Spectral concentration estimates for trees

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(joint work with W. Kirsch, M. Krishna, W. Spitzer)

This talk is a preliminary report on work in progress. Let T be a $(d+1)$-regular infinite tree and let

$$H = \Delta + \kappa q$$

be an Anderson Hamiltonian acting on $\ell^2(T)$. Specifically, Δ is the adjacency operator for T, q is an i.i.d. random potential whose single site distribution has bounded support, and $\kappa \geq 0$ is a coupling constant.

Let X_n be a $(d+1)$-regular labeled graph with $|X_n| = n$, chosen uniformly at random and let

$$H_n = \Delta_n + \kappa q$$

acting on $\ell^2(X_n)$, where Δ_n is the adjacency matrix for X_n.

We wish to determine how well H_n approximates H as n tends to infinity by comparing the density of states for these operators. Pick vertices $0 \in T$ and $0 \in X_n$ and define e_0 in $\ell^2(T)$ and also in $\ell^2(X_n)$ by

$$e_0(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

Let χ_I denote the indicator function of the interval I. For H, the density of states at the site 0 for the energy interval I is defined as

$$\begin{aligned} \mathbb{E}\left[\operatorname{tr}(|e_0\rangle\langle e_0|\chi_I(H))\right] &= \mathbb{E}\left[\langle e_0, \chi_I(H)e_0\rangle\right] \\ &= \mathbb{E}\left[\int \chi_I(x)d\mu(x)\right] \end{aligned}$$

where $d\mu$ is the spectral measure for e_0. The density of states does not depend on the choice of 0. For H_n the density of states is defined as

$$\begin{aligned} \mathbb{E}\left[\operatorname{tr}(|e_0\rangle\langle e_0|\chi_I(H_n))\right] &= \mathbb{E}\left[\langle e_0, \chi_I(H_n)e_0\rangle\right] \\ &= \frac{1}{n}\mathbb{E}\left[\operatorname{tr}\chi_I(H_n)\right] \\ &= \frac{1}{n}\#\{\sigma(H_n) \cap I\} \\ &= \mathbb{E}\left[\int \chi_I(x)d\rho_n(x)\right]. \end{aligned}$$

Here $d\rho_n(x) = \frac{1}{n}\sum_{\lambda\in\sigma(H_n)} \delta(x-\lambda)$ is the empirical counting measure.

To compare the density of states, we introduce the respective Green functions

$$G(E + i\epsilon) = \langle e_0, (H - E - i\epsilon)^{-1} e_0 \rangle$$
$$G_n(E + i\epsilon) = \langle e_0, (H_n - E - i\epsilon)^{-1} e_0 \rangle,$$

whose expectations are the Stieltjes transforms of the density of states measures $\mathbb{E}d\mu$ and $\mathbb{E}d\rho_n$. In both cases their distribution does not depend on the choice of 0.

Our main observation is that since a point in X_n typically has a tree neighbourhood with $\log(n)/\log(d)$ levels [4], we can compare the Green functions for X_n and T using contraction estimates similar to those that have been used to prove the existence of absolutely continuous spectrum for H [2, 3]. For fixed ϵ and small κ an ϵ dependent contraction estimate shows that

$$\mathbb{E}\,|G_n(E + i\epsilon) - G(E + i\epsilon)| \to 0$$

as $n \to \infty$. This implies that the density of states measure for H_n converges vaguely to that of H. When $\kappa = 0$ this is a classical result of McKay [5]. Our goal is to go beyond this and show that for a sequence $\epsilon_n \to 0$ we have

$$\mathbb{E}\,|G_n(E + i\epsilon_n) - G(E + i\epsilon_n)| \to 0,$$

for $|E| < 2\sqrt{d}$ and κ small. This would imply spectral concentration estimates. We are able to do this if we let the coupling constant $\kappa = \kappa_n$ also depend on n with $\kappa_n \to 0$. In this case G can be replaced by Green function of the tree without a potential. This is the (deterministic) Stieltjes transform of the Kesten-McKay law. We can also consider the case where the co-ordination number increases with n. If $d = d_n$ with $d_n \to \infty$ and if we scale H_n by $1/\sqrt{d_n}$ a similar estimate is true where G is replaced by Stieltjes transform of the semi-circle law. The random graph case where $\kappa = 0$ and $d_n \to \infty$ has been the subject of recent activity and stronger results are known [1].

References

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