{E}(f)\Big]^{ heta},$$
 with  $heta = n/(n+2).$ 

◆□▶ ◆□▶ ◆臣▶ ◆臣▶ 臣 の�?

Introduction

### Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

# Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

$$\|f\|_{2}^{2} \leqslant \|f\|_{1}^{2(1-\theta)} \Big[\|f\|_{2}^{2} + C\mathcal{E}(f)\Big]^{\theta},$$
 with  $\theta = n/(n+2).$ 

Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

Nash from Sobolev : use Holder's inequality (same constants).

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

# Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# $\|f\|_{2}^{2} \leq \|f\|_{1}^{2(1-\theta)} \left[\|f\|_{2}^{2} + C\mathcal{E}(f)\right]^{\theta},$ with $\theta = n/(n+2)$ . Nash from Sobolev : use Holder's inequality (same constants).

Sobolev from Nash : use slicing : apply to  $(f - 2^k)_+ \wedge 2^k$  and add.

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

One looses on the constants, but same exponent n.

# Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# with $\theta = n/(n+2)$ . Nash from Sobolev : use Holder's inequality (same constants). Sobolev from Nash : use slicing : apply to $(f - 2^k)_+ \wedge 2^k$ and add.

 $\|f\|_{2}^{2} \leq \|f\|_{1}^{2(1-\theta)} \Big[\|f\|_{2}^{2} + C\mathcal{E}(f)\Big]^{\theta},$ 

One looses on the constants, but same exponent n.

Generalised Nash  $(N(\Psi))$ 

$$\frac{\|f\|_2^2}{\|f\|_1^2} \leqslant \Psi\big(\frac{\mathcal{E}(f)}{\|f\|_1^2}\big),$$

with  $\Psi$  increasing and concave.

# Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# with $\theta = n/(n+2)$ . Nash from Sobolev : use Holder's inequality (same constants). Sobolev from Nash : use slicing : apply to $(f - 2^k)_+ \wedge 2^k$ and add.

 $\|f\|_{2}^{2} \leq \|f\|_{1}^{2(1-\theta)} \Big[\|f\|_{2}^{2} + C\mathcal{E}(f)\Big]^{\theta},$ 

One looses on the constants, but same exponent n.

Generalised Nash  $(N(\Psi))$ 

$$\frac{\|f\|_2^2}{\|f\|_1^2} \leqslant \Psi\big(\frac{\mathcal{E}(f)}{\|f\|_1^2}\big),$$

with  $\Psi$  increasing and concave. Usual Nash :  $\Psi(x) = (1 + x)^{\theta}$ ,  $0 < \theta < 1$ .

# Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# with $\theta = n/(n+2)$ . Nash from Sobolev : use Holder's inequality (same constants). Sobolev from Nash : use slicing : apply to $(f - 2^k)_+ \wedge 2^k$ and add.

 $\|f\|_{2}^{2} \leq \|f\|_{1}^{2(1-\theta)} \Big[\|f\|_{2}^{2} + C\mathcal{E}(f)\Big]^{\theta},$ 

One looses on the constants, but same exponent n.

Generalised Nash  $(N(\Psi))$ 

$$\frac{\|f\|_2^2}{\|f\|_1^2} \leqslant \Psi\big(\frac{\mathcal{E}(f)}{\|f\|_1^2}\big),$$

with  $\Psi$  increasing and concave. Usual Nash :  $\Psi(x) = (1+x)^{\theta}$ ,  $0 < \theta < 1$ . Equivalently, with  $\Psi(r) \leq rx + \beta(r)$  $(SPI) \int f^2 d\mu \leq r \mathcal{E}(f) + \beta(r) \left( \int_{e^{-1}} |f| d\mu \right)^2$ .

# Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# with $\theta = n/(n+2)$ . Nash from Sobolev : use Holder's inequality (same constants). Sobolev from Nash : use slicing : apply to $(f - 2^k)_+ \wedge 2^k$ and add.

 $\|f\|_{2}^{2} \leq \|f\|_{1}^{2(1-\theta)} \Big[\|f\|_{2}^{2} + C\mathcal{E}(f)\Big]^{\theta},$ 

One looses on the constants, but same exponent n.

Generalised Nash  $(N(\Psi))$ 

$$\frac{\|f\|_2^2}{\|f\|_1^2} \leqslant \Psi\big(\frac{\mathcal{E}(f)}{\|f\|_1^2}\big),$$

with  $\Psi$  increasing and concave. Usual Nash :  $\Psi(x) = (1+x)^{\theta}$ ,  $0 < \theta < 1$ . Equivalently, with  $\Psi(r) \leq rx + \beta(r)$  $(SPI) \int f^2 d\mu \leq r \mathcal{E}(f) + \beta(r) \left( \int_{e^{-1}} |f| d\mu \right)^2$ .

# Nash and Ultracontractivity from Nash

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super
Poincaré
Inequalities

Weighted Nash Inequalities

Examples

◆□▶ ◆□▶ ◆臣▶ ◆臣▶ 臣 の�?

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# Ultracontractivity from Nash Assume $N(\Psi)$ with $\int_{-\infty}^{\infty} \frac{\Psi'(x)}{x} dx < \infty$ . Then

 $\|P_tf\|_{\infty} \leqslant K(t)\|f\|_1,$ 

### with

$$\mathcal{K}^{-1}(s) = \int_{\Psi^{-1}(s)}^{\infty} \frac{\Psi'(x)}{x} dx.$$

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# Ultracontractivity from Nash Assume $N(\Psi)$ with $\int_{-\infty}^{\infty} \frac{\Psi'(x)}{x} dx < \infty$ . Then

$$\|P_tf\|_{\infty} \leqslant K(t)\|f\|_1,$$

### with

$$\mathcal{K}^{-1}(s) = \int_{\Psi^{-1}(s)}^{\infty} \frac{\Psi'(x)}{x} dx.$$

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

For classical Nash,  $(\Psi(x) = Cx^{n/(n+2)})$ , this gives  $Ct^{-n/2}$ .

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# Ultracontractivity from Nash Assume $N(\Psi)$ with $\int_{x}^{\infty} \frac{\Psi'(x)}{x} dx < \infty$ . Then

$$\|P_tf\|_{\infty} \leqslant K(t)\|f\|_1$$

### with

$$\mathcal{K}^{-1}(s) = \int_{\Psi^{-1}(s)}^{\infty} \frac{\Psi'(x)}{x} dx.$$

## Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

For classical Nash, 
$$(\Psi(x) = Cx^{n/(n+2)})$$
, this gives  $Ct^{-n/2}$ .  
Conversely if  
 $\|P_t f\|_2 \leqslant K(t) \|f\|_1$ ,

then  $N(\Psi)$  with

$$\Psi^{-1}(x) = \sup_{t>0} \frac{x}{2t} \ln \frac{x}{\mathcal{K}^2(t)}.$$

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# Ultracontractivity from Nash Assume $N(\Psi)$ with $\int_{-\infty}^{\infty} \frac{\Psi'(x)}{x} dx < \infty$ . Then

$$\|P_tf\|_{\infty} \leq K(t)\|f\|_1$$

### with

$$\mathcal{K}^{-1}(s) = \int_{\Psi^{-1}(s)}^{\infty} \frac{\Psi'(x)}{x} dx.$$

### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

For classical Nash, 
$$(\Psi(x) = Cx^{n/(n+2)})$$
, this gives  $Ct^{-n/2}$ .  
Conversely if  
 $\|P_t f\|_2 \leqslant K(t) \|f\|_1$ ,

then  $N(\Psi)$  with

$$\Psi^{-1}(x) = \sup_{t>0} \frac{x}{2t} \ln \frac{x}{\mathcal{K}^2(t)}.$$

Sobolev case :  $\mathcal{K}(t) = ct^{-n/2}$  (0 < t < 1) then  $\Psi = C x^{n/(n+2)} (x \to \infty)$ .

### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super
Poincaré
Inequalities

Weighted Nash Inequalities



D. Bakry						
Joint works						
with F.						
Bolley, I.						
Gentil, P.						
Maheux						

### Hint on the proof :

Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super
Poincaré
Inequalities

Weighted Nash Inequalities

Examples

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Hint on the proof : Ultracontractivity from Nash

#### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities



D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# Hint on the proof : Ultracontractivity from Nash

with  $f \ge 0$ ,  $\int f d\mu = 1$ :  $H(t) = \|P_t f\|_2^2$ ,  $H'(t) = -2\mathcal{E}(P_t f)$ .

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Hint on the proof : Ultracontractivity from Nash ١ 1

with 
$$f \ge 0$$
,  $\int f d\mu = 1$ :  $H(t) = ||P_t f||_2^2$ ,  $H'(t) = -2\mathcal{E}(P_t f)$ .  
Apply  $N(\Psi)$  to  $P_t f$ .

Introduction

**Classical Nash** and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### Hint on the proof : Ultracontractivity from Nash with $f \ge 0$ , $\int f d\mu = 1$ : $H(t) = ||P_t f||_2^2$ , $H'(t) = -2\mathcal{E}(P_t f)$ .

Apply  $N(\Psi)$  to  $P_t f$ .

$$N(\Psi) \Longrightarrow H \leqslant \Psi(-H'/2) \iff rac{dH}{\Psi^{-1}(H)} \leqslant -2dt.$$

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### Hint on the proof : Ultracontractivity from Nash

with  $f \ge 0$ ,  $\int f d\mu = 1$ :  $H(t) = ||P_t f||_2^2$ ,  $H'(t) = -2\mathcal{E}(P_t f)$ . Apply  $N(\Psi)$  to  $P_t f$ .

$$N(\Psi) \Longrightarrow H \leqslant \Psi(-H'/2) \iff \frac{dH}{\Psi^{-1}(H)} \leqslant -2dt.$$

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

Uses the fact that 
$$\int P_t f d\mu = \int f d\mu$$

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### Hint on the proof : Ultracontractivity from Nash with $f \ge 0$ , $\int f d\mu = 1$ : $H(t) = ||P_t f||_2^2$ , $H'(t) = -2\mathcal{E}(P_t f)$ .

Apply  $N(\Psi)$  to  $P_t f$ .

 $N(\Psi) \Longrightarrow H \leqslant \Psi(-H'/2) \iff \frac{dH}{\Psi^{-1}(H)} \leqslant -2dt.$ 

Uses the fact that  $\int P_t f d\mu = \int f d\mu$ . Differential equation, from which a bound  $\|P_t f\|_2^2 \leq K_1(t) \|f\|_1$ .

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# Hint on the proof : Ultracontractivity from Nash

with  $f \ge 0$ ,  $\int f d\mu = 1$ :  $H(t) = ||P_t f||_2^2$ ,  $H'(t) = -2\mathcal{E}(P_t f)$ . Apply  $N(\Psi)$  to  $P_t f$ .

$$N(\Psi) \Longrightarrow H \leqslant \Psi(-H'/2) \iff \frac{dH}{\Psi^{-1}(H)} \leqslant -2dt.$$

Uses the fact that  $\int P_t f d\mu = \int f d\mu$ . Differential equation, from which a bound  $\|P_t f\|_2^2 \leq K_1(t) \|f\|_1$ .

Nash from Ultracontractivity:

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### Hint on the proof : Ultracontractivity from Nash with $f \ge 0$ , $\int f d\mu = 1$ : $H(t) = ||P_t f||_2^2$ , $H'(t) = -2\mathcal{E}(P_t f)$ . Apply $N(\Psi)$ to $P_t f$ .

$$N(\Psi) \Longrightarrow H \leqslant \Psi(-H'/2) \iff rac{dH}{\Psi^{-1}(H)} \leqslant -2dt.$$

Uses the fact that  $\int P_t f d\mu = \int f d\mu$ . Differential equation, from which a bound  $\|P_t f\|_2^2 \leq K_1(t) \|f\|_1$ .

Nash from Ultracontractivity: use the convexity of  $\ln H$ : from  $\|P_t f\|_2 \leq K(t/2)^{1/2} \|f\|_1$ ,

 $H(t) \leqslant H(0)^{1-\alpha t} K(2\alpha)^{\alpha t/2}$ 

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# Hint on the proof : Ultracontractivity from Nash

with  $f \ge 0$ ,  $\int f d\mu = 1$ :  $H(t) = ||P_t f||_2^2$ ,  $H'(t) = -2\mathcal{E}(P_t f)$ . Apply  $N(\Psi)$  to  $P_t f$ .

$$N(\Psi) \Longrightarrow H \leqslant \Psi(-H'/2) \iff \frac{dH}{\Psi^{-1}(H)} \leqslant -2dt.$$

Uses the fact that  $\int P_t f d\mu = \int f d\mu$ . Differential equation, from which a bound  $\|P_t f\|_2^2 \leq K_1(t) \|f\|_1$ .

Nash from Ultracontractivity: use the convexity of  $\ln H$ : from  $||P_t f||_2 \leq K(t/2)^{1/2} ||f||_1$ ,

$$H(t) \leqslant H(0)^{1-lpha t} K(2lpha)^{lpha t/2}$$

take asymptotics in t = 0 and optimize in  $\alpha$ .

Nash	and
spect	rum

# Super Poincaré

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# Super Poincaré

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# Beyond spectral gap : bottom of the essential spectrum

Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

▲□▶ ▲圖▶ ★園▶ ★園▶ - 園 - のへで

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# Super Poincaré

Beyond spectral gap : bottom of the essential spectrum If

$$\int f^2 d\mu \leqslant r \mathcal{E}(f) + \left(\int \Pi(f) d\mu\right)^2 \tag{1}$$

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

### Introduction

Classical Nash and Sobolev Inequalities

## where $\Pi(f)$ is the projection onto a finite dimensional space. Then, $\sigma_{ess} \in [\frac{1}{r}, \infty)$ .

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# Super Poincaré

Beyond spectral gap : bottom of the essential spectrum If

$$\int f^2 d\mu \leqslant r \mathcal{E}(f) + \left(\int \Pi(f) d\mu\right)^2 \tag{1}$$

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

where  $\Pi(f)$  is the projection onto a finite dimensional space. Then,  $\sigma_{ess} \in [\frac{1}{r}, \infty)$ . Conversely, if  $\sigma_{ess} \subset (\frac{1}{r}, \infty)$ , then (1) holds.

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# Super Poincaré

Beyond spectral gap : bottom of the essential spectrum If

$$\int f^2 d\mu \leqslant r \mathcal{E}(f) + \left(\int \Pi(f) d\mu\right)^2 \tag{1}$$

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

### Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

where  $\Pi(f)$  is the projection onto a finite dimensional space. Then,  $\sigma_{ess} \in [\frac{1}{r}, \infty)$ . Conversely, if  $\sigma_{ess} \subset (\frac{1}{r}, \infty)$ , then (1) holds. Other version : Super Poincaré Inequalities

### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# Super Poincaré

Beyond spectral gap : bottom of the essential spectrum If

$$\int f^2 d\mu \leqslant r \mathcal{E}(f) + \left(\int \Pi(f) d\mu\right)^2 \tag{1}$$

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

where  $\Pi(f)$  is the projection onto a finite dimensional space. Then,  $\sigma_{ess} \in [\frac{1}{r}, \infty)$ . Conversely, if  $\sigma_{ess} \subset (\frac{1}{r}, \infty)$ , then (1) holds. Other version : Super Poincaré Inequalities If  $\sigma_{ess} = \emptyset$ , then there exists  $w \in \mathcal{L}^2(\mu)$  and  $r \mapsto \beta(r)$  on  $(0, \infty)$  such that

$$(SPI)\int f^2d\mu\leqslant r\mathcal{E}(f)+\beta(r)\Big(\int |fw|d\mu\Big)^2.$$

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# Super Poincaré

Beyond spectral gap : bottom of the essential spectrum If

$$\int f^2 d\mu \leqslant r \mathcal{E}(f) + \left(\int \Pi(f) d\mu\right)^2 \tag{1}$$

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

where  $\Pi(f)$  is the projection onto a finite dimensional space. Then,  $\sigma_{ess} \in [\frac{1}{r}, \infty)$ . Conversely, if  $\sigma_{ess} \subset (\frac{1}{r}, \infty)$ , then (1) holds. Other version : Super Poincaré Inequalities If  $\sigma_{ess} = \emptyset$ , then there exists  $w \in \mathcal{L}^2(\mu)$  and  $r \mapsto \beta(r)$  on  $(0, \infty)$  such that

$$(SPI)\int f^2d\mu\leqslant r\mathcal{E}(f)+\beta(r)\Big(\int |fw|d\mu\Big)^2.$$

Typically  $w = \sum_{i} a_{i} f_{i}$ , where  $f_{i}$  eigenvectors and  $(a_{i})$  decreasing such that  $\sum_{i} |a_{i}| < \infty$ ; then  $\beta(r) = n(r)/a_{n(r)}^{2}$ , such that  $(f_{1}, \dots, f_{n(r)})$  span the spectral space  $E_{1/r}$ .

Nash and spectrum	From	(SPI)	to discr	ete spectru	m : V	Vang's	theorem
D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux							
Introduction							
Classical Nash and Sobolev Inequalities							
Ultracontractivity							
Super Poincaré Inequalities							
Weighted Nash Inequalities							
Examples							

### Nash and From (SPI) to discrete spectrum : Wang's theorem spectrum D. Bakry Joint works with F. Bolley, I. If (SPI) holds for some function $r \mapsto \beta(r)$ and $w \in \mathcal{L}^2(\mu)$ and Gentil, P. Maheux Introduction Classical Nash and Sobolev Ultracontractivity Super Poincaré Inequalities Weighted Nash Inequalities

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux From (SPI) to discrete spectrum : Wang's theorem

If (SPI) holds for some function  $r\mapsto \beta(r)$  and  $w\in \mathcal{L}^2(\mu)$  and

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

• Either  $P_t$  has a density for some t > 0.

Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# If (SPI) holds for some function $r\mapsto eta(r)$ and $w\in \mathcal{L}^2(\mu)$ and

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

From (SPI) to discrete spectrum : Wang's theorem

• Either  $P_t$  has a density for some t > 0.

Either the operator L is Persson

Classical Nash and Sobolev Inequalities

Introduction

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

# If (SPI) holds for some function $r \mapsto \beta(r)$ and $w \in \mathcal{L}^2(\mu)$ and

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

From (SPI) to discrete spectrum : Wang's theorem

- Either  $P_t$  has a density for some t > 0.
- Either the operator L is Persson

Then, the essential spectrum is empty.

### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

### Super Poincaré Inequalities

Weighted Nash Inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# From (SPI) to discrete spectrum : Wang's theorem

If (SPI) holds for some function  $r\mapsto eta(r)$  and  $w\in \mathcal{L}^2(\mu)$  and

- Either  $P_t$  has a density for some t > 0.
- Either the operator L is Persson

Then, the essential spectrum is empty. Persson : There exists an increasing sequence  $(A_k)$  such that  $\cup A_k = E$  and  $\mu(A_k) < \infty$  such that  $\inf \sigma_{ess} \ge \sup_k \lambda_k$ , where

$$\lambda_k = \inf \left\{ \frac{cE(f)}{\int f^2 d\mu}, f \text{ supported in } A_k^c \right\}.$$

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# From (SPI) to discrete spectrum : Wang's theorem

If (SPI) holds for some function  $r \mapsto \beta(r)$  and  $w \in \mathcal{L}^2(\mu)$  and

- Either  $P_t$  has a density for some t > 0.
- Either the operator L is Persson

Then, the essential spectrum is empty. Persson : There exists an increasing sequence  $(A_k)$  such that  $\cup A_k = E$  and  $\mu(A_k) < \infty$  such that  $\inf \sigma_{ess} \ge \sup_k \lambda_k$ , where

$$\lambda_k = \inf \left\{ \frac{cE(f)}{\int f^2 d\mu}, \quad f \text{ supported in } A_k^c \right\}.$$

Holds as soon as the  $A_k$  are nicely separated and the embedding from  $\mathcal{H}^1(A_k)$  into  $\mathcal{L}^2(A_k)$  is compact.

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Introduction Other Version of (SPI) : Weighted Nash Inequalities N(w, Ψ) Classical Nash and Sobolev Inequalities Other Version of (SPI) : Weighted Nash Inequalities N(w, Ψ) Ultracontractivity Super Poincaré Inequalities Weighted Nash Inequalities Keighted Nash Inequalities Examples Veighted Nash Inequalities N(w, Ψ)

Weighted Nash inequalities

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Weighted Nash inequalities

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

Other Version of (SPI) : Weighted Nash Inequalities  $N(w, \Psi)$ For some  $w \in \mathcal{L}^2(\mu)$ ,  $w \ge 0$  and increasing concave  $\Psi$  with  $\lim_{r\to\infty} \Psi(r)/r = 0$ 

$$\frac{\|f\|_2^2}{(\int |wf|d\mu]^2} \leqslant \Psi\Big(\frac{\mathcal{E}(f)}{(\int |wf|d\mu)^2}\Big).$$

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# Other Version of (SPI) : Weighted Nash Inequalities $N(w, \Psi)$ For some $w \in \mathcal{L}^2(\mu)$ , $w \ge 0$ and increasing concave $\Psi$ with $\lim_{r\to\infty} \Psi(r)/r = 0$

Weighted Nash inequalities

$$\frac{\|f\|_2^2}{(\int |wf|d\mu]^2} \leqslant \Psi\Big(\frac{\mathcal{E}(f)}{(\int |wf|d\mu)^2}\Big).$$

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

: Question : what is the relation with Ultracontractivity?

Nash and spectrum	From	Weighted	Nash	to	control	on	the	spect	rum
D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux									
Introduction									
Classical Nash and Sobolev Inequalities									
Ultracontractivity									
Super Poincaré Inequalities									
Weighted Nash Inequalities									
Examples									

## From Weighted Nash to control on the spectrum

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

 $\frac{\|f\|_2^2}{(\int w|f|d\mu)^2} \leqslant \Psi\Big(\frac{\mathcal{E}(f)}{(\int w|f|d\mu)^2}\Big).$ 

Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities



### Nash and From Weighted Nash to control on the spectrum spectrum D. Bakry Joint works with F. Bolley, I. Gentil, P. $\frac{\|f\|_2^2}{(\int w|f|d\mu)^2} \leqslant \Psi\Big(\frac{\mathcal{E}(f)}{(\int w|f|d\mu)^2}\Big).$ Maheux Introduction Classical Nash $\int_{-\infty}^{\infty} \frac{\Psi'(x)}{x} dx < \infty$ , $\int w^2 d\mu = 1$ and $Lw \leq cw$ . and Sobolev Inequalities Ultracontractivity Super Poincaré

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

Inequalities

Weighted Nash Inequalities

# Nash and spectrum From Weighted Nash to control on the spectrum

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

$$\frac{\|f\|_2^2}{(\int w|f|d\mu)^2} \leqslant \Psi\Big(\frac{\mathcal{E}(f)}{(\int w|f|d\mu)^2}\Big).$$

Introduction Classical Nash and Sobolev Inequalities

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

$$Q_t(f) = w^{-1}P_t(wf)$$
 bounded from  $\mathcal{L}^1(w^2d\mu)$  to  $\mathcal{L}^2(w^2d\mu)$ .

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

 $\int_{-\infty}^{\infty} \frac{\Psi'(x)}{x} dx < \infty$ ,  $\int w^2 d\mu = 1$  and  $Lw \leq cw$ . Then,

## Nash and spectrum From Weighted Nash to control on the spectrum

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

$$\frac{\|f\|_2^2}{(\int w|f|d\mu)^2} \leqslant \Psi\Big(\frac{\mathcal{E}(f)}{(\int w|f|d\mu)^2}\Big).$$

Classical Nash and Sobolev Inequalities

Introduction

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

$$\int^\infty rac{\Psi'(x)}{x} dx < \infty$$
,  $\int w^2 d\mu = 1$  and  $Lw \leqslant cw$ . Then,

 $Q_t(f) = w^{-1}P_t(wf)$  bounded from  $\mathcal{L}^1(w^2d\mu)$  to  $\mathcal{L}^2(w^2d\mu)$ . Bound on the spectrum and the density

$$\sum_{n} e^{-\lambda_{n}t} \leqslant K^{2}(t)e^{ct}, \ p_{t}(x,y) \leqslant K(t)e^{ct}w(x)w(y)$$

$$\mathcal{K}^{-1}(s) = \int_{\Psi^{-1}(s)}^{\infty} \frac{\Psi'(x)}{x} dx.$$

◆□ > ◆□ > ◆豆 > ◆豆 > ̄豆 \_ のへぐ

Nash and spectrum	From	Weighted	Nash	Inequalities and	back
D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux					
Introduction					
Classical Nash and Sobolev Inequalities					
Ultracontractivity					
Super Poincaré Inequalities					
Weighted Nash Inequalities					
Examples					

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

### Introduction From Nash to bounds Classical Nash and Sobolev Inequalities Ultracontractivity Super Poincaré Inequalities Weighted

Nash Inequalities

Examples

# From Weighted Nash Inequalities and back

### Nash and From Weighted Nash Inequalities and back spectrum D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux From Nash to bounds Introduction Hint : as before : differential equation on $\frac{\|P_t f\|_2^2}{(\int w P_t f d\mu)^2}$ . Classical Nash and Sobolev Ultracontractivity Poincaré Weighted Nash Inequalities Examples

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

Nash and spectrum	From Weighted Nash Inequalities and back					
D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux						
Introduction	From Nash to bounds					
Classical Nash and Sobolev Inequalities	Hint : as before : differential equation on $\frac{\ P_t f\ _2^2}{(\int wP_t f d\mu)^2}$ . Replace invariance by $\int wP_t f d\mu \leq e^{ct} \int wf d\mu$ due to $Lw \leq cw$ .					
Ultracontractivity						
Super Poincaré Inequalities						
Weighted Nash Inequalities						
Examples						

▲□▶ ▲□▶ ▲三▶ ▲三▶ ▲□ ● ● ●

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

## From Weighted Nash Inequalities and back

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### From Nash to bounds Hint : as before : differential equation on $\frac{\|P_t f\|_2^2}{(\int w P_t f d\mu)^2}$ . Replace

invariance by  $\int wP_t f d\mu \leq e^{ct} \int wf d\mu$  due to  $Lw \leq cw$ . Gives  $\|P_t f\|_2^2 \leq K(t) (\int |fw| d\mu)^2$ , and  $\|w^{-1}P_t(fw)\|_{2,w^2d\mu} \leq \|f\|_{2,w^2d\mu}$ .

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

## From Weighted Nash Inequalities and back

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### From Nash to bounds

Hint : as before : differential equation on  $\frac{||P_t f||_2^2}{(\int w P_t f d\mu)^2}$ . Replace

・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・

invariance by  $\int wP_t f d\mu \leq e^{ct} \int wf d\mu$  due to  $Lw \leq cw$ . Gives  $\|P_t f\|_2^2 \leq K(t) (\int |fw| d\mu)^2$ , and  $\|w^{-1}P_t(fw)\|_{2,w^2 d\mu} \leq \|f\|_{2,w^2 d\mu}$ . From bounds to weighted Nash

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

## From Weighted Nash Inequalities and back

From Nash to bounds

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

## Hint : as before : differential equation on $\frac{\|P_t f\|_2^2}{(\int wP_t fd\mu)^2}$ . Replace invariance by $\int wP_t fd\mu \leq e^{ct} \int wfd\mu$ due to $Lw \leq cw$ . Gives $\|P_t f\|_2^2 \leq K(t)(\int |fw|d\mu)^2$ , and $\|w^{-1}P_t(fw)\|_{2,w^2d\mu} \leq \|f\|_{2,w^2d\mu}$ . From bounds to weighted Nash If $\|P_t f\|_2 \leq K(t)\|fw\|_1$ , then $N(\Phi)$ with $\Phi(x) = \sup\{\frac{x}{2t} \ln \frac{x}{K^2(t)}, t > 0\}$ .

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

## From Weighted Nash Inequalities and back

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### From Nash to bounds Hint : as before : differential equation on $\frac{\|P_t f\|_2^2}{(\int w P_t f d\mu)^2}$ . Replace invariance by $\int w P_t f d\mu \leq e^{ct} \int w f d\mu$ due to $Lw \leq cw$ . Gives $\|P_t f\|_2^2 \leq K(t) (\int |fw| d\mu)^2$ , and $\|w^{-1} P_t(fw)\|_{2,w^2 d\mu} \leq \|f\|_{2,w^2 d\mu}$ . From bounds to weighted Nash If $\|P_t f\|_2 \leq K(t) \|fw\|_1$ , then $N(\Phi)$ with $\Phi(x) = \sup\{\frac{x}{2t} \ln \frac{x}{K^2(t)}, t > 0\}$ . Hint Use convexity of $t \mapsto \ln \|P_t f\|_2$ .

Nash	and
spect	rum

### **Examples**

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

(ロ)、(型)、(E)、(E)、 E) の(の)

### Examples

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

Model examples on the real line, with  $\mu(dx) = e^{-W} dx$  and  $\Gamma(f) = f^{2}$ .

Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities



Examples

#### D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

Model examples on the real line, with  $\mu(dx) = e^{-W} dx$  and  $\Gamma(f) = f'^2$ .

W = a|x|: Poincaré ; no discrete spectrum.

#### Introduction

Classical Nash and Sobolev Inequalities

### Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities



Examples

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

Model examples on the real line, with  $\mu(dx) = e^{-W} dx$  and  $\Gamma(f) = f'^2$ .

W = a|x| : Poincaré ; no discrete spectrum.

Introduction Classical Nash

and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

 $W = ax^2$ : Logarithmic Sobolev ; no ultracontractivity (so no direct bound on the spectrum).

Examples

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

# Model examples on the real line, with $\mu(dx) = e^{-W} dx$ and $\Gamma(f) = f^{2}$ .

W = a|x| : Poincaré ; no discrete spectrum.

 $W = ax^2$ : Logarithmic Sobolev ; no ultracontractivity (so no direct bound on the spectrum).

(Log-Sob does not imply discrete spectrum : in infinite dimension, infinite products of Gaussian measures satisfy logarithmic Sobolev but no discrete spectrum).

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### Examples

Model examples on the real line, with  $\mu(dx) = e^{-W} dx$  and  $\Gamma(f) = f'^2$ .

W = a|x| : Poincaré ; no discrete spectrum.

 $W = ax^2$ : Logarithmic Sobolev ; no ultracontractivity (so no direct bound on the spectrum).

(Log-Sob does not imply discrete spectrum : in infinite dimension, infinite products of Gaussian measures satisfy logarithmic Sobolev but no discrete spectrum).

 $W = a |x|^{lpha}$  with lpha > 2 : ultracontractivity and  $N(\Psi)$  with

$$\Psi(x) = C \big( 1 + x (\ln x)^{-(2\alpha-2)/\alpha} \big).$$

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities

Examples

### Examples

Model examples on the real line, with  $\mu(dx) = e^{-W} dx$  and  $\Gamma(f) = f'^2$ .

W = a|x|: Poincaré ; no discrete spectrum.

 $W = ax^2$ : Logarithmic Sobolev ; no ultracontractivity (so no direct bound on the spectrum).

(Log-Sob does not imply discrete spectrum : in infinite dimension, infinite products of Gaussian measures satisfy logarithmic Sobolev but no discrete spectrum).

 $W = a |x|^{lpha}$  with lpha > 2 : ultracontractivity and  $N(\Psi)$  with

$$\Psi(x) = C (1 + x(\ln x)^{-(2\alpha - 2)/\alpha}).$$

 $W = c|x|^{\alpha}$ ,  $1 < \alpha < 2$ . Not ultracontractive but Weighted Nash with  $\Psi(x) = C(1+x)^{\lambda}$  ( $0 < \lambda < 1$ ) and  $w = e^{W/2}|x|^{-\gamma}$ ,  $\gamma > 0$ .

D. Bakry Joint works with F. Bolley, I. Gentil, P. Maheux

#### Introduction

Classical Nash and Sobolev Inequalities

# Thank you for your attention

▲ロト ▲帰ト ▲ヨト ▲ヨト - ヨ - の々ぐ

Ultracontractivity

Super Poincaré Inequalities

Weighted Nash Inequalities