The Boltzmann equation for plane Couette flow

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- In the context of **fluid dynamics**, for instance, the fluid motion is governed by the compressible/incompressible Euler or Navier-Stokes equations, an interesting topic is to understand the *dynamical* stability/instability of **parallel shear flows** of the form
$$(U(y), 0, 0),$$
i.e., a flow in the x-direction that varies with y.
- A specific form of the *wavelike* disturbance leads to solve the *Rayleigh* equation. In case of inviscid flow (I.E.), the classical inflection point criterion
$$U^{''}(U - U_s) < 0$$
by Rayleigh (1880) and Fjortoft (1950) gives the necessary condition for instability.
- The significant mathematical progress on stability, in particular, for the **Couette flow** $U(y) = y$ or even a monotone shear flow in 2D $(\mathbb{T}_x \times \mathbb{R}_y)$ or 3D $(\mathbb{T}_x \times \mathbb{R}_y \times \mathbb{T}_z)$, has been extensively made by Bedrossian with his collaborators (Germain, Masmoudi, Vicol, F Wang, Zelati) as well as many others (Y. Deng, Ionescu, H. Jia, T. Li, CC Lin, Z Lin, Masmoudi, D. Wei,C Zeng, ZF Zhang, W Zhao, Zillinger,...).

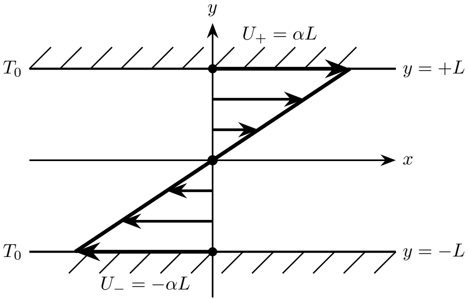
- In particular, the case of domain $\mathbb{T}_x \times [-1, 1]$ with physical boundary was treated independently by Ionescu-Jia and Masmoudi-Zhao. We will consider a similar setting of domains.
- Remark: The results for inviscid damping are also inspired by the famous work on the *Landau damping* for the Vlasov-Poisson system by Mouhot and Villani (2011).
- In the context of **kinetic theory**, the motion for a rarefied gas flow is governed by the nonlinear Boltzmann equation with the finite Knudsen number, in order to determine the density distribution function

$$F(t, x, y, z, v_x, v_y, v_z) \geq 0$$

  of particles with position (x, y, z) and velocity $v = (v_x, v_y, v_z)$ at t.
- An analogue topic in the kinetic regime is to ask
  - existence of a steady motion with shear effect arising from differences of temperatures (*heat transfer*) or velocities (*Couette flow*) at boundaries or from pressure gradient (*Poiseuille flow*) or temperature gradient (*thermal transpiration*) in a nozal, see the books by Kogan, Cercignani, Garzó-Santos or Sone.
  - dynamical stability/instability of those kinetic simple flows.
  - asymptotic limit as Kn is vanishing or sufficiently large.

- Such topic in kinetic theory is far from being understood in mathematics and few rigorous results are known, but we may refer to
    - Esposito-Lebowitz-Marra: to be discussed later
    - Arkeryd-Nouri: Couette flow between two coaxial rotating cylinders, Taylor-Couette type bifurcation,...
    - Arkeryd-Esposito-Marra-Nouri: Plane Couette flow, ghost effect by curvature.
- **The first step for us is to understand the qualitative and quantitative properties of the stationary flows at the kinetic level.**

The object of this talk is concerned with the **plane Couette flow** (see the books by Kogan, Cercignani, Garzó-Santos or Sone):



$$F(y, v_x, v_y, v_z) \geq 0, \quad -L < y < +L, v \in \mathbb{R}^3.$$

The rarefied gas is between two parallel infinite plates at the same uniform temperature $T_0 > 0$, one at $y = +L$ is moving with velocity $(U_+, 0, 0)$ with $U_+ = \alpha L$ and the other at $y = -L$ is moving with velocity $(U_-, 0, 0)$ with $U_- = -\alpha L$, where $\alpha > 0$ is a a parameter denoting the shear rate. Moreover, we assume that the gas molecules are of the Maxwellian type and reflected diffusely on the plates $y = \pm L$.

The **steady motion** is determined by solving the BVP:

$$v_y \partial_y F = \frac{1}{\mathrm{Kn}} Q(F, F), \quad -L < y < +L, v \in \mathbb{R}^3,$$

subject to the diffuse reflection boundary conditions at $y = \pm L$:

$$F(\pm L, v) = \mathcal{M}_{T_0}(v_x - U_\pm, v_y, v_z) \int_{v_y \gtrless 0} F(\pm L, v)|v_y|\, dv \quad \text{for } v_y \lessgtr 0,$$

as well as a given total mass

$$\frac{1}{2L} \int_{-L}^{L} \int_{\mathbb{R}^3} F(y, v)\, dvdy = m > 0.$$

Here,

$$\mathcal{M}_{T_0}(v) = \frac{1}{2\pi T_0^2} e^{-\frac{|v_x|^2 + |v_y|^2 + |v_z|^2}{2T_0}}, \quad v = (v_x, v_y, v_z) \in \mathbb{R}^3.$$

Note: $\int_{v_y \lessgtr 0} \mathcal{M}_{T_0}(v)|v_y|\, dv = 1$ and $\int_{\mathbb{R}^3} \mathcal{M}_{T_0}(v) dv = (2\pi/T_0)^{1/2}$.

For the **Maxwell molecule model**, the collision operator operator Q, which is bilinear and acts only on velocity variable, takes the form of

$$Q(F_1, F_2)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0(\cos\theta)[F_1(v_*')F_2(v') - F_1(v_*)F_2(v)]\, d\omega dv_*,$$

where the velocity pairs (v_*, v) and (v_*', v') satisfy the relation

$$v_*' = v_* - [(v_* - v)\cdot\omega]\omega, \quad v' = v + [(v_* - v)\cdot\omega]\omega,$$

according to conservations of momentum and energy of two particles before and after an elastic collision, i.e.,

$$v_* + v = v_*' + v', \quad |v_*|^2 + |v|^2 = |v_*'|^2 + |v'|^2.$$

We assume that the collision kernel $B_0(\cos\theta)$ with $\cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\omega$ satisfies the Grad's angular cutoff assumption

$$0 \le B_0(\cos\theta) \le C|\cos\theta|,$$

for a generic constant $C > 0$.

A quick look at solutions to (BVP) and how to construct:

$$\begin{cases} v_y \partial_y F = \dfrac{1}{\mathrm{Kn}} Q(F,F), \quad -L < y < +L, v \in \mathbb{R}^3, \\ F(\pm L, v) = \mathcal{M}_{T_0}(v_x - U_\pm, v_y, v_z) \displaystyle\int_{v_y \gtrless 0} F(\pm L, v)|v_y|\, dv \quad \text{for } v_y \lessgtr 0, \\ \dfrac{1}{2L} \displaystyle\int_{-L}^{L} \int_{\mathbb{R}^3} F(y,v)\, dv dy = m > 0. \end{cases}$$

Q.: Existence and dynamical stability of stationary solutions for any small enough $|U_+ - U_-|$.

- If $U_+ = U_- := U$ then the boundary distribution for incoming particles becomes uniform in $y = \pm L$, so the global Maxwellian
$$m\sqrt{\frac{T_0}{2\pi}}\mathcal{M}_{T_0}(v_x - U, v_y, v_z)$$
is an equilibrium solution.
- If $U_+ \neq U_-$ then two boundaries induce the relative shearing motion in the tangent direction, and this drives the system to get away from a homogeneous equilibrium, so a *spatially inhomogeneous* **non-equilibrium stationary** solution must be generated.

- Thus, for a **fixed** $\mathrm{Kn}>0$, one may expect to construct the stationary solutions via a perturbation approach with respect to

$$|U_+ - U_-| > 0 \text{ small enough.}$$

- Indeed, the approach by R. Esposito, Y. Guo, C. Kim and R. Marra (CMP, 2013) can be applicable. Specifically, let $U_\pm$ be both around zero, i.e, $|U_\pm| \ll 1$, then corresponding to the linearization of boundary Maxwellians at $y = \pm L$:

$$\mathcal{M}(v_x + U, v_y, v_z) = \mathcal{M}(v) + U\mathcal{M}_1(v) + U^2\mathcal{M}_2(v) + \cdots,$$

  one may construct solutions around global Maxwellians of the form

$$F(y, v) = \mathcal{M}(v) + \sqrt{\mathcal{M}(v)}(Ug_1 + U^2 g_2 + \cdots).$$

- Obviously, in such way, the truncated approximation solution up to any finite order does not satisfy the kinetic boundary, in particular, the reference profile $\mathcal{M}(v)$ is a global Maxwellian as a reference equilibrium profile that fails to fulfill BC.

- **The question in mind is whether one can construct approximation solutions matching the BC in an exact way or one can capture the shearing solution at the zero-order level!**

- Alternative ways to construct approximation solutions in terms of size of Kn are also possible:
  - **strong collision regime**: $0 <\mathrm{Kn} \ll 1$, fluid dynamic limit;
  - **weak collision regime**: $\mathrm{Kn} \gg 1$, free molecule flow limit.
- For instance, for a free molecule flow with the same boundary condition, the solution is given in terms of characteristics by

$$F(y,v) = \begin{cases} \beta_+ \mathcal{M}_{T_0}(v_x - U_+, v_y, v_z), & \text{for } v_y < 0, \\ \beta_- \mathcal{M}_{T_0}(v_x - U_-, v_y, v_z), & \text{for } v_y > 0, \end{cases}$$

  where $\beta_+ = \beta_-$ are chosen to satisfy a given total mass. It is direct to see that the solution is discontinuous at $v_y = 0$ (velocity normal to wall) if $U_+ \neq U_-$. The same thing should occur to the case of finite Knudsen number that gives the nonlocal collision terms.
- In case $0 <\mathrm{Kn} \ll 1$, Esposito-Lebowitz-Marra (CMP 1994, JSP 1995) and Di Meo-Esposito (JSP 1996) gave the hydrodynamic description of the steady rarefied gas flow via the approximation of the corresponding **compressible Navier-Stokes** equations with **no-slip** boundary conditions.

Theorem (Esposito-Lebowitz-Marra, Di Meo-Esposito). Consider

$$\begin{cases} v_y \partial_y F + \epsilon G \partial_{v_x} F = \frac{1}{\epsilon} Q(F,F), \quad -L < y < +L, v \in \mathbb{R}^3, \\ F(\pm L, v) = \mathcal{M}_{T_\pm}(v - U_\pm) \int_{v_y \gtrless 0} F(\pm L, v)|v_y|\, dv \quad \text{for } v_y \lessgtr 0, \\ \frac{1}{2L} \int_{-L}^{L} \int_{\mathbb{R}^3} F(y,v)\, dv dy = m > 0, \end{cases}$$

where G is a constant force parallel to the boundary, $T_\pm$ boundary temperatures and $U_\pm$ (vectors!) boundary velocities in the tangent (x,z)-plane. There are positive constants ϵ_0, θ_0 and q_0 such that if $\epsilon < \epsilon_0$, $\theta < \theta_0$ and

$$q := \max\{|G|, |T_+ - T_-|, |U_+ - U_-|\} < q_0,$$

then the BVP admits a solution of the form

$$F = M_{[\rho, U, T]} + \sum_{n=1}^{6} \epsilon F_n + \epsilon^3 F_R,$$

where

- $M_{[\rho,U,T]}$ is the local Maxwellian with parameters

$$\rho = \rho(y),\ \ U = (u(y), 0, w(y)),\ \ T = T(y)$$

given by the solution to stationary hydrodynamic equations

$$\begin{aligned} (\rho T)_y &= 0, \\ (\eta(T)u_y)_y + \rho G = 0, \quad (\eta(T)w_y)_y &= 0, \\ (\kappa(T)T_y)_y + \eta(T)(u_y^2 + w_y^2) &= 0, \end{aligned}$$

supplemented with the no-slip BCs $U(\pm L) = U_\pm$ and $T(\pm L) = T_\pm$ and $\int \rho dy = m$.

- $F_n = B_n + (b_n^+ + b_n^-)$, where B_n describes F in the bulk and $b_n^\pm$ give boundary layer corrections.

- For any positive integer r there is a constant $c > 0$ such that

$$\|F_R\|_{r,\theta} \le c\epsilon^{\frac{3}{2}}\|A\|_{r,\theta}$$

where A is induced by $M_{[\rho,U,T]}$ and F_n $(0 \le n \le 6)$ with $\|A\|_{r,\theta} < \infty$, and

$$\|f\|_{r,\theta} = \sup_{-L \le y \le L} \sup_{v \in \mathbb{R}^3} (1 + |v|)^r e^{\theta|v|^2}|f(y,v)|. \quad \square$$

Remark: Weight depends on type of collision kernel ($\gamma = 1$ vs $0 \le \gamma < 1$).

Remarks:

- Recently D.-Liu-Yang-Zhang (preprint) revisited the only heat transfer problem (i.e., $G = 0$, $U_\pm = 0$) with the no-slip BC for fluid equations replaced by the **slip BC** so that the kinetic diffusive reflection BC can be satisfied up to $O(\epsilon)$. **Dynamical stability of stationary solutions uniform in Knudsen number and shear strength** so far still remains a big open problem.
- In the meantime, for the corresponding IBVP in a general bounded domain with homogeneous diffusive reflection BC, D.-Liu (TAMS, 2021) constructed the compressible Navier-Stokes approximation solutions uniform in time and Knudsen number, but the long time behavior or small Knudsen limit is trivial.
- Back to the plane Couette flow under consideration, the theorem by Esposito-Lebowitz-Marra and Di Meo-Esposito tells that in the case of Maxwell molecules, $G = 0$, $T_\pm = T_0$ and $U_\pm = (U_\pm, 0, 0)$, one has that $w(y) \equiv 0$, $u(y)$ **connects** $U_\pm$ **linearly and** $T(y)$ **is a parabola with maximum at** $y = 0$ **and** T_0 **at** $y = \pm L$. If we set $U_\pm = \pm\alpha L$ corresponding to a shear rate $\alpha > 0$, then $u(y) = \alpha y$. This motivates us to consider the **peculiar velocity** $v_x \to v_x - \alpha y$ in the spirit of the Lagrangian coordinates!

Our formulation of the problem: Let stationary solutions take the specific form

$$F_{st}(y, v_x - \alpha y, v_y, v_z),$$

then we are reduced to solve the BVP

$$\begin{cases} v_y \partial_y F_{st} - \alpha v_y \partial_{v_x} F_{st} = Q(F_{st}, F_{st}), \\ \qquad\qquad\qquad\qquad y \in (-1,1),\ v = (v_x, v_y, v_z) \in \mathbb{R}^3, \\ F_{st}(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi}\mu \displaystyle\int_{v_y \gtrless 0} F_{st}(\pm 1, v)|v_y| dv,\ v \in \mathbb{R}^3, \\ \dfrac{1}{2}\displaystyle\int_{-1}^{1}\int_{\mathbb{R}^3} F_{st}(y, v)\, dvdy = 1. \end{cases}$$

Remark: As such, the BC becomes homogeneous while there appears a shear force that drives the system to get far from the global Maxwellian μ corresponding to the solution in case $\alpha = 0$. The reference profile in the original setting becomes a local Maxwellian

$$\mu(v_x - \alpha y, v_y, v_z)$$

which exactly matches the original BC at the zero-order level. □

Now we set

$$F_{st} = \mu + \sqrt{\mu}\{\alpha G_1 + \alpha^2 G_R\},$$

with

$$\int_{-1}^{1}\int_{\mathbb{R}^3} \sqrt{\mu} G_1 \, dvdy = \int_{-1}^{1}\int_{\mathbb{R}^3} \sqrt{\mu} G_R \, dvdy = 0.$$

- The first-order correction term G_1 satisfies

$$\begin{cases} v_y \partial_y G_1 + L G_1 = -v_x v_y \sqrt{\mu}, \\ G_1(\pm 1, v)|_{v_y \lesseqgtr 0} = \sqrt{2\pi\mu} \displaystyle\int_{v_y \gtrless 0} \sqrt{\mu} G_1(\pm 1, v)|v_y| \, dv, \end{cases}$$

where due to symmetry, the BC can be replaced by

$$G_1(\pm 1, v)|_{v_y \lesseqgtr 0} = 0.$$

- The remainder G_R satisfies

$$\begin{cases} v_y \partial_y G_R - \alpha v_y \partial_{v_x} G_R + \dfrac{\alpha}{2} v_x v_y G_R + L G_R = v_y \partial_{v_x} G_1 - \dfrac{1}{2} v_x v_y G_1 \\ \qquad\qquad + \Gamma(G_1, G_1) + \alpha\{\Gamma(G_R, G_1) + \Gamma(G_1, G_R)\} + \alpha^2 \Gamma(G_R, G_R), \\ G_R(\pm 1, v)|_{v_y \lesseqgtr 0} = \sqrt{2\pi\mu} \displaystyle\int_{v_y \gtrless 0} \sqrt{\mu} G_R(\pm 1, v)|v_y| \, dv. \end{cases}$$

Theorem (D.-Liu-Yang arXiv:2107.02458)

Assume that the Boltzmann collision kernel is of the Maxwell molecule type. Then, the BVP on the steady Couette flow admits a unique steady solution $F_{st} = F_{st}(y, v) \geq 0$ of the above form with the following estimates on G_1 and G_R, respectively. Let $w_q = (1 + |v|^2)^q$.

(i) *For any integers $m \geq 0$ and $q \geq 0$,*

$$\|w_q \partial_{v_x}^m G_1\|_{L^\infty} \leq \tilde{C}_1,$$

where $\tilde{C}_1 > 0$ is a constant depending only on m and q.

(ii) *There is an integer $q_0 > 0$ such that for any integer $q \geq q_0$, there is $\alpha_0 = \alpha_0(q) > 0$ depending on q such that for any $\alpha \in (0, \alpha_0)$ and any integer $m \geq 0$, it holds for $\widetilde{G}_R := \sqrt{\mu} G_R$ that*

$$\|w_q \partial_{v_x}^m \widetilde{G}_R\|_{L^\infty} \leq \tilde{C}_{m,q},$$

where $\tilde{C}_{m,q} > 0$ is a constant depending only on m and q but independent of α.

To establish the **positivity** of the stationary profile $F_{st}(y,v)$, we are further devoted to studying the following initial boundary value problem of the **1D** Boltzmann equation with a shear force

$$\begin{cases} \partial_t F + v_y \partial_y F - \alpha v_y \partial_{v_x} F = Q(F,F), \\ \qquad\qquad t>0,\ y \in (-1,1),\ v=(v_x,v_y,v_z)\in\mathbb{R}^3, \\ F(0,y,v) = F_0(y,v),\ y\in(-1,1),\ v\in\mathbb{R}^3, \\ F(t,\pm1,v)|_{v_y \lessgtr 0} = \sqrt{2\pi}\mu \displaystyle\int_{v_y \gtrless 0} F(t,\pm 1,v)|v_y|dv,\ t\geq 0,\ v\in\mathbb{R}^3. \end{cases}$$

Theorem (D.-Liu-Yang arXiv:2107.02458)

Let $F_{st}(y,v)$ be the steady state obtained before corresponding to any shear rate $\alpha\in(0,\alpha_0)$. There are constants $\varepsilon_0>0$, $\lambda_0>0$ and $C>0$, independent of α, such that if initial data $F_0(y,v)\geq 0$ satisfy

$$\|w_q[F_0(y,v)-F_{st}(y,v))]\|_{L^\infty}\leq\varepsilon_0,\quad \int_{-1}^{1}\int_{\mathbb{R}^3}[F_0(y,v)-F_{st}(y,v)]\,dvdy=0,$$

then the IBVP admits a unique solution $F(t,y,v)\geq 0$ satisfying

$$\|w_q[F(t,y,v)-F_{st}(y,v)]\|_{L^\infty}\leq Ce^{-\lambda_0 t}\|w_q[F_0(y,v)-F_{st}(y,v)]\|_{L^\infty},\forall\, t\geq 0.$$

Remark 1:

• The main extra obstacle in analysis basing on the $L^\infty \cap L^2$ approach developed by Guo (ARMA 2010) and Esposito-Guo-Kim-Marra (CMP 2013) is the appearance of the shear force whose balance with collisions in case of Maxwell molecules gives that the solution can have the only polynomial tail at large velocities, for instance, the linear term $\frac{1}{2}\alpha v_x v_y G_R$ makes a severe difficulty such that the estimates are out of control for the large velocities, as the dissipation is of the form $\int G_R^2$ for the Maxwell molecule model.

• To overcome it, we introduce the Caflisch's decomposition

$$\sqrt{\mu} G_R = G_{R,1} + \sqrt{\mu} G_{R,2}$$

such that $G_{R,1}$ and $G_{R,2}$ satisfy the coupling boundary value problems

$$\begin{cases} v_y \partial_y G_{R,1} - \alpha v_y \partial_{v_x} G_{R,1} + \nu_0 G_{R,1} = \underbrace{\mathbf{1}_{|v|\geq M} \mathcal{K} G_{R,1}}_{\text{small}} - \frac{1}{2}\alpha \sqrt{\mu} v_x v_y G_{R,2} + \mathcal{F}_1, \\ G_{R,1}(\pm 1, v)|_{v_y \lessgtr 0} = 0, \end{cases}$$

and

$$\begin{cases} v_y \partial_y G_{R,2} - \alpha v_y \partial_{v_x} G_{R,2} + L G_{R,2} = \mathbf{1}_{|v| \le M} \mu^{-\frac{1}{2}} \mathcal{K} G_{R,1} + \mathcal{F}_2, \\ G_{R,2}(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi\mu} \int_{v_y \gtrless 0} \sqrt{\mu} G_R(\pm 1, v) |v_y| dv, \end{cases}$$

respectively.

• To treat G_1 or $G_{R,2}$ in $L^2 \cap L^\infty$, we borrowed an iterative method from D.-Huang-Wang-Zhang (ARMA 19) to replace L by

$$L_\sigma = \nu - \sigma K, \quad 0 \le \sigma \le 1.$$

□

Remark 2: The current problem is also closely related to the Boltzmann equation for **uniform shear flow** without boundary (y-independent):

$$\partial_t F - \alpha v_y \partial_{v_x} F = Q(F, F),$$

which has been also studied in the context of the homoenergetic solutions by Galkin (1956), Truesdell (1956), **James-Nota-Velázquez** (ARMA 2019), and **Bobylev-Nota-Velázquez** (CMP 2020), and D.-Liu (to appear in ARMA). Different from the case with boundary, the heat that the shear force produces can not be dissipative but rather accumulates and grows to reach infinity at the infinite time. The large time behavior starting from initial data of finite energy is determined by the self-similar solution

$$F(t, v) \to e^{-3\beta t} G(\frac{v}{e^{\beta t}}),$$

with $\beta = -\frac{\alpha}{3} \int_{\mathbb{R}^3} v_x v_y G \, dv$ and G given by

$$\boxed{\frac{1}{3} \int_{\mathbb{R}^3} v_x v_y G \, dv \, \nabla_v \cdot (vG) - v_y \partial_{v_x} G = \frac{1}{\alpha} Q(G, G).}$$

Here G is far from an equilibrium as it has only the polynomial tail in large velocities. This is a formulation essentially different from previous works. □

Prospectives:

- Characterize polynomial tail (indicated by numerics but a rigorous proof is still missing)
- Stability in 2D and 3D (in case with boundaries singularity occurs around boundary, so a low-regularity framework has to be used...with the help of kinetic effect.)
- Fluid limit or weak-collision limit
- ...

Thank you!