The Cauchy problem and the continuous limit for the multilayer model in geophysical fluid dynamics.

T.Colin[1]

Mathématiques Appliquées de Bordeaux, CNRS ERS 123 et Université Bordeaux 1, 351 cours de la libération, 33405 Talence cedex, France.

Abstract: We study a multilayer model in geophysical fluid dynamics that governs approximately the large-scale motions of the atmosphere or the ocean. The model consists of n two-dimensional Euler equations which represent the evolution of n layers of liquid. These equations are written using the potential vorticity. The potential vorticity in each layer is obtained from the velocity potential of the adjacent layers. We show that the Cauchy problem for this model is globally well-posed in time for smooth initial data. In a second part, we let the number of layers tend to infinity while their thickness tends to zero. We write the system as a suitable finite-element approximation of a continuous model and show the convergence of this approximation to the classical quasi-geostrophic model.

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Abbreviated title: Multilayer model in geophysical dynamics.

1 Introduction, setting of the problem and statement of results.

1.1 The model.

Consider a fluid which is formed by the superposition of a finite number n of homogeneous layers with uniform density within each layer, the density being different from one layer to another. The multilayered quasi-geostrophic model describes concervation of potential vorticity ζ_l in each layer with the $\beta-$plane approximation. The thikness of each layer has the same value D_0 and we denote by ρ_l the density in the l^{th} layer, $l = 1, ..., n$. The nondimensional model reads (see [9] p.421)

$$\left\{ \begin{array}{l} (\frac{\partial}{\partial t} + u_l \frac{\partial}{\partial x} + v_l \frac{\partial}{\partial y})\zeta_l = -a v_l, \; l = 1 \text{ to } n. \\ \\ u_l = -\frac{\partial \psi_l}{\partial y}, \; v_l = \frac{\partial \psi_l}{\partial x}, \end{array} \right. \tag{1}$$

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$$
\begin{cases}
\zeta_1 &= \Delta_{x,y}\psi_1 + f_{r1,2}(\psi_2 - \psi_1), \\
\zeta_l &= \Delta_{x,y}\psi_l - f_{rl,1}(\psi_l - \psi_{l-1}) + f_{rl,2}(\psi_{l+1} - \psi_l), \ 2 \le l \le n-1, \\
\zeta_n &= \Delta_{x,y}\psi_n - f_{rn,1}(\psi_n - \psi_{n-1}),
\end{cases} \tag{2}
$$

where

$$
f_{rl,1} = \frac{DF}{D_0}\frac{\rho_0}{\rho_l - \rho_{l-1}},
$$

and

$$
f_{rl,2} = \frac{DF}{D_0}\frac{\rho_0}{\rho_{l+1} - \rho_l}.
$$

In the above equations, (u_l, v_l) is the velocity field within the l^{th} layer. All functions only depend on the horizontal variables x and y and on the time t. The term $-av_l$ in (1) corresponds to the action of the Coriolis force in the β-plane approximation [9]. Note that the function ψ_l is in fact the physical pressure inside the l^{th} layer, see [9] p. 421. ρ_0 is a characteristic (constant) value for the density of the fluid and $F = \frac{f_0^2 L^2}{gD}$, where L (resp. D) is the characteristic horizontal (resp. vertical) scale of the fluid; f_0 is the Coriolis parameter and g the gravitational acceleration. We introduce the following notation:

$$
\beta_l = \frac{DF\rho_0}{D_0(\rho_l - \rho_{l-1})}\varepsilon^2,
$$

where ε is a small (nondimensional) parameter, and we suppose that:

$$
\frac{1}{\delta} \ge \beta_l \ge \delta, \ \text{for } l = 2, ..., n, \tag{3}
$$

where δ is a fixed positive number. This last relation is taken to be satisfied independently of n. The parameters $f_{rl,1}$ and $f_{rl,2}$ therefore read:

$$
f_{rl,1} = \frac{1}{\varepsilon^2}\beta_l, \ f_{rl,2} = \frac{1}{\varepsilon^2}\beta_{l+1}.
$$

With this notation, system (2) takes the form

$$
\begin{cases}
\zeta_1 &= \Delta_{x,y}\psi_1 + \frac{1}{\varepsilon^2}\beta_2(\psi_2 - \psi_1), \\
\zeta_l &= \Delta_{x,y}\psi_l - \frac{1}{\varepsilon^2}\beta_l(\psi_l - \psi_{l-1}) + \frac{1}{\varepsilon^2}\beta_{l+1}(\psi_{l+1} - \psi_l), \ 2 \le l \le n-1, \\
\zeta_n &= \Delta_{x,y}\psi_n - \frac{1}{\varepsilon^2}\beta_n(\psi_n - \psi_{n-1}),
\end{cases} \tag{4}
$$

A derivation of this model and its physical meaning can be found in [9] p.416-422. See also the appendix of [6].

We will use periodic boundary conditions corresponding to $(x, y) \in \mathbf{T}^2$ where $\mathbf{T}^2$ denotes the two-dimensional torus. The aim of this work is to show that the system (1)-(4) is globally well-posed for smooth initial data and to perform the limit $n \to \infty$ in a suitable sense.

The "natural" limit system is the quasi-geostrophic equation

$$(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y})(\Delta_{x,y}\psi + \frac{\partial}{\partial z}(\beta(z)\frac{\partial \psi}{\partial z})) = -a\frac{\partial \psi}{\partial x}. \tag{5}$$

In this equation, z is a spatial variable while the function $z \mapsto \beta(z)$ is proportional to

$$\frac{1}{\frac{\partial \rho(z)}{\partial z}},$$

i.e. to the inverse of the square of the Brunt-Väisälä frequency (see [9] p. 354-358).

This sytem is of considerable physical interest because of the qualitative results that are readily deduced from it (see [9], [11]). For its mathematical treatement see [1] in which the asymptotic expansion leading from some set of primitive equations to (5) is justified, and [2] where the viscous case is considered. See also [7] and [4] for analysis of models without vertical stratification.

1.2 Notations and statement of the results.

In section 2, we prove that for fixed n, system (1)-(4) is globally well-posed. Our result in this direction is the following (see Theorem 1).

Let $s > 2$ and $(\zeta_1^0, ..., \zeta_n^0) \in (H^s(\mathbf{T}^2))^n$ be such that

$$\sum_{l=1}^{n} \int_{\mathbf{T}^2} \zeta_l^0 = 0.$$

Then there exists a unique solution $(\zeta_1, ...\zeta_n) \in (\mathcal{C}(\mathbf{R}^+, H^s(\mathbf{T}^2)))^n$ and $((u_1, v_1), ..., (u_n, v_n)) \in (\mathcal{C}(\mathbf{R}^+, H^{s-1}(\mathbf{T}^2)))^{2n}$ to (1) and (4) such that $(\zeta_1, ...\zeta_n)(t = 0) = (\zeta_1^0, ..., \zeta_n^0)$.

In the third section, we show how (1)-(4) may be viewed as a finite-element approximation in the vertical direction to the continuous system (5) and how the limit $n \to \infty$ can be performed. Indeed, system (4) can be viewed as a finite-element approximation in only one direction of a continuous 3-dimensional elliptic equation. This leads us to introduce the functions $\Psi_1^\varepsilon(x, y, z, t)$, $\Psi_3^\varepsilon(x, y, z, t)$ and $Z_2^\varepsilon(x, y, z, t)$ below in terms of the classical basis functions. However, equation (1) is not a finite-element approximation of the corresponding continuous equation. We therefore need to introduce other auxilliary functions ($\Psi_2^\varepsilon(x, y, z, t)$ and $Z_1^\varepsilon(x, y, z, t)$) in order to write system (1)-(4) as a coherent approximation of the quasi-geostrophic model (5).

Specifically, we denote by $\phi_j^\varepsilon(z)$ the functions defined on $[0, 1]$ such that:

$$\phi_1^\varepsilon(z) = 1 - \frac{z}{\varepsilon} \text{ if } 0 \leq z \leq \varepsilon, \text{ 0 elsewhere,}$$

$$\phi_i^\varepsilon(z) = 1 - \frac{|z - i\varepsilon|}{\varepsilon} \text{ if } z \in [(i-1)\varepsilon, (i+1)\varepsilon], \, 0 \text{ elsewhere}, \, i = 2, ..., n-1,$$

$$\phi_n^\varepsilon(z) = 1 - \frac{1-z}{\varepsilon} \text{ if } z \in [1-\varepsilon, 1], \, 0 \text{ elsewhere},$$

where we assume that $\varepsilon = \frac{1}{n-1}$. We assume that there exists a smooth function $\beta(z)$ such that

$$\beta_l^\varepsilon = \frac{1}{\varepsilon} \int_{(l-2)\varepsilon}^{(l-1)\varepsilon} \beta(z) dz, \text{ for } l = 2, ..., n,$$

and then introduce

$$\Psi_1^\varepsilon(x, y, z, t) = \sum_{l=1}^{n} \psi_l^\varepsilon(x, y, t) \phi_l^\varepsilon(z). \tag{6}$$

We also construct the piecewise constant function $\Psi_2^\varepsilon(x, y, z, t)$ which is equal to $\psi_l^\varepsilon(x, y, t)$ on $[(l-1)\varepsilon, l\varepsilon[$ for $l = 1, ..., n-1$, namely

$$\Psi_2^\varepsilon(x, y, z, t) = \sum_{l=1}^{n-1} \psi_l^\varepsilon(x, y, t) \mathbf{1}_{[(l-1)\varepsilon, l\varepsilon[}(z), \tag{7}$$

where $\mathbf{1}_{[a,b]}(z)$ denotes the characteristic function of $[a, b]$.
In order to solve (4), we need to consider the function Ψ_3^ε of the following form

$$\Psi_3^\varepsilon = \sum_{i=1}^{n} f_i^\varepsilon \phi_i^\varepsilon. \tag{8}$$

Moreover, we impose that Ψ_3^ε satisfies

$$\int_0^1 \Psi_3^\varepsilon(z) \phi_i^\varepsilon(z) dz = \varepsilon \psi_i^\varepsilon, \text{ for all } i = 1, ..., n. \tag{9}$$

Thus (9) reads

$$\int_0^1 \Psi_3^\varepsilon \phi_k^\varepsilon(z) dz = \int_0^1 \sum_{i=1}^{n} f_i^\varepsilon \phi_i^\varepsilon(z) \phi_k^\varepsilon(z) dz = \varepsilon \psi_k^\varepsilon.$$

Using the explicit values of ϕ_l^ε, one find that the coefficients (f_i^ε) are given by

$$\begin{pmatrix} \psi_1^\varepsilon \\ ... \\ \psi_n^\varepsilon \end{pmatrix} = B_n \begin{pmatrix} f_1^\varepsilon \\ ... \\ f_n^\varepsilon \end{pmatrix}, \tag{10}$$

B_n being the $n \times n$ matrix

$$B_n = \frac{1}{6} \begin{pmatrix} 2 & 1 & 0 & & & & \\ 1 & 4 & 1 & 0 & & 0 & \\ 0 & ... & ... & ... & 0 & & \\ & 0 & 1 & 4 & 1 & 0 & \\ & & 0 & ... & ... & ... & 0 \\ & 0 & & 0 & 1 & 4 & 1 \\ & & & & 0 & 1 & 2 \end{pmatrix}.$$

In the same direction, we define the piecewise constant function Z_1^ε by

$$Z_1^\varepsilon(x,y,z,t) = \sum_{l=1}^{n-1} \zeta_l^\varepsilon(x,y,t)\mathbf{1}_{[(l-1)\varepsilon,l\varepsilon[}(z). \tag{11}$$

We also introduce $Z_2^\varepsilon(x,y,z,t)$ defined in the same way as Ψ_3^ε:

$$Z_2^\varepsilon(x,y,z,t) = \sum_{i=1}^{n} g_i^\varepsilon \phi_i^\varepsilon, \tag{12}$$

where (g_i^ε) are given by

$$\begin{pmatrix} \zeta_1^\varepsilon \\ \ldots \\ \zeta_n^\varepsilon \end{pmatrix} = B_n \begin{pmatrix} g_1^\varepsilon \\ \ldots \\ g_n^\varepsilon \end{pmatrix}.$$

With this notation, the original system can be written exactly as follows:

$$\frac{\partial}{\partial t} Z_1^\varepsilon + \nabla.((U_2^\varepsilon, V_2^\varepsilon)(Z_1^\varepsilon + ay)) = 0, \tag{13}$$

and for all $k = 1, \ldots, n$:

$$\int_0^1 Z_2^\varepsilon \phi_k^\varepsilon(z)dz = \int_0^1 \Delta_{x,y}\Psi_3^\varepsilon(z)\phi_k^\varepsilon(z)dz - \int_0^1 \beta(z)\frac{\partial \Psi_1^\varepsilon}{\partial z}\frac{\partial \phi_k^\varepsilon}{\partial z}dz, \tag{14}$$

where

$$U_2^\varepsilon = -\frac{\partial \Psi_2^\varepsilon}{\partial y} \text{ and } V_2^\varepsilon = \frac{\partial \Psi_2^\varepsilon}{\partial x}. \tag{15}$$

The result that we obtain is (for a precise statement, see Theorem 2):

The sequences Ψ_1^ε, Ψ_2^ε and Ψ_3^ε converge to the same limit Ψ which is a solution of (3).

2 Global existence of strong solution for fixed n.

We aim to solve (4) on the torus $\mathbf{T}^2$ with the additional restriction

$$\sum_{l=1}^{n} \int_{\mathbf{T}^2} \psi_l = 0.$$

We need this condition in order to ensure uniqueness of the solution of the elliptic system (4). Indeed, this system corresponds to a discretization of a continuous problem with homogeneous Neuman boundary conditions.

One of the principal goals of this section is to prove the following result:

Theorem 1 *Let $s > 2$ and $(\zeta_1^0, ..., \zeta_n^0) \in (H^s(\mathbf{T}^2))^n$ be such that*

$$\sum_{l=1}^{n} \int_{\mathbf{T}^2} \zeta_l^0 = 0.$$

Then there exists a unique solution $(\zeta_1, ...\zeta_n) \in (\mathcal{C}(\mathbf{R}^+, H^s(\mathbf{T}^2)))^n$ and $((u_1, v_1), ..., (u_n, v_n)) \in (\mathcal{C}(\mathbf{R}^+, H^{s+1}(\mathbf{T}^2)))^{2n}$ to (1) and (4) such that $(\zeta_1, ...\zeta_n)(t = 0) = (\zeta_1^0, ..., \zeta_n^0)$.

To prove this result, we use a classical energy method to obtain local in time existence. Then the solution is shown to extend globally using *a priori* estimates. The constants occuring in this section may depend on n.

2.1 Local in time existence.

Proposition 1 *Let $(\zeta_1^0, ..., \zeta_n^0) \equiv \zeta^0 \in (H^s(\mathbf{T}^2))^n$ with $s > 2$ be such that $|\zeta^0|_{(H^s)^n} \leq M$. Then for sufficiently small T, there exists a unique solution $\zeta = (\zeta_1, ..., \zeta_n)$ to (1)-(4) such that $\zeta \in \mathcal{C}([0, T], (H^s(\mathbf{T}^2))^n)$ and $|\zeta|_{L^\infty([0,T],(H^s(\mathbf{T}^n))^n)} \leq 2M$.*

Proof: For $\zeta \in \mathcal{C}([0, T], (H^s(\mathbf{T}^2))^n)$, we construct $\tilde{\psi}$ satisfying

$$\left\{ \begin{array}{rcl} \zeta_1 &=& \Delta_{x,y}\tilde{\psi}_1 + \frac{1}{\varepsilon^2}\beta_2(\tilde{\psi}_2 - \tilde{\psi}_1), \\ \zeta_l &=& \Delta_{x,y}\tilde{\psi}_l - \frac{1}{\varepsilon^2}\beta_l(\tilde{\psi}_l - \tilde{\psi}_{l-1}) + \frac{1}{\varepsilon^2}\beta_{l+1}(\tilde{\psi}_{l+1} - \tilde{\psi}_l), \ 2 \leq l \leq n-1, \\ \zeta_n &=& \Delta_{x,y}\tilde{\psi}_n - \frac{1}{\varepsilon^2}\beta_n(\tilde{\psi}_n - \tilde{\psi}_{n-1}). \end{array} \right. \tag{16}$$

Let $\tilde{u}_l = -\frac{\partial\tilde{\psi}_l}{\partial y}$, $\tilde{v}_l = \frac{\partial\tilde{\psi}_l}{\partial x}$ and consider the solution $\tilde{\zeta}$ to

$$\left\{ \begin{array}{c} (\frac{\partial}{\partial t} + \tilde{u}_l\frac{\partial}{\partial x} + \tilde{v}_l\frac{\partial}{\partial y})\tilde{\zeta}_l = -a\tilde{v}_l, \ l = 1 \text{ to } n. \\ \tilde{\zeta}(0) = \zeta_0. \end{array} \right. \tag{17}$$

We denote by $\mathcal{T}$ the mapping that carries ζ into $\tilde{\zeta}$. We want to prove that $\mathcal{T}$ is a contraction in a suitable space $\mathcal{C}(0, T, X)$ for sufficiently small T. Let us first solve the elliptic system. Introduce the operator on $\mathbf{R}^n$ whose matrix is

$$\mathcal{A}_n = - \begin{pmatrix} -\frac{\beta_2}{\varepsilon^2} & \frac{\beta_2}{\varepsilon^2} & & & 0 \\ & ... & ... & & \\ & \frac{\beta_l}{\varepsilon^2} & -\frac{\beta_l+\beta_{l+1}}{\varepsilon^2} & \frac{\beta_{l+1}}{\varepsilon^2} & \\ & & ... & ... & \\ 0 & & & \frac{\beta_n}{\varepsilon^2} & -\frac{\beta_n}{\varepsilon^2} \end{pmatrix}.$$

The matrix $\mathcal{A}_n$ is symmetric and

$$(\mathcal{A}_n X, X) = -\sum_{i=1}^{n-1} \beta_{i+1}(\frac{x_{i+1}-x_i}{\varepsilon})^2$$

where $X = (x_1, ..., x_n)$, so that, thanks to (3), 0 is a simple eigenvalue. Hence, if $\zeta \in (\mathcal{C}([0,T],(H^s)))^n$ then there exists a unique $\tilde{\psi} \in (\mathcal{C}([0,T],(H^{s+2})))^n$ satisfying (16) such that $\sum_{i=1}^n \int_{\mathbf{T}^2} \psi_i dxdy = 0$ and

$$|\tilde{\psi}|_{(\mathcal{C}([0,T],(H^{s+2})))^n} \leq C|\zeta|_{(\mathcal{C}([0,T],(H^s)))^n}. \tag{18}$$

Let us now apply ∂_α^s to (17) and form the product with $\partial_\alpha^s \tilde{\zeta}_l$; an integration yields

$$\frac{\partial}{\partial t}\int_{\mathbf{T}^2} |\partial_\alpha^s \tilde{\zeta}_l|^2 + \int_{\mathbf{T}^2} \partial_\alpha^s((\tilde{u}_l,\tilde{v}_l).\nabla\tilde{\zeta}_l)\partial_\alpha^s\tilde{\zeta}_l = -a\int_{\mathbf{T}^2}\partial_\alpha^s\tilde{v}_l\partial_\alpha^s\tilde{\zeta}_l. \tag{19}$$

On the other hand, since $\frac{\partial}{\partial x}\tilde{u}_l + \frac{\partial}{\partial y}\tilde{v}_l = 0$, $\int_{\mathbf{T}^2}(\tilde{u}_l,\tilde{v}_l).\nabla\partial_\alpha^s\tilde{\zeta}_l\partial_\alpha^s\tilde{\zeta}_l = 0$ for all l, and hence (19) becomes

$$\frac{\partial}{\partial t}\int_{\mathbf{T}^2} |\partial_\alpha^s \tilde{\zeta}_l|^2 + \int_{\mathbf{T}^2}(\partial_\alpha^s\nabla((\tilde{u}_l,\tilde{v}_l)\tilde{\zeta}_l) - (\tilde{u}_l,\tilde{v}_l).\nabla\partial_\alpha^s\tilde{\zeta}_l)\partial_\alpha^s\tilde{\zeta}_l = -a\int_{\mathbf{T}^2}\partial_\alpha^s\tilde{v}_l\partial_\alpha^s\tilde{\zeta}_l. \tag{20}$$

The classical commutator estimate (see [3]) implies

$$\frac{\partial}{\partial t}\int_{\mathbf{T}^2} |\partial_\alpha^s \tilde{\zeta}_l|^2 \leq C(|(\tilde{u}_l,\tilde{v}_l)|_{H^{s+1}}|\tilde{\zeta}_l|_{L^\infty} + |\nabla(\tilde{u}_l,\tilde{v}_l)|_{L^\infty}|\tilde{\zeta}_l|_{H^s})|\tilde{\zeta}_l|_{H^s} + a|\tilde{v}_l|_{H^s}|\tilde{\zeta}_l|_{H^s}. \tag{21}$$

Furthermore $|\tilde{\zeta}_l|_{L^\infty} \leq C|\tilde{\zeta}_l|_{H^s}$ as soon as $s > 1$ and $|\nabla(\tilde{u}_l,\tilde{v}_l)|_{L^\infty} \leq C|(\tilde{u}_l,\tilde{v}_l)|_{H^{s+1}}$. Hence equation (21) leads to

$$\frac{\partial}{\partial t}\int_{\mathbf{T}^2} |\partial_\alpha^s \tilde{\zeta}_l|^2 \leq C(|(\tilde{u}_l,\tilde{v}_l)|_{H^{s+1}}|\tilde{\zeta}_l|_{H^s}^2) + a|\tilde{v}_l|_{H^s}|\tilde{\zeta}_l|_{H^s}. \tag{22}$$

Thanks to (18), we obtain

$$\frac{\partial}{\partial t}|\tilde{\zeta}_l|_{H^s}^2 \leq C_1(1+M)|\tilde{\zeta}_l|_{H^s}^2 + C_2$$

if $|\zeta|_{(L^\infty(0,T,H^s))^n} \leq 2M$. This yields:

$$\begin{aligned} |\tilde{\zeta}_l|_{H^s}^2 &\leq e^{C_1(1+M)t}|\zeta_0|_{H^s}^2 + (e^{C_1(1+M)t}-1)\tfrac{C_2}{C_1(1+M)}, \\ &\leq e^{C_1(1+M)t}M + (e^{C_1(1+M)t}-1)\tfrac{C_2}{C_1(1+M)}. \end{aligned} \tag{23}$$

Choose T sufficiently small that the right-hand side of (9) is less than or equal to $2M$. If B_M denotes the ball of radius $2M$ in $(\mathcal{C}([0,T],H^s))^n$, then for sufficiently small T,

$$\mathcal{T} \text{ maps } B_M \text{ into itself.} \tag{24}$$

We next show that $\mathcal{T}$ is a contraction in the $(\mathcal{C}([0,T],L^2))^n$ norm, provided that T is sufficiently small. Indeed, for ζ^1 and ζ^2, we have

$$|\tilde{\psi}^1 - \tilde{\psi}^2|_{(L^\infty(0,T,H^{s+2}))^n} \leq C|\zeta^1 - \zeta^2|_{(L^\infty(0,T,H^s))^n}. \tag{25}$$

On the other hand

$$\frac{\partial}{\partial t}(\tilde{\zeta}_l^1 - \tilde{\zeta}_l^2) + (\tilde{u}_l^1, \tilde{v}_l^1).\nabla(\tilde{\zeta}_l^1 - \tilde{\zeta}_l^2) + ((\tilde{u}_l^1, \tilde{v}_l^1) - (\tilde{u}_l^2, \tilde{v}_l^2)).\nabla\tilde{\zeta}_l^2 = -a(\tilde{v}_l^1 - \tilde{v}_l^2). \tag{26}$$

Multiply (12) by $(\tilde{\zeta}_l^1 - \tilde{\zeta}_l^2)$ and integrate to obtain

$$\begin{aligned} \tfrac{\partial}{\partial t}\textstyle\int_{\mathbf{T}^2} |\tilde{\zeta}_l^1 - \tilde{\zeta}_l^2|^2 &\leq C'|\nabla\tilde{\zeta}_l^2|_{L^\infty}|\tilde{\zeta}_l^1 - \tilde{\zeta}_l^2|_{L^2}|\zeta^1 - \zeta^2|_{(L^2)^n} \\ &\leq C'|\tilde{\zeta}_l^1 - \tilde{\zeta}_l^2|_{L^2}|\zeta^1 - \zeta^2|_{(L^2)^n} \end{aligned}$$

as soon as $s > 2$,where (11) has been used. One deduces

$$|\tilde{\zeta}^1 - \tilde{\zeta}^2|_{(L^\infty(0,T,L^2))^n} \leq C'T|\zeta^1 - \zeta^2|_{(L^\infty(0,T,L^2))^n}.$$

If T is such that $C'T < 1$, then $\mathcal{T}$ becomes a contraction from B_R into itself and therefore it has a unique fixed point $\zeta \in (L^\infty(0,T,H^s))^n$. Since $\frac{\partial}{\partial t}\zeta \in (L^\infty(0,T,H^{s-1}))^n$, we get $\zeta \in (\mathcal{C}([0,T],H^{s-\eta}))^n$ for all $\eta > 0$. We still have to prove that $\zeta \in (\mathcal{C}([0,T],H^s))^n$. To this end, use the fact that $(u,v) \in (L^\infty(0,T,H^{s+1}))^n$ and write ζ in term of caracteristics. This concludes the proof of the proposition 1. ■

2.2 Globalization

In order to show that the solution is global, it is sufficient to prove that $|\zeta|_{(H^s)^n}(t)$ can not tend to infinity in finite time. To do that, we use the fact that (22) can be refined by Youdovitch's techniques [12] for the 2-D Euler equation (see also [3] or [8]). Namely, we have $|\zeta(t)|_{(L^\infty)^n} \leq C|\zeta_0|_{(L^\infty)^n}$. It follows that $|(u,v)(t)|_{\mathcal{C}^1_*} \leq C|\zeta_0|_{(L^\infty)^n}$, where $\mathcal{C}^1_*$ denotes Zygmund's class (see [3]). Then, for all $\varepsilon > 0$, on obtains

$$|\nabla(u,v)(t)|_{(L^\infty)^{2n}} \leq \frac{C}{\varepsilon}|(u,v)|_{\mathcal{C}^1_*}\log(e + \frac{|(u,v)|_{\mathcal{C}^{1+\varepsilon}}}{|(u,v)|_{\mathcal{C}^1_*}})$$

(see [3] for a proof of this inequality). In the present context, this gives:

$$\begin{aligned} |\nabla(u,v)(t)|_{(L^\infty)^{2n}} &\leq C\log(e + |(u,v)|_{(H^{s+1})^n}) \\ &\leq C\log(e + |\zeta|_{(H^s)^n}), \end{aligned}$$

as soon as $s > 2$. Estimate (21) leads to

$$\frac{\partial}{\partial t}|\zeta|^2_{(H^s)^n} \leq C|\zeta|^2_{(H^s)^n}\log(e + |\zeta|_{(H^s)^n}),$$

which gives $|\zeta|_{(H^s)^n} \leq Ce^{Ce^t}$. Hence, the solution is global. ■

3 Continuous stratification limit.

The above results give some bounds on the solution. However, these bounds depend on the number of layers n and therefore they are not directly helpful in performing the limit $n \to \infty$. Here, we will derive some bounds which are independent of the number of layers and introduce functions depending of a vertical variable z.

Remark 1 *The different constants occuring in this section, and that we denote generically by C,* **do not depend on** n.

We will use in this section the notations introduced in the introduction, especially (6)-(15).

Let $\zeta_0(x,y,z)$ be defined on $\mathbf{T}^2 \times [0,1]$ and let V_ε be the subspace engendered by $\{(\phi_i^\varepsilon(z)), i = 1, ..., n\}$ and Π_ε the projector onto V_ε in L^2. We assume that $\zeta_0(x,y,z) \in L^2 \cap \mathcal{C}^0(\mathbf{T}^2 \times [0,1])$ and $\zeta_0^\varepsilon = \Pi_\varepsilon \zeta_0$.

The result reads as follows.

Theorem 2 *Let $\zeta_0(x,y,z) \in L^2 \cap \mathcal{C}^0(\mathbf{T}^2 \times [0,1])$. The following convergences hold for all $0 < T < \infty$ when ε tends to 0:*

$$i)(U_2^\varepsilon, V_2^\varepsilon) \to (U,V), \ \text{ in } \mathcal{C}([0,T], L^2(\mathbf{T}^2 \times [0,1])) \text{ strongly.}$$

ii) Z_1^ε and Z_2^ε converge to the same limit Z in $L^p(0,T,L^2(\mathbf{T}^2 \times [0,1]))$ strongly for all $p < \infty$ and in $L^\infty(0,T,L^2 \cap L^\infty(\mathbf{T}^2 \times [0,1]))$ weakly.

$$iii)\Psi_3^\varepsilon \rightharpoonup \Psi \ \text{ in } L^\infty(0,T,L^2(\mathbf{T}^2 \times [0,1])) \text{ weakly,}$$

$$iv)\nabla^\perp \Psi_1^\varepsilon \to \nabla^\perp \Psi \ \text{ in } \mathcal{C}([0,T], L^2(\mathbf{T}^2 \times [0,1]))$$
$$\text{strongly and in } L^\infty(0,T,H^1(\mathbf{T}^2 \times [0,1])) \text{ weakly.}$$

Moreover (U,V), Ψ and Z satisfy

$$\frac{\partial Z}{\partial t} + \nabla.((U,V)(Z+ay)) = 0,$$

$$U = -\frac{\partial \Psi}{\partial y} \ \text{ and } V = \frac{\partial \Psi}{\partial x},$$

and

$$Z = \Delta_{x,y}\Psi + \frac{\partial}{\partial z}(\beta(z)\frac{\partial \Psi}{\partial z}),$$

with $\frac{\partial \Psi}{\partial z} = 0$ in $z = 0$ and $z = 1$,

$$Z(t=0) = \zeta_0.$$

We have used the classical notation $\nabla^\perp h$ for the vector field $(-\frac{\partial h}{\partial y}, \frac{\partial h}{\partial x})$. The remaining of this section is devoted to the proof of this result.

3.1 A priori estimates.

We define the sequences Z_{01}^{ε} and Z_{02}^{ε} by (11) and (12). Then Z_{01}^{ε} and Z_{02}^{ε} converge to ζ_0 in $L^2(\mathbf{T}^2 \times [0,1])$ strongly. Moreover, since the functions $\zeta_l^{\varepsilon} + ay$ are transporting by a mesure preserving flow in time, we have:

$$\text{the sequences } Z_1^{\varepsilon} \text{ and } Z_2^{\varepsilon} \text{ are bounded in } L^{\infty}(0,+\infty, L^2 \cap L^{\infty}(\mathbf{T}^2 \times [0,1])) \tag{27}$$

and moreover the following equality holds:

$$|ay + \zeta_l^{\varepsilon}(x,y,t)|_{L^2\cap L^{\infty}(\mathbf{T}^2)} = |ay + \zeta_{0l}^{\varepsilon}(x,y)|_{L^2\cap L^{\infty}(\mathbf{T}^2)}. \tag{28}$$

Definition 1 *Let us introduce the space $H_n^s(\mathbf{T^2})$ as follows.*

$$H_n^s(\mathbf{T^2}) = \Big\{ \psi = (\psi_1, ..., \psi_n) \in (H^s(\mathbf{T^2}))^n, \text{ such that } \int_{\mathbf{T^2}} \sum_{i=1}^{n} \psi_i = 0 \text{ and}$$

$$\frac{1}{n-1} \sum_{i=1}^{n} \int_{\mathbf{T^2}} |(-\Delta_{x,y})^{s/2}\psi_i|^2 + \int_{\mathbf{T^2}} \frac{1}{n-1} |(\mathcal{A}_n)^{s/2}\psi|^2 < \infty, \Big\}$$

where $\mathcal{A}_n$ is the matrix introduced in section 2.1.

The space $H_n^s(\mathbf{T^2})$ is endowed with its natural norm.

Remark 2 *The matrix $\mathcal{A}_n$ correspond to a discretization of the operator*

$$\frac{\partial}{\partial z}(\beta(z)\frac{\partial \Psi}{\partial z})$$

with homogeneous Neuman boundary conditions, see for example [10].

If we denote by λ_l the lth eigenvalue of $-\frac{\partial}{\partial z}(\beta(z)\frac{\partial\Psi}{\partial z})$ and by λ_l^{ε} the lth eigenvalue of $\mathcal{A}_n$, the Min-Max principle (see [5]) implies $\lambda_l^{\varepsilon} \geq \lambda_l$. Moreover $\lambda_0 = 0$ and $\lambda_1 > 0$, hence $\mathcal{A}_n$ has 0 as simple eigenvalue, the corresponding eigenvector is $(1, ..., 1)$. Therefore $(\frac{1}{n-1}\sum_{i=1}^{n}\int_{\mathbf{T^2}} |(-\Delta_{x,y})^{s/2}\psi_i|^2 + \int_{\mathbf{T^2}} \frac{1}{n-1}|(\mathcal{A}_n)^{s/2}\psi|^2)^{1/2}$ is a norm on $H_n^s(\mathbf{T^2})$.

Lemma 1 *The sequence (ψ_l^{ε}) satisfies*

$$|(\psi_1^{\varepsilon}, ..., \psi_n^{\varepsilon})|_{L^{\infty}(\mathbf{R}^+, H_n^2(\mathbf{T^2}))} \leq C,$$

where the constant C is independent of n.

Proof: We form the scalar product of (4) with $(\Delta_{x,y}\psi_l^{\varepsilon})$ and with $(\mathcal{A}_n(\psi_l^{\varepsilon}))$ and we use the fact that (ζ_l^{ε}) is bounded in $L^{\infty}(\mathbf{R}^+, H_n^0(\mathbf{T^2}))$. This yields the result. We obtain moreover

$$\frac{1}{n-1}\int_{\mathbf{T^2}} |\nabla_{x,y}\mathcal{A}_n^{1/2}(\psi_l^{\varepsilon})|^2 \in L^{\infty}(\mathbf{R}^+). \tag{29}$$

■

Lemma 2 *The sequence* (ζ_l^ε) *satisfies*

$$\frac{1}{n-1}\sum_{l=1}^{n}|\frac{\partial\zeta_l^\varepsilon}{\partial t}|_{H^{-1}(\mathbf{T}^2)} \leq K$$

and $\dfrac{\partial\Psi_1^\varepsilon}{\partial t}$ *is bounded in* $L^\infty(\mathbf{R}^+, H^1(\mathbf{T}^2\times[0,1]))$.

Proof: For all l we have

$$\frac{\partial}{\partial t}\zeta_l^\varepsilon + \nabla_{x,y}.(\nabla_{x,y}^\perp\psi_l^\varepsilon(\zeta_l^\varepsilon + ay)) = 0.$$

Furthermore

$$\begin{aligned}|\nabla_{x,y}.(\nabla_{x,y}^\perp\psi_l^\varepsilon(\zeta_l^\varepsilon + ay))|_{H^{-1}(\mathbf{T}^2)} &\leq C|\psi_l^\varepsilon|_{H^1(\mathbf{T}^2)}|\zeta_l^\varepsilon|_{L^\infty(\mathbf{T}^2)},\\ &\leq C|\psi_l^\varepsilon|_{H^1(\mathbf{T}^2)}.\end{aligned}$$

These two last inequalities imply

$$|\frac{\partial\zeta_l^\varepsilon}{\partial t}|_{H^{-1}(\mathbf{T}^2)} \leq C|\psi_l^\varepsilon|_{H^1(\mathbf{T}^2)}.$$

The first part of the lemma then follows from (29). For the second part, differentiate (4) with respect to t and multiply it by (ψ_l^ε), one obtains that

$$\frac{\partial\psi_l^\varepsilon}{\partial t} \text{ is bounded in } L^\infty(0,T,H_n^1(\mathbf{T}^2)),$$

it then follows that $\dfrac{\partial\Psi_1^\varepsilon}{\partial t}$ is bounded in $L^\infty(\mathbf{R}^+, H^1(\mathbf{T}^2\times[0,1]))$ by the definition (6) of Ψ_1^ε. ■

Lemma 3 *The sequences* Ψ_3^ε *and* Z_2^ε *are bounded in* $L^\infty(\mathbf{R}^+, L^2(\mathbf{T}^2\times[0,1]))$.

Proof: It is sufficient to show that the matrix B_n occuring in system (10) is invertible, the norm of its inverse beeing bounded independantly of n. The matrix B_n is clearly symmetric and irreductible. Gerschgörin's theorem implies that its eigenvalues are included in the union of the disks

$$|\lambda - \frac{1}{3}| \leq \frac{1}{6} \text{ and } |\lambda - \frac{2}{3}| \leq \frac{1}{3}.$$

The lemma follows. ■

3.2 End of the proof of the theorem.

Let us consider $\nabla^{\perp}_{x,y}\Psi_1^{\varepsilon} = \sum_{l=1}^{n} \nabla^{\perp}_{x,y}\psi_l^{\varepsilon}(x,y,t)\phi_l^{\varepsilon}(z)$. The vector field $\nabla^{\perp}_{x,y}\Psi_1^{\varepsilon}$ is bounded in $L^{\infty}(\mathbf{R}^+, H^1(\mathbf{T}^2 \times [0,1]))$ by (29) and $\frac{\partial}{\partial t}\nabla^{\perp}_{x,y}\Psi_1^{\varepsilon}$ is bounded in $L^{\infty}(\mathbf{R}^+, L^2(\mathbf{T}^2 \times [0,1]))$ thanks to lemma 2. We therefore can extract a subsequence that converges in $\mathcal{C}([0,T], L^2(\mathbf{T}^2 \times [0,1]))$ strongly and in $L^{\infty}(0,T, H^1(\mathbf{T}^2 \times [0,1]))$ weakly for all $T < \infty$ to a vector field $\nabla^{\perp}_{x,y}\Psi$. Moreover $Z_1^{\varepsilon} \rightharpoonup Z$ in $L^{\infty}(0,T, L^2 \cap L^{\infty})$ weakly. We will now show that the limits of the sequences Z_1^{ε} and Z_2^{ε} are the same, and that it is also the case for those of Ψ_1^{ε}, Ψ_2^{ε} and Ψ_3^{ε}.

Function Ψ_1^{ε} is related to a P1 finite element approximation in the z-direction of Ψ, while Ψ_2^{ε} is a piecewise constant approximation of Ψ_1^{ε}. It will therefore be possible to show that $\nabla^{\perp}_{x,y}\Psi_2^{\varepsilon} - \nabla^{\perp}_{x,y}\Psi_1^{\varepsilon} \to_{\varepsilon\to 0} 0$ in $L^{\infty}([0,T], L^2(\mathbf{T}^2 \times [0,1]))$ strongly; this is related to the fact that $\nabla^{\perp}_{x,y}\Psi_1^{\varepsilon}$ is bounded in $L^{\infty}([0,T], H^1(\mathbf{T}^2 \times [0,1]))$.

Functions Ψ_3^{ε} and Z_2^{ε} are also some piecewise constant approximations of Ψ and Z, but these approximations are obtained versus the mass matrix B_n, and we can not use directly the H^1 bound of Ψ_1^{ε} and therefore the convergences $\Psi_3^{\varepsilon} - \Psi \to 0$ and $Z_2^{\varepsilon} - Z \to 0$ are obtained only in $L^{\infty}([0,T], L^2(\mathbf{T}^2 \times [0,1]))$ weakly.

These results are stated precisely in the next two propositions and proved by some explicit computations.

Proposition 2 *The sequence $\nabla^{\perp}_{x,y}\Psi_2^{\varepsilon}$ converges to $\nabla^{\perp}_{x,y}\Psi$ in $\mathcal{C}([0,T], L^2(\mathbf{T}^2 \times [0,1]))$ strongly.*

Proof: We compute the $L^2(\mathbf{T}^2 \times [0,1])$-norm of the difference between $\nabla^{\perp}_{x,y}\Psi_2^{\varepsilon}$ and $\nabla^{\perp}_{x,y}\Psi_1^{\varepsilon}$ and we get

$$|\nabla^{\perp}_{x,y}\Psi_2^{\varepsilon} - \nabla^{\perp}_{x,y}\Psi_1^{\varepsilon}|_{L^2(\mathbf{T}^2\times[0,1])}$$

$$= \int_{\mathbf{T}^2}\int_0^1 |\sum_{l=1}^{n} \nabla^{\perp}_{x,y}\psi_l^{\varepsilon}(x,y,t)\phi_l^{\varepsilon}(z) - \sum_{l=1}^{n-1} \nabla^{\perp}_{x,y}\psi_l^{\varepsilon}(x,y,t)\mathbf{1}_{[(l-1)\varepsilon, l\varepsilon[}(z)|^2 dxdydz.$$

Taking into account the supports of ϕ_l^{ε}, we obtain

$$\int_0^1 |\sum_{l=1}^{n} \nabla^{\perp}_{x,y}\psi_l^{\varepsilon}(x,y,t)\phi_l^{\varepsilon}(z) - \sum_{l=1}^{n-1} \nabla^{\perp}_{x,y}\psi_l^{\varepsilon}(x,y,t)\mathbf{1}_{[(l-1)\varepsilon, l\varepsilon[}(z)|^2 dz$$

$$= \int_0^1 \sum_{l=1}^{n} |\nabla^{\perp}_{x,y}\psi_l^{\varepsilon}|^2 |\phi_l^{\varepsilon}|^2 dz + \frac{1}{n-1}\sum_{l=1}^{n-1} |\nabla^{\perp}_{x,y}\psi_l^{\varepsilon}|^2$$

$$-2\int_0^1 \sum_{l=1}^{n-1} \nabla^{\perp}_{x,y}\psi_l^{\varepsilon}\phi_l^{\varepsilon}\nabla^{\perp}_{x,y}\psi_l^{\varepsilon}\mathbf{1}_{[(l-1)\varepsilon, l\varepsilon[}$$

$$-2\int_0^1 \sum_{l=2}^{n-1} \nabla^{\perp}_{x,y}\psi^{\varepsilon}_l \phi^{\varepsilon}_l \nabla^{\perp}_{x,y}\psi^{\varepsilon}_{l-1}\mathbf{1}_{[(l-2)\varepsilon,(l-1)\varepsilon[}$$

$$+2\int_0^1 \sum_{l=1}^{n-1} \nabla^{\perp}_{x,y}\psi^{\varepsilon}_l \phi^{\varepsilon}_l \nabla^{\perp}_{x,y}\psi^{\varepsilon}_{l+1}\phi^{\varepsilon}_{l+1}.$$

An explicit computation of the integrales yields

$$\int_0^1 |\sum_{l=1}^{n} \nabla^{\perp}_{x,y}\psi^{\varepsilon}_l(x,y,t)\phi^{\varepsilon}_l(z) - \sum_{l=1}^{n-1} \nabla^{\perp}_{x,y}\psi^{\varepsilon}_l(x,y,t)\mathbf{1}_{[(l-1)\varepsilon,l\varepsilon[}(z)|^2 dz$$

$$= \frac{2\varepsilon}{3}\sum_{l=2}^{n-2} \nabla^{\perp}_{x,y}\psi^{\varepsilon}_l.(\nabla^{\perp}_{x,y}\psi^{\varepsilon}_l - \nabla^{\perp}_{x,y}\psi^{\varepsilon}_{l-1}) \tag{30}$$

$$-\frac{2\varepsilon}{3}\nabla^{\perp}_{x,y}\psi^{\varepsilon}_n.\nabla^{\perp}_{x,y}\psi^{\varepsilon}_{n-1} + \frac{\varepsilon}{3}|\nabla^{\perp}_{x,y}\psi^{\varepsilon}_1|^2 + \frac{\varepsilon}{3}|\nabla^{\perp}_{x,y}\psi^{\varepsilon}_n|^2.$$

Then, since $(\nabla^{\perp}_{x,y}\psi^{\varepsilon}_1)_l$ is bounded in $L^{\infty}(\mathbf{R}^+, H^1_n(\mathbf{T}^2))$, we have

$$|\int_{\mathbf{T}^2} \frac{2\varepsilon}{3}\sum_{l=2}^{n-2} \nabla^{\perp}_{x,y}\psi^{\varepsilon}_l.(\nabla^{\perp}_{x,y}\psi^{\varepsilon}_l - \nabla^{\perp}_{x,y}\psi^{\varepsilon}_{l-1})| \leq C\frac{2\varepsilon}{3} \rightarrow_{\varepsilon\rightarrow 0} 0$$

in $L^{\infty}(0,T)$. We still have to deal with the following terms in (30):

$$-\frac{2\varepsilon}{3}\nabla^{\perp}_{x,y}\psi^{\varepsilon}_n.\nabla^{\perp}_{x,y}\psi^{\varepsilon}_{n-1} + \frac{\varepsilon}{3}|\nabla^{\perp}_{x,y}\psi^{\varepsilon}_1|^2 + \frac{\varepsilon}{3}|\nabla^{\perp}_{x,y}\psi^{\varepsilon}_n|^2.$$

Since $\nabla^{\perp}_{x,y}\Psi^{\varepsilon}_1$ is bounded in $L^{\infty}(\mathbf{R}^+, H^1(\mathbf{T}^2\times[0,1]))$, for allmost every x, y and for all i, we have

$$|\nabla^{\perp}_{x,y}\psi^{\varepsilon}_i|^2_{L^{\infty}(0,1)} \leq C|\nabla^{\perp}_{x,y}\psi^{\varepsilon}_i|^2_{H^1(0,1)}.$$

Hence

$$\int_{\mathbf{T}^2} |\nabla^{\perp}_{x,y}\psi^{\varepsilon}_i|^2 \leq C|\nabla^{\perp}_{x,y}\Psi^{\varepsilon}_1|^2_{H^1((0,1)\times\mathbf{T}^2)},$$

and

$$\int_{\mathbf{T}^2}(-\frac{2\varepsilon}{3}\nabla^{\perp}_{x,y}\psi^{\varepsilon}_n.\nabla^{\perp}_{x,y}\psi^{\varepsilon}_{n-1} + \frac{\varepsilon}{3}|\nabla^{\perp}_{x,y}\psi^{\varepsilon}_1|^2 + \frac{\varepsilon}{3}|\nabla^{\perp}_{x,y}\psi^{\varepsilon}_n|^2) \rightarrow_{\varepsilon\rightarrow 0} 0$$

in $L^{\infty}(0,T)$. Plugging this result in (30) and integrating on $\mathbf{T}^2$ conclude the proof of the proposition. ∎

Concerning the sequences Ψ^{ε}_3 and Z^{ε}_2, we have the following result.

Proposition 3 *The sequences Ψ^{ε}_3 and Z^{ε}_2 converges respectively to Ψ and Z in $L^{\infty}(0,T,L^2(\mathbf{T}^2\times[0,1]))$ weakly.*

Proof: Since the functions Ψ_3^ε and Z_2^ε are obtained by the same construction, it is enough to show the result for one of them. We will work with Z_2^ε and we will show that $Z_1^\varepsilon - Z_2^\varepsilon \rightharpoonup 0$ in $L^\infty(0,T,L^2(\mathbf{T}^2 \times [0,1]))$.

For any interval $[a,b] \subset [0,1]$, for fixed n, there exists two integers k_1 and k_2 such that $\frac{k_1-1}{n-1} < a \leq \frac{k_1}{n-1}$ and $\frac{k_2}{n-1} \leq b < \frac{k_2+1}{n-1}$. Then

$$\int_a^b (Z_1^\varepsilon - Z_2^\varepsilon)dz = \int_a^{\frac{k_1}{n-1}} (Z_1^\varepsilon - Z_2^\varepsilon)dz + \int_{\frac{k_1}{n-1}}^{\frac{k_2}{n-1}} (Z_1^\varepsilon - Z_2^\varepsilon)dz + \int_{\frac{k_2}{n-1}}^{b} (Z_1^\varepsilon - Z_2^\varepsilon)dz. \quad (31)$$

Let us compute each term of the right-hand side of (31)

$$i) \int_{\frac{k_1}{n-1}}^{\frac{k_2}{n-1}} (Z_1^\varepsilon - Z_2^\varepsilon)dz = \int_{\frac{k_1}{n-1}}^{\frac{k_2}{n-1}} \sum_{l=k_1+1}^{k_2} \zeta_l^\varepsilon \mathbf{1}_{[(l-1)\varepsilon, l\varepsilon[} dz - \int_{\frac{k_1}{n-1}}^{\frac{k_2}{n-1}} \sum_{l=k_1-1}^{k_2} g_l^\varepsilon \phi_l^\varepsilon dz$$

$$= \varepsilon \sum_{l=k_1+1}^{k_2} \zeta_l^\varepsilon - g_{k_1-1}^\varepsilon \frac{\varepsilon}{2} - g_{k_2}^\varepsilon \frac{\varepsilon}{2} - \varepsilon \sum_{l=k_1}^{k_2-1} g_l^\varepsilon,$$

using the relationship between ζ_l^ε and g_l^ε, namely $\zeta_l^\varepsilon = \frac{1}{6}(g_{l-1}^\varepsilon + 4g_l^\varepsilon + g_{l+1}^\varepsilon)$ for $1 < l < n$, we get after simplification

$$\int_{\frac{k_1}{n-1}}^{\frac{k_2}{n-1}} (Z_1^\varepsilon - Z_2^\varepsilon)dz = \varepsilon\zeta_{k_2}^\varepsilon - \frac{\varepsilon}{3}g_{k_2}^\varepsilon - \frac{\varepsilon}{3}g_{k_2-1}^\varepsilon - \frac{\varepsilon}{2}g_{k_1-1}^\varepsilon - \frac{\varepsilon}{3}g_{k_1+1}^\varepsilon - \frac{5\varepsilon}{6}g_{k_1}^\varepsilon. \quad (32)$$

Therefore, equation (32) yields

$$|\int_{\frac{k_1}{n-1}}^{\frac{k_2}{n-1}} (Z_1^\varepsilon - Z_2^\varepsilon)dz| \leq C\sqrt{\varepsilon}(\varepsilon \sum_{l=1}^n g_l^{\varepsilon 2})^{1/2}; \quad (33)$$

the constant C do not depend on n, neither on $[a,b]$.

ii) We proceed in the same way for the terms $\int_a^{\frac{k_1}{n-1}}$ et $\int_{\frac{k_2}{n-1}}^b$ and we obtain finally as in (33)

$$|\int_a^b (Z_1^\varepsilon - Z_2^\varepsilon)dz| \leq C\sqrt{\varepsilon}(\varepsilon \sum_{l=1}^n g_l^{\varepsilon 2})^{1/2}. \quad (34)$$

Take now $\phi \in L^1(0,T,L^2(\mathbf{T}^2 \times [0,1]))$ and construct

$$\phi^N = \sum_{i=1}^N \alpha_i^N(x,y,t)\mathbf{1}_{[a_i(x,y,t),b_i(x,y,t)[}(z)$$

such that

$$|\phi - \phi^N|_{L^1(0,T,L^2(\mathbf{T}^2\times[0,1]))} \rightarrow_{N\rightarrow\infty} 0.$$

Then,

$$\int_0^T \int_{\mathbf{T}^2} \int_0^1 (Z_1^\varepsilon - Z_2^\varepsilon)\phi dxdydzdt =$$

$$\int_0^T \int_{\mathbf{T}^2} \int_0^1 (Z_1^\varepsilon - Z_2^\varepsilon)(\phi - \phi^N) + \int_0^T \int_{\mathbf{T}^2} \int_0^1 (Z_1^\varepsilon - Z_2^\varepsilon)\phi^N.$$

The first estimate we have is

$$|\int_0^T \int_{\mathbf{T}^2} \int_0^1 (Z_1^\varepsilon - Z_2^\varepsilon)\phi| \leq C|\phi - \phi^N|_{L^1(0,T,L^2(\mathbf{T}^2\times[0,1]))} + |\int_0^T \int_{\mathbf{T}^2} \int_0^1 (Z_1^\varepsilon - Z_2^\varepsilon)\phi^N|.$$

Hence, by (24),

$$|\int_0^T \int_{\mathbf{T}^2} \int_0^1 (Z_1^\varepsilon - Z_2^\varepsilon)\phi^N| \leq \int_0^T \int_{\mathbf{T}^2} C\sqrt{\varepsilon}(\sum_{i=1}^N |\alpha_i^N(x,y,t)|)(\varepsilon \sum_{l=1}^n g_l^{\varepsilon 2})^{1/2},$$

$$\leq \sqrt{\varepsilon}|(\sum_{i=1}^N |\alpha_i^N(x,y,t)|)|_{L^2((0,T)\times\mathbf{T}^2)},$$

since $(\varepsilon \sum_{l=1}^n g_l^{\varepsilon 2})^{1/2}$ is bounded in $L^\infty((0,T), L^2(\mathbf{T}^2))$. We therefore obtain

$$|\int_0^T \int_{\mathbf{T}^2} \int_0^1 (Z_1^\varepsilon - Z_2^\varepsilon)\phi| \leq C|\phi - \phi^N|_{L^1(0,T,L^2(\mathbf{T}^2\times[0,1]))}$$

$$+C\sqrt{\varepsilon}|(\sum_{i=1}^N |\alpha_i^N(x,y,t)|)|_{L^2((0,T)\times\mathbf{T}^2)}.$$

It follows that

$$\limsup_{\varepsilon\to 0} |\int_0^T \int_{\mathbf{T}^2} \int_0^1 (Z_1^\varepsilon - Z_2^\varepsilon)\phi| \leq C|\phi - \phi^N|_{L^1(0,T,L^2(\mathbf{T}^2\times[0,1]))}.$$

Letting $N \to \infty$ in this expression, we obtain the result. ∎

We can now perform the limit process on (13), (14) and (15). Thanks to proposition 2, (15) gives directly:

$$U = -\frac{\partial \Psi}{\partial y} \text{ and } V = \frac{\partial \Psi}{\partial x}.$$

On the other hand, since $Z_1^\varepsilon \rightharpoonup Z$ in $L^\infty(0,T,L^2\cap L^\infty)$ weakly and since $(U_2^\varepsilon, V_2^\varepsilon) \to (U,V)$ in $L^\infty(0,T,L^2)$ strongly, we have

$$(U_2^\varepsilon, V_2^\varepsilon)(Z_1^\varepsilon + ay) \rightharpoonup (U,V)(Z+ay) \text{ in } \mathcal{D}',$$

and (13) gives

$$\frac{\partial Z}{\partial t} + \nabla.((U,V)(Z+ay)) = 0.$$

In order to perform the limit on the elliptic equation (14), one use proposition 3 and the classical result of variational approximation.

Moreover, since the L^2 norm of $Z + ay$ is conserved by the flow, and since $Z_{01}^\varepsilon \to \zeta_0$ in L^2 strongly, for all t the convergences of $Z_1^\varepsilon(t)$ and $Z_2^\varepsilon(t)$ are strong in L^2, hence $Z_1^\varepsilon(t)$ and $Z_2^\varepsilon(t)$ converge to $Z(t)$ in $L^p(0,T,L^2(\mathbf{T}^2\times[0,1]))$ strongly for all $p < \infty$. ∎

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