Some remarks on the derivation of the Sverdrup relation.

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Abstract

We are interested in the derivation of the Sverdrup relation by an asymptotic analysis on the one layer stationary quasigeostrophic equation arising in geophysics. This last equation being of order 4, we have to choose two boundary conditions to close the model. We study the influence of the choice of the second boundary condition on this derivation. The boundary conditions that are considered are that given by J. Pedlosky in [14].

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1 Introduction

A simplified nondimensional stationary model describing the homogeneous oceanic circulation is given in a domain $\Omega \subset \mathbb{R}^2$ by

$$E\,\Delta^2\Psi - \mu\Delta\Psi - \varepsilon\,\mathbf{J}(\Psi,\Delta\Psi) - \partial_x\Psi = f \tag{1}$$

with $\mathbf{J}$ the jacobian defined by

$$\mathbf{J}(\Psi, \Delta\Psi) = \nabla^{\perp}\Psi \cdot \nabla\Delta\Psi,$$

and

$$\varepsilon = (\frac{\delta_I}{L})^2, \quad \mu = \frac{\delta_S}{L}, \quad E = (\frac{\delta_M}{L})^3.$$

Function Ψ denotes the stream function so that the horizontal velocity field is given by $(-\partial_y\Psi, \partial_x\Psi)$.
This equation is called the quasigeostrophic equation. The dimensionless constants ε, μ, E are related to the relative importance of the nonlinearity, of the bottom friction and of the lateral diffusion. The constants L, δ_I, δ_S and δ_M represent respectively the different motion scales (which can be different from the lake scale that can be considered: the inertia scale, the Stommel scale and the Munk one). The exterior force f is given by $f = -k \cdot \mathrm{curl}\tau$ where $k = (0, 0, 1)$ and $\tau = (\tau_x, \tau_y, \tau_z)$ is the Cauchy stress due to the wind.

For a formal derivation of this model from the Navier–Stokes equations, we refer the reader to [13], [7] for example. This model represents the vertical mean value of the circulation without stratification (homogeneous fluid) and under the β-plan hypothesis.

We have to choose for equation (1) boundary conditions. On the boundary of the basin, the normal velocity has to be prescribed. If the basin is closed and if we consider that there is no water which going out, the flux correponding to the horizontal velocity v has to vanish on the boundary. Therefore, we impose

$$\overline{v} \cdot n = \nabla^{\perp}\Psi \cdot n = 0$$

where n is the unit boundary normal vector, which is equivalent to write that Ψ is constant on each connected boundary component. If we assume that there is no islands, we may choose

$$\Psi = 0 \text{ on } \partial\Omega. \tag{2}$$

The presence of the diffusion term $E\Delta^2\Psi$ implies to choose another boundary condition. J. Pedlosky proposed in [14] to consider one of the following three conditions on $\partial\Omega$:

$$\nabla\Psi \cdot n = 0, \quad \Delta\Psi = 0, \quad \nabla(\Delta\Psi) \cdot n + \varepsilon^{-1} j \cdot n = 0,$$

where j is the unit vector field correponding to the y axis. The first condition corresponds to noslip boundary condition, the second corresponds to a condition of a vanishing tangential stress, the last one corresponds to a relation beetween the eddy flux of the total vorticity and the large scale vorticity field, see [14] page 34. What condition on the boundary due to the diffusion term, do we have to choose ? It the dissipation is small, which means $E \ll 1$, do this choice has some influence on the global circulation character ? This choice is not so clear as it is pointed out by J. Pedlosky in [14] page 35.

Therefore, in a first step, we have to study the influence of the boundary conditions on the convergence. In [13], It is noted that if E, ε, μ are all small coefficients, the equilibrium equation (1) is reduced, in all these cases, to

$$-\partial_x \Psi = f$$

which correponds to the Sverdrup relation proposed in oceanography in 1947 by Sverdrup. In a first step we will justify this convergence by a method exposed in [11] and used by [7] but for mixed boundary conditions of the type

$$\Delta\Psi = 0 \text{ on } \Gamma_1, \quad \nabla\Psi \cdot n = 0 \text{ on } \Gamma_2 \tag{3}$$

with $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$ and Ω given figure 1 where g_{East} and g_{West} are graphs of class $\mathcal{C}^2$ on $(0, 1)$ correponding to the East coast and the West coast. These conditions are usually used, see for example [9] pages 2183, 2184. In a second step, we will study the case

$$\nabla(\Delta\Psi) \cdot n = 0 \text{ on } \partial\Omega \tag{4}$$

in a domain of the form $\Omega = S^1 \times (0, 1)$.

The study for small parameters on the stationary equation (1) has been realized for example by [1] with ε and μ fixed and E which tends to 0 with the supplementary boundary condition $\Delta\Psi = 0$ on $\partial\Omega$ and $E \leq \mu^3/8$. The Sverdrup relation derivation from (1) has been justified in [7] with the supplementary boundary condition $\nabla\Psi = 0$ on $\partial\Omega$ and assuming that μ has the same order than ε and assuming a ratio beetween E and ε. The study in [7] used strongly that $\nabla\Psi \cdot n = 0$ on $\partial\Omega$ and the fact that μ is of order ε. We will prove (Theorem 1) that μ can tend to 0 without ratio hypothesis between ε and E with eventually $\mu = 0$. For this we will give more precise estimates than in [7].

By this way, we will answer to a question proposed by J.–L. Lions page 403 [11] which is the study of (1) with $\mu = 0$ for ε, $E \to 0$ and the boundary conditions $\Delta\Psi = 0$ on Γ_{South}, Γ_{North} and $\nabla\Psi \cdot n = 0$ on Γ_{East}, Γ_{West} where Ω is defined as below.

Figure 1: The domain.

We can find this kind of equation and boundary conditions in [4], [5], [6] where a physical analysis is performed and in [2], [3] et [13] page 310–311 where some numerical simulations are given in a square domain.

The method used in [11] for the equation $\varepsilon\,\Delta^2\Psi - \partial_x\Psi = f$ has been used by [7] for the equation (1) with the supplementary boundary condition $\nabla\Psi \cdot n = 0$. The method in order to prove that we have convergence close to the boundary where $n(x) > 0$ seems to not work if we consider $\Delta\Psi = 0$ on this part of the boundary because of the added term coming from the integration by part that we have to control. Thus, we will, in a second part (part (ii) theorem 1), change the proof of [7] in order to be more general for a domain as in figure 1 which represents the North Atlantic. We will assume for this f more regular than $f \in L^2(\Omega)$. This is the case in all the examples considered in Oceanography. We refer the interested reader to [13] page 31 and page 65 where $\Omega =]0, L_1[^2$ with $f = -W_0 \sin(\pi\, y/L_1)$ and $f = -W_0 \sin(\pi x/L_1)\, \sin(\pi\, y/L_1)$ is considered. Thus we will prove a direct convergence with a better range of coefficients than in [7].

We will prove (Part (iii) Theorem 1) the convergence for $\Delta\Psi = 0$ on $\partial\Omega$ without the hypothesis $E \le \mu^3/8$ given in [1]. The nonlinear term will be dominated by the friction coefficient and the lateral diffusion term will be independent on μ and ε.

We will finish by Theorem 2 where we will prove that, with the supplementary condition (4) and in a domain $\Omega = S \times (0,1)$, we can obtain better result.

The first result is the following

Theorem 1 *Let Ω as in figure 1 with g_{East} and g_{West} of class $\mathcal{C}^2$.*

(i) *Let μ, ε and E go to 0 with $\varepsilon \ll E$ and $f \in L^2(\Omega)$. Let us consider $(\Psi)_{\varepsilon,\mu,E}$ solutions in $H^3(\Omega)$ of (1)–(3). Then there exists a subsequence of $(\Psi)_{\varepsilon,\mu,E}$, still denoted by $(\Psi)_{\varepsilon,\mu,E}$, which converges to Ψ_0 in $L^2(\Omega)$ weak with Ψ_0 satisfying, in the distribution sense, the equation*

$$-\partial_x \Psi_0 = f. \tag{5}$$

Moreover, we have

$$\int_\Omega |\Psi|^2 + E\int_\Omega |\Delta\Psi|^2 + \mu \int_\Omega |\nabla\Psi|^2 \le C \tag{6}$$

where C is a constant independent of μ, E and ε.

(ii) *Let μ, ε and E go to 0 with $\varepsilon \ll E$ and $f \in H^2(\Omega)$. Thus $\eta\Psi$ converges to $\eta\psi_I$ in $L^2(\Omega)$ where Ψ_I is solution of*

$$-\partial_x \Psi_I = f, \quad \Psi_I = 0 \text{ on } \{x \in \partial\Omega : n_x > 0\} \tag{7}$$

for all $\eta \in \mathcal{C}^2(\overline{\Omega})$ with $\eta \equiv 0$ in a neighborhood of $\{n_x \le 0\}$ where n_x is the horizontal component of the normal to $\partial\Omega$.

(iii) *Let μ, ε and E converge to 0 and $f \in L^2(\Omega)$. Let $(\Psi)_{\varepsilon,\mu,E}$ solutions in $H^3(\Omega)$ of (1)–(3) with $\Gamma_2 = \emptyset$. There exists c_Ω such that if $\varepsilon \ll c_\Omega \mu^3$, there exists a subsequence of $(\Psi)_{\varepsilon,\mu,E}$, still denoted $(\Psi)_{\varepsilon,\mu,E}$, converging to Ψ_0 in $L^2(\Omega)$ weak with Ψ_0 solution of (5).* $\square$

Remark. Let us remark that, Point (i), Ψ_0 is not unique. This explain why we obtain only the convergence of a subsequence. On the contrarly, point (ii), since $a = (-1, 0)$ is a vector field with all characteristic lines crossing the boundary of the domain Ω, The solution Ψ_I of (7) is unique in $L^2(\Omega)$. $\square$

Remark. Point iii) is valid even if $E = 0$.

Finally, we will study the problem with the supplementary condition (4) in the linear case which means $\varepsilon = 0$ and we will give more precise estimates than before. More precisely, we will prove the following result which will give uniform estimates on the gradient of Ψ in weighted spaces.

Theorem 2 *Let $\Omega = (0,1) \times S^1$ and $\varepsilon = 0$. Let μ, E tend to 0 and $f \in H^1(\Omega)$. Let us assume $(\Psi)_{\varepsilon,\mu,E}$ solutions in $H^4(\Omega)$ of (1), (2) and (4).*

(i) *If $\int_0^1 f(\xi, y)\, d\xi = 0$, there exists c_Ω such that if $E \ll c_\Omega \mu^3$ then*

$$\Psi_{\mu,E} \rightharpoonup \Psi_1 \text{ in } H^1(\Omega) \text{ weak} \tag{8}$$

where Ψ_1 is given by $\Psi_1 = \int_0^x f(\xi, y)\, d\xi$.

(ii) *If $\int_0^1 f(\xi, y)\, d\xi \neq 0$, there exists c_Ω such that if $E \ll c_\Omega \mu^{7/2}$ then*

$$\begin{aligned} \Psi_{\mu,E} &\rightharpoonup \Psi_2 \text{ in } L^2(\Omega) \text{ weak} \\ x^{1/2}\partial_x(\Psi_{\mu,E}) &\rightharpoonup x^{1/2}\partial_x \Psi_2 \text{ in } L^2(\Omega) \text{ weak} \end{aligned} \tag{9}$$

where Ψ_2 is solution of

$$\partial_x \Psi_2 = f \text{ in } \Omega, \quad \Psi_2 = 0 \text{ on } \{1\} \times S^1. \ \square \tag{10}$$

Remark. The convergence for $\mu = 0$ and $E \to 0$ with ε fixed remains an open problem. The convergences for μ fixed different from 0 and E converging to 0 for $\nabla\Psi \cdot n = 0$ is also an open problem. □

Remark. Theorem 2 show the eventual existence of a boundary layer corresponding to the western intensification of currents as for example the Gulf Stream in the North Atlantic, the Kuroshio in the Pacific Ocean, the alguhas currents in the North East off Africa. □

Remark. We should take into account the effect due to the bathymetry with an added term denoted η_B in the equation (1) as in [7]. This term has no influence in the mathematical analysis and therefore we omit it here. The interested reader is referred to [13] and [7]. □

We give here some results on an homogeneous model coming from oceanography in the stationary case. We refer the reader interested by an asymptotic analysis in the nonstationary case in the same area to [8], [10] and references cited therein.

2 Proof of Theorem 1

Part (i)**:**

We prove as in [7] by mulptiplying (1) by Ψ that

$$E\int_\Omega |\Delta\Psi|^2 + \mu\int_\Omega |\nabla\Psi|^2 \leq \|f\|_{L^2(\Omega)}\|\Psi\|_{L^2(\Omega)}. \tag{11}$$

It remains to obtain an uniform estimate on $\|\Psi\|_{L^2(\Omega)}$ in order to deduce the inequality (6). We have the following estimates :

$$\begin{aligned}
-\int_\Omega \partial_x\Psi \mathrm{e}^x\Psi &= \frac{1}{2}\int_\Omega \Psi^2\mathrm{e}^x\\
-\int_\Omega \Delta\Psi \mathrm{e}^x\Psi &= \int_\Omega |\nabla\Psi|^2\mathrm{e}^x - \frac{1}{2}\int_\Omega \Psi^2\mathrm{e}^x\\
\int_\Omega \Delta^2\Psi \mathrm{e}^x\Psi &= \int_\Omega |\Delta\Psi|^2\mathrm{e}^x + 2\int_\Omega \Delta\Psi\partial_x\Psi\mathrm{e}^x + \int_\Omega \Delta\Psi\mathrm{e}^x\Psi\\
\int_\Omega \mathbf{J}(\Psi,\Delta\Psi)\Psi\mathrm{e}^x &= \int_\Omega \partial_y\Psi\Delta\Psi\Psi\mathrm{e}^x.
\end{aligned}$$

Thus let us multiply (1) by the test function $\mathrm{e}^x\Psi$ as in [7] in order to obtain with the help of the equalities above

$$\begin{aligned}
&\frac{1-cE-\mu}{2}\int_\Omega \Psi^2\mathrm{e}^x + \frac{E}{2}\int_\Omega |\Delta\Psi|^2\mathrm{e}^x + \mu\int_\Omega |\nabla\Psi|^2\mathrm{e}^x\\
&\qquad\leq \Big|\varepsilon\int_\Omega \partial_y\Psi\Delta\Psi\Psi\mathrm{e}^x - 2E\int_\Omega \Delta\Psi\partial_x\Psi\mathrm{e}^x + \int_\Omega f\Psi\mathrm{e}^x\Big|.
\end{aligned} \tag{12}$$

In [7], the first term in the right coming from the jacobian is estimated splitting it in two terms on the form

$$\int_\Omega \partial_y\Psi\Delta\Psi\Psi\mathrm{e}^x = -\frac{1}{2}\int_\Omega \Psi_y\,|\nabla\Psi|^2\mathrm{e}^x - \int_\Omega \Psi_y\Psi_x\Psi\mathrm{e}^x.$$

The first term in the expression above show the necessity to use δ_S and then to make the hypothesis that δ_I and δ_S have the same order. We can, in fact, estimate directly the term coming from the jacobian and we obtain :

$$\varepsilon\,|\int_\Omega \partial_y\Psi\Delta\Psi\Psi\mathrm{e}^x| \leq c\,\varepsilon\,E^{-1}\|\Psi\|^2_{L^2(\Omega)}. \tag{13}$$

We use for this

$$\|\Psi\|_{\mathrm{H}^1_0(\Omega)} \leq c\|\Psi\|^{1/2}_{L^2(\Omega)}\|\Delta\Psi\|^{1/2}_{L^2(\Omega)}. \tag{14}$$

Indeed

$$\int_\Omega |\Psi\,\partial_y\Psi\,\Delta\Psi| \leq \|\Psi\|_{L^4(\Omega)}\|\partial_y\Psi\|_{L^4(\Omega)}\|\Delta\Psi\|_{L^2(\Omega)}.$$

Therefore by Gagliardo–Nirenberg Inequality

$$(15)\qquad \|\cdot\|_{L^4(\Omega)} \leq c\,\|\cdot\|_{L^2(\Omega)}^{1/2}\|\nabla(\cdot)\|_{(L^2(\Omega))^2}^{1/2}$$

which is satisfied by for Ψ and $\partial_y\Psi$ since $\Psi = 0$ on $\partial\Omega$ and $\int_\Omega \partial_y\Psi = 0$, we obtain

$$(16)\qquad |\int_\Omega \Psi\,\partial_y\Psi\,\Delta\Psi| \leq \|\Psi\|_{L^2(\Omega)}^{1/2}\|\nabla\Psi\|_{(L^2(\Omega))^2}\|\Delta\Psi\|_{L^2(\Omega)}^{3/2}.$$

This gives the desired estimate (13) with the help of (14) and then (11). Then, it remains to estimate $\int_\Omega \Delta\Psi\partial_x\Psi e^x$ in (12). In [7], this terms is estimated after an integration by part using the fact that $\nabla\Psi = 0$ on $\partial\Omega$. Here $\nabla\Psi$ is not equal to 0 on $\partial\Omega$ but we can obtain an estimate directly observing

$$(17)\qquad |\int_\Omega \Delta\Psi\partial_x\Psi e^x| \leq c\,\|\Delta\Psi\|_{L^2(\Omega)}^2 \leq cE^{-1}\|f\|_{L^2(\Omega)}\|\Psi\|_{L^2(\Omega)}.$$

This uses the equivalence of the $\|\Delta\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^2(\Omega)}$ norm since $\Psi = 0$ on $\partial\Omega$ and Inequality (11).

Thus we obtain the inequality (6) with the help of (12), (13), (17) and therefore the existence of a subsequence Ψ converging to Ψ_0 in $L^2(\Omega)$ weak. It just remains now to prove that Ψ_0 satisfies (5) in the distribution sense. The only term which can get some problems is the term coming from the jacobian. In [7], to prove that $\varepsilon\int_\Omega \mathbf{J}(\Psi,\varphi)\,\Delta\Psi \to 0$ the fact that δ_S and δ_I have the same order is used. In fact we can directely make an estimation without use the coefficient δ_S. We have $\left|\varepsilon\int_\Omega \mathbf{J}(\Psi,\varphi)\,\Delta\Psi\right| \leq c\,\varepsilon\|\Delta\Psi\|_{L^2(\Omega)}^2\|\nabla\varphi\|_{(L^\infty(\Omega))^2}$ and therefore by using the estimate (6), we obtain the convergence to 0 when $\varepsilon/E \to 0$. □

Remark. By this proof, we generalize the result obtained in [7] since we can study the problem proposed page 403 by J. – L. Lions where $\delta_S = 0$ and mixed boundary conditions without lack of range between ε and E as in [7]. □

Part (ii)**:**

Let us denote $\Gamma_{East} = \{x \in \partial\Omega : n_x > 0\}$ and $\Gamma_- = \partial\Omega / \overline{\Gamma}_{East}$. Let Ψ_I given by $-\partial_x \Psi_I = f$ with $\Psi_I = 0$ on Γ_{East} and Φ defined by

$$\Phi = \int_{g_{\mathrm{West}}(y)}^{x} \eta(x_\star, y)\, dx_\star$$

where $\eta \in \mathcal{C}^2(\overline{\Omega})$, $\eta = 0$ in a neighborhood of Γ_- and $\eta \geq 0$ in Ω. We have $\Phi \geq 0$ in Ω and $\Phi = 0$ in a neighborhood of Γ_- and $\Phi \in \mathcal{C}^2(\overline{\Omega})$ since g_{East} is a graph of class $\mathcal{C}^2$ on $(0,1)$.

We have

$$\int_\Omega |\Psi - \Psi_I|^2 \eta = \int_\Omega |\Psi - \Psi_I|^2 \partial_x \Phi.$$

But

$$\frac{1}{2}\int_\Omega |\Psi - \Psi_I|^2 \partial_x \Phi = -\int_\Omega (\Psi - \Psi_I)\partial_x(\Psi - \Psi_I)\Phi$$

and $\Psi - \Psi_I$ satisfy

$$-\partial_x(\Psi - \Psi_I) + E\,\Delta^2\Psi - \mu\,\Delta\Psi - \varepsilon\,\mathbf{J}(\Psi, \Delta\Psi) = 0,$$

therefore multiplying by $(\Psi - \Psi_I)\Phi$ we will have

(18)
$$\frac{1}{2}\int_\Omega |\Psi - \Psi_I|^2 \eta + E\int_\Omega \Delta^2\Psi(\Psi - \Psi_I)\Phi - \mu\int_\Omega \Delta\Psi(\Psi - \Psi_I)\Phi - \varepsilon\int_\Omega \mathbf{J}(\Psi, \Delta\Psi)(\Psi - \Psi_I)\Phi = 0.$$

But

$$\begin{aligned}\int_\Omega \Delta^2\Psi(\Psi - \Psi_I)\Phi = {} & \langle \Delta\Psi, \Phi\nabla\Psi_I \cdot n\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \\ & + \int_\Omega |\Delta\Psi|^2\Phi - \int_\Omega \Delta\Psi\Delta(\Psi_I\Phi) \\ & + 2\int_\Omega \Delta\Psi\nabla\Psi \cdot \nabla\Phi + \int_\Omega \Delta\Psi\Psi\Delta\Phi\end{aligned} \tag{19}$$

since $\Delta\Psi = 0$ or $\nabla\Psi \cdot n = 0$ on $\partial\Omega$, $\Phi = 0$ in a neighborhood of Γ_- and $\Psi - \Psi_I = 0$ on Γ_{East}.

We have too

(20)
$$-\int_\Omega \Delta\Psi(\Psi - \Psi_I)\Phi = \int_\Omega |\nabla\Psi|^2\Phi - \int_\Omega \nabla\Psi \cdot \nabla(\Psi_I\Phi) + \int_\Omega \nabla\Psi \cdot \nabla\Phi\,\Psi.$$

Thus by (18), (19) and (20)

(21)

$$\frac{1}{2}\int_\Omega |\Psi-\Psi_I|^2\eta + E\int_\Omega |\Delta\Psi|^2\Phi + \mu\int_\Omega |\nabla\Psi|^2\Phi =$$

$$\begin{aligned} &- E\,\langle \Delta\Psi, \Phi\nabla\Psi_I\cdot n\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} + E\int_\Omega \Delta\Psi\Delta(\Psi_I\Phi)\\ &- 2\,E\int_\Omega \Delta\Psi\nabla\Psi\cdot\nabla\Phi - \,E\int_\Omega \Delta\Psi\Psi\Delta\Phi - \mu\int_\Omega \nabla\Psi\cdot\nabla\Phi\,\Psi\\ &+ \mu\int_\Omega \nabla\Psi\cdot\nabla(\Psi_I\Phi) - \varepsilon\int_\Omega \mathbf{J}(\Psi,(\Psi-\Psi_I)\Phi)\Delta\Psi. \end{aligned}$$

Let us estimate each terms in the right-hand side and let us prove that they converge to 0 when μ, ε, $E \to 0$ and $\varepsilon\,E^{-1}\to 0$. By [7] page 167

$$\|\Psi\|_{H^{5/2}(\Omega)} \le C\varepsilon E^{-2}.$$

therefore

$$E|\,\langle \Delta\Psi, \Phi\nabla\Psi_I\cdot n\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}| \le C\varepsilon E^{-1}$$

and then this term tends to 0 when $\varepsilon\,E^{-1}\to 0$. We have

$$E|\int_\Omega \Delta\Psi\Delta(\Psi_I\Phi)| \le E\|\Delta\Psi\|_{L^2(\Omega)}\|\Delta(\Psi_I\Phi)\|_{L^2(\Omega)}.$$

Then, by (6), this term converges to 0 when $E\to 0$. Moreover we have

$$E|\int_\Omega \Delta\Psi\nabla\Psi\cdot\nabla\Phi| \le cE\|\Delta\Psi\|_{L^2(\Omega)}\|\nabla\Psi\|_{(L^2(\Omega))^2}.$$

Therefore using the inequalities (14) and (6), this term converges to 0 when $E\to 0$. We conclude in a similar way for $E\int_\Omega \Delta\Psi\Psi\Delta\Phi$ and $\mu\int_\Omega \nabla\Psi\cdot\nabla\Phi\,\Psi$. We have

$$\mu|\int_\Omega \nabla\Psi\cdot\nabla(\Psi_I\Phi)| \le \mu\|\nabla\Psi\|_{(L^2(\Omega))^2}.$$

This term converges to 0 when $\mu\to 0$. Let us now estimate the term coming from the jacobian. We have

$$|\varepsilon\int_\Omega \mathbf{J}(\Psi,(\Psi-\Psi_I)\Phi)\Delta\Psi| \le c\varepsilon(\|\nabla\Psi\|^2_{(L^4(\Omega))^2}\|\Delta\Psi\|_{L^2(\Omega)} + \|\nabla\Psi\|_{(L^2(\Omega))^2}\|\Delta\Psi\|_{L^2(\Omega)}).$$

We conclude that this term converges to 0 when E and $\varepsilon\,E^{-1}\to 0$ by using (15) and (6).

Therefore we have by (21)

$$\int_\Omega |\Psi - \Psi_I|^2 \eta \to 0$$

when E, μ, ε and $\varepsilon\, E^{-1}$ tend to 0 for all η such that $\eta = 0$ in a neightborhood of Γ_- and $\eta \geq 0$. We obtain therefore the result since Φ is positive. $\square$

Remark. By assuming f more regular than L^2, we precise the result obtain in [7]. We obatin a better range of coefficients for the strong convergence and we can choose $\Delta\Psi = 0$ on the East coastal part. $\square$

Part (iii)**:**

We will study the convergence when the bottom friction coefficient dominates the nonlinearity coefficient with the diffusion coefficient independent on μ and ε. Let us recall that (11) and (12) are satisfied here. Let us multiply the equation (1) by $\Delta\Psi$, we obtain

$$\|\Delta\Psi\|^2_{L^2(\Omega)} \leq 2(\|\nabla\Psi\|^2_{(L^2(\Omega))^2} + \|f\|^2_{L^2(\Omega)})/\mu^2 \tag{22}$$

and therefore, by (11),

$$\|\Delta\Psi\|^2_{L^2(\Omega)} \leq 2\,(\|f\|_{L^2(\Omega)}\|\Psi\|_{L^2(\Omega)}/\mu + \|f\|^2_{L^2(\Omega)})/\mu^2.$$

Using (16) (which is still valid here) and (11), one gets:

$$\varepsilon|\int_\Omega \Psi\,\partial_y\Psi\,\Delta\Psi| \leq \frac{c\varepsilon}{\mu^3}(\|f\|_{L^2(\Omega)}\|\Psi\|_{(L^2(\Omega))^2} + \mu\|f\|^2_{L^2(\Omega)})\|\Psi\|_{L^2(\Omega)}.$$

We conclude as in the proof of (i). $\square$

Remark. Let us remark that the hypothesis $E \leq \mu^3/8$ made in [1] is not necessary. The authors had not establish the estimates above but used the following inequality

$$\|\nabla\Psi\|^2_{(L^2(\Omega))^2} \leq (2E/\mu)\|\Delta\Psi\|^2_{L^2(\Omega)} + \frac{\|f\|^2_{L^2(\Omega)}}{\mu^2\lambda_1} \leq \frac{\mu^2}{4}\|\Delta\Psi\|^2_{L^2(\Omega)} + \frac{\|f\|^2_{L^2(\Omega)}}{\mu^2\lambda_1}$$

if $E \leq \mu^3/8$ and (22) but in fact we have directly by (11)

$$\|\nabla\Psi\|^2_{(L^2(\Omega))^2} \leq \frac{\|f\|^2_{L^2(\Omega)}}{\mu^2\lambda_1}. \ \square$$

3 Proof of Theorem 2

The Case $\int_0^1 f(\xi,y)\,d\xi = 0$:

Thanks to Equation (1) we obtain

$$\int_0^1 \Delta\Psi(\xi,y)\,d\xi = 0 \tag{23}$$

since $g(y) = \displaystyle\int_0^1 \Delta\Psi(\xi,y)d\xi$ satsifies

$$Eg'' - \mu g = 0$$

which only periodic solution is zero.
Therefore, multiplying Equation (1) by $-\int_0^x \Delta\Psi(\xi,y)\,d\xi$, one gets

We have

$$\int_\Omega \partial_x\Psi\,(\int_0^x \Delta\Psi) = -\int_\Omega \Psi\Delta\Psi = \int_\Omega |\nabla\Psi|^2,$$

and

$$\int_\Omega \Delta\Psi\,(\int_0^x \Delta\Psi) = \int_\Omega \partial_x(\int_0^x \Delta\Psi)(\int_0^x \Delta\Psi) = \frac{1}{2}\int_{\partial\Omega} |\int_0^x \Delta\Psi|^2 n_x = 0$$

by (23). In addition, we have by using (23)

$$\begin{aligned}
\int_\Omega \Delta^2\Psi\,(\int_0^x \Delta\Psi) &= \int_\Omega \partial_x(\int_0^x \Delta^2\Psi)\,(\int_0^x \Delta\Psi) \\
&= -\int_\Omega (\int_0^x \Delta^2\Psi)\,\partial_x(\int_0^x \Delta\Psi) \\
&= -\int_\Omega \Delta(\int_0^x \Delta\Psi)\,\partial_x(\int_0^x \Delta\Psi) \\
&= -\int_{\partial\Omega} \nabla(\int_0^x \Delta\Psi)\cdot n\,\partial_x(\int_0^x \Delta\Psi) + \int_\Omega \nabla(\int_0^x \Delta\Psi)\cdot\partial_x\nabla(\int_0^x \Delta\Psi) \\
&= -(|\Delta\Psi|^2|_{\Gamma_{\mathrm{East}}} - |\Delta\Psi|^2|_{\Gamma_{\mathrm{West}}}) + \frac{1}{2}\int_\Omega |\nabla(\int_0^x \Delta\Psi)|^2 n_x \\
&= -\frac{1}{2}(|\Delta\Psi|^2|_{\Gamma_{\mathrm{East}}} - |\Delta\Psi|^2|_{\Gamma_{\mathrm{West}}}).
\end{aligned}$$

therefore we obtain the following equality

$$\int_\Omega |\nabla\Psi|^2 = -\int_\Omega f\,(\int_0^x \Delta\Psi) - \frac{E}{2}(|\Delta\Psi|^2|_{\Gamma_{\mathrm{East}}} - |\Delta\Psi|^2|_{\Gamma_{\mathrm{West}}}). \tag{24}$$

But

$$\begin{aligned}(25)\qquad \int_\Omega \nabla(\Delta\Psi)\cdot\nabla(\partial_x\Psi) &= \int_{\partial\Omega} \Delta\Psi\nabla(\partial_x\Psi)\cdot n - \int_\Omega \Delta\Psi\partial_x\Delta\Psi \\ &= \frac{1}{2}(|\Delta\Psi|^2|_{\Gamma_{\text{East}}} - |\Delta\Psi|^2|_{\Gamma_{\text{West}}})\end{aligned}$$

and

$$\begin{aligned}(26)\qquad \int_\Omega f\,(\int_0^x \Delta\Psi) = \int_\Omega \partial_x(\int_0^x f)\,(\int_0^x \Delta\Psi) &= -\int_\Omega(\int_0^x f)\,\Delta\Psi \\ &= -\int_\Omega \nabla\Psi\cdot\nabla(\int_0^x f).\end{aligned}$$

since $\int_0^1 f(\xi,y)\,d\xi = 0$. Let us denote $F = \int_0^x f$. We will obtain from (24), (25) and (26)

$$\frac{3}{4}\|\nabla\Psi\|^2_{(L^2(\Omega))^2} \leq \|\nabla F\|^2_{(L^2(\Omega))^2} + E\|\nabla\Delta\Psi\|_{(L^2(\Omega))^2}\|\nabla\partial_x\Psi\|_{(L^2(\Omega))^2}.$$

We have $\int_\Omega \partial_x\Psi = 0$ therefore

$$\frac{3}{4}\|\nabla\Psi\|^2_{(L^2(\Omega))^2} \leq \|\nabla F\|^2_{(L^2(\Omega))^2} + c_\Omega E\|\nabla\Delta\Psi\|^{3/2}_{(L^2(\Omega))^2}\|\partial_x\Psi\|^{1/2}_{(L^2(\Omega))^2}.$$

This gives

$$\frac{1}{2}\|\nabla\Psi\|^2_{(L^2(\Omega))^2} \leq \|\nabla F\|^2_{(L^2(\Omega))^2} + \frac{3}{4}c_\Omega E^{4/3}\|\nabla\Delta\Psi\|^2_{(L^2(\Omega))^2}.$$

Multiplying (1) by $\mu\Delta\Psi$,

$$(27)\qquad \|\nabla(\Delta\Psi)\|^2_{(L^2(\Omega))^2} \leq 2(\|f\|^2_{L^2(\Omega)} + \|\partial_x\Psi\|^2_{(L^2(\Omega))})/E\mu$$

then

(28)

$$\frac{1}{2}\|\nabla\Psi\|^2_{(L^2(\Omega))^2} \leq \|\nabla F\|^2_{(L^2(\Omega))^2} + \frac{c_\Omega E^{1/3}}{\mu}(\|f\|^2_{L^2(\Omega)} + \|\partial_x\Psi\|^2_{(L^2(\Omega))}).$$

thus there exists c_Ω such that if $E < c_\Omega\mu^3$, we obtain the convergence (8). We prove that the limit is given by $\Psi = \int_0^x f(\xi,y)\,d\xi$ by convergence in the distribution sense using (27) et (28).

The case $\int_0^1 f(\xi,y)\,d\xi \neq 0$:

Let us multiply the equation (1) by Ψe^x, we obtain
(29)
$$\frac{1-\mu}{2}\int_\Omega |\Psi|^2 e^x + \mu\int_\Omega |\nabla\Psi|^2 e^x = \int_\Omega f\,\Psi e^x + E\int_\Omega \nabla(\Delta\Psi)\cdot\nabla(\Psi e^x).$$

Indeed, we have

$$-\mu\int_\Omega \Delta\Psi\Psi e^x = \mu\int_\Omega |\nabla\Psi|^2 e^x - \frac{\mu}{2}\int_\Omega |\Psi|^2 e^x,$$

$$-\int_\Omega \partial_x\Psi\Psi e^x = \frac{1}{2}\int_\Omega |\Psi|^2 e^x$$

and
$$E\int_\Omega \Delta^2\Psi\,\Psi e^x = -E\int_\Omega \nabla\Delta\Psi\cdot\nabla(\Psi e^x),$$
since $\partial_y\Psi = 0$ on $\partial\Omega$. Let us multiply now the equation by $x\partial_x\Psi$, we get
(30)
$$\int_\Omega x|\partial_x\Psi|^2 + \frac{\mu}{2}\int_{\partial\Omega} x|\nabla\Psi|^2 n_x - \frac{\mu}{2}(\int_\Omega |\partial_x\Psi|^2 - \int_\Omega |\partial_y\Psi|^2) = -\int_\Omega x\,f\partial_x\Psi$$
$$- E\int_\Omega \partial_x(\Delta\Psi)\partial_x\Psi - \frac{E}{2}\int_{\partial\Omega} x|\Delta\Psi|^2 n_x - \frac{E}{2}\int_\Omega |\Delta\Psi|^2 + E\int_\Omega \Delta\Psi\partial_x^2\Psi.$$

Indeed, we have

$$-\mu\int_\Omega \Delta\Psi\partial_x\Psi x = -\frac{\mu}{2}\int_\Omega x|\partial_x\Psi|^2 n_x + \frac{\mu}{2}\Big(\int_\Omega |\partial_x\Psi|^2 - \int_\Omega |\partial_y\Psi|^2\Big),$$

and

$$E\int_\Omega \Delta^2\Psi\partial_x\Psi x = -\frac{E}{2}\int_\Omega |\Delta\Psi|^2 + E\int_\Omega \Delta\Psi\partial_x^2\Psi - E\int_\Omega \partial_x\Delta\Psi\partial_x\Psi - \frac{E}{2}\int_{\partial\Omega} x|\Delta\Psi|^2 n_x.$$

Adding (29) and (30), we get

$$\frac{3-4\mu}{8}\int_\Omega |\Psi|^2 e^x + \frac{\mu}{2}\int_\Omega |\nabla\Psi|^2 e^x + \frac{7}{8}\int_\Omega x|\partial_x\Psi|^2$$
$$\leq c\|f\|^2_{L^2(\Omega)} + c_\Omega E(\|\nabla(\Delta\Psi)\|_{(L^2(\Omega))^2}\|\Psi\|_{H^1(\Omega)} + \|\Delta\Psi\|^2_{L^2(\Omega)}) =$$
$$\leq c\|f\|^2_{L^2(\Omega)} + c_\Omega E((\|\nabla(\Delta\Psi)\|_{(L^2(\Omega))^2}\|\Delta\Psi\|^{1/2}_{L^2(\Omega)}\|\Psi\|^{1/2}_{L^2(\Omega)} + \|\Delta\Psi\|^2_{L^2(\Omega)})$$

since $\int_{\partial\Omega} x|\Delta\Psi|^2 n_x \geq 0$ and $\int_{\partial\Omega} x|\nabla\Psi|^2 n_x \geq 0$. Therefore we obtain

$$\frac{1-4\mu}{8}\int_\Omega |\Psi|^2 \mathrm{e}^x + \frac{\mu}{2}\int_\Omega |\nabla\Psi|^2 \mathrm{e}^x + \frac{7}{8}\int_\Omega x|\partial_x\Psi|^2$$
$$\leq c\|f\|^2_{L^2(\Omega)} + c_\Omega(E^{4/3}\|\nabla(\Delta\Psi)\|^{4/3}_{(L^2(\Omega))^2}\|\Delta\Psi\|^{2/3}_{L^2(\Omega)} + E\|\Delta\Psi\|^2_{L^2(\Omega)}).$$

Multiplying the equation (1) by $\mu\Delta\Psi$,

$$\|\Delta\Psi\|^2_{L^2(\Omega)} \leq 2(\|f\|^2_{L^2(\Omega)} + \|\partial_x\Psi\|^2_{(L^2(\Omega))})/\mu^2$$

then with (27)

(31)
$$\frac{1-4\mu}{8}\int_\Omega |\Psi|^2 \mathrm{e}^x + \frac{\mu}{2}\int_\Omega |\nabla\Psi|^2 \mathrm{e}^x + \frac{7}{8}\int_\Omega x|\partial_x\Psi|^2$$
$$\leq c\|f\|^2_{L^2(\Omega)} + c_\Omega(\|f\|^2_{L^2(\Omega)} + \|\partial_x\Psi\|^2_{(L^2(\Omega))})\Big((\frac{E}{\mu^2})^{2/3} + \frac{E}{\mu^2}\Big).$$

Thus there exists c_Ω such that if $E < c_\Omega\mu^{7/2}$, we have (9). We obtain (10) by convergence in the distribution sense by using (27) and (31). □

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