Phase-space and Optimal Transport formulation of the Einstein equations in vacuum

Yann Brenier, CNRS,
LMO, Orsay, Université Paris-Saclay
(just moving from DMA-ENS Paris)

Webinar Kinetic and Fluid equations, 17 Dec 2021
French-Korean IRL in Mathematics

General Relativity (in vacuum)

'Free fall' follows geodesic curves $s \in \mathbb{R} \to x(s) \in \mathbb{R}^4$

i.e. critical points of $\displaystyle\int g_{ij}(x(s)) \frac{dx^i(s)}{ds} \frac{dx^j(s)}{ds} ds$

where g is a Lorentzian metric over $\mathbb{R}^4$,

General Relativity (in vacuum)

'Free fall' follows geodesic curves $s \in \mathbb{R} \to x(s) \in \mathbb{R}^4$

$$\text{i.e. critical points of } \int g_{ij}(x(s)) \frac{dx^i(s)}{ds} \frac{dx^j(s)}{ds} ds$$

where g is a Lorentzian metric over $\mathbb{R}^4$, which reads

$$\frac{dx^i(s)}{ds} = \xi^i(s), \quad \frac{d\xi^i(s)}{ds} = -\Gamma^i_{jk}(x(s))\xi^j(s)\xi^k(s),$$

$$2g_{mi}\Gamma^i_{jk} + \partial_m g_{jk} - \partial_j g_{mk} - \partial_k g_{mj} = 0.$$

General Relativity (in vacuum)

'Free fall' follows geodesic curves $s \in \mathbb{R} \to x(s) \in \mathbb{R}^4$

$$\text{i.e. critical points of } \int g_{ij}(x(s)) \frac{dx^i(s)}{ds} \frac{dx^j(s)}{ds} ds$$

where g is a Lorentzian metric over $\mathbb{R}^4$, which reads

$$\frac{dx^i(s)}{ds} = \xi^i(s), \quad \frac{d\xi^i(s)}{ds} = -\Gamma^i_{jk}(x(s))\xi^j(s)\xi^k(s),$$

$$2g_{mi}\Gamma^i_{jk} + \partial_m g_{jk} - \partial_j g_{mk} - \partial_k g_{mj} = 0.$$

Einstein equations in vacuum
just means that g has zero Ricci curvature.

Phase-space formulation of the Ricci curvature

Key idea: view Γ as a collection of 4 vector fields over the phase space $(x, \xi) \in \mathbb{R}^8$ which are linear in ξ:
$V_k^j(x, \xi) = -\Gamma^j_{k\gamma}(x)\xi^\gamma$,

Phase-space formulation of the Ricci curvature

Key idea: view Γ as a collection of 4 vector fields over the phase space $(x, \xi) \in \mathbb{R}^8$ which are linear in ξ: $V_k^j(x, \xi) = -\Gamma_{k\gamma}^j(x)\xi^\gamma,$ so that the Riemann and the Ricci curvatures just read as commutators:

$$R_{jkm}^n(x)\xi^m = \left((\partial_{x^k} + V_k^\gamma \partial_{\xi^\gamma})V_j^n - (\partial_{x^j} + V_j^\gamma \partial_{\xi^\gamma})V_k^n\right)(x, \xi)$$

Phase-space formulation of the Ricci curvature

Key idea: view Γ as a collection of 4 vector fields over the phase space $(x, \xi) \in \mathbb{R}^8$ which are linear in ξ: $V_k^j(x, \xi) = -\Gamma^j_{k\gamma}(x)\xi^\gamma,$ so that the Riemann and the Ricci curvatures just read as commutators:

$$R^n_{jkm}(x)\xi^m = \left((\partial_{x^k} + V_k^\gamma \partial_{\xi^\gamma})V_j^n - (\partial_{x^j} + V_j^\gamma \partial_{\xi^\gamma})V_k^n\right)(x, \xi)$$

$$= \partial_{x^k} V_j^n + \partial_{\xi^j}(V_k^\gamma V_\gamma^n) - \partial_{x^j} V_k^n - \partial_{\xi^k}(V_i^\gamma V_\gamma^n),$$

Phase-space formulation of the Ricci curvature

Key idea: view Γ as a collection of 4 vector fields over the phase space $(x, \xi) \in \mathbb{R}^8$ which are linear in ξ: $V_k^j(x,\xi) = -\Gamma_{k\gamma}^j(x)\xi^\gamma,$ so that the Riemann and the Ricci curvatures just read as commutators:

$$R_{jkm}^n(x)\xi^m = \left((\partial_{x^k} + V_k^\gamma \partial_{\xi^\gamma})V_j^n - (\partial_{x^j} + V_j^\gamma \partial_{\xi^\gamma})V_k^n\right)(x,\xi)$$

$$= \partial_{x^k} V_j^n + \partial_{\xi^j}(V_k^\gamma V_\gamma^n) - \partial_{x^j} V_k^n - \partial_{\xi^k}(V_i^\gamma V_\gamma^n),$$

$$R_{km}(x)\xi^m = \partial_{x^k} V_j^j + \partial_{\xi^j}(V_k^\gamma V_\gamma^j) - \partial_{x^j} V_k^j - \partial_{\xi^k}(V_j^\gamma V_\gamma^j).$$

The main result

Theorem: Let (g, Γ) be a smooth solution to the Einstein equations in vacuum. Let us define

$$V^j_k(x,\xi) = -\Gamma^j_{k\gamma}(x)\xi^\gamma, \quad (x,\xi) \in \mathbb{R}^8,$$

$$C^j_k(x,\xi) = \partial_{\xi^k} A^j(x,\xi) - \partial_{\xi^q} A^q(x,\xi)\ \delta^j_k,$$

$$A^j(x,\xi) = \xi^j \,\mathrm{det} g(x)\ \cos\Big(\frac{g_{\alpha\beta}(x)\xi^\alpha\xi^\beta}{2}\Big).$$

The main result

Theorem: Let (g, Γ) be a smooth solution to the Einstein equations in vacuum. Let us define

$$V_k^j(x,\xi) = -\Gamma^j_{k\gamma}(x)\xi^\gamma, \quad (x,\xi) \in \mathbb{R}^8,$$

$$C_k^j(x,\xi) = \partial_{\xi^k} A^j(x,\xi) - \partial_{\xi^q} A^q(x,\xi)\ \delta_k^j,$$

$$A^j(x,\xi) = \xi^j\ \mathrm{det} g(x)\ \cos(\frac{g_{\alpha\beta}(x)\xi^\alpha\xi^\beta}{2}).$$

Then (C, V) is a solution of the following generalized matrix-valued optimal transport problem:

Find a pair $(C, V)(x, \xi)$ of 4×4 matrix-valued fields over the 'phase-space' $(x, \xi) \in \mathbb{R}^8$ critical point of

$$\int \mathrm{trace}(C(x, \xi) V^2(x, \xi)) dx d\xi$$

Find a pair $(C, V)(x, \xi)$ of 4×4 matrix-valued fields over the 'phase-space' $(x, \xi) \in \mathbb{R}^8$ critical point of

$$\int \mathrm{trace}(C(x, \xi) V^2(x, \xi)) dx d\xi$$

subject to the 'generalized continuity equation'

$$\partial_{x^j} C^j_k + \partial_{\xi^j} (CV + VC)^j_k = 0$$

Find a pair $(C, V)(x, \xi)$ of 4×4 matrix-valued fields over the 'phase-space' $(x, \xi) \in \mathbb{R}^8$ critical point of

$$\int \mathrm{trace}(C(x,\xi)V^2(x,\xi))dxd\xi$$

subject to the 'generalized continuity equation'

$$\partial_{x^j} C^j_k + \partial_{\xi^j}(CV + VC)^j_k = 0$$

and the linear symmetry constraints $\partial_{\xi^i} V^k_j = \partial_{\xi^j} V^k_i$,

$$\partial_{\xi^m} C^\gamma_\gamma \, \delta^j_k - 3\partial_{\xi^m} C^j_k = \partial_{\xi^k} C^\gamma_\gamma \, \delta^j_m - 3\partial_{\xi^k} C^j_m.$$

ref: https://hal.archives-ouvertes.fr/hal-03311171

Relation with the (quadratic) Monge OT problem

$$Monge_2(\rho_0, \rho_1)^2 = \inf \int_{\mathbb{R}^d} |T(x) - x|^2 \rho_0(x) dx,$$

for all Borel maps T for which $\rho_1(y)dy$ is the image by $y = T(x)$ of $\rho_0(x)dx$.

Relation with the (quadratic) Monge OT problem

$$Monge_2(\rho_0, \rho_1)^2 = \inf \int_{\mathbb{R}^d} |T(x) - x|^2 \rho_0(x) dx,$$

for all Borel maps T for which $\rho_1(y)dy$ is the image by $y = T(x)$ of $\rho_0(x)dx$.

Then, one can show:

$$Monge_2(\rho_0, \rho_1)^2 = \inf \int_0^1 dt \int_{\mathbb{R}^d} \rho(t,x) |v(t,x)|^2 dx,$$

where (ρ, v) is subject to $\partial_t \rho + \nabla \cdot (\rho v) = 0, \ \ \rho(0, \cdot) = \rho_0 \ \ \rho(1, \cdot) = \rho_1.$

(Benamou-B. 2000, see also Otto 2001, Ambrosio-Gigli-Savaré 2005.)

Relation with the (quadratic) Monge OT problem

$$Monge_2(\rho_0, \rho_1)^2 = \inf \int_{\mathbb{R}^d} |T(x) - x|^2 \rho_0(x) dx,$$

for all Borel maps T for which $\rho_1(y)dy$ is the image by $y = T(x)$ of $\rho_0(x)dx$.

Then, one can show:

$$Monge_2(\rho_0, \rho_1)^2 = \inf \int_0^1 dt \int_{\mathbb{R}^d} \rho(t,x)|v(t,x)|^2 dx,$$

where (ρ, v) is subject to $\partial_t \rho + \nabla \cdot (\rho v) = 0, \;\; \rho(0,\cdot) = \rho_0 \;\; \rho(1,\cdot) = \rho_1$.

(Benamou-B. 2000, see also Otto 2001, Ambrosio-Gigli-Savaré 2005.)

N.B. Optimality equations read: $v = \nabla \phi, \;\; \partial_t v + \nabla(|v|^2/2) = 0.$

General Relativity GR and Optimal Transport OT

Recent works linking GR and OT:
A. Mondino, S. Suhr arXiv:1810.13309, 2018,
R. McCann, Camb. J. Math. 2020, based on
Lott-Sturm-Villani OT definition of Ricci curvature.

General Relativity GR and Optimal Transport OT

Recent works linking GR and OT:
A. Mondino, S. Suhr arXiv:1810.13309, 2018,
R. McCann, Camb. J. Math. 2020, based on
Lott-Sturm-Villani OT definition of Ricci curvature.
Our approach is different and related to the hydrodynamical formulation of OT (Benamou-B. 2000).

ref: Y.B. CRAS 2021 (https://hal.archives-ouvertes.fr/hal-03311171).

A toy model : the Hamilton-Jacobi equation (1/4)

$$\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 = 0, \quad \phi = \phi(t, x), \ \ x \in \mathbb{T}^d,$$

written as a 'conservation law' for $V = \nabla \phi$,

$$\partial_t V + \nabla(\frac{|V|^2}{2}) = 0, \quad V = V(t, x) \in \mathbb{R}^d, \ \ x \in \mathbb{T}^d.$$

A toy model : the Hamilton-Jacobi equation (1/4)

$$\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 = 0, \quad \phi = \phi(t,x), \ \ x \in \mathbb{T}^d,$$

written as a 'conservation law' for $V = \nabla \phi$,

$$\partial_t V + \nabla(\frac{|V|^2}{2}) = 0, \quad V = V(t,x) \in \mathbb{R}^d, \ \ x \in \mathbb{T}^d.$$

Ignoring BC, let us look for critical points (A, V) of

$$\int \left(-\partial_t A \cdot V - \frac{(\nabla \cdot A)|V|^2}{2} \right) dxdt, \quad A = A(t,x) \in \mathbb{R}^d.$$

A toy model : the Hamilton-Jacobi equation (2/4)

Critical points (A, V) of

$$\mathcal{I}(A, V) = \int \left(-\partial_t A \cdot V - \frac{(\nabla \cdot A)|V|^2}{2} \right) dxdt.$$

$$\partial_A \mathcal{I}(A, V) = 0 \ \Rightarrow \ \ (1) \ \ \partial_t V + \nabla(\frac{|V|^2}{2}) = 0$$

(as expected),

A toy model : the Hamilton-Jacobi equation (2/4)

Critical points (A, V) of

$$\mathcal{I}(A,V) = \int \left(-\partial_t A \cdot V - \frac{(\nabla \cdot A)|V|^2}{2} \right) dxdt.$$

$$\partial_A \mathcal{I}(A,V) = 0 \;\Rightarrow\; (1) \quad \partial_t V + \nabla(\frac{|V|^2}{2}) = 0$$

(as expected),

$$\partial_V \mathcal{I}(A,V) = 0 \;\Rightarrow\; (2) \quad \partial_t A + V(\nabla \cdot A) = 0$$

(additional information that we are now going to use).

A toy model : the Hamilton-Jacobi equation (3/4)

We use (2) $\partial_t A + V(\nabla \cdot A) = 0$ to rewrite $\mathcal{I}(A, V)$ as:

$$\mathcal{I}_2(A, V) = \int \frac{(\nabla \cdot A)|V|^2}{2} dxdt.$$

A toy model : the Hamilton-Jacobi equation (3/4)

We use (2) $\partial_t A + V(\nabla \cdot A) = 0$ to rewrite $\mathcal{I}(A, V)$ as:

$$\mathcal{I}_2(A, V) = \int \frac{(\nabla \cdot A)|V|^2}{2} dxdt.$$

Claim: whenever (A, V) is critical for $\mathcal{I}(A, V)$, then (A, V) is also critical for $\mathcal{I}_2(A, V)$, but subject to (2).

A toy model : the Hamilton-Jacobi equation (3/4)

We use (2) $\partial_t A + V(\nabla \cdot A) = 0$ to rewrite $\mathcal{I}(A, V)$ as:

$$\mathcal{I}_2(A, V) = \int \frac{(\nabla \cdot A)|V|^2}{2} dxdt.$$

Claim: whenever (A, V) is critical for $\mathcal{I}(A, V)$, then (A, V) is also critical for $\mathcal{I}_2(A, V)$, but subject to (2).

Proof: Let us introduce Lagrangian $\mathcal{L}(A, V, B) = \mathcal{I}_2(A, V) - \int B \cdot (\partial_t A + V(\nabla \cdot A))$.

A toy model : the Hamilton-Jacobi equation (3/4)

We use (2) $\partial_t A + V(\nabla \cdot A) = 0$ to rewrite $\mathcal{I}(A, V)$ as:

$$\mathcal{I}_2(A, V) = \int \frac{(\nabla \cdot A)|V|^2}{2} dxdt.$$

Claim: whenever (A, V) is critical for $\mathcal{I}(A, V)$, then (A, V) is also critical for $\mathcal{I}_2(A, V)$, but subject to (2).

Proof: Let us introduce Lagrangian $\mathcal{L}(A, V, B) = \mathcal{I}_2(A, V) - \int B \cdot (\partial_t A + V(\nabla \cdot A))$.
The corresponding optimality equations read:
$\partial_B \mathcal{L}(A, V, B) = 0 \Rightarrow$ (2) (of course), $\partial_V \mathcal{L}(A, V, B) = 0 \Rightarrow (\nabla \cdot A)V - B(\nabla \cdot A) = 0$,
$\partial_A \mathcal{L}(A, V, B) = 0 \Rightarrow -\nabla(|V|^2/2) + \partial_t B + \nabla(B \cdot V) = 0$.

A toy model : the Hamilton-Jacobi equation (3/4)

We use (2) $\partial_t A + V(\nabla \cdot A) = 0$ to rewrite $\mathcal{I}(A, V)$ as:

$$\mathcal{I}_2(A, V) = \int \frac{(\nabla \cdot A)|V|^2}{2} dxdt.$$

Claim: whenever (A, V) is critical for $\mathcal{I}(A, V)$, then (A, V) is also critical for $\mathcal{I}_2(A, V)$, but subject to (2).

Proof: Let us introduce Lagrangian $\mathcal{L}(A, V, B) = \mathcal{I}_2(A, V) - \int B \cdot (\partial_t A + V(\nabla \cdot A))$.
The corresponding optimality equations read:
$\partial_B \mathcal{L}(A, V, B) = 0 \Rightarrow$ (2) (of course), $\partial_V \mathcal{L}(A, V, B) = 0 \Rightarrow (\nabla \cdot A)V - B(\nabla \cdot A) = 0$,
$\partial_A \mathcal{L}(A, V, B) = 0 \Rightarrow -\nabla(|V|^2/2) + \partial_t B + \nabla(B \cdot V) = 0$.

Assuming that (A, V) is critical for $\mathcal{I}(A, V)$, we have

$\partial_t A + V(\nabla \cdot A) = 0$ and $\partial_t V + \nabla(|V|^2/2) = 0$.

A toy model : the Hamilton-Jacobi equation (3/4)

We use (2) $\partial_t A + V(\nabla \cdot A) = 0$ to rewrite $\mathcal{I}(A, V)$ as:

$$\mathcal{I}_2(A, V) = \int \frac{(\nabla \cdot A)|V|^2}{2} dxdt.$$

Claim: whenever (A, V) is critical for $\mathcal{I}(A, V)$, then (A, V) is also critical for $\mathcal{I}_2(A, V)$, but subject to (2).

Proof: Let us introduce Lagrangian $\mathcal{L}(A, V, B) = \mathcal{I}_2(A, V) - \int B \cdot (\partial_t A + V(\nabla \cdot A))$.
The corresponding optimality equations read:
$\partial_B \mathcal{L}(A, V, B) = 0 \Rightarrow$ (2) (of course), $\partial_V \mathcal{L}(A, V, B) = 0 \Rightarrow (\nabla \cdot A)V - B(\nabla \cdot A) = 0$,
$\partial_A \mathcal{L}(A, V, B) = 0 \Rightarrow -\nabla(|V|^2/2) + \partial_t B + \nabla(B \cdot V) = 0$.

Assuming that (A, V) is critical for $\mathcal{I}(A, V)$, we have

$\partial_t A + V(\nabla \cdot A) = 0$ and $\partial_t V + \nabla(|V|^2/2) = 0$. Setting $B = V$, we are just in business!

A toy model : the Hamilton-Jacobi equation (4/4)

Let us now write everything in terms of $(\rho = \nabla \cdot A, V)$:

$$(2) \quad \partial_t A + V(\nabla \cdot A) = 0 \;\Rightarrow\; \partial_t \rho + \nabla \cdot (\rho V) = 0,$$

$$\mathcal{I}_2(A, V) = \int \frac{(\nabla \cdot A)|V|^2}{2} dxdt \;\Rightarrow\; \int \frac{\rho |V|^2}{2} dxdt.$$

A toy model : the Hamilton-Jacobi equation (4/4)

Let us now write everything in terms of $(\rho = \nabla \cdot A, V)$:

$$(2)\quad \partial_t A + V(\nabla \cdot A) = 0 \;\Rightarrow\; \partial_t \rho + \nabla \cdot (\rho V) = 0,$$

$$\mathcal{I}_2(A, V) = \int \frac{(\nabla \cdot A)|V|^2}{2} dxdt \;\Rightarrow\; \int \frac{\rho |V|^2}{2} dxdt.$$

So, we just find the usual quadratic optimal problem in its BB formulation (up to BC which are ignored here)!

BB formulation of General Relativity in vacuum

We now treat the zero-Ricci 'phase-space' equation

$$\partial_{x^k} V_j^j + \partial_{\xi^j}(V_k^\gamma V_\gamma^j) - \partial_{x^j} V_k^j - \partial_{\xi^k}(V_j^\gamma V_\gamma^j) = 0$$

just as the HJ equation with the following dictionary

BB formulation of General Relativity in vacuum

We now treat the zero-Ricci 'phase-space' equation

$$\partial_{x^k} V_j^j + \partial_{\xi^j}(V_k^\gamma V_\gamma^j) - \partial_{x^j} V_k^j - \partial_{\xi^k}(V_j^\gamma V_\gamma^j) = 0$$

just as the HJ equation with the following dictionary

$$C_k^j(x,\xi) \quad \Leftarrow \quad \rho(t,x)$$

BB formulation of General Relativity in vacuum

We now treat the zero-Ricci 'phase-space' equation

$$\partial_{x^k} V_j^j + \partial_{\xi^j}(V_k^\gamma V_\gamma^j) - \partial_{x^j} V_k^j - \partial_{\xi^k}(V_j^\gamma V_\gamma^j) = 0$$

just as the HJ equation with the following dictionary

$$C_k^j(x,\xi) \quad \Leftarrow \quad \rho(t,x)$$

$$\partial_{x^j} C_k^j + \partial_{\xi^j}(CV + VC)_k^j = 0 \quad \Leftarrow \quad \partial_t \rho + \nabla \cdot (\rho V) = 0,$$

BB formulation of General Relativity in vacuum

We now treat the zero-Ricci 'phase-space' equation

$$\partial_{x^k} V_j^j + \partial_{\xi^j}(V_k^\gamma V_\gamma^j) - \partial_{x^j} V_k^j - \partial_{\xi^k}(V_j^\gamma V_\gamma^j) = 0$$

just as the HJ equation with the following dictionary

$$C_k^j(x,\xi) \quad \Leftarrow \quad \rho(t,x)$$

$$\partial_{x^j} C_k^j + \partial_{\xi^j}(CV + VC)_k^j = 0 \quad \Leftarrow \quad \partial_t \rho + \nabla \cdot (\rho V) = 0,$$

$$\int \mathrm{trace}(C(x,\xi) V^2(x,\xi)) dx d\xi \quad \Leftarrow \quad \int \rho |V|^2 dx dt$$

BB formulation of General Relativity in vacuum

We now treat the zero-Ricci 'phase-space' equation

$$\partial_{x^k} V_j^j + \partial_{\xi^j}(V_k^\gamma V_\gamma^j) - \partial_{x^j} V_k^j - \partial_{\xi^k}(V_j^\gamma V_\gamma^j) = 0$$

just as the HJ equation with the following dictionary

$$C_k^j(x,\xi) \quad \Leftarrow \quad \rho(t,x)$$

$$\partial_{x^j} C_k^j + \partial_{\xi^j}(CV + VC)_k^j = 0 \quad \Leftarrow \quad \partial_t \rho + \nabla \cdot (\rho V) = 0,$$

$$\int \mathrm{trace}(C(x,\xi)V^2(x,\xi))dxd\xi \quad \Leftarrow \quad \int \rho |V|^2 dxdt$$

and subject to $\quad \partial_{\xi^i} V_j^k = \partial_{\xi^j} V_i^k, \quad \partial_{\xi^m} C_\gamma^\gamma \ \delta_k^j - 3\partial_{\xi^m} C_k^j = \partial_{\xi^k} C_\gamma^\gamma \ \delta_m^j - 3\partial_{\xi^k} C_m^j.$

Finally, we check that, whenever (g, Γ) is a smooth solution to the Einstein equations in vacuum, then

$$V_k^j(x,\xi) = -\Gamma^j_{k\gamma}(x)\xi^\gamma,$$

$$C_k^j(x,\xi) = \partial_{\xi^k} A^j(x,\xi) - \partial_{\xi^q} A^q(x,\xi)\ \delta^j_k,$$

$$A^j(x,\xi) = \xi^j \operatorname{det} g(x)\ \cos\left(\frac{g_{\alpha\beta}(x)\xi^\alpha\xi^\beta}{2}\right).$$

defines a solution (C, V) to the BB-OT formulation.

ref: https://hal.archives-ouvertes.fr/hal-03311171

Finally, we check that, whenever (g, Γ) is a smooth solution to the Einstein equations in vacuum, then

$$V_k^j(x,\xi) = -\Gamma^j_{k\gamma}(x)\xi^\gamma,$$

$$C_k^j(x,\xi) = \partial_{\xi^k} A^j(x,\xi) - \partial_{\xi^q} A^q(x,\xi)\ \delta_k^j,$$

$$A^j(x,\xi) = \xi^j\ \mathrm{det} g(x)\ \cos(\frac{g_{\alpha\beta}(x)\xi^\alpha\xi^\beta}{2}).$$

defines a solution (C, V) to the BB-OT formulation.

ref: https://hal.archives-ouvertes.fr/hal-03311171

END OF PROOF!

THANKS FOR YOUR ATTENTION!

THANKS FOR YOUR ATTENTION!

**Thanks to Bernard Julia
and François-Xavier Vialart for their help!**