GDT "GALOIS THEORY FOR SCHEMES"
CHAPTER 1
GALOIS THEORY OF FIELDS

Exercise 1 [Ex. 1.9, [L]] Let $S = \varprojlim S_i$, an inverse limit of sets.

a) Suppose that all sets S_i are endowed with a compact Hausdorff topology, that all S_i are non-empty, and that all maps f_{ij} are continuous. Prove that S is non-empty and compact.
b) Suppose that all sets S_i are finite and non-empty. Prove that $S \neq 0$.
c) Suppose that I is countable, that all S_i are non-empty, and that all maps f_{ij} are surjective. Prove that $S \neq 0$.
d) Let I be a collection of all finite subsets of $\mathbb{R}$, and let I be partially ordered by inclusion. For each $i \in I$, let S_i be the set of injective maps $\psi : i \longrightarrow \mathbb{Z}$, and let $f_{ij} : S_i \longrightarrow S_j$ (for $j \subset i$) map ψ to its restriction $\psi|j$. Prove that this defines a projective system in which all S_i are non-empty and all f_{ij} are surjective, but that the projective limit S is empty.

Exercise 2 [Ex. 1.2, [T]] Let G be a profinite group, p a prime number. A pro-p-Sylow subgroup of G is a pro-p-group whose image in each finite quotient of G is of index prime to p. Show that pro-p-Sylow subgroups exist and they are conjugate in G. [Hint: Apply the previous exercise to the inverse system formed by the sets of p-Sylow subgroups in each finite quotient of G.]

Exercise 3 [Ex. 1.14, [L]] Prove that there is an isomorphism $\widehat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$ of topological rings, where $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

Exercise 4 [Ex. 1.5, [T]] Let G be a profinite group acting via field automorphisms on a field K. Assume that the action is continuous when K carries the discrete topology and that each non-trivial element in G acts non-trivially on K. Show that $G = \mathrm{Gal}(K|k)$, where $k = K^G$.

Exercise 5 [Ex. 1.7, [T]] Let k be a field, and A a finite étale k-algebra equipped with an action of a finite group G via k-algebra automorphisms; we call such algebras G- algebras. We moreover say that A is Galois with group G if $\dim_k(A)$ equals the order of G and $A^G = k$.

a) Consider the G-algebra structure on $A \otimes_k \bar{k}$ given by $g(a \otimes \alpha) = g(a) \otimes \alpha$. Prove that A is Galois with group G if and only if $A \otimes_k \bar{k}$ is isomorphic to the group algebra $\bar{k}[G]$ as a G-algebra.
b) Making G act on $\mathrm{Hom}_k(A, k_s)$ via $\psi \longrightarrow \psi \circ g$, show that in the correspondence of Theorem 1.5.4 Galois algebras with group G correspond to finite continuous $\mathrm{Gal}(k)$-sets with simply transitive G-action.