Remarks on an homogeneous model of ocean circulation.

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Abstract : We consider a simplified model of ocean circulation. We give a mathematical justification of a singular limit process which leads to the Sverdrup relation. This singular limit implies the existence of boundary layers that correspond to currents intensification near the western coast of oceans.

1 The homogeneous model.

1.1 Quasigeostrophic approximation.

Large-scaled motions for atmosphere and ocean have been recently studied from the mathematical point of view particulary in [3] and [4]. In this papers serie, the authors present models for global circulation phenomenas and give mathematical theories corresponding to these models. The aim of this work is to present the formal derivation of a simplified model of ocean circulation and to study it mathematicaly. This model is said homogeneous since we do not take into account the effect of stratification.

We present in this section the derivation of the homogeneous model. This derivation is more or less a unified approach of several calculations contained in the book of Pedlosky [6] in chapter 3,4 and 5. The ocean is represented

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by a layer of homogeneous fluid in a rotating frame. The equations are:

$$u_t + uu_x + vu_y + wu_z - (f_0 + \beta_0 y)v = -\frac{1}{\rho}p_x + A_V u_{zz} + A_H(u_{xx} + u_{yy}), \quad (1)$$

$$v_t + uv_x + vv_y + wv_z + (f_0 + \beta_0 y)u = -\frac{1}{\rho}p_y + A_V v_{zz} + A_H(v_{xx} + v_{yy}), \quad (2)$$

$$w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho}p_z - g + A_V w_{zz} + A_H(w_{xx} + w_{yy}), \quad (3)$$

$$u_x + v_y + w_z = 0. \quad (4)$$

(u, v, w) are the three components of the velocity of the fluid, and p is the pressure. g is the acceleration of the gravity. A_V and A_H are respectively the vertical and horizontal eddy viscosities. ρ is a **constant** parameter which value is the density of the fluid. These equations are valid in the domain $(x, y, z) \in \mathbf{R}^2 \times [-h_B(x, y), h(x, y, t)]$ where $-h_B(x, y)$ is the elevation of the bottom of the ocean and is a given function, while $h(x, y, t)$ is the elevation of the free surface of the ocean and has to be determined. Note that the $\beta-$plane approximation has already been used: the horizontal equations of motion have been projected on the tangent plane and the Coriolis parameter, which is $2\Omega \sin\theta$ (where θ is the latitude) has been linearized around a mean latitude θ_0. That is $f = f_0 + \beta_0 y$, with $f_0 = 2\Omega \sin\theta_0$ and $\beta_0 = \frac{2\Omega}{r_0}\cos\theta_0$, r_0 being the earth's radius. In order to complete the equations, we assume that

$$(u, v, w) = 0 \text{ at } z = -h_B(x, y), \quad (5)$$

i.e. the velocity field vanishes at the bottom of the ocean.

At the free surface $z = h(x, y, t)$, the following relations hold:

$$p(x, y, h(x, y, t)) = Cte, \quad (6)$$

$$w = h_t + uh_x + vh_y \text{ at } z = h(x, y, t), \quad (7)$$

$$\begin{cases} \tau^{(x)} &= \rho A_V u_z + \rho A_H w_x, \\ \tau^{(y)} &= \rho A_V v_z + \rho A_H w_y, \\ \tau^{(z)} &= \rho A_H (v_x + u_y), \end{cases} \quad (8)$$

at $z = h(x, y, t)$. Equation (6) means the continuity of the pressure since there is no surface tension, while (7) is the usual kinematic condition for a free surface. The set of equations (8) comes from the existence of an applied stress on the free surface which components are $(\tau^{(x)}, \tau^{(y)}, \tau^{(z)})$. These three functions of x, y and t are given data of the problem. (8) modelized the action of the wind on the surface of the ocean. We now give a nondimensional form of this system and we will introduce small parameters.

We denote with a prime the new variables and unknowns. Let

$$\left\{ \begin{array}{ccc} (x,y) = L(x',y'), & z = Dz', & (u,v) = U(u',v'), \\ w = \frac{UD}{L} w', & t = \frac{L}{U} t', & p = -\rho g z + \rho f_0 U L p'. \end{array} \right. \tag{9}$$

L and D denote respectively the caracteristic horizontal and vertical lenght of the motion and U is the caracteristic horizontal velocity. The scaling on w comes from the incompressibility condition (4). The scaling on p is a choice in order to ensure a balance between the coriolis force and the horizontal pressure gradient.

We also rescale h_B and h in the following way:

$$\eta_B = h_B \frac{f_0 L}{DU}, \; h = Dh',$$

and the surface stress τ is rescaled as follows:

$$\tau = \tau_0 \tau'.$$

Putting these new functions and variables in (1)-(8) and omiting the prime leads to :

$$\varepsilon(u_t + uu_x + vu_y + wu_z) - (1 + \varepsilon\beta y)v = -p_x + \frac{1}{2} E_V u_{zz} + \frac{1}{2} E_H (u_{xx} + u_{yy}), \tag{10}$$

$$\varepsilon(v_t + uv_x + vv_y + wv_z) + (1 + \varepsilon\beta y)u = -p_y + \frac{1}{2} E_V v_{zz} + \frac{1}{2} E_H (v_{xx} + v_{yy}), \tag{11}$$

$$\varepsilon\delta^2(w_t + uw_x + vw_y + ww_z) = -p_z + \delta^2(\frac{1}{2} E_V w_{zz} + \frac{1}{2} E_H (w_{xx} + w_{yy})), \tag{12}$$

$$u_x + v_y + w_z = 0. \tag{13}$$

$$(u, v, w) = 0 \text{ at } z = \varepsilon \eta_B, \tag{14}$$

$$w = h_t + u h_x + v h_y \text{ at } z = h, \tag{15}$$

$$\begin{cases} \frac{D\tau_0}{\rho A_V U} \tau^{(x)} &= u_z + \frac{E_H}{E_V} w_x, \\ \frac{D\tau_0}{\rho A_V U} \tau^{(y)} &= v_z + \frac{E_H}{E_V} w_y, \\ \frac{D\tau_0}{\rho A_H U} \frac{1}{\delta} \tau^{(z)} &= (v_x + u_y), \end{cases} \tag{16}$$

at $z = h$. The parameters are given by:
$\varepsilon = \frac{U}{f_0 L}$ is the Rossby number, $\delta = \frac{D}{L}$ is the aspect ratio, $E_V = \frac{2A_V}{f_0 D^2}$ and $E_H = \frac{2A_H}{f_0 L^2}$ are respectively the vertical and horizontal Ekman numbers and $\beta = \frac{\beta_0 L^2}{U}$. The four numbers ε, δ, E_V and E_H are small parameters while β is $O(1)$. We impose moreover the following relationships:

$$\frac{E_H}{\varepsilon} = \frac{2}{R_e}, \tag{17}$$

where R_e is the Reynolds number which is $O(1)$ and

$$\frac{E_V^{1/2}}{\varepsilon} = 2r, \tag{18}$$

where r is $O(1)$ also.

We now perform the asymptotic expansion. We introduce the following ansatz:

$$u = u_0 + \varepsilon u_1 + ..., \; v = v_0 + \varepsilon v_1 + ..., \; w = w_0 + \varepsilon w_1 + ..., \; p = p_0 + \varepsilon p_1 + ...$$

The order 0 terms in (10)-(13) satisfy

$$u_0 = -p_{0y}, \tag{19}$$

$$v_0 = p_{0x}, \tag{20}$$

$$0 = p_{0z}, \tag{21}$$

$$u_{0x} + v_{0y} + w_{0z} = 0. \tag{22}$$

This set of equations shows that the order 0 solution is independent of z. That means that we can not use the boundary conditions on (u_0, v_0, w_0) unless accepting the identicaly vanishing solution. Let us now compute the equations satisfied by the order 1 terms:

$$u_{0t} + u_0 u_{0x} + v_0 u_{0y} - \beta y v_0 - v_1 = -p_{1x} + \frac{1}{R_e}(u_{0xx} + u_{0yy}), \tag{23}$$

$$v_{0t} + u_0 v_{0x} + v_0 v_{0y} + \beta y u_0 + u_1 = -p_{1y} + \frac{1}{R_e}(v_{0xx} + v_{0yy}), \tag{24}$$

$$p_{1z} = 0, \tag{25}$$

$$u_{1x} + v_{1y} + w_{1z} = 0. \tag{26}$$

We now eliminate p_1 between (23) and (24) by cross differentiation and we get

$$-\zeta_{0t} - u_0\zeta_{0x} - v_0\zeta_{0y} = (u_{1x} + v_{1y}) - \frac{1}{R_e}(\zeta_{0xx} + \zeta_{0yy}) + \beta v_0,$$

where

$$\zeta_0 = v_{0x} - u_{0y}.$$

Using (26), we obtain:

$$\frac{d}{dt}(\zeta_0 + \beta y) = \frac{\partial w_1}{\partial z} + \frac{1}{R_e}\Delta\zeta_0, \tag{27}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y},$$

and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Integrating (27) from $\varepsilon\eta_B$ to $\frac{h}{D}$ we get at leading order

$$\frac{d}{dt}(\zeta_0 + \beta y) = w_1(x, y, \frac{h}{D}, t) - w_1(x, y, \varepsilon\eta_B, t) + \frac{1}{R_e}\Delta\zeta_0. \tag{28}$$

We still have to determine accurate boundary conditions for w_1. Because of the relation (18), this asymptotic expansion on w correspond to a zero viscosity limit. Therefore, in order to obtain the "correct" boundary conditions on w_1, we have to compute the boundary layer and to apply the matching principle (see [6] chap. 4 or [8]). In the context of rotating fluids, this boundary layer take the name of Ekman layer.

i) Ekman layer at the bottom of the ocean:

We treat here the case of a flat bottom *i.e.* the case when $h_B \equiv 0$ and we introduce $l \equiv$ the thickness of the Ekman layer and $\zeta = \frac{z}{l}$ denotes the vertical coordinate inside the boundary layer. Denoting by $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}$ the fonctions inside the boundary layer, equations (10)-(13) read as follows.

$$\varepsilon(\tilde{u}_t+\tilde{u}\tilde{u}_x+\tilde{v}\tilde{u}_y+\frac{\tilde{w}}{l}\tilde{u}_\zeta)-(1+\varepsilon\beta y)\tilde{v} = -\tilde{p}_x+\frac{1}{2}\frac{E_V}{l^2}\tilde{u}_{\zeta\zeta}+\frac{1}{2}E_H(\tilde{u}_{xx}+\tilde{u}_{yy}), \quad (29)$$

$$\varepsilon(\tilde{v}_t+\tilde{u}\tilde{v}_x+\tilde{v}\tilde{v}_y+\frac{\tilde{w}}{l}\tilde{v}_\zeta)+(1+\varepsilon\beta y)\tilde{u} = -\tilde{p}_y+\frac{1}{2}\frac{E_V}{l^2}\tilde{v}_{\zeta\zeta}+\frac{1}{2}E_H(\tilde{v}_{xx}+\tilde{v}_{yy}), \quad (30)$$

$$\delta^2\varepsilon(\tilde{w}_t+\tilde{u}\tilde{w}_x+\tilde{v}\tilde{w}_y+\frac{\tilde{w}}{l}\tilde{w}_\zeta) = -\frac{1}{l}\tilde{p}_\zeta+\frac{\delta^2}{l^2}\frac{E_V}{2}\tilde{w}_{\zeta\zeta}+\frac{\delta^2}{2}E_H(\tilde{w}_{xx}+\tilde{w}_{yy}), \quad (31)$$

$$\tilde{u}_x+\tilde{v}_y+\frac{1}{l}\tilde{w}_\zeta = 0, \quad (32)$$

$$(\tilde{u}, \tilde{v}, \tilde{w}) = 0 \text{ at } \zeta = 0. \quad (33)$$

We now impose $l = E_V^{1/2}$ in order to take into account the bounadary layer (see [6] p. 200 for more details) and according to (32) we rescale $\tilde{w}$ in $\tilde{w} = E_V^{1/2}\tilde{w}_0 + ...$ Denoting by $\tilde{u}_0, \tilde{v}_0, \tilde{p}_0$ the zero order limit of $\tilde{u}_0, \tilde{v}_0, \tilde{p}_0$ as ε goes to zero we obtain the following set of equation:

$$-\tilde{v}_0 = -\tilde{p}_{0x}+\frac{1}{2}\tilde{u}_{0\zeta\zeta}, \quad (34)$$

$$\tilde{u}_0 = -\tilde{p}_{0y}+\frac{1}{2}\tilde{v}_{0\zeta\zeta}, \quad (35)$$

$$0 = -\tilde{p}_{0\zeta}, \tag{36}$$

$$\tilde{w}_{0\zeta} = -(\tilde{u}_{0x} + \tilde{v}_{0y}). \tag{37}$$

The matching principle consists in saying that for any unknown f, if we denote by $\tilde{f}$ its value inside the boundary layer, we impose:

$$\lim_{\zeta\to\infty} \tilde{f}(\zeta) = \lim_{z\to 0} f(z).$$

Applying this principle in our case, since $\tilde{p}_0$ is independent of ζ, we obtain:

$$\tilde{p}_{0x} = p_{0x} = v_0,$$

$$\tilde{p}_{0y} = p_{0y} = -u_0,$$

and

$$\begin{cases} \frac{1}{2}\tilde{u}_{0\zeta\zeta} + \tilde{v}_0 &= v_0, \\ \frac{1}{2}\tilde{v}_{0\zeta\zeta} - \tilde{u}_0 &= -u_0, \\ \tilde{w}_{0\zeta} &= -(\tilde{u}_{0x} + \tilde{v}_{0v}). \end{cases}$$

We can solve this set of equation and we get

$$\tilde{w}_0(x, y, \zeta, t) = -\frac{1}{2}(v_{0x} - u_{0y})e^{-\zeta}(\cos\zeta + \sin\zeta) + C(x, y, t).$$

The boundary condition $\tilde{w}_0(\zeta = 0) = 0$ enables us to calculate the constant $C(x, y, t)$ and we finally obtain

$$\tilde{w}_0(x, y, \zeta, t) = \frac{1}{2}(v_{0x} - u_{0y})(1 - e^{-\zeta}(\cos\zeta + \sin\zeta)).$$

It follows that

$$\lim_{\zeta\to\infty} \tilde{w}_0(x, y, \zeta, t) = \frac{1}{2}\zeta_0.$$

Hence using (18), we get

$$w_1(x, y, 0, t) = \frac{r}{2}\zeta_0.$$

In [6] p. 219, the case of a bottom with a constant slope in treated analyticaly and one find that the contribution of the slope is

$$(u_0\frac{\partial}{\partial x}+v_0\frac{\partial}{\partial y})\eta_B.$$

The author assume that this condition can be used even with a non-constant slope, and therefore the boundary condition that we take for w_1 is

$$w_1(x,y,\varepsilon\eta_B,t)=\frac{r}{2}\zeta_0+(u_0\frac{\partial}{\partial x}+v_0\frac{\partial}{\partial y})\eta_B. \quad (38)$$

ii) Ekman layer at the free boundary.
The calculation follows the same lines as the preeceding one (see [6] p. 228-229) and one obtains:

$$w_1(x,y,\frac{h}{D},t)=\frac{d}{dt}(\frac{h}{\varepsilon D})+\frac{\tau_0}{\varepsilon\rho f_0UD}\mathbf{k}.curl\tau, \quad (39)$$

where $\mathbf{k}$ is the vertical unitary vector and τ denotes the vector $(\tau^{(x)},\tau^{(y)},\tau^{(z)})$ which is given. We still have to determine the value of $\frac{h}{\varepsilon D}$. Note that (9) applied at the free surface and using (6) leads to:

$$\rho f_0Up_{0x}=\frac{\rho gD}{L}h_x,$$

$$\rho f_0Up_{0y}=\frac{\rho gD}{L}h_y.$$

It follows that

$$\frac{d}{dt}(\frac{h}{\varepsilon D})=\frac{f_0^2L^2}{gD}\frac{d}{dt}p_0.$$

Introducing $F=\frac{f_0^2L^2}{gD}$, we obtain the following boundary condition on w_1:

$$w_1(x,y,\frac{h}{D},t)=\frac{d}{dt}(Fp_0)+\frac{\tau_0}{\varepsilon\rho f_0UD}\mathbf{k}.curl\tau. \quad (40)$$

Again, these calculation are done in [6] in the case of a flat free surface (which gives the term $\frac{\tau_0}{\varepsilon\rho f_0UD}\mathbf{k}.curl\tau$) while in the case of a non flat free surface the kinematic condition (7) leads to introduce the term $\frac{d}{dt}(\frac{h}{\varepsilon D})$.

Now, equations (28), (38), (40) lead to:

$$\left\{ \begin{array}{rcl} \frac{d}{dt}(\zeta_0 + \beta y - F p_0 + \eta_B) & = & \frac{\tau_0}{\varepsilon \rho f_0 U D}\mathbf{k}.curl\tau - \frac{r}{2}\zeta_0 + \frac{1}{R_e}\Delta\zeta_0, \\ u_0 = -p_{0y}, \;\; v_0 = p_{0x}, & & \zeta_0 = v_{0x} - u_{0y} = \Delta p_0, \\ \frac{d}{dt} & = & \frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}. \end{array} \right. \tag{41}$$

The system (41) is the homogeneous model of ocean circulation. Note that this derivation is far away from being justified. Let us note that there exists a rigourous justification of the derivation of the quasigeostrophic model on all $\mathbf{R^3}$ (or with periodic boundary conditions) in Chemin [1]. The main difference with what we present here is that Chemin takes into account the variation of the density (*i.e.* ρ in equations (1)-(3) is not constant) so that the limit equation are three dimensionals ones.

1.2 Boundary conditions and the Sverdrup relation.

We now choose the velocity scaling U such that the coefficient of $\mathbf{k}.curl\tau$ is equal to 1 in order to ensure a balance between the wind-stress curl and the $\beta-$term. Rewriting (41) in terms of $\psi = p_0$ leads to

$$\frac{1}{\beta}(\frac{\partial}{\partial t} - \psi_y\frac{\partial}{\partial x} + \psi_x\frac{\partial}{\partial y})(\Delta\psi - F\psi + \eta_B) + \psi_x = \mathbf{k}.curl\tau - \frac{r}{\beta}\Delta\psi + \frac{1}{\beta R_e}\Delta^2\psi. \tag{42}$$

If we restrict ourself to a bounded open set Ω, the natural boundary conditions are

$$\psi,\ \nabla\psi = 0 \text{ on } \partial\Omega.$$

Physically, $\frac{1}{\beta}, \frac{r}{\beta}, \frac{1}{\beta R_e}$ can be considered as being small parameters (for instance, a typical value for β is 100), so that (42) can be approximated by

$$\psi_x = \mathbf{k}.curl\tau \tag{43}$$

which is the Sverdrup relation. For a detailled discussion of this relation, see [6] p. 264. The limit process from (42) to (43) is singular and a boundary layer exists for ψ. This boundary layer may be interpreted as an intensification of the currents in the western coast of the oceans ([6] p. 278).

In the remaining of this paper we give a mathematical justification of the limit process leading from (42) to (43).

Let us mention that a mathematical theory of the stationnary problem corresponding to (43) has been done in [2] in the case where $\frac{1}{\beta R_e} = 0$. The authors construct approximate solutions satisfying (43) and let $\frac{1}{\beta R_e}$ tend to 0 using uniform in $\frac{1}{\beta R_e}$ estimates.

2 A rapid theory for the evolution equation.

Equation (42) is very similar to the bidimensional Navier-Stokes equations and the Cauchy problem theory is a straightforward application of the Galerkin method. We therefore only sketch the proof. We introduce the following functions space

$$H_0^2 = \{\psi \in H^2,\ \psi,\ \nabla\psi = 0 \text{ sur } \partial\Omega\}.$$

We then write a variational formulation of (42), multiplying it by $v \in \mathcal{C}_0^\infty(\Omega)$:

$$\begin{aligned} &-\partial_t(\textstyle\int_\Omega \nabla\psi.\nabla v + F\psi v) + \int_\Omega J(\psi,\Delta\psi)v + \int_\Omega J(\psi,\eta_B)v + \beta\int_\Omega \psi_x v \\ &-\tfrac{r}{2}\textstyle\int_\Omega \nabla\psi.\nabla v - \frac{1}{R_e}\int_\Omega \Delta\psi\Delta v = \int_\Omega \beta\mathbf{k}.\mathrm{curl}\tau, \end{aligned} \tag{44}$$

where $J(f,g)$ denotes the Jacobian of the two functions f and g.

We then remark that

$$\forall\psi, v \in H_0^2 \cap \mathcal{C}^\infty(\bar{\Omega}),\ \int_\Omega J(\psi,\Delta\psi)v = \int_\Omega J(v,\psi)\Delta\psi, \tag{45}$$

so that (44) can be rewritten under the form

$$\begin{aligned} &-\partial_t(\textstyle\int_\Omega \nabla\psi.\nabla v + F\psi v) + \int_\Omega J(v,\psi)\Delta\psi + \int_\Omega J(\psi,\eta_B)v + \beta\int_\Omega \psi_x v \\ &-\tfrac{r}{2}\textstyle\int_\Omega \nabla\psi.\nabla v - \frac{1}{R_e}\int_\Omega \Delta\psi\Delta v = \int_\Omega \beta\mathbf{k}.\mathrm{curl}\tau\ . \end{aligned} \tag{46}$$

The variational problem that we consider reads as follows :

$$(P1) \text{ Find } \psi \in L^\infty(0,T,H_0^1) \cap L^2(0,T,H_0^2) \text{ such that}$$

$$\begin{aligned} \partial_t(\int_\Omega \nabla\psi.\nabla v + F\psi v) - \int_\Omega J(v,\psi)\Delta\psi - \int_\Omega J(\psi,\eta_B)v - \beta\int_\Omega \psi_x v \\ +\frac{r}{2}\int_\Omega \nabla\psi.\nabla v + \frac{1}{R_e}\int_\Omega \Delta\psi\Delta v = -\int_\Omega \beta\mathbf{k}.\mathrm{curl}\tau,\ \forall v \in \mathcal{D}(\Omega) \\ \text{and } \psi(x,0) = \psi_0(x). \end{aligned}$$

The result that we obtain is

Proposition 1 *Let us suppose that* $\mathbf{k}.curl\tau \in L^2(0,T,L^2)$, $\forall T > 0$. *There exists a unique global solution to* $(P1)$, *and moreover*

$$\psi_t \in L^{4/3}(0,T,L^2),\ \forall T > 0.$$

Proof : We apply the Galerkin method with the following a priori estimate

$$(\int_\Omega |\nabla\psi|^2 + F|\psi|^2)(T) + \int_0^T \int_\Omega (\frac{r}{2}|\nabla\psi|^2 + \frac{1}{R_e}|\Delta\psi|^2)$$
$$\leq c\int_0^T \int_\Omega |\mathbf{k}.curl\tau|^2 + \int_\Omega |\nabla\psi_0|^2 + F|\psi_0|^2.$$

In order to conclude, we apply a compactness result. ■

3 Stationary problem and Sverdrup relation.

Let us recall equation (42) in the time-independent case :

$$\frac{1}{\beta}(\psi_x\frac{\partial}{\partial y} - \psi_y\frac{\partial}{\partial x})(\Delta\psi + \eta_B) + \psi_x = \mathbf{k}.\mathrm{curl}\tau - \frac{r}{2\beta}\Delta\psi + \frac{1}{\beta R_e}\Delta^2\psi. \quad (47)$$

In order to obtain the limit expression of (47), we will work on the following domain: $(x,y) \in [0,1] \times S^1$ with the following boundary conditions :

$$\psi,\ \nabla\psi = 0 \text{ on } \{0\} \times S^1 \text{ and } \{1\} \times S^1.$$

The particular choice of this domain does not have any importance from mathematical point of view for the results that we prove. We choose it only in order to be "near" the reality. However, this particular choice would be important if we want to find a corrector for this singular limit (see the final remark). One can prove, as for time-independent Navier-Stokes equations (see [7]), that there exists at least one solution to (47). We now write (47) under the form

$$\frac{\delta_I^2}{L^2}J(\psi, \Delta\psi + \eta_B) + \psi_x = \mathbf{k}.curl\tau - \frac{\delta_S}{L}\Delta\psi + \frac{\delta_M^3}{L^3}\Delta^2\psi$$

and $\delta_I,\ \delta_S,\ \delta_M << L$. We will work in the case of the Munk layer which correspond to the asumption $\delta_M >> \delta_I,\ \delta_S$ (see [6]). From now on, all convergence processes will correspond to the limit $\frac{\delta_I}{L},\ \frac{\delta_S}{L},\ \frac{\delta_M}{L} \to 0$.

Our first result reads as follows :

Proposition 2 *Let us suppose that* $\frac{\delta_I}{L}$ *and* $\frac{\delta_S}{L}$ *have the same order of magnitude and that* $\frac{\delta_I}{L} << (\frac{\delta_M}{L})^{3/2}$. *There exists a constant* C *independant of* $\delta_I,\ \delta_S,\ \delta_M$ *such that*

$$\int \psi^2 + \frac{\delta_S}{L}\int |\nabla\psi|^2 + (\frac{\delta_M}{L})^3 \int |\Delta\psi|^2 \leq C.$$

Moreover $\psi \rightharpoonup \psi_0$ *in* $L^2(\Omega)$ *weakly, where* ψ_0 *is a solution to*

$$\frac{\partial \psi_0}{\partial x} = \mathbf{k}.curl\tau.$$

Proof : the method that we use is exposed in [5]. Here, we have to deal with two other terms, namely the jacobian and the term $\Delta\psi$. The first step consists in multiplying (47) by ψ and integrating over Ω :

$$0 = \int \mathbf{k}.\mathrm{curl}\tau\psi + \frac{\delta_S}{L}\int |\nabla\psi|^2 + (\frac{\delta_M}{L})^3 \int |\Delta\psi|^2,$$

this yields

$$\frac{\delta_S}{L}\int |\nabla\psi|^2 + (\frac{\delta_M}{L})^3 \int |\Delta\psi|^2 \leq |\mathbf{k}.\mathrm{curl}\tau|_{L^2}|\psi|_{L^2}. \tag{48}$$

In a second step we find estimates with a weight. More precisely, one multiplies (47) by $e^x\psi$ and integrates over Ω. Let us estimate each term.
$i)$ The term given by ψ_x :

$$\int \psi_x e^x \psi = -\frac{1}{2}\int \psi^2. \tag{49}$$

$ii)$ The term given by $\Delta\psi$:

$$\int \Delta\psi e^x\psi = -\int |\psi_x|^2 e^x - \int \psi_x\psi e^x - \int |\psi_y|^2 e^x,$$

it follows that

$$\int \Delta\psi e^x\psi = -\int |\nabla\psi|^2 e^x + \frac{1}{2}\int |\psi|^2 e^x. \tag{50}$$

$iii)$ The calculation for the bilaplacian term is technicaly more difficult :

$$\int \Delta^2\psi\psi e^x = -\int (\Delta\psi)_x e^x \psi_x - \int (\Delta\psi)_x e^x\psi + \int \Delta\psi\psi_{yy}e^x,$$

$$= \int \Delta\psi e^x \psi_{xx} + 2\int \Delta\psi\psi_x e^x + \int \Delta\psi\psi_{yy} e^x + \int \Delta\psi e^x \psi.$$

We then obtain

$$\int \Delta^2\psi\psi e^x = \int |\Delta\psi|^2 + 2\int \Delta\psi\psi_x e^x + \int \Delta\psi e^x \psi. \qquad (51)$$

Moreover

$$\int \Delta\psi\psi_x e^x = -\frac{1}{2}\int |\psi_x|^2 e^x + \frac{1}{2}\int |\psi_y|^2 e^x.$$

Equation (51) used with (50) leads to

$$\int \Delta^2\psi\psi e^x = \int |\Delta\psi|^2 e^x - \int |\psi_x|^2 e^x + \int |\psi_y|^2 e^x - \int |\nabla\psi|^2 e^x + \frac{1}{2}\int \psi^2 e^x.$$

Finally

$$\int \Delta^2\psi\psi e^x = \int |\Delta\psi|^2 e^x - 2\int |\psi_x|^2 e^x + \frac{1}{2}\int \psi^2 e^x. \qquad (52)$$

iv) Let us now estimate the term containing the jacobian :

$$\int J(\psi,\Delta\psi)\psi e^x = -\int \psi_{xy}\Delta\psi\psi e^x - \int \psi_x \Delta\psi\psi_y e^x$$

$$+ \int \psi_{xy}\Delta\psi\psi e^x + \int \psi_x\Delta\psi\psi_y e^x + \int \psi_y\Delta\psi\psi e^x,$$

hence

$$\int J(\psi,\Delta\psi)\psi e^x = \int \psi_y \Delta\psi\psi e^x. \qquad (53)$$

On the other hand

$$\int \psi_y\Delta\psi\psi e^x = \int \psi_y\psi_{xx}\psi e^x + \int \psi_y\psi_{yy}\psi e^x,$$

$$= -\int \psi_{xy}\psi_x\psi e^x - \int \psi_y|\psi_x|^2 e^x - \int \psi_y\psi_x\psi e^x - \frac{1}{2}\int \psi_y^3 e^x.$$

Together with (53), this inequality leads to

$$\int J(\psi,\Delta\psi)\psi e^x = -\frac{1}{2}\int \psi_y|\nabla\psi|^2 e^x - \int \psi_y\psi_x\psi e^x. \qquad (54)$$

Making use of (49), (50), (51) and (52) in (47), we get

$$
\begin{aligned}
&(\tfrac{\delta_M^3}{2L^3} + \tfrac{1}{2} - \tfrac{\delta_S}{2L}) \textstyle\int \psi^2 e^x \\
&+ \tfrac{\delta_S}{L} \textstyle\int |\nabla\psi|^2 e^x + \tfrac{\delta_M^3}{L^3} \int |\Delta\psi|^2 e^x \\
&= 2\tfrac{\delta_M^3}{L^3} \textstyle\int |\psi_x|^2 e^x - \int \mathbf{k}.\mathrm{curl}\tau\psi e^x - \tfrac{\delta_I^2}{2L^2} \int \psi_y |\nabla\psi|^2 e^x - \tfrac{\delta_I^2}{L^2} \int \psi_y \psi_x \psi e^x - \tfrac{\delta_I^2}{2L^2} \int \psi^2 \eta_{By} e^x.
\end{aligned}
\tag{55}
$$

We now estimate the right-hand side of (55).

Let us first recall the interpolation estimate

$$|\psi|_{H_0^1} \leq C|\psi|_{L^2}^{1/2} |\Delta\psi|_{L^2}^{1/2},$$

we then obtain

$$2\frac{\delta_M^3}{L^3} \int |\psi_x|^2 e^x \leq 2\frac{\delta_M^3}{L^3} |\Delta\psi|_{L^2}^2 + C\frac{\delta_M^3}{L^3} |\psi|_{L^2}^2. \tag{56}$$

Therefore equation (55) with (56) leads to

$$
\begin{aligned}
&(\tfrac{\delta_M^3}{2L^3} + \tfrac{1}{2} - \tfrac{\delta_S}{2L} - C\tfrac{\delta_M^3}{L^3} - C\tfrac{\delta_I^2}{2L^2} - \tfrac{1}{4}) \textstyle\int \psi^2 e^x + \tfrac{\delta_S}{L} \int |\nabla\psi|^2 e^x + \tfrac{\delta_M^3}{2L^3} \int |\Delta\psi|^2 e^x \\
&\leq K - \tfrac{\delta_I^2}{2L^2} (\textstyle\int \psi_y |\nabla\psi|^2 e^x + 2\int \psi_y \psi_x \psi e^x).
\end{aligned}
\tag{57}
$$

We still have to estimate the two terms of the right-hand side of (57). We first have

$$
\begin{aligned}
|\int \psi_y \psi_x \psi e^x| &\leq C(\int \psi^2)^{1/2} (\int \psi_x^2 \psi_y^2)^{1/2}, \\
&\leq \frac{L^2}{8\delta_I^2} \int \psi^2 + C\frac{\delta_I^2}{L^2} \int \psi_x^2 \psi_y^2, \\
&\leq \frac{L^2}{8\delta_I^2} \int \psi^2 + C\frac{\delta_I^2}{L^2} \int |\nabla\psi|^2 \int |\Delta\psi|^2.
\end{aligned}
$$

Thanks to (48), we obtain

$$|\int \psi_y \psi_x \psi e^x| \leq \frac{L^2}{8\delta_I^2} \int \psi^2 + C\frac{\delta_I^2}{L^2} \frac{L}{\delta_S} \frac{L^3}{\delta_M^3} \int \psi^2,$$

which leads to

$$|\int \psi_y \psi_x \psi e^x| \leq (\frac{L^2}{8\delta_I^2} + C\frac{\delta_I^2}{L^2} \frac{L}{\delta_S} \frac{L^3}{\delta_M^3}) \int \psi^2. \tag{58}$$

On the other hand, we have

$$\frac{\delta_I^2}{L^2}|\int \psi_y|\nabla\psi|^2 e^x| \leq \frac{\delta_I^2}{L^2}(\int |\nabla\psi|^2)^{1/2}(\int |\nabla\psi|^4)^{1/2},$$

$$\leq \frac{\delta_S}{L}\int |\nabla\psi|^2 + C\frac{\delta_I^4}{L^4}\frac{L}{\delta_S}\int |\nabla\psi|^4,$$

$$\leq \frac{\delta_S}{L}\int |\nabla\psi|^2 + C\frac{\delta_I^4}{L^4}\frac{L}{\delta_S}\int |\nabla\psi|^2\int |\Delta\psi|^2,$$

$$\leq \frac{\delta_S}{L}\int |\nabla\psi|^2 + C\frac{\delta_I^4}{L^4}\frac{L}{\delta_S}\frac{L}{\delta_S}\frac{L^3}{\delta_M^3}\int \psi^2$$

thanks to (48). We therefore obtained

$$\frac{\delta_I^2}{L^2}|\int \psi_y|\nabla\psi|^2 e^x| \leq \frac{\delta_S}{L}\int |\nabla\psi|^2 + C\frac{\delta_I^4 L}{\delta_S^2\delta_M^3}\int \psi^2. \tag{59}$$

Equations (58) and (59) give in (57) :

$$(\frac{1}{6} - \frac{1}{8} - C\frac{\delta_I^4 L}{\delta_S^2\delta_M^3} - C\frac{\delta_I^4}{\delta_S\delta_M^3})\int \psi^2 + \frac{\delta_S}{2L}\int |\nabla\psi|^2 e^x + \frac{\delta_M^3}{2L^3}\int |\Delta\psi|^2 e^x \leq K.$$

If δ_I and δ_S are of the same order of magnitude, then

$$\frac{\delta_I^4 L}{\delta_S^2\delta_M^3} \approx \frac{\delta_I^2 L}{\delta_M^3},$$

and

$$\frac{\delta_I^4}{\delta_S\delta_M^3} \approx (\frac{\delta_I}{\delta_M})^3.$$

We then impose $\frac{\delta_I^2 L}{\delta_M^3} \leq \epsilon$, ϵ fixed and sufficiently small, which is equivalent to $\frac{\delta_I}{L} \leq \epsilon'(\frac{\delta_M}{L})^{3/2}$, and this leads to the estimate of proposition 2.

We deduce that $\psi \rightharpoonup \psi_0$ in L^2 weakly. We still have to determine the equation satisfied by ψ_0. The only problem is the nonlinear term. Let ϕ be a test function. We have

$$\frac{\delta_I^2}{L^2}\int J(\psi, \Delta\psi)\phi = -\frac{\delta_I^2}{L^2}\int J(\psi, \phi)\Delta\psi.$$

But $|\Delta\psi|_{L^2} \le C(\frac{L}{\delta_M})^{3/2}$ and $|\nabla\psi|_{L^2} \le C(\frac{L}{\delta_S})^{1/2}$, that leads to

$$\frac{\delta_I^2}{L^2}|\int J(\psi,\Delta\psi)\phi| \le C(\frac{L}{\delta_M})^{3/2}|\nabla\phi|_{L^\infty}(\frac{L}{\delta_S})^{1/2}\frac{\delta_I^2}{L^2},$$

$$\le C(\frac{L}{\delta_M})^{3/2}|\nabla\phi|_{L^\infty}(\frac{\delta_I^2}{L^2})^{3/2} = C(\frac{\delta_I}{\delta_M})^{3/2}|\nabla\phi|_{L^\infty} \to 0,$$

the convergence toward 0 follows from the hypothesis of proposition 2. ■

We still have to find a boundary condition on ψ_0 in order to characterize this function.

Proposition 3 *Let us assume that for an $\epsilon > 0$, we have $\frac{\delta_I}{L} \le (\frac{\delta_M}{L})^{9/4+\epsilon}$. Then*

$$\psi \to \psi_0 \textit{ in } L^2_{loc}(]0,1] \times S^1),$$

and $\psi_0 = 0$ on $\{1\} \times S^1$.

Remark: The hypothesis that we make on the size of the parameters ($\frac{\delta_I}{L} \le (\frac{\delta_M}{L})^{9/4+\epsilon}$) is stronger than that made in [6] p.273, which is $\delta_M >> \delta_I, \delta_S$.

Proof : We multiply (47) by $\psi_x\phi$ with ϕ to be chosen later one. Let us compute each term.

i) The term $\int |\psi_x|^2\phi$ has already the right form.

ii) The term concerning $\Delta\psi$:

$$\int \Delta\psi\psi_x\phi = -\frac{1}{2}\int |\psi_x|^2\phi_x - \int \psi_y\psi_{xy}\phi - \int \psi_y\psi_x\phi_y,$$

hence

$$\int \Delta\psi\psi_x\phi = -\frac{1}{2}\int |\psi_x|^2\phi_x + \frac{1}{2}\int |\psi_y|^2\phi_x - \int \psi_y\psi_x\phi_y. \tag{60}$$

iii)The term concerning $\Delta^2\psi$:

$$\int \Delta^2\psi\psi_x\phi = \int \Delta\psi\Delta(\psi_x\phi) - \int_{\partial\Omega} \Delta\psi\frac{\partial}{\partial\nu}(\psi_x\phi),$$

$$= \int \Delta\psi(\Delta\psi)_x\phi + 2\int \Delta\psi\nabla\psi_x.\nabla\phi + \int \Delta\psi\psi_x\Delta\phi - \int_{\partial\Omega} \Delta\psi\frac{\partial}{\partial\nu}(\psi_x)\phi$$

since ψ_x is equal to zero on the boundary. Here "$\frac{\partial}{\partial\nu}$" denote the derivative with respect to the exterior normal to Ω. Hence we get

$$\int \Delta^2\psi\psi_x\phi = -\frac{1}{2}\int |\Delta\psi|^2\phi_x + 2\int \Delta\psi\nabla\psi_x.\nabla\phi$$

$$+\int \Delta\psi\psi_x\Delta\phi - \int_{\partial\Omega}\Delta\psi\frac{\partial}{\partial\nu}(\psi_x)\phi + \frac{1}{2}\int_{\partial\Omega}|\Delta\psi|^2\phi\nu_x.$$

But on $\partial\Omega$, since $\psi = 0$ and $\nabla\psi = 0$, we have $\Delta\psi = \frac{\partial^2\psi}{\partial\nu^2}$ and $\frac{\partial}{\partial\nu}\psi_x = \nu_x\frac{\partial^2\psi}{\partial\nu^2}$ where ν_x denote the x-component of the exterior normal. Here the value of ν_x is + or − 1. We then obtain

$$\begin{aligned}&\int \Delta^2\psi\psi_x\phi + \int_{\partial\Omega}|\Delta\psi|^2\phi\nu_x = -\tfrac{1}{2}\int|\Delta\psi|^2\phi_x\\ &+2\int\Delta\psi\nabla\psi_x.\nabla\phi + \int\Delta\psi\psi_x\Delta\phi + \tfrac{1}{2}\int_{\partial\Omega}\nu_x(\tfrac{\partial^2\psi}{\partial\nu^2})^2\phi.\end{aligned} \tag{61}$$

iv) The term concerning the jacobian :

$$\int J(\psi,\Delta\psi)\psi_x\phi = -\int \Delta\psi J(\psi,\psi_x\phi). \tag{62}$$

Assembling (60), (61) and (62) we obtain

$$\begin{aligned}&-\tfrac{\delta_I^2}{L^2}\int\Delta\psi J(\psi,\psi_x\phi) + \tfrac{\delta_I^2}{L^2}\int J(\psi,\eta_B)\psi_x\phi + \int|\psi_x|^2\phi\\ &= \int \mathbf{k}.\mathrm{curl}\tau\psi_x\phi + \tfrac{\delta_S}{2L}\int|\psi_x|^2\phi_x - \tfrac{\delta_S}{2L}\int|\psi_y|^2\phi_x + \tfrac{\delta_S}{L}\int\psi_y\psi_x\phi_y\\ &-\tfrac{\delta_M^3}{2L^3}\int|\Delta\psi|^2\phi_x + 2\tfrac{\delta_M^3}{2L^3}\int\Delta\psi\nabla\psi_x.\nabla\phi + \tfrac{\delta_M^3}{2L^3}\int\Delta\psi\psi_x\Delta\phi - \tfrac{\delta_M^3}{2L^3}\int_{\partial\Omega}\nu_x(\tfrac{\partial^2\psi}{\partial\nu^2})^2\phi.\end{aligned} \tag{63}$$

Proposition 2 implies that $\frac{\delta_M^3}{L^3}\int|\Delta\psi|^2$ et $\frac{\delta_S}{L}\int|\nabla\psi|^2$ are bounded and (63) leads to

$$\begin{aligned}&\int|\psi_x|^2\phi + \tfrac{\delta_M^3}{2L^3}\int_{\partial\Omega}\nu_x(\tfrac{\partial^2\psi}{\partial\nu^2})^2\phi \le K|\phi|_{\mathcal{C}^2(\bar\Omega)}\\ &+\int\mathbf{k}.\mathrm{curl}\tau\psi_x\phi + \tfrac{\delta_I^2}{L^2}\int\Delta\psi J(\psi,\psi_x\phi) - \tfrac{\delta_I^2}{L^2}\int J(\psi,\eta_B)\psi_x\phi.\end{aligned} \tag{64}$$

Since δ_I and δ_S are of the same order of magnitude, we get

$$\frac{\delta_I^2}{L^2}\int J(\psi,\eta_B)\psi_x\phi \le K. \tag{65}$$

It follows that equations (64) and (65) lead to

$$\int |\psi_x|^2\phi + K'\frac{\delta_M^3}{2L^3}\int_{\partial\Omega}\nu_x(\frac{\partial^2\psi}{\partial\nu^2})^2\phi \le K|\phi|_{\mathcal{C}^2(\bar\Omega)} + \frac{\delta_I^2}{L^2}\int \psi_x\phi J(\psi,\Delta\psi). \quad (66)$$

We still have to estimate the term containing the jacobian, more precisely we get

$$\frac{\delta_I^2}{L^2}\int \psi_x\phi J(\psi,\Delta\psi) = -\frac{\delta_I^2}{L^2}\int \Delta\psi J(\psi,\psi_x)\phi - \frac{\delta_I^2}{L^2}\int \Delta\psi J(\psi,\phi)\psi_x.$$

It follows that

$$\frac{\delta_I^2}{L^2}|\int \psi_x\phi J(\psi,\Delta\psi)| \le \frac{\delta_I^2}{L^2}|\Delta\psi|_{L^2}^2|\nabla\psi|_{L^\infty} + \frac{\delta_I^2}{L^2}|\Delta\psi|_{L^2}^3. \quad (67)$$

We then multiply (47) by a test function χ and we integrate

$$-\frac{\delta_I^2}{L^2}\int \Delta\psi J(\psi,\chi) - \int \psi\chi_x + \frac{\delta_I^2}{L^2}\int \psi J(\chi,\eta_B)$$
$$-\int \mathbf{k}.\mathrm{curl}\tau\chi - \frac{\delta_S}{L}\int \nabla\psi.\nabla\chi + \frac{\delta_M^3}{L^3}\int \Delta^2\psi\chi = 0.$$

this gives

$$|\frac{\delta_M^3}{L^3}\int \Delta^2\psi\chi| \le K\frac{\delta_I^2}{L^2}(\frac{L}{\delta_M})^3|\chi|_{H^{3/2}},$$

and hence

$$|\psi|_{H^{5/2}} \le K\frac{\delta_I^2}{L^2}(\frac{L}{\delta_M})^6.$$

Relation (67) then implies

$$\frac{\delta_I^2}{L^2}|\int \psi_x\phi J(\psi,\Delta\psi)| \le K_\epsilon\frac{\delta_I^2}{L^2}(\frac{L}{\delta_M})^3(\frac{L}{\delta_M})^{3/2+\epsilon}. \quad (68)$$

(66) and (68) lead to

$$\int |\psi_x|^2\phi + K'\frac{\delta_M^3}{2L^3}\int_{\partial\Omega}\nu_x(\frac{\partial^2\psi}{\partial\nu^2})^2\phi \le K|\phi|_{\mathcal{C}^2(\bar\Omega)} + K_\epsilon\frac{\delta_I^2}{L^2}(\frac{L}{\delta_M})^{9/2+\epsilon}.$$

We now impose that $\phi \ge 0$ and $\phi \equiv 0$ in the neighborhood of $\{x = 0\} = \{\nu_x \le 0\}$. We obtain $\int |\psi_x|^2\phi \le K|\phi|_{\mathcal{C}^2(\bar\Omega)}$, the result of the proposition follows. ■

Remark : One can formally calculate a corrector for this singular limit process, see [6]. However, in the nonlinear case, the rigorous proof of the fact that it is a corrector is certainly very technical. In the linear case, the form of Ω that we consider is essential to compute the corrector, see [5].

References

[1] J.Y. Chemin, *A propos d'un problème de pénalisation de type anti-symétrique.* Preprint.

[2] V. Barcilon, P. Constantin and E.S. Titi, *Existence of solutions to the Stommel-Charney model of the gulf stream*, SIAM J. Math. Anal., vol 19, No 6, November 1988.

[3] J.L. Lions, R. Temam et S. Wang, *Models of the coupled atmosphere and ocean (CAO I), Numerical analysis of the coupled atmosphere-ocean models (CAO II)* , Computational Mechanics Advances 1 (1993) 5-54, 55-119 .

[4] J.L. Lions, R. Temam et S. Wang, *Mathematical theory for the coupled atmosphere-ocean models (CAO III)*, to appear in journal de Math. pures et Appliquées.

[5] J.L. Lions, *Perturbations singulières dans les problèmes aux limites et en contrôle optimal*, Lecture Notes in Mathematics, vol. 323, Springer-Verlag, Berlin, 1973.

[6] J. Pedlosky, *Geophysical fluid dynamics*, second edition, Springer Verlag, 1987.

[7] R. Temam, *Navier-Stokes equations* , North-Holland, 1984.

[8] R.K. Zeytounian, *Modélisation asymptotique en mécanique des fluides newtoniens*, Collection Mathématiques et Applications, Springer-Verlag, 1994.