Functional Analysis

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1 Metric spaces

1.1 Definitions

Definition 1.1 (Distance). *Given a set X, we say that a function $d : X \times X \to \mathbb{R}^+$ is a distance on X if*

1. $d(x, y) = 0$ *if and only if* $x = y$;
2. $d(x, y) = d(y, x)$;
3. *For any* x, y, $z \in X$, $d(x, y) \leqslant d(x, z) + d(y, z)$.

Then we say that (X, d) is a metric space.

For any x in X and any $r > 0$, we denote

$$B(x, r) := \{y \in X : \ d(x, y) < r\} \quad (\text{resp. } \overline{B}(x, r) := \{y \in X : \ d(x, y) \leqslant r\})$$

the open (resp. closed) ball of center x and radius r. A subset is said to be bounded if it is contained in a ball of finite radius.

Definition 1.2 (**Open/closed sets**). *We say that*

1. *$U \subset X$ is open if for any $x \in U$, there exists $r > 0$ such that $B(x,r) \subset X$,*

2. *$V \subset X$ is a neighbourhood of $x \in X$ if there is $U \subset V$ which contains x,*

3. *F is closed if $X \setminus U$ is open.*

Remark 1.1. *The set of all the open sets $U \subset X$ is called the topology associated with the metric space (X,d).*

Definition 1.3 (**Interior and closure**). *Given a set $A \subset X$, we define*

1. *the interior of A by $A^\circ := \{x \in A :$ there exists U open such that $x \in U \subset A\}$;*

2. *the closure of A by $\overline{A} := \{x \in X :$ for any U open with $x \in U$, then $U \cap A \neq \emptyset\}$.*

We say that $x \in \overline{A}$ is an adherent point and $x \in A^\circ$ an interior point.

Let us observe that $A^\circ \subset A \subset \overline{A}$.

Definition 1.4 (**Density**). *Given a set $A \subset X$, we say that A is dense in X if $\overline{A} = X$.*

Definition 1.5 (**Limit of a sequence**). *We say that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to x in X if for any open set U with $x \in U$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geqslant n_0$.*

Proposition 1.1. *The following statements hold:*

1. *A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to x in X if and only if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, then $d(x,x_n) < \varepsilon$;*

2. *A subset F of X is closed if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subset F$ converging to x in X, then $x \in F$.*

Definition 1.6 (**Continuity**). *Given two metric spaces (X_1,d_1) and (X_2,d_2), we say that a map $f : X_1 \to X_2$ is continuous at $x_1 \in X_1$ if for all open set U in X_2 such that $f(x_1) \in U$, then $f^{-1}(U)$ is an open set in X_1.*

Proposition 1.2. *Let (X_1,d_1) and (X_2,d_2) be two metric spaces, $f : X_1 \to X_2$ and $x_1 \in X_1$. Then the following statements are equivalent:*

1. *f is continuous at x_1;*

2. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in X_1$ is such that $d_1(x_1,x) < \delta$, then $d_2(f(x_1),f(x)) < \varepsilon$;*

3. *For any sequence $(x_n)_{n\in\mathbb{N}} \subset X_1$ converging to x_1, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$ in X_2.*

Definition 1.7 (**Uniform continuity**). *Let (X_1,d_1) and (X_2,d_2) be two metric spaces. We say that an map $f : X_1 \to X_2$ is uniformly continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if x and $y \in X_1$ satisfy $d_1(x,y) < \delta$, then $d_2(f(x),f(y)) < \varepsilon$.*

It is clear from the definitions above that uniform continuity implies continuity. We will see in Theorem 1.2 that the converse statement holds true when the space (X_1,d_1) is compact.
Let us now see a particular case of metric space.

Definition 1.8 (Normed vector space). *A normed vector space over* $\mathbb{R}$ *is a pair* $(V, \|\cdot\|)$ *where* V *is a real vector space and* $\|\cdot\|$ *is a norm on* X *that is a function from* X *to* $\mathbb{R}_+$ *satisfying*

1. $\|u\| = 0$ *if and only* $u = 0$ *(positive definiteness),*
2. *for any* u *in* V*, for any* $\lambda \in \mathbb{R}$, $\|\lambda u\| = |\lambda|\|u\|$ *(positive homogeneity),*
3. *for any* u, v *in* V, $\|u + v\| \leqslant \|u\| + \|v\|$ *(triangle inequality or subadditivity).*

If the first item above is not satisfied then $\|\cdot\|$ *is called a seminorm.*

Remark 1.2. *We define in a similar way a normed vector space over* $\mathbb{C}$ *by considering a complex vector space and by extending the second property above to any* $\lambda \in \mathbb{C}$.

Remark 1.3. *We can easily associate a distance with the norm of a normed vector space, through the formula* $d(u, v) := \|u - v\|$.

1.2 Completeness

Completeness is an important notion in topology and in functional analysis because it enables one to characterize converging sequences without the knowledge of their limit. We first define the Cauchy property.

Definition 1.9 (Cauchy sequence). *Given a metric space* (X, d)*, we say that a sequence* $(u_n)_{n\in\mathbb{N}} \subset X$ *is a Cauchy sequence if for any* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* n, $n' \geqslant n_0$ *then* $d(u_n, u_{n'}) < \varepsilon$.

Definition 1.10 (Completeness). *A metric space* (X, d) *is complete if any Cauchy sequence converges in* X.

Let us give as a first example the set $\mathbb{R}$ endowed with the usual metric $d(x, y) := |x - y|$. It is also useful to notice that a closed subset of a complete metric space is complete.

An important application of the notion of completeness is given by the following theorem.

Theorem 1.1 (Banach, Picard). *Given a complete metric space* (X, d) *and* $f : X \to X$*. Assume that* f *is a contraction, i.e. that there exists a constant* $\theta \in (0, 1)$ *such that for all* x *and* $y \in X$*, then* $d(f(x), f(y)) \leqslant \theta d(x, y)$*. Then there exists a unique fixed point* $x^* \in X$ *such that* $f(x^*) = x^*$.

Proof. Let $x_0 \in X$ and let $(x_n)_{n\in\mathbb{N}}$ the associated sequence defined by the relation

$$x_{n+1} = f(x_n). \tag{1}$$

By iteration we have

$$d(x_{n+1}, x_n) \leqslant \theta^n d(x_1, x_0).$$

For any $n' > n$,

$$d(x_{n'}, x_n) \leqslant \sum_{k=1}^{n'-n} d(x_{n+k}, x_{n+k-1}) \leqslant d(x_1, x_0) \sum_{k=1}^{n'-n} \theta^{n+k-1} \leqslant \frac{\theta^n}{1-\theta} d(x_1, x_0).$$

Therefore $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and by completeness, it converges to an element $x^* \in X$. Since f is continuous, we have $f(x^*) = x^*$ by passing to the limit in (1). The uniqueness follows from the contraction assumption. □

The previous theorem is useful in the proof of the Cauchy-Lipschitz theorem in the theory of ordinary differential equations, and also in the proof of the local inversion theorem.

Definition 1.11 (**Banach space/algebra**). *Let $(V, \|\cdot\|)$ be a normed vector space. If V is complete then we say that $(V, \|\cdot\|)$ is a Banach space. If in addition V is an associative algebra whose multiplication law is compatible with the norm in the sense that $\|u \cdot v\| \leqslant \|u\| \cdot \|v\|$ for any u, v in V, then we say that $(V, \|\cdot\|)$ is a Banach algebra.*

1.3 Compactness

Several notions of compactness are available. Let us start with the following one.

Definition 1.12 (**Compactness**). *We say that a metric space (X, d) is compact if any open cover has a finite subcover, i.e. for every arbitrary collection $\{U_i\}_{i \in I}$ of open subsets of X such that $X \subset \cup_{i \in I} U_i$, there is a finite subset $J \subset I$ such that $X \subset \cup_{i \in J} U_i$.*

It is a good exercise to prove the following theorem in order to understand the power of the previous definition.

Theorem 1.2 (**Heine**). *Every continuous image of a compact set is compact. Moreover a continuous function on a compact set is uniformly compact.*

Definition 1.13 (**Limit point**). *Let S be a subset of a metric space X. We say that a point $x \in X$ is a limit point of S if every open set containing x also contains a point of S other than x itself. It is equivalent to requiring that every neighbourhood of x contains infinitely many points of S.*

Let us also define what we mean by a totally bounded space.

Definition 1.14 (**Totally boundedness**). *We say that a metric space (X, d) is totally bounded if for every $\varepsilon > 0$, there exists a finite cover of X by open balls of radius less than ε.*

Since for every $\varepsilon > 0$, $\cup_{x \in X} B(x, \varepsilon)$ is an open cover of X, it follows from Definitions 1.12 and 1.14 that a compact metric space is totally bounded.

Proposition 1.3. *A metric space is compact if and only if every sequence has a limit point.*

Next proposition gives another criterion of compactness for metric spaces.

Proposition 1.4. *A metric space is compact if and only if it is complete and totally bounded.*

1.4 Separability

Definition 1.15 (**Separability**). *We say that a metric space (X, d) is separable if it contains a countable dense subset, i.e., there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X such that every nonempty open subset of the space contains at least one element of the sequence.*

Any metric space which is itself finite or countable is separable. An important example of uncountable separable space is the real line (with its usual topology), in which the rational numbers form a countable dense subset.

Proposition 1.5. *Every compact metric space is separable.*

Proof. Let (X, d) be a compact metric space. For any $k \in \mathbb{N}^*$, $\cup_{x \in X} B(x, 1/k)$ is a open cover of X. By compactness, there exists $x_1^k, \ldots, x_{n_k}^k \in X$ such that $X = \cup_{j=1}^{n_k} B(x_{n_j}^k, 1/k)$. Then the collection $\bigcup_{k \in \mathbb{N}^*} \bigcup_{j=1}^{n_k} \{x_{n_j}^k\}$ is a countable dense subset of X. □

Let us finish with the following useful criterion for a metric space to be not separable.

Proposition 1.6. *If a metric space (X, d) contains a uncountable subset Y such that $\delta := \inf\{d(y, y') : \ y, y' \in Y, \ y \neq y'\} > 0$, then X is not separable.*

Proof. We argue by contradiction. Let us assume that (X, d) is separable and therefore contains a countable dense subset $(x_n)_{n\in\mathbb{N}}$. We can then define a map by associating to any $y \in Y$ the smallest $n \in \mathbb{N}$ such that $d(y, x_n) < \delta/3$. This map turns out to be injective because if $d(y, x_n) < \delta/3$ and $d(y', x_n) < \delta/3$, then $d(y, y') < 2\delta/3$ which is possible (when y and $y' \in Y$) only if $y = y'$. We deduce that Y is countable which is the absurd. □

2 Spaces of continuous functions

2.1 Basic definitions

Definition 2.1 (Bounded and continuous functions). *Let be given two metric spaces (X_1, d_1) and (X_2, d_2). We denote by*

$$\mathcal{B}(X_1; X_2) := \{f : X_1 \to X_2 : \ f(X_1) \textit{ is a bounded subset of } X_2\}, \tag{2}$$

$$\mathcal{C}_b(X_1; X_2) := \{f \in \mathcal{B}(X_1; X_2) \textit{ which are continuous}\}. \tag{3}$$

For any f_1 and $f_2 \in \mathcal{B}(X_1; X_2)$, we denote the uniform distance by

$$d_u(f_1, f_2) := \sup_{x\in X_1} d_2(f_1(x), f_2(x)). \tag{4}$$

Endowed with the distance d_u, $\mathcal{B}(X_1; X_2)$ is a metric space. When (X_1, d_1) is compact, a continuous mapping $f : X_1 \to X_2$ is bounded thanks to Heine's theorem (Theorem 1.2). In this case we simply denote $\mathcal{C}(X_1; X_2)$ instead of $\mathcal{C}_b(X_1; X_2)$.

Proposition 2.1. *The space $\mathcal{C}_b(X_1; X_2)$ is closed in $\mathcal{B}(X_1; X_2)$.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}_b(X_1; X_2)$ converging to f in $\mathcal{B}(X_1; X_2)$. Let us prove that f is continuous at x in X_1. By Proposition 1.2, it suffices to consider a sequence $(x_n)_{n\in\mathbb{N}}$ in X_1 converging to x and to prove that $f(x_n)$ converges to $f(x)$. Let $\varepsilon > 0$. There exists n_0 such that for any $n \geqslant n_0$, $d_u(f, f_n) < \varepsilon/3$. Since f_n is continuous there exists $\delta > 0$ such that for any y in X_1 with $d(x, y) < \delta$, then $d(f_n(x), f_n(y)) < \varepsilon/3$. For n large enough, $d(x_n, x) < \delta$, so that by the triangle inequality we get $d(f(x), f(x_n)) < \varepsilon$. Hence f is continuous at x. □

2.2 Completeness

Theorem 2.1. *Let (X_1, d_1) and (X_2, d_2) be two metric spaces, the latter being complete. Then $\mathcal{B}(X_1; X_2)$ and $\mathcal{C}_b(X_1; X_2)$ are complete.*

Proof. Since by Proposition 2.1 $\mathcal{C}_b(X_1; X_2)$ is closed in $\mathcal{B}(X_1; X_2)$ it suffices to prove that the latter is complete to prove the result. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}(X_1; X_2)$. It follows from the definition of d_u that for any x in X_1, the sequence $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in X_2. Since X_2 is complete the sequence $(f_n(x))_{n\in\mathbb{N}}$ has a limit that we call $f(x)$. It then remains to verify that it defines a function f in $\mathcal{B}(X_1; X_2)$ and that $(f_n)_{n\in\mathbb{N}}$ actually converges to f in $\mathcal{B}(X_1; X_2)$. Since $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}(X_1; X_2)$, there exists n_0 such that for any n, $n' \geqslant n_0$, $d_u(f_n, f_{n'}) < 1$. Passing to the limit $n \to +\infty$ yields $d_u(f, f_{n'}) < 1$. Since $f_{n'}$ is in $\mathcal{B}(X_1; X_2)$ there exists $x_2 \in X_2$ and $r > 0$ such that $f_{n'}(X_1) \subset B(x_2, r)$. Therefore $f(X_1) \subset B(x_2, r+1)$ and thus $f \in \mathcal{B}(X_1; X_2)$. To prove that $(f_n)_{n\in\mathbb{N}}$ converges to f in $\mathcal{B}(X_1; X_2)$ it is sufficient to pass to the limit in the Cauchy property. □

2.3 Compactness

The following result gives some sufficient conditions for a collection of continuous functions on a compact metric space to be relatively compact (*i.e.* whose closure is compact). In particular this could allow to extract an uniformly convergent subsequence from a sequence of continuous functions. The main condition is the equicontinuity which was introduced at around the same time by Ascoli (1883 – 1884) and Arzelà (1882 – 1883).

Theorem 2.2 (Ascoli). *Let (X_1, d_1) be a compact metric space and (X_2, d_2) be a complete metric space. Let $\mathcal{A}$ be a subset of $\mathcal{C}(X_1; X_2)$ such that*

1. *$\mathcal{A}$ is uniformly equicontinuous: for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_1(x, y) < \delta$, then $\sup_{f\in\mathcal{A}} d_2(f(x), f(y)) < \varepsilon$;*

2. *$\mathcal{A}$ is pointwise relatively compact, i.e., for all $x \in X_1$, the set $\overline{\{f(x) : f \in \mathcal{A}\}}$ is compact in X_2.*

Then $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{C}(X_1; X_2)$.

3 Continuous linear maps

3.1 Space of continuous linear maps

This section is devoted to $\mathcal{L}_c(X, Y)$ the space of the bounded linear operators between normed linear spaces X and Y. Let us recall that a linear operator T is bounded if one of the following assertion is satisfied:

1. T is bounded on every ball,

2. T is bounded on some ball,

3. T is continuous at every point,

4. T is continuous at some point.

5. T is uniformly continuous.

6. T is Lipschitz.

Theorem 3.1. *If X and Y are some normed linear spaces, then $\mathcal{L}_c(X, Y)$ is a normed linear space with the norm*

$$\begin{aligned}
\|T\|_{\mathcal{L}_c(X,Y)} &:= \sup_{x\neq x'\in X} \frac{\|Tx - Tx'\|_Y}{\|x - x'\|_X} \\
&= \sup_{x\in X\setminus 0} \frac{\|Tx\|_Y}{\|x\|_X}, \\
&= \sup_{\|x\|_X \leqslant 1} \|Tx\|_Y, \\
&= \sup_{\|x\|_X = 1} \|Tx\|_Y.
\end{aligned}$$

If moreover Y is a Banach space then $\mathcal{L}_c(X, Y)$ is a Banach space.

If $T \in \mathcal{L}_c(X,Y)$ and $U \in \mathcal{L}_c(Y,Z)$, then $UT = U \circ T \in \mathcal{L}_c(X,Z)$ and $\|UT\|_{\mathcal{L}_c(X,Z)} \leq \|U\|_{\mathcal{L}_c(Y,Z)}\|T\|_{\mathcal{L}_c(X,Y)}$. In particular, $\mathcal{L}_c(X) := \mathcal{L}_c(X,X)$ is a algebra, *i.e.*, it has an additional "multiplication" operation which makes it a non-commutative algebra, and this operation is continuous.

The dual space of X is $X' := \mathcal{L}_c(X,\mathbb{R})$ (or $\mathcal{L}_c(X,\mathbb{C})$ for complex vector spaces). According to the previous proposition it is a Banach space (whether X is or not).

Definition 3.1 (Weak and strong onvergences). *Given some normed linear spaces X and Y, and a sequence $(u_n)_n$ a sequence in $\mathcal{L}_c(X,Y)$, we say that*

1. *$(u_n)_n$ converges strongly to u in $\mathcal{L}_c(X,Y)$ if $\|u_n - u\|_{\mathcal{L}_c(X,Y)} \to 0$ when $n \to \infty$,*
2. *$(u_n)_n$ converges weakly* to u in $\mathcal{L}_c(X,Y)$ if for any $x \in X$, $(u_n(x))_n$ converges to $u(x)$ in Y.*

3.2 Uniform boundedness principle–Banach-Steinhaus theorem

Proposition 3.1. *Let X be a normed vector space and Y be a Banach space. Consider a dense subset A of X and $(u_n)_n$ a sequence in $\mathcal{L}_c(X,Y)$ such that*

1. $\sup_n \|u_n\|_{\mathcal{L}_c(X,Y)} < \infty$,
2. *for any x in A, $(u_n(x))_n$ converges.*

Then there exists a unique u in $\mathcal{L}_c(X,Y)$ such that $(u_n)_n$ converges weakly to u in $\mathcal{L}_c(X,Y)$. Moreover*

$$\|u\|_{\mathcal{L}_c(X,Y)} \leqslant \liminf_{n\to\infty} \|u_n\|_{\mathcal{L}_c(X,Y)}. \tag{5}$$

Theorem 3.2 (Uniform boundedness principle). *Let X be a Banach space and Y be a normed vector space. Suppose that B is a collection of continuous linear operators from X to Y. The uniform boundedness principle states that if for all x in X we have*

$$\sup_{u\in B} \|u(x)\|_Y < \infty. \tag{6}$$

Then

$$\sup_{u\in B} \|u\|_{\mathcal{L}_c(X,Y)} < \infty. \tag{7}$$

We infer from Theorem 3.2 the following corollary.

Corollary 3.1 (Banach-Steinhaus). *Let X be a Banach space and Y be a normed vector space. If $(u_n)_n$ is a sequence of $\mathcal{L}_c(X,Y)$ which converges weakly* to u, then u is in $\mathcal{L}_c(X,Y)$ and $\|u\|_{\mathcal{L}_c(X,Y)} \leqslant \liminf_{n\to\infty} \|u_n\|_{\mathcal{L}_c(X,Y)}$.*

In the particular case where Y is $\mathbb{R}$ or $\mathbb{C}$, we denote by $X' := \mathcal{L}_c(X,Y)$, and we call X' the *dual space* of X. Applying the uniform boundedness principle yields the following result:

Corollary 3.2. *Let X be a Banach space. Then any weakly* converging sequence in X' is bounded.*

One advantage of the weak* convergence is that the following partial converse is available.

Theorem 3.3. *Let X be a separable Banach space. Then any bounded sequence in X' admits a weakly* converging subsequence.*

Proof. Let $(u_n)_n$ be a bounded sequence of X'. Let $(x_k)_{k\in\mathbb{N}}$ a dense sequence in X. Using Cantor's diagonal argument there exists a subsequence $(u_{n_j})_j$ of $(u_n)_n$ such that $(u_{n_j}(x_k))_j$ converges for any $k \in \mathbb{N}$. We then apply Proposition 3.1. □

4 Hilbert analysis

A Hilbert space, named after David Hilbert, is a vector space possessing the structure of an inner product which is complete for the norm associated with its inner product. It generalizes the notion of Euclidean space. In particular the Pythagorean theorem and parallelogram law hold true in a Hilbert space.

4.1 Inner product space

In the real case an inner product on a vector space is a positive definite, symmetric bilinear form from $X \times X$ to $\mathbb{R}$. In the complex case it is positive definite, Hermitian symmetric, sesquilinear form from $X \times X$ to $\mathbb{C}$.

Definition 4.1 (Inner product). *Let X be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We say that a map $(\cdot, \cdot)$ from $X \times X$ to K is a inner product if*

1. $\forall u,\ v,\ w \in X$, $\forall \alpha, \beta$ *in* K, $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$,
2. $\forall u,\ v \in X$, $(u, v) = \overline{(v, u)}$,
3. $\forall u \in X \setminus \{0\}$, $(u, u) > 0$.

Endowed with $(\cdot, \cdot)$, X is a inner product space (or pre-Hilbert space).

The following lemma is quite famous.

Lemma 4.1 (Cauchy-Schwarz inequality). *Let X be a pre-Hilbert space. Then*

$$\forall u, v \in X,\ |(u, v)| \leqslant \sqrt{(u, u)}\sqrt{(v, v)}. \tag{8}$$

Proof. Let θ be a real number such that $(u, v) = |(u, v)|e^{i\theta}$. Let us define $\tilde{u} := e^{-i\theta}u$. Let $P(\lambda) := (\tilde{u} + \lambda v, \tilde{u} + \lambda v)$ for every $\lambda \in \mathbb{R}$. Since $P \geqslant 0$, the discriminant of the quadratic equation $P(\lambda) = 0$ is nonpositive. Since

$$P(\lambda) = (\tilde{u}, \tilde{u}) + 2\lambda\Re(\tilde{u}, v) + \lambda^2(v, v) = (u, u) + 2\lambda|(u, v)| + \lambda^2(v, v),$$

this yields the Cauchy-Schwarz inequality. □

Next lemma states that an inner product gives rise to a norm. An inner product space is thus a special case of a normed linear space.

Lemma 4.2. *Let X be a pre-Hilbert space. Then the map $\|\cdot\| : u \in X \mapsto \sqrt{(u, u)}$ defined a norm on X.*

Proof. The main point is to prove the triangle inequality, what can be done thanks to the Cauchy-Schwarz inequality: for any $u, v \in X$, we have

$$\begin{aligned} \|u + v\|^2 &= \|u\|^2 + 2\Re(u, v) + \|v\|^2, \\ &\leqslant \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2. \end{aligned} \tag{9}$$

□

Conversely the following polarization identities expresses the norm of an inner product space in terms of the inner product.

Proposition 4.1. *For real (respectively complex) inner product spaces we have for any $u, v \in X$,*

$$(u,v) = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2), \text{ resp. } (u,v) = \frac{1}{4}(\|u+v\|^2 + i\|u+iv\|^2 - \|u-v\|^2 - i\|u-iv\|^2).$$

Proof. It suffices to apply (9) to v, $-v$, iv and $-iv$ and to combine the resulting equalities. □

In inner product spaces we also have (straightforwardly) the following parallelogram law.

Proposition 4.2. *Let X be a real or complex inner product space, then for any $u, v \in X$,*

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

This gives a criterion for a normed space to be an inner product space. Any norm coming from an inner product satisfies the parallelogram law and, conversely, if a norm satisfies the parallelogram law, we can show (but not so easily) that the polarization identity defines an inner product, which gives rise to the norm. This is a result by von Neumann.

Now, let see a few general properties satisfied by an inner product space.

Lemma 4.3. *Let X be a pre-Hilbert space. In addition the scalar product is a continuous bilinear mapping from $X \times X$ to $\mathbb{C}$.*

Definition 4.2 (Orthonormal sequence). *A family $(e_i)_{i\in I}$ in X is said an orthonormal sequence if for any $i, j \in I$, $(e_i, e_j) = \delta_{i,j}$.*

Let us continue with the following Pythagore equality.

Lemma 4.4. *Let X be a inner product space and $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in X and $(\alpha_n)_{n\in\mathbb{N}}$ some complex numbers. Then, for any $n, m \in \mathbb{N}$ with $n > m$,*

$$\left\|\sum_{k=m}^{n} \alpha_k e_k\right\|^2 = \sum_{k=m}^{n} |\alpha_k|^2.$$

The proof, quite easy, is left to the reader. We now give the following result.

Lemma 4.5. *Let X be a inner product space and $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in X. Then, for any $u \in X$,*

1. *for any $k \in \mathbb{N}$,*

$$\|u\|^2 = \sum_{n=0}^{k} |(u, e_n)|^2 + \|u - \sum_{n=0}^{k}(u, e_n)e_n\|^2,$$

2. **Bessel's inequality:**

$$\sum_{n=0}^{\infty} |(u, e_n)|^2 \leqslant \|u\|^2.$$

Proof. We have, using Lemma 4.4,

$$\begin{aligned}
\|u - \sum_{n=0}^{k}(u, e_n)e_n\|^2 &= (u - \sum_{n=0}^{k}(u, e_n)e_n, u - \sum_{n=0}^{k}(u, e_n)e_n) \\
&= \|u\|^2 - 2\sum_{n=0}^{k} |(u, e_n)|^2 + \sum_{0\leqslant n\leqslant k} |(u, e_n)|^2 \\
&= \|u\|^2 - \sum_{n=0}^{k} |(u, e_n)|^2,
\end{aligned}$$

what yields (1). To obtain (2) it suffices to pass to the limit in the inequality $\sum_{n=0}^{k} |(u, e_n)|^2 \leqslant \|u\|^2$. ☐

Remark 4.1. Note that in general the sequence $u - \sum_{n=0}^{k}(u, e_n)e_n$ does not converge to 0 when n goes to infinity. We will go back to this issue in the next section with two more assumptions.

Remark 4.2. Let $\mathcal{E}$ be an orthonormal set of arbitrary cardinality. It follows from Bessel's inequality that for $\epsilon > 0$ and $u \in X$, $\{ e \in \mathcal{E} : (u, e) \geq \epsilon \}$ is finite, and hence that $\{ e \in \mathcal{E} : (u, e) > 0 \}$ is countable. We can thus extend Bessel's inequality to an arbitrary orthonormal set: $\sum_{e\in\mathcal{E}}(u, e)^2 \leqslant \|u\|^2$, where the sum is just a countable sum of positive terms.

4.2 Hilbert spaces

Definition 4.3 (Hilbert space)**.** *A Hilbert space is an inner product space which is complete.*

A Hilbert space is therefore a special case of a Banach space.

Theorem 4.1. *Let X be a Hilbert space and $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in X. The series $\sum_{n\in\mathbb{N}} \alpha_n e_n$ converges in X if and only if the sequence $(\alpha_n)_{n\in\mathbb{N}}$ belongs to $\ell^2(\mathbb{N})$. Moreover when the series $\sum_{n\in\mathbb{N}} \alpha_n e_n$ converges in X, then*

$$\left\| \sum_{n\in\mathbb{N}} \alpha_n e_n \right\|^2 = \sum_{\in\mathbb{N}} |\alpha_n|^2. \tag{10}$$

Proof. Since the space X is complete, the series $\sum_{n\in\mathbb{N}} \alpha_n e_n$ converges in X if and only if it satisfies the Cauchy property. Therefore using Lemma 4.4, we obtain that $\sum_{n\in\mathbb{N}} \alpha_n e_n$ satisfies the Cauchy property in X if and only if the series $\sum_{k=0}^{n} |\alpha_k|^2$ satisfies the Cauchy property in $\mathbb{R}$. Since $\mathbb{R}$ is also complete, this yields the first part of the result.

To prove (10) it is sufficient to use the continuity of the norm and the Pythagore equality:

$$\left\| \sum_{n\in\mathbb{N}} \alpha_n e_n \right\|^2 = \left\| \lim_{k\to\infty} \sum_{n=0}^{k} \alpha_n e_n \right\|^2 = \lim_{k\to\infty} \left\| \sum_{n=0}^{k} \alpha_n e_n \right\|^2 = \lim_{k\to\infty} \sum_{n=0}^{k} |\alpha_n|^2 = \sum_{n\in\mathbb{N}} |\alpha_n|^2.$$

☐

Combining Lemma 4.5 (2) and Theorem 4.1 we obtain the following result.

Corollary 4.1. *Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space X, and let $u \in X$. Then the series $\sum_{n\in\mathbb{N}}(u, e_n)e_n$ converges in X.*

Given an orthonormal sequence $(e_n)_{n\in\mathbb{N}}$ in X, we define the linear mapping

$$\Phi : u \in X \mapsto ((u, e_n))_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}).$$

Notice that the range of Φ is contained in $\ell^2(\mathbb{N})$ according to (1). Combining Lemma 4.3 and Corollary 4.1 we get that if $u \in X$ satisfies $u = \sum_{n\in\mathbb{N}} \alpha_n e_n$, then $\Phi(u) = (\alpha_n)_{n\in\mathbb{N}}$.

Definition 4.4 (Hilbert basis). *An orthonormal sequence $(e_n)_{n\in\mathbb{N}}$ in X is said total or a Hilbert basis if Φ is injective, i.e., if $(u, e_n) = 0$ for every $n \in \mathbb{N}$ implies that $u = 0$.*

Let us observe that it follows from the continuity of the scalar product that for every $u \in X$, $\Phi(u) = \Phi(\sum_{n\in\mathbb{N}}(u, e_n)e_n)$.

Theorem 4.2. *Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal family in an Hilbert space X. Then the following statements are equivalent:*

(i) The family $(e_n)_{n\in\mathbb{N}}$ is total;

(ii) For every $u \in X$, $u = \sum_{n\in\mathbb{N}}(u, e_n)e_n$;

(iii) For every $u \in X$, $\|u\|^2 = \sum_{n\in\mathbb{N}} |(u, e_n)|^2$.

Proof. Let us first assume (i). Since Φ is injective by assumption, (ii) follows. Let us now assume that (ii) holds. Then, using Pythagore's equality, we obtain that for every $u \in X$,

$$\|u\|^2 = \left\|\lim_{k\to\infty}\sum_{n=0}^{k}(u, e_n)e_n\right\|^2 = \lim_{k\to\infty}\left\|\sum_{n=0}^{k}(u, e_n)e_n\right\|^2 = \lim_{k\to\infty}\sum_{n=0}^{k}|(u, e_n)|^2 = \sum_{n\in\mathbb{N}}|(u, e_n)|^2.$$

Finally we assume that (iii) holds. Then, if $u \in X$ is such that $(u, e_n) = 0$ for every $n \in \mathbb{N}$ then clearly $u = 0$. ☐

As a consequence of Zorn's lemma, every Hilbert space admits an orthonormal basis; furthermore, any two orthonormal bases of the same space have the same cardinality, called the Hilbert dimension of the space.

4.3 Projection on a closed convex set

An essential property of Hilbert space is that the distance of a point to a closed convex set is alway attained.

Theorem 4.3. *Let X be a Hilbert space, K a closed convex subset, and $u \in X$. Then there exists a unique $\bar{u} \in K$ such that*

$$\|u - \bar{u}\| = \inf_{v\in K}\|u - v\|.$$

Moreover $\bar{u}$ is the unique element of K which satisfies $\Re(u - \bar{u}, v - \bar{u}) \leqslant 0$ for any $v \in K$.

Proof. Translating, we may assume that $u = 0$, and so we must show that there is a unique element of K of minimal norm. Let $d = \inf_{v\in K}\|v\|$ and chose $u_n \in K$ with $\|u_n\| \to d$. Then the parallelogram law gives

$$\left\|\frac{u_n - u_m}{2}\right\|^2 = \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\|u_m\|^2 - \left\|\frac{u_n + u_m}{2}\right\|^2 \leqslant \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\|u_m\|^2 - d^2,$$

where we have used convexity to infer that $(u_n + u_m)/2 \in K$. Thus (u_n) is a Cauchy sequence and so has a limit $\bar{u}$, which must belong to K, since K is closed. Since the norm is continuous, $\|\bar{u}\| = \lim_n \|u_n\| = d$.

For uniqueness, note that if $\|\bar{u}\| = \|\tilde{u}\| = d$, then $\|(\bar{u}+\tilde{u})/2\| = d$ and the parallelogram law gives

$$\|\bar{u} - \tilde{u}\|^2 = 2\|\bar{u}\|^2 + 2\|\tilde{u}\|^2 - \|\bar{u} + \tilde{u}\|^2 = 2d^2 + 2d^2 - 4d^2 = 0.$$

Let now prove the characterization of $\bar{u}$ through obtuse angles. Let v be in K, $\lambda \in (0, 1)$ and let $z := (1 - \lambda)\bar{u} + \lambda v$ which is in K by convexity. Therefore

$$\|u - \bar{u}\|^2 \leqslant \|u - z\|^2 = \|(u - \bar{u}) - \lambda(v - \bar{u})\|^2 = \|u - \bar{u}\|^2 + 2\lambda\Re(u - \bar{u}, \bar{u} - v) + \lambda^2\|v - \bar{u}\|^2.$$

Thus,

$$2\Re(u-\bar{u},v-\bar{u})-\lambda\|v-\bar{u}\|^2 \leqslant 0$$

and then, by letting λ tend to 0^+, we obtain that $\Re(u-\bar{u},v-\bar{u}) \leqslant 0$ for any $v \in K$.

Conversely if $\bar{u}$ is an element of K which satisfies $\Re(u-\bar{u},v-\bar{u}) \geqslant 0$ for any $v \in K$, then we have

$$\|(1-\lambda)\bar{u}+\lambda v-u\|^2 \geqslant \|u-\bar{u}\|^2+\lambda^2\|v-\bar{u}\|^2.$$

Letting λ goes to 1 yields $\|v-u\|^2 \geqslant \|v-\bar{u}\|^2$. □

The unique nearest element to u in K is often denoted $P_K u$, and referred to as the projection of u onto K. It satisfies $P_K \circ P_K = P_K$, the definition of a projection. This terminology is especially used when K is a closed linear subspace of X, in which case P_K is a linear projection operator.

Theorem 4.4. *Let X be a Hilbert space, Y a closed subspace, and $x \in X$. Then there exists a continuous linear mapping P_Y from X onto Y with $\|P_Y\| \leqslant 1$ such that for any $v \in Y$,*

$$\|u-P_Y u\| = \inf_{v\in Y}\|u-v\|.$$

Moreover $P_Y u$ is the unique element of Y which satisfies $(u-P_Y u,v)=0$ for any $v \in Y$.

We say that P_Y is the orthogonal projection onto Y.

Proof. The existence of P_Y is given by the previous theorem. We now prove the characterization of $P_Y u$ as the unique element of Y which satisfies $(u-\bar{u},v)=0$ for any $v \in Y$. Using the characterization of the previous theorem with $v+P_Y u$ instead of v, we have that $P_Y u$ satisfies $\Re(u-P_Y u,v) \leqslant 0$ for any $v \in Y$. Using this last inequality with $-v$, iv and $-iv$ instead of v yields $(u-P_Y u,v)=0$ for any $v \in Y$. The converse is straightforward: if an element $\bar{u}$ in Y satisfies $(u-\bar{u},v)=0$ for any $v \in Y$ then it satisfies the characterization of the previous theorem so it is the projection of u onto Y.

From this characterization we infer that P_Y is linear. Now to prove that P_Y is continuous with $\|P_Y\| \leqslant 1$ it suffices to apply the Cauchy-Schwarz inequality to the characterization with $v = P_Y u$. □

Definition 4.5 (Orthogonal space). *If S is any subset of a inner product space X, let*

$$S^\perp = \{\, u \in X : (u,s)=0 \text{ for all } s \in S \,\}.$$

Then we have the following.

Lemma 4.6. *We have*

1. *$S^\perp$ is a closed subspace of X,*
2. *$S \cap S^\perp = 0$,*
3. *$S \subset S^{\perp\perp}$,*
4. *if $S_1 \subset S_2$ then $S_2^\perp \subset S_1^\perp$.*

Proof. The first point follows from the continuity of the inner product. We have that $S \cap S^\perp = \{0\}$ since $u \in S \cap S^\perp$ implies $\|u\|^2 = (u,u) = 0$. The third point is a consequence of the (hermitian) symmetry of an inner product and the fourth is straightforward. □

Now if we assume that the space is complete we get the following.

Lemma 4.7. *If X is a Hilbert space and S is a closed subspace of X, then $X = S \bigoplus S^{\perp}$.*

Proof. In addition for any u in X, $u = P_S u + (u - P_S u)$ provides a decomposition in $S \bigoplus S^{\perp}$, according to the previous theorem. $\square$

Corollary 4.2. *If X is a Hilbert space and S is a subspace of X, then $\overline{S} = X$ if and only if $S^{\perp} = \{0\}$.*

Proof. Suppose that $\overline{S} = X$ and let u be in $S^{\perp}$. Then there exists $(u_n)_n$ in S converging to u. For any n, we have $(u_n, u) = 0$, and since the scalar product is continuous, passing to the limit yields $\|u\|^2 = (u, u) = 0$.

Conversely if we assume now that $S^{\perp} = \{0\}$ then from the previous lemma applied to $\overline{S}$ we infer that $X = \overline{S} \bigoplus \overline{S}^{\perp}$. But $\overline{S}^{\perp} = \{0\}$ so that $\overline{S} = X$. $\square$

4.4 Duality and weak convergence

The identification of the dual space of Hilbert spaces is easy.

Theorem 4.5 (**Riesz Representation Theorem**). *If X is a real Hilbert space, define $j : X \to X'$ by $j_y(x) = (x, y)$. This map is a linear isometry of X onto X'. For a complex Hilbert space it is a antilinear (or conjugate-linear) isometry (it satisfies $j_{\alpha y} = \bar{\alpha} j_y$ for any $\alpha \in \mathbb{C}$).*

Proof. It is easy to infer from the Cauchy-Schwarz inequality that j is an isometry of X into X'. Therefore the main issue is to show that any $f \in X'$ can be written as j_y for some y. We may assume that $f \neq 0$, so $\ker(f)$ is a proper closed subspace of X. Let $y_0 \in [\ker(f)]^{\perp}$ be of norm 1 and set $y = \overline{f(y_0)} y_0$. For all $x \in X$, we clearly have that $f(y_0)x - f(x)y_0 \in \ker(f)$, so

$$j_y(x) = (x, y) = (x, \overline{f(y_0)} y_0) = (f(y_0)x, y_0) = (f(x)y_0, y_0) = f(x).$$

$\square$

Via the map j we can define an inner product on X', so it is again a Hilbert space. The Riesz map j actually identifies X and X'.

Proposition 4.3. *If X is a real Hilbert space, then its dual space X' is also a Hilbert space. Moreover a sequence $(u_n)_n$ in X weakly converges to u if and only for any v in X, $(u_n, v) \to (u, v)$.*

Then as a consequence of Corollary 3.1 (respectively Theorem 3.3), we have the following results:

Proposition 4.4. *Let X be a Hilbert space and $(u_n)_n$ be a weakly converging sequence in X. Then $(u_n)_n$ is bounded.*

Theorem 4.6. *Let X be a Hilbert space and $(u_n)_n$ be a bounded sequence in X. Then there exists a subsequence $(u_{n_k})_k$ which weakly converges to some u in X.*

Proof. Let us introduce $Y := \overline{\text{Vect } (u_n)_{n \in \mathbb{N}}}$ which is, for the topology induced by X, a separable Hilbert space. Therefore since $(u_n)_n$ is a bounded sequence in Y, there exists a subsequence

$(u_{n_k})_k$ which weakly converges to u in Y. Let us now consider the orthogonal projection P on Y. We have, for any v in X,

$$\begin{array}{rcl} (u_n, v) & = & (u_n, Pv) + (u_n, (Id - P)v) \\ & = & (u_n, Pv) \quad \text{since } (Id - P)v \in Y^{\perp} \\ & \to & (u, Pv) \quad \text{when } n \to +\infty \\ & & = (u, v). \end{array}$$

□