On the generalized Diophantine m-tuples

by

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1)

Can you see anything from $\{1, 3, 8, 120\}$?

One of the properties:

$$1 \cdot 3 = 3 = 2^2 - 1$$

$$3 \cdot 8 = 24 = 5^2 - 1$$

$$8 \cdot 120 = 960 = 31^2 - 1$$

$$1 \cdot 120 = 120 = 11^2 - 1$$

$$\vdots$$

2)

DEF

For any integer $n \neq 0$, a set of natural numbers

$$\{a_1, a_2, \dots a_m\}$$

is a Diophantine m-tuples with property $D(n)$ if $a_i a_j + n$ is a perfect square for all $i \neq j$.

Remark

- The previous example $\{1, 3, 8, 120\}$ was discovered by Fermat. w/ property $D(1)$

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- The name comes from Diophantus, who discovered $\{1, 33, 68, 105\}$ with property $D(256)$
- Euler discovered the infinitely many quadruples with property $D(1)$, namely

$$\{a, b, ab+2r, 4r(r+a)(r+b)\}$$

where $ab+1=r^2$.

- Anything beyond?

How about anything beyond quadruples?

THM (Baker–Davenport, 1969)

Fermat's example is the only extension of $\{1,3,8\}$ with property $D(1)$.

THM (Dujella, 2004) There are no $D(1)$-sextuples and there are only finitely many $D(1)$-quintuples.

THM (He, Togbé, Ziegler. 2019)

There's no $D(1)$-quintuples.

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On the other hand, for general $n \neq 1$, there are $D(n)$-quintuples, for instance,

$$\{1, 33, 105, 320, 18240\}$$

is a Diophantine quintuple of $D(256)$.

There are known examples of $D(n)$-sextuples, but no septuple is known.

Q. what is the largest tuple with property $D(n)$?

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Observation: why is it hard to find a large $D(n)$-tuple?

If $S = \{a_1, a_2, a_3\}$ is a set of $D(n)$, then an extra a_i for $i > 3$ yields an integer point on the elliptic curve

$$y^2 = (a_1 x + n)(a_2 x + n)(a_3 x + n)$$

for all $i > 3$.

$\Rightarrow$ By Siegel's theorem. the number of integer points any elliptic curve is finite.

For this talk, we extend this definition and study:

<u>DEF</u> (Generalized Diophantine m-tuples & $M_k(n)$)

- For $k \geq 2$, a set of natural numbers

$$\{a_1, a_2, \ldots, a_m\}$$

is said to satisfy the property $D_k(n)$ if $a_i a_j + n$ is a k^{th}-power for all $i \neq j$.

- Define

$$M_k(n) := \sup\{ |S| : S \text{ satisfy } D_k(n)\}.$$

$\rightsquigarrow$ Previous example was $k=2$.

- Unconditionally Bugeaud & Dujella showed

$$M_3(1) \le 7, \quad M_4(1) \le 5, \quad M_k(1) \le 4, \text{ for } k \geq 5.$$

- The Caporaso-Harris conjecture implies that $M_k(n)$ is uniformly bounded independent of n.

THM (DIXIT – KIM – MURTY)

The following holds for large enough n:

a) For $L \geqslant 3$,

$$M_k(n;L) := \sup \{ |S \cap [n^L, \infty)| : S \text{ has } D_k(n) \} \ll 1,$$

where the implied constant depends on k & L, but is independent of n.

b) Unconditionally, $M_k(n) \ll_k \log n$.

c) Assuming the Paley graph conjecture, for any $\varepsilon > 0$,

$$M_k(n) \ll_{k,\varepsilon} (\log n)^{\varepsilon}.$$

PALEY GRAPHS

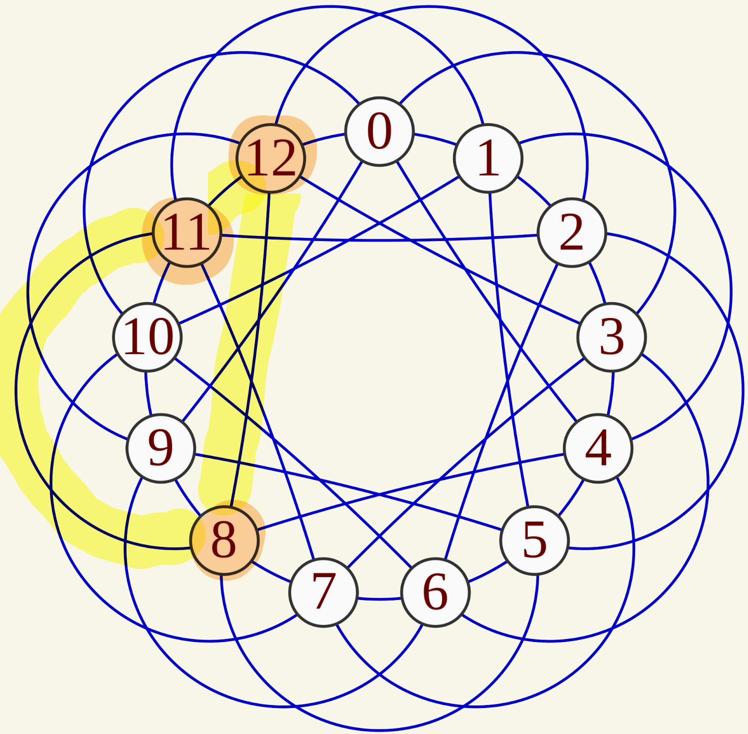
A Paley graph is a graph with

- vertex set $V = \mathbb{F}_p$
- edge set E

Such that $(a, b) \in E$ iff

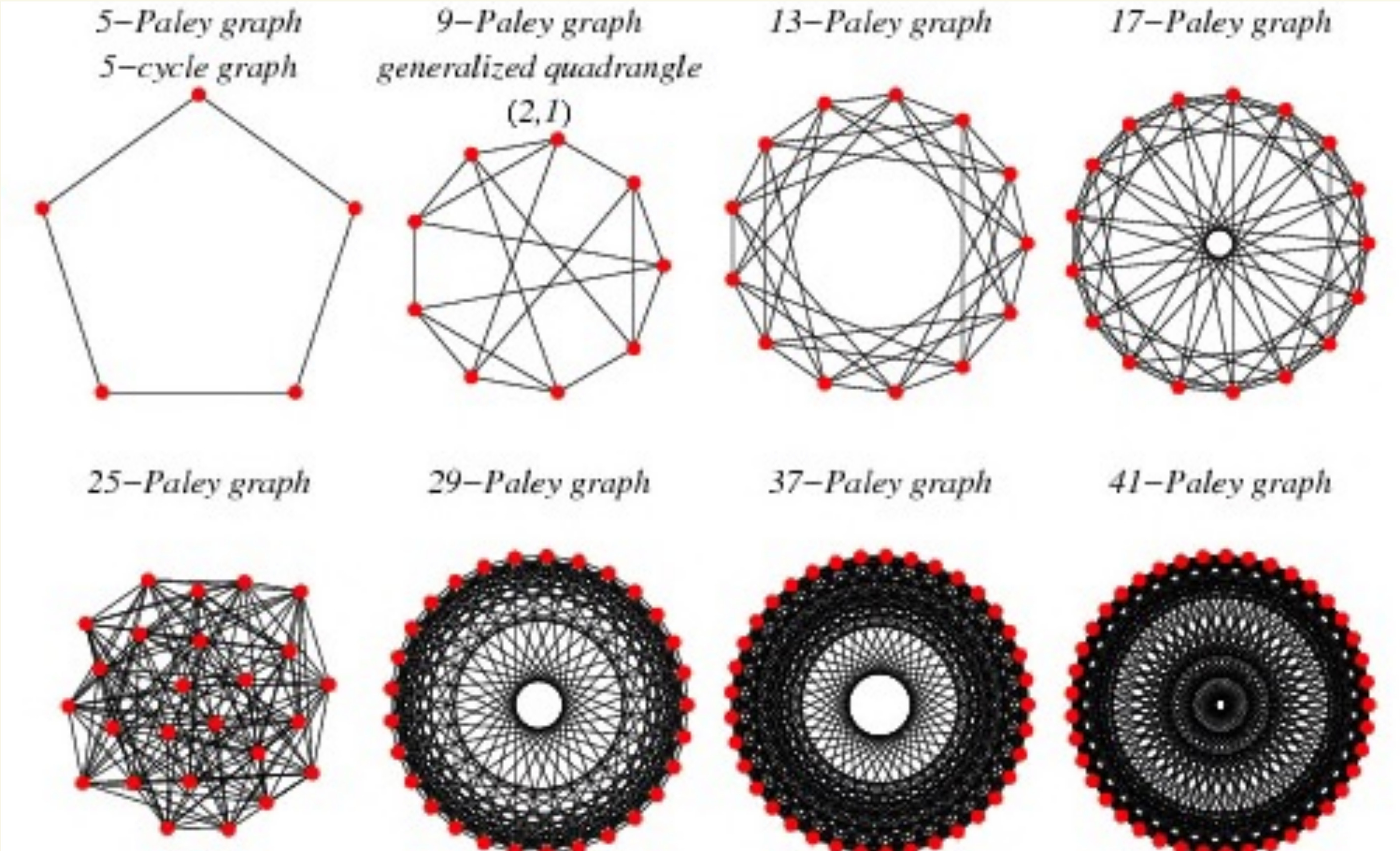
$a - b$ is quadratic residue mod p.

Paley in $\mathbb{F}_{13}$



http://cs.indstate.edu/ge/Paley/cliques.html

PALEY GRAPHS...



From Wolfram Alpha

Paley graph conjecture

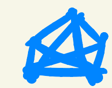
Let $\varepsilon > 0$, and $S, T \subset \mathbb{F}_p$ for an odd prime p with $|S|, |T| > p^{\varepsilon}$, and χ be any nontrivial multiplicative character modulo p. Then there is some number $\delta = \delta(\varepsilon)$ for which

$$\left| \sum_{\substack{a \in S \\ b \in T}} \chi(a+b) \right| \leq p^{-\delta} |S||T|$$

holds for primes p larger than some constant $C(\varepsilon)$.

What's the implication of the conjecture on Paleys?

$T = -S$, χ : quadratic character.

The clique number of a graph := the maximal complete graph 

Sketch of the proof of (c):

<u>Step 1</u> $M_k(n; 3)$ is bounded. (NON TRIVIAL)

Thus it is sufficient to consider an m-tuple

$$S = \{a_1, a_2, \dots, a_m\}$$

with property $D_k(n)$ in $[1, n^3]$.

<u>Step 2</u> For each prime p, consider $S_p = S \pmod{p}$. &

define for $i = 0, 1, \dots, k-1$,

$$T_i = \{a \in S_p \mid \chi(a) = \zeta_k^i\}.$$

Then

$$|S_p| \leq |T_0| + |T_1| + \cdots + |T_{k-1}|.$$

Assuming the Paley graph conjecture, we deduce

$$|S_p| \leq 1 + kp^{\epsilon}$$

for a fixed $\epsilon > 0$.

<u>Step 3</u> Using Gallagher's large Sieve, we have

$$|S| \leq \frac{\sum_{p \leq Q} \log p - \log N}{\sum_{p \leq Q} \frac{\log p}{|S_p|} - \log N} \implies |S| \leq (\log n)^{\epsilon}.$$