Decidable problems on integral SL_2-characters

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Q1. Is $\begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}$ a commutator in $SL_2(\mathbb{Z})$? NO A product of two commutators? YES

Q2. Let $G = \left\langle \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \right\rangle \leq SL_2(\mathbb{Z})$. Is $623 \in \mathrm{tr}(G) \subseteq \mathbb{Z}$? YES

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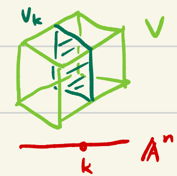
§1. Diophantine Motivation

Consider a parametric family of Diophantine equations: $(g_i, f_j \in \mathbb{Z}[x_1, \dots, x_m])$

$$V: g_1(x_1, \dots, x_m) = \cdots = g_\ell(x_1, \dots, x_m) = 0$$

$$\begin{cases} f_1(x_1, \dots, x_m) = k_1 \\ \quad \vdots \qquad\qquad\qquad \vdots \\ f_n(x_1, \dots, x_m) = k_n \end{cases}$$

$$\begin{array}{c} V \subseteq \mathbb{A}^m \\ \Big\downarrow {\scriptstyle f = (f_1, \dots, f_n)} \\ \mathbb{A}^n \end{array} \qquad V_k := f^{-1}(k).$$



Basic Questions:

① (Base). Is $f(V(\mathbb{Z})) \subseteq \mathbb{Z}^n$ decidable?

② (Fiber). Structure of each $V_k(\mathbb{Z})$? (e.g. finite, fin. gen.)

③ (Family) Behavior of $k \mapsto V_k(\mathbb{Z})$?

Thm (Matiyasevich) $\exists\, f: V \to \mathbb{A}^n$ st. $f(V(\mathbb{Z})) \subseteq \mathbb{Z}^n$ is undecidable.

Goal. Find new examples of $f: V \to \mathbb{A}^n$ with $f(V(\mathbb{Z})) \subseteq \mathbb{Z}^n$ decidable.

Examples.

① $f : \mathbb{A}^m \to \mathbb{A}^n$ linear

② $f : \mathbb{A}^m \to \mathbb{A}^1$ quadratic form.

③ $f : \mathbb{A}^2 \to \mathbb{A}^1$ degree $d \geq 3$ form (Thue, Baker)

④ $f : \mathbb{A}^m \to \mathbb{A}^m /\!\!/ G$ affine homogeneous varieties $V_k := f^{-1}(k)$.

$G \curvearrowright \mathbb{A}^m$ rat'l linear rep. of reductive alg. gp $G/\mathbb{Q}$.

$f = (f_1, \dots, f_n)$, $f_i \in \mathbb{Z}[\mathbb{A}^m]$ st. $\mathbb{C}[\mathbb{A}^m]^G = \langle f_1, \dots, f_n \rangle$.

Thm (Borel-Harish-Chandra) If V_k is a closed G-orbit, then $V_k(\mathbb{Z}) = G(\mathbb{Z}) \cdot S$ for some computable finite S.

Ex. $V = \{ \text{BQFs } Q(x,y) = ax^2 + bxy + cy^2 \} \simeq \mathbb{A}^3 \circlearrowleft SL_2$ action by linear change of variables (x,y)

$\mathrm{disc}(a,b,c) = b^2 - 4ac : V \to \mathbb{A}^1$. $V_k = \mathrm{disc}^{-1}(k)$.

Thm (Lagrange, Gauss). If $k \neq 0$, then $h(k) := |SL_2(\mathbb{Z}) \backslash V_k(\mathbb{Z})| < \infty$. ($k \mapsto h(k)$ actively studied).

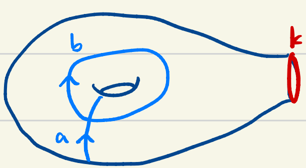
Goal. Go beyond the above examples, utilizing nonlinear symmetry

Markoff equation

- Space $\quad X = \mathbb{A}^3$
- Invariants $\quad M: X \to \mathbb{A}^1, \quad M(x,y,z) = x^2+y^2+z^2-xyz-2. \quad X_k := M^{-1}(k).$
- Dynamics $\quad X_k \circlearrowleft \Gamma$ group of nonlinear transformations generated by:
  - permutation of coordinates (x,y,z),
  - Vieta involution $(x,y,z) \mapsto (x,y,xy-z)$.

Thm (Markoff 1880) If $k \neq 2$, then $|\Gamma \backslash X_k(\mathbb{Z})| < \infty$. $\quad$ Ghosh-Sarnak

Let $\Sigma = \Sigma_{1,1}$ surface of genus 1 with 1 boundary curve. $\pi_1 \Sigma = \langle a, b \rangle \cong F_2$.



$X = \mathrm{Hom}(\pi_1\Sigma, SL_2) /\!\!/ \mathrm{Inn}(SL_2) \quad (\mathrm{tr}_a, \mathrm{tr}_b, \mathrm{tr}_{ab}) : X \xrightarrow{\sim} \mathbb{A}^3.$

$\mathrm{tr}_{[a,b]} = M(\mathrm{tr}_a, \mathrm{tr}_b, \mathrm{tr}_{ab}) : X \longrightarrow \mathbb{A}^1 \quad X_k = \mathrm{tr}_{[a,b]}^{-1}(k).$ (Rmk. $X_2 = X_{red}$).

Γ-action is (essentially) action of mapping class group of Σ.

Goal. Generalize to SL_2-character varieties of other groups.

§2. Reduction theory.

$\Sigma = \Sigma_{g,n}$ surface of type (g,n), $\partial\Sigma = c_1 \sqcup \cdots \sqcup c_n$. $(3g+n-3>0)$.

- Space $\quad X = X(\Sigma) := \mathrm{Hom}(\pi_1\Sigma, SL_2)/\!/\mathrm{Inn}(SL_2)$ $\quad X_{red} := \{[\rho] \in X(\mathbb{C}) : \rho \text{ reducible}\}$.

  $X(\mathbb{Z}) = \{[\rho] \in X(\mathbb{C}) : \mathrm{tr}\,\rho(\alpha) \in \mathbb{Z}\ \forall \alpha \in \pi_1\Sigma\}$.

- Invariants. $\quad f = (\mathrm{tr}_{c_1}, \dots, \mathrm{tr}_{c_n}) : X \longrightarrow \mathbb{A}^n$. $\quad X_k = f^{-1}(k)$. irred affine var. of dim $6g+2n-6$.

- Dynamics. $\quad X_k \circlearrowleft \Gamma = \pi_0 \mathrm{Diff}^+(\Sigma, \partial\Sigma)$ mapping class group of Σ.

Main Thm (W.) Assume $X_k \cap X_{red} = \emptyset$. Then $X_k(\mathbb{Z})$ is effectively finitely generated.

More precisely, $X_k(\mathbb{Z}) = \Gamma \cdot \bigcup_{i=1}^{n} g_i(G_i(\mathbb{Z}))$

for some computable $g_i : G_i \to X_k$ from alg gps G_i with eff. f.g. lattice $G_i(\mathbb{Z}) \leq G_i(\mathbb{R})$.

Cor. $f(X(\mathbb{Z})) \subseteq \mathbb{Z}^n$ is decidable.

Cor. There is an algorithm that decides, given $A \in SL_2(\mathbb{Z})$ and $g \geq 1$, whether or not A is a product of g commutators in $SL_2(\mathbb{Z})$.

Remarks

① In general, $|\Gamma \backslash X_k(\mathbb{Z})|$ can be infinite.

Def. $\cdot$ X_k^{deg} := union of images of nonconstant morphisms $\mathbb{A}^1 \to X_k$.

$\cdot$ $X_k(\mathbb{Z})^* := X_k(\mathbb{Z}) \backslash X_k^{deg}$.

Motivation: each X_k is log CY (W.)

Thm. (W.) We have $|\Gamma \backslash X_k(\mathbb{Z})^*| < \infty$, and $\exists$ Zariski closed $Z \subsetneq X_k$ st. $X_k^{deg} = \Gamma \cdot Z$.

(Cf. Lang's Conj. Let $V/\mathbb{Q}$ be a proj. variety of general type

Let V^{ex} = union of images of nonconstant rat'l maps $\mathbb{P}^1$, Ab. var $\dashrightarrow X_k$.

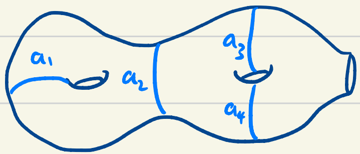
Then $V^{ex} \subsetneq V$ is Zariski closed, and $V(\mathbb{Q}) \backslash V^{ex}$ is finite.)

② We have inclusions of Γ-invariant subsets

$$X_k(\mathbb{Z})^{Teich} \subseteq X_k(\mathbb{Z})^{inj} \subseteq X_k(\mathbb{Z})^*.$$

Proof of Main Thm.

Sp. $P = a_1 \amalg \cdots \amalg a_{3g+n-3} \subset \Sigma$ pants decomposition.



$$X_k \xrightarrow{tr_P = (tr_{a_i})} \mathbb{A}^{3g+n-3} \qquad X_{k,t}^P := tr_P^{-1}(t).$$

Generically, $X_{k,t}^P \approx (\mathbb{C}^\times)^{3g+n-3}$

$X_{k,t}^P \circlearrowleft \mathbb{Z}^{3g+n-3} \cong \Gamma_P := \underbrace{\langle \tau_{a_1}, \dots, \tau_{a_{3g+n-3}} \rangle}_{\text{Dehn twists along } a_i.} \leq \Gamma$

Let $\rho \in X_k(\mathbb{Z})$.

A. Find $P \subset \Sigma$ st. $\|tr_P(\rho)\| = O_k(1)$.

B. Reduction theory on $X_{k,t}^P$: Each $X_{k,t}^P$ is essentially an <u>affine homogeneous variety</u>.

<u>Prop.</u> If $X_{k,t}^P \cap X_{red} = \phi$, then $X_{k,t}^P(\mathbb{Z}) = \bigcup_{i=1}^r g_i(G_i(\mathbb{Z}))$

for some computable $g_i : G_i \to X_{k,t}^P$ from alg groups G_i with eff. f.g. lattice $G_i(\mathbb{Z})$.

C. Up to Γ-action, $\exists$ at most fintely many $P \subset \Sigma$.

A. Find $\rho \subset \Sigma$ s.t. $\|\mathrm{tr}_\rho(\rho)\| = O_k(1)$.

For $g \in SL_2(\mathbb{C}) \curvearrowright \mathbb{H}^3$, let $\mathrm{length}(g) = \inf_{x \in \mathbb{H}^3} d(x, g \cdot x)$. ($\approx \log|\mathrm{tr}(g)|$)

Def. For $\rho : \pi_1 \Sigma \to SL_2(\mathbb{C})$, let $\mathrm{sys}(\rho) = \inf\{\mathrm{length}\, \rho(a) : a \subset \Sigma$ essential simple closed curve$\}$.

By induction on topological type of Σ, suffices to prove:

Thm. (W.) sys is bounded on $X_k(\mathbb{C})$.

Pf. Let $\rho : \pi_1 \Sigma \to SL_2(\mathbb{C})$ be given. Fix σ_0 hyperbolic metric on Σ.

$$\begin{array}{c} \tilde{\Sigma} \times_\rho \mathbb{H}^3 \\ s \curvearrowright \big\downarrow \pi \\ \Sigma \end{array}$$

Let $\sigma(\varepsilon) = s^*(\varepsilon \sigma_0 + \sigma_{\mathbb{H}^3})$ for $\varepsilon > 0$.

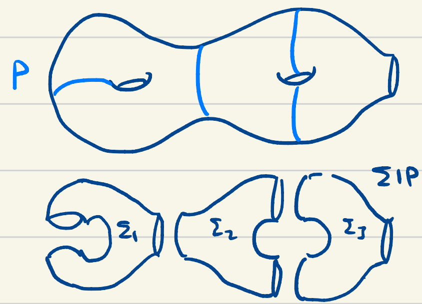
$$\mathrm{sys}(\rho) \le \mathrm{sys}(\Sigma, \sigma(\varepsilon))$$:= inf length of essential closed geodesic on Σ

$$\ll \mathrm{Vol}(\Sigma, \sigma(\varepsilon))^{\frac{1}{2}} + \mathrm{length}(\partial\Sigma, \sigma(\varepsilon))$$ (Loewner, Gromov, ...)

If s is harmonic (+ ∂ cond) then RHS bdd via $|\chi(\Sigma)|$ & k as $\varepsilon \to 0$. (cf. Milnor-Wood)

Harmonic section exists (for ρ with $\mathbb{Z}$-dense image) by Donaldson, Hamilton, Corlette. $\square$

B. Reduction theory on $X^P_{k,t}$.



Let $\Sigma | P = \Sigma_1 \amalg \cdots \amalg \Sigma_{2g+n-2}$. $\Sigma_i \approx \Sigma_{0,3}$.

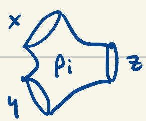
Let G be an form of SL_2 over $\mathbb{Q}$.

$$\begin{array}{ccc} \mathrm{Hom}(\pi_1\Sigma, G) & \overset{\pi}{\twoheadrightarrow} & X(\Sigma) \\ \downarrow \tilde{f} & & \downarrow f \\ \prod_i \mathrm{Hom}(\pi_1\Sigma_i, G) & \overset{\pi}{\twoheadrightarrow} & \prod_i X(\Sigma_i). \end{array}$$

① **Prop.** Let $[\rho] \in X(\Sigma)(\mathbb{Z})$ be irreducible. Then $\exists!$ $G/\mathbb{Q}$ eff. det, $[\rho] = [\rho']$, $\rho' : \pi_1\Sigma \to G(\mathbb{Z}) \leq SL_2(\mathbb{C})$.

Cor Assume $X_k \cap X_{red} = \phi$. Then there are at most finitely many $G/\mathbb{Q}$ st. $\pi^{-1}(X_k)(\mathbb{Z}) \neq \phi$.

Pf Let $\rho \in X_k(\mathbb{Z})$. By A, $\exists\, P \subset \Sigma$ st. $\|\mathrm{tr}_P(\rho)\| = O_k(1)$.



Since $X_k \cap X_{red} = \phi$, some $\rho_i = \rho|_{\Sigma_i}$ is irreducible.

$\rho \sim \rho' : \pi_1\Sigma \to G(\mathbb{Z})$ where G = norm one units of $(x^2-4, M(x,y,z)-2)_{\mathbb{Q}}$. □

② **Prop.** Let $P \in \prod_i X(\Sigma_i)(\mathbb{Z})$. ($+\varepsilon$ if $G = SL_{2,\mathbb{Q}}$). Then $\pi^{-1}(P)(\mathbb{Z})$ belongs to a closed G^{2g+n-2}-orbit.

For each $\tilde{P} \in \pi^{-1}(P)$, the fiber $\tilde{f}^{-1}(\tilde{P})$ is a closed $\prod_{j=1}^{3g+n-3} H_j$-orbit where H_j = Centralizer of $\rho(a_j)$ in G.

⇝ Use Borel–Harish-Chandra reduction theory to conclude.

§5. Representation problem. Let F_m denote the free gp on $m \geq 1$ generators.

Thm (W., in progress.) Let $\rho : F_m \to SL_2(\mathbb{C})$ with integral character. Then $\mathrm{Im}(\mathrm{tr}\,\rho) \subseteq \mathbb{Z}$ is decidable.

Pf sketch. There are four steps.

① Bounding topology.

Assume $\bar{\rho} : F_m \xrightarrow{\rho} SL_2(\mathbb{R}) \twoheadrightarrow PSL_2(\mathbb{R})$ nonelementary (i.e. does not fix a point in $\overline{\mathbb{H}^2}$.)

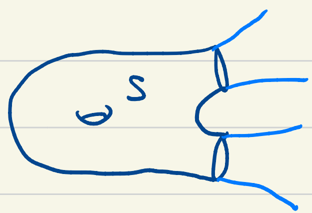
Let $\pi = \mathrm{Im}(\bar{\rho}) \leq PSL_2(\mathbb{R})$. Then π is a Fuchsian group.

Say $\mathbb{H}/\pi$ is a surface of genus g with finitely many orbifold points of orders $m_1, \dots, m_r$ $(m_i \leq 3)$,

$s \in \mathbb{Z}_{\geq 0}$ cusps, and $t \in \mathbb{Z}_{\geq 0}$ flares.

For simplicity, assume $r = 0$. Let $n = s + t$.

Let $S \subset \mathbb{H}/\pi$ be the convex core.



Note. $2g + n \leq 1 + m$.

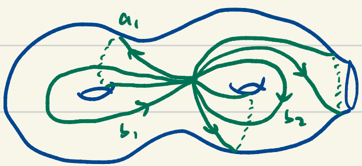
Thus, there is a finite list L of topological types (g, n) possible for π.

② <u>Standard form.</u>

Suppose we know $\sigma = \{a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n\} \subseteq \pi$ giving standard presentation of $\pi = \pi_1(S)$.

<u>Prop.</u> The equation $\mathrm{tr}(g) = t$ holds for $g \in \pi$ in at most finitely many effectively determined $g \in \pi$ up to conjugacy.

<u>Idea.</u> (a) Construct a pants decomposition $P \subset S$ with $\mathrm{length}_S(P)$ bounded in terms of σ.

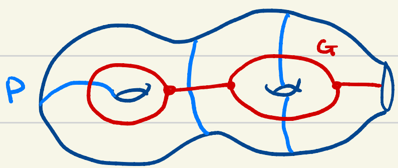


Let $S \setminus P = \coprod_i S_i$. Let $G \subset \Sigma$ be dual graph.

(b) Suppose $C \subset S$ is a closed geodesic st $\mathrm{tr}(C) = t$, so $\mathrm{length}_S(C) \approx \log|t|$.

Then $\exists$ upper bound on:

- $\#(C \cap P)$,
- $\#(C \cap G)$,
- $\#$(self-intersections of each arc on $C \cap S_i$).



(c) Use data from (b) to write down an element of π whose conjugacy class corresponds to C. $\square$

It remains to find an algorithm to produce standard generators for $\pi = \pi_1(S)$.

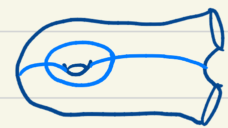
③ Bounding boundary.

Fact. Let $\langle \gamma_1, \dots, \gamma_m \rangle = F_m$ generators. Then $\mathbb{C}[X(F_m)] = \langle \operatorname{tr} \gamma : \gamma \in \underbrace{\{ \gamma_{i_1} \cdots \gamma_{i_k} : 1 \le i_1 < \dots < i_k \le m \}_{1 \le k \le 3}}_{\gamma_1, \dots, \gamma_N} \rangle$.

The conjugacy classes of $\bar{\rho}(\gamma_1), \dots, \bar{\rho}(\gamma_N) \in \pi$ determine closed geodesics $\alpha_1, \dots, \alpha_N$ on S.

Prop. For each component c of ∂S, we have

$$\operatorname{Length}_S(c) \le 2 \sum_{i=1}^{N} \operatorname{Length}_S(\alpha_i).$$



Pf sketch. Let $T \subset S$ be a closed tubular neighborhood of $\bigcup_{i=1}^{N} \alpha_i$.

Then each component of ∂S is isotopic on S to a component of ∂T. □

④ Reduction theory.

Combining ① & ③ and reduction theory: there is a finite collection $C = \{ \rho_i : \pi_1(\Sigma_{g,n}) \to SL_2(\mathbb{R}) \}$

and marked hyperbolic str. with integral character, with $\Sigma_{g,n}$ ranging over a finite list, s.t.

$\operatorname{Im}(\rho) = \operatorname{Im}(\rho_i)$ for some $\rho_i \in C$. Use ② for each ρ_i to finish. QED.

effectively determine using ②