Rankin-Selberg integrals in positive characteristic

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Rankin-Selberg integrals in positive characteristic and its connection to Langlands functoriality

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I. Motivation

N: a positive integer

$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$: a Dirichlet character

$\chi(m) = 0$ for $(m, N) \neq 1$

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \frac{\chi(1)}{1^s} + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \cdots$$

It converges for $\mathrm{Re}(s) > 1$.

It has a meromorphic continuation.

If χ is trivial,

$$\zeta(s) := L(s, 1)$$

Euler Product

$$L(s,\chi) = \prod_{p:\text{ prime}} \left(1 + \frac{\chi(p^s)}{p^s} + \frac{\chi(p^{2s})}{p^{2s}} + \cdots\right)$$

factorization

$$= \prod_{p:\text{ prime}} \frac{1}{1-\chi(p)p^{-s}}$$

The uniqueness of prime factorization

$$n^{-s} = (p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k})^{-s}$$

Rankin-Selberg method

Langlands-Shahidi method

The functional equation

The standard Gamma function

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx, \quad \mathrm{Re}(s) > 0$$

The complete (global) Dirichlet L-function

$$\Lambda(s,\chi) := \pi^{-\frac{s+\varepsilon}{2}} \Gamma\left(\frac{s+\varepsilon}{2}\right) L(s,\chi)$$

$$= \underbrace{\pi^{-\frac{s+\varepsilon}{2}}\Gamma\left(\frac{s+\varepsilon}{2}\right)}_{\text{archimedean factor}} \underbrace{\prod_{p \mid N} 1}_{\text{non-archimedean ramified factor}} \underbrace{\prod_{p \nmid N} \frac{1}{1-\chi(p)p^{-s}}}_{\text{non-archimedean unramified factor}}$$

$\varepsilon \in \{0,1\}$ such that $\chi(-1) = (-1)^{\varepsilon}$

$$\boxed{\Lambda(s,\chi) = (-i)^{\varepsilon}\tau(\chi) N^{-s} \Lambda(1-s,\bar{\chi})}$$

$$i = \sqrt{-1}$$

$$\tau(\chi) := \sum_{n \bmod N} \chi(n) e^{\frac{2\pi i n}{N}},$$

Gauss Sum

Global ε-functions

$$\varepsilon(s,\chi) = (-i)^{\varepsilon}\tau(\chi) N^{-s}$$

$\left\{ \begin{array}{l} \text{The Dirichlet Character} \\ \qquad - GL(1) \\ \text{The classical holomorphic} \\ \text{cusp forms (Hecke eigenforms)} \\ \qquad - GL(2) \end{array} \right.$

$\rightsquigarrow$ "automorphic representations and L-functions" and L-functions

II. Tate Thesis

F: non-archimedean local fields

ex) $F = \mathbb{Q}_p$, $\mathbb{F}_p((T))$

$\mathcal{O}$; a ring of integers for F

$\mathcal{O}^\times$; units in $\mathcal{O}$

ϖ; a uniformizer with $q = |\varpi|^{-1}$

χ, μ; characters of $F^\times$

$S(F)$: Bruhat-Schwartz functions on F.

For $\Phi \in S(F)$, we define

$$\Psi(s, \chi, \mu, \Phi) = \int_{F^\times} (\chi\mu)(a)\Phi(a)|a|^s d^\times a$$

It converges absolutely for $\mathrm{Re}(s) \gg 0$

Tate L-factor

$$L(s, \chi \times \mu) := \begin{cases} \dfrac{1}{1-\chi\mu(\varpi)q^{-s}}, & \text{If } \chi \text{ and } \mu \text{ are unramified} \\ 1, & \text{otherwise} \end{cases}$$

Observation

$L(s, \chi \times \mu)$ has a pole at $s=0$
if and only if $\chi\mu \equiv 1$ $(\Leftrightarrow \chi = \mu^{-1})$

Fourier Transform

ψ: a non-trivial additive character on F

$$\hat{\Phi}(y) = \int_F \Phi(x)\psi(xy)\,dx.$$

- γ-factor

$$\Psi(1-s, \chi^{-1}, \mu^{-1}, \hat{\Phi})$$

[illegible]

$$= \gamma(s, \chi\times\mu, \psi)\Psi(s, \chi, \mu, \Phi)$$

$$\gamma(s, \chi\times\mu, \psi) \in \mathbb{C}(q^{-s})$$

- ε-factor

$$\varepsilon(s, \chi\times\mu, \psi) = \gamma(s, \chi\times\mu, \psi)\frac{L(s, \chi\times\mu)}{L(1-s, \chi^{-1}\times\mu^{-1})}$$

$$\varepsilon(s, \chi\times\mu, \psi) \in \mathbb{C}[q^{\pm s}]$$

III. The Local Functoriality

Assume that $\mathrm{char}(F)=0$, ex) $F = \mathbb{Q}_p$

$\mathcal{A}^g_0(SO_{2r+1})$;

the set of irreducible generic supercuspidal representations of $SO_{2r+1}(F)$ (up to equivalence)

$\mathcal{A}_\ell(GL_{2r})$;
the set of all Π of $GL_{2r}(F)$ of the form

$$\Pi \simeq \mathrm{Ind}(\Pi_1 \otimes \cdots \otimes \Pi_d),$$

where each Π_i is an irreducible, supercuspidal, (self-dual representation) of some $GL_{2r_i}(F)$ such that
$L(s, \Pi_i, \wedge^2)$ has a pole at $s=0$
and $\Pi_i \not\simeq \Pi_j$ for $i \neq j$.

<u>Theorem</u> (Jiang and Soudry)
There exists a unique bijection taking $\mathcal{A}_0^g(SO_{2r+1}) \longrightarrow \mathcal{A}_\ell(GL_{2r})$,

$$\pi \longmapsto \Pi$$

such that

$$L(s, \pi \times \tau) = L(s, \Pi \times \tau)$$
$$\varepsilon(s, \pi \times \tau, \psi) = \varepsilon(s, \Pi \times \tau, \psi)$$

for all irreducible supercuspidal representations τ of $GL(F)$

representations τ of $GL_n(F)$ for all n.

① Existence ⟵ global functorial lifting (Cogdell, Kim, Piatetski-Shapiro and Shahidi)

② Injective ⟵ local converse thm "local descent theory" + ⋯ (backward lifting) Ginzburg, Rallis, and Soudry

③ Surjective

⟵ "local descent theory"

<u>Remark</u> (Jiang and Soudry)

It can be extended to all of $\mathcal{A}^g(SO_{2r+1})$, the set of all irreducible admissible generic representations of $SO_{2r+1}(F)$

<u>Theorem</u> (Jiang, Nien and Qin)

(1) $L_{LS}(s, \pi, \wedge^2)$ has a pole at $s=0$.

(2) π has a Shalika functional.

(3) Π is a local functorial transfer from π.

IV. Local Exterior Square L-functions

Assume that $\mathrm{char}(F) > 2$, ex) $F = \mathbb{F}_p((T))$

$N_r(F)$: the subgroup of upper triangular matrices with 1 in diagonals

$M_r(F)$: $r \times r$ matrices;

$N_r(F)$: the subgroups of $M_r(F)$ consisting of upper triangular matrices

Perfect shuffled permutation

$$\sigma_{2r} = \left(\begin{array}{cccc|cccc} 1 & 2 & \cdots & r & r+1 & r+2 & \cdots & 2r \\ 1 & 3 & \cdots & 2r-1 & 2 & 4 & \cdots & 2r \end{array}\right)$$

$$\sigma_{2r+1} = \left(\begin{array}{cccc|ccccc} 1 & 2 & \cdots & r & r+1 & r+2 & \cdots & 2r & 2r+1 \\ 1 & 3 & \cdots & 2r-1 & 2 & 4 & \cdots & 2r & 2r+1 \end{array}\right)$$

$m = 2r$ or $2r+1$

π: an irreducible admissible generic (complex) representation of $GL_m(F)$

$W(\pi, \psi)$: Whittaker model for π

$S(F^r)$: Bruhat-Schwartz functions on F^r

$e_r = (0, 0, \cdots, 1)$

$W \in \mathcal{W}(\pi, \psi)$, $\Phi \in S(F^r)$

$\boxed{m=2r}$

$$J(s, W, \Phi) = \int_{N_r(F)\backslash GL_r(F)} \int_{N_r(F)\backslash M_r(F)} W\left(\sigma_{2r} \begin{pmatrix} I_r & X \\ & I_r \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix}\right) \Phi(e_r g) \psi^{-1}(\mathrm{Tr} X) |\det g|^s dX\, dg$$

$\boxed{m=2r+1}$

$$J(s, W, \Phi) = \int_{N_r(F)\backslash GL_r(F)} \int_{N_r(F)\backslash M_r(F)} \int_{F^r} W\left(\sigma_{2r+1} \begin{pmatrix} I_r & X & \\ & I_r & \\ & & 1 \end{pmatrix} \begin{pmatrix} g & & \\ & g & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_r & & \\ & I_r & \\ & y & 1 \end{pmatrix}\right) \Phi(y) \psi^{-1}(\mathrm{Tr} X) |\det g|^{s-1} dy\, dX\, dg$$

<u>Theorem</u> (Jacquet and Shalika)

① For $W \in \mathcal{W}(\pi, \psi)$ and $\Phi \in S(F^r)$, $J(s, W, \Phi) \in \mathbb{C}(q^{-s})$.

Hence, it admits a meromorphic continuation.

② $\langle J(s, W, \Phi) \rangle$ is a $\mathbb{C}[q^{\pm s}]$-fractional ideal in $\mathbb{C}(q^{-s})$.

③ $\langle J(s,W,\Phi)\rangle = \left\langle \frac{1}{P(q^{-s})} \right\rangle$ such that

$P(X)\in\mathbb{C}[X]$ and $P(0)=1$.

Def $L(s,\pi,\wedge^2) = \frac{1}{P(q^{-s})}$

Theorem (J.)

π; irreducible admissible representation of $GL_m(F)$

$$L(s,\pi,\wedge^2) = L_{JS}(s,\pi,\wedge^2)$$

· char$(F)=0$, 2018

$\leftarrow$ Kewat-Raghunathan

Shalika subgroup

$$S_{2r} = \left\{ \begin{pmatrix} I_r & Z \\ & I_r \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix} \;\middle|\; \begin{matrix} Z\in M_r(F) \\ g\in GL_r(F) \end{matrix} \right\}$$

Shalika functionals

A linear form Λ on $\mathcal{W}(\pi,\psi)$ is called a Shalika functional if

$$\Lambda\left(\pi\left(\begin{pmatrix} I_r & Z \\ & I_r \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right) W\right)$$

$= \psi(\mathrm{Tr}\,Z)\,\Lambda(W)$ for $W\in\mathcal{W}(\pi,\psi)$

$= \psi(\mathrm{Tr}\, Z)\Lambda(W)$ for $W = W(\ldots)$

$(0\neq)\ \Lambda \in \mathrm{Hom}_{S_{2r}}(\pi, \Theta) \neq 0$

Shalika Character

$$\Theta\left(\begin{pmatrix} I_r & Z \\ & I_r \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right) = \psi(\mathrm{Tr}\, Z)$$

Example

Δ: a discrete series representation

$L(s, \Delta, \wedge^2)$ has a pole at $s=0$

$\iff \Delta$ has a Shalika functional

Question (Cogdell, Piatetski-Shapiro and Matringe)

π will have a Shalika functional precisely when it is a functorial lifting from $SO_{2r+1}(F)$.

Theorem (Lomelí)

There exists a map

$\mathcal{A}_0^g(SO_{2r+1}) \to \mathcal{A}_e(GL_{2r})$, satisfying

$\pi \longmapsto \Pi$ (with $\pi \in \mathcal{A}_0^g(SO_{2r+1})$)

$$L(s, \pi\times\tau) = L(s, \Pi\times\tau)$$

$$\varepsilon(s, \pi\times\tau, \psi) = \varepsilon(s, \Pi\times\tau, \psi)$$

for all irreducible supercuspidal representations τ of $GL_n(F)$ for all n.

<u>Theorem</u> (Matringe)

$\pi = \mathrm{Ind}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$;
an irreducible generic representation of GL_{2r}, where Δ_i are discrete series representation. Then
π admits a (S_{2r}, Θ)-Shalika functional
$\Longleftrightarrow$ there is a reordering of the

Δ_i's and an integer n between 1 and $[\frac{t}{2}]$ such that
① $\Delta_{i+1} \simeq \Delta_i^\vee$ for $i = 1, 3, \cdots, 2n-1$
and
② Δ_i admits a (S_{2r_i}, Θ)-Shalika functional for $i > 2n$.
$(\Longleftrightarrow L(s, \Delta_i, \Lambda^2)$ has a pole at $s=0$.)

<u>Take away</u>

<u>Theorem</u> (Kaplan)

$\mathrm{char}(F) = 0$

$\pi = \mathrm{Ind}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$: an irreducible generic representation of $GL_m(F)$.

π is θ-distinguished

$\Longleftrightarrow$ there is a reordering of Δ_i's and an integer n between 1 and $[\frac{t}{2}]$ such that

① $\Delta_{i+1} = \Delta_i^\vee$ for $i = 1, 3, \cdots, 2n-1$

② Δ_i is θ-distinguished for $i > 2n$.

($\Longleftrightarrow L(s, \pi, Sym^2)$ has a pole

at $s = (\ ).)$

Question (Savin, Kable, and Kaplan)

π is θ-distinguished precisely when it is a functorial lifting from $SO_{2r}(F)$ / $Sp_{2r}(F)$.

V. Local Rankin-Selberg γ-factors for $SO_{2r+1}(F) \times GL_n(F)$

Theorem (J.)

$char(F) > 0$

π_1, π_2 irreducible admissible generic representations of $SO_{2r+1}(F)$

Suppose that

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)$$

for all irreducible supercuspidal τ of $GL_n(F)$ for $1 \leq n \leq r$.

Then $\pi_1 \simeq \pi_2$.

<u>Corollary</u>

The map
$$\mathcal{A}_0^g(SO_{2r+1}) \to \mathcal{A}_\ell(GL_{2r})$$
$$\pi \longmapsto \Pi$$
is injective. "local descent theory"

<u>Remark</u> $\mathrm{char}(F)=0$

- Jiang and Soudry + Jacquet's Conjecture (Liu-Jacquet, Chai)
- $Sp_{2r}(F)$, $U_{2r+1}(F)$, $U_{2r}(F)$ — Q. Zhang
- Finite fields — Nien, Liu and Zhang
- Chen, Henniart, ...