Positivity in Arakelov geometry and arithmetic Okounkov bodies

François Ballaÿ

Laboratoire de Mathématiques Nicolas Oresme
Université Caen Normandie, France

France-Korea IRL Webinar in Number Theory
November 7th, 2022

1 Positivity in Arakelov geometry
- Positivity in algebraic geometry
- Arakelov theory
- Positivity in Arakelov geometry

2 Arithmetic Okounkov bodies
- Motivation and definition
- Results
- Applications

Positivity in algebraic geometry

- X a smooth projective variety over a field $k = \overline{k}$
- $\mathrm{Pic}(X)$: group of isomorphism classes of line bundles

Let $L \in \mathrm{Pic}(X)$. Assume that $H^0(X, L) \neq 0$. We have a rational map

$$\phi_L \colon X \dashrightarrow \mathbb{P}(H^0(X, L))$$

sending closed points to one-dimensional quotients of $H^0(X, L)$.

Positivity in algebraic geometry

We say that L is

1. **very ample** if $\phi_L\colon X \hookrightarrow \mathbb{P}(H^0(X,L))$ is a closed immersion
2. **ample** if $L^{\otimes n}$ is very ample for some $n \geq 1$
3. **big** if $\phi_{L^{\otimes n}}$ is birational onto its image for some $n \geq 1$. L is big if and only if

$$\mathrm{vol}(L) := \limsup_{n\to\infty} \frac{\dim_k H^0(X, L^{\otimes n})}{n^{\dim X}/(\dim X)!} > 0.$$

4. **nef** if $L^{\dim Y} \cdot Y \geq 0$ for every subvariety $Y \subset X$.

Nakai-Moishezon criterion:

$$L \text{ is ample} \Leftrightarrow L^{\dim Y} \cdot Y > 0 \text{ for every subvariety } Y \subset X.$$

Diophantine equation	$\leftrightsquigarrow$	Algebraic variety
$f(X) = 0,\ f \in \mathbb{Z}[X_1, \ldots, X_n]$		$X = Z(f)$ over $\mathbb{Q}$

- **Goal:** Study $X(\mathbb{Q}),\ X(\overline{\mathbb{Q}})$
- **Fundamental tool:** Height functions

$$h : X(\overline{\mathbb{Q}}) \to \mathbb{R}$$

measuring the "size" of points.

Theorem (Mordell conjecture - Faltings, 1983)

Let C be an algebraic curve of genus $g \geq 2$ over $\mathbb{Q}$. Then $C(\mathbb{Q})$ is finite.

Theorem (Bogomolov conjecture - Ullmo, 1998)

Let C be an algebraic curve of genus $g \geq 2$ over $\mathbb{Q}$. There exists a a $\varepsilon > 0$ such that

$$\{x \in C(\overline{\mathbb{Q}}) \mid \widehat{h}(x) < \varepsilon\}$$

is finite.

- Classical algebraic geometry:

$$\text{geometry of } X \leftrightsquigarrow \mathrm{Pic}(X)$$

- Arakelov geometry:

$$\text{arithmetic geometry of } X/\mathbb{Q} \leftrightsquigarrow \widehat{\mathrm{Pic}}(X)$$

$\widehat{\mathrm{Pic}}(X)$: arithmetic analogue of $\mathrm{Pic}(X)$

Let X be a projective variety over $\mathbb{Q}$ and let $L \in \mathrm{Pic}(X)$.

- $X_\infty := X(\mathbb{C})$: complex variety
- $L_\infty \in \mathrm{Pic}(X_\infty)$ pull-back of L

Definition

A ∞-adic metric on L is

- a continuous metric on L_∞,
- invariant by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Given a global section $s \in H^0(X, L)$, a ∞-adic metric $\|.\|_\infty$ on L gives a map

$$x \in X(\mathbb{C}) \mapsto \|s(x)\|_\infty \in \mathbb{R}$$

Non-archimedean analogue: similarly, we can define p-adic metrics on L for every prime number p.

We denote by $\mathcal{P}$ be the set of prime numbers.

Definition

An adelic metric $\|.\|$ on L is a collection $\|.\| = (\|.\|_v)_{v\in\mathcal{P}\cup\{\infty\}}$ such that

- for every $v \in \mathcal{P} \cup \{\infty\}$, $\|.\|_v$ is a v-adic metric on L
- for almost all $p \in \mathcal{P}$, $\|.\|_p$ is induced by a fixed model of (X, L) over $\mathrm{Spec}\,\mathbb{Z}$

Definition

An adelic line bundle $\overline{L}$ on X is a pair $(L, \|.\|)$ where

- $L \in \mathrm{Pic}(X)$
- $\|.\|$ is an adelic metric on L.

We denote by $\widehat{\mathrm{Pic}}(X)$ the group of adelic line bundles on X.

Important convention: In the sequel, all the metrics are assumed semi-positive

Definition

The height of a point $x \in X(\mathbb{Q})$ with respect to $\overline{L}$ is the quantity

$$h_{\overline{L}}(x) = - \sum_{v \in \mathcal{P} \cup \{\infty\}} \ln \|s(x)\|_v$$

where s is any section with $s(x) \neq 0$.

This construction can be generalized to define:

- the height $h_{\overline{L}}(x)$ of a point $x \in X(\overline{\mathbb{Q}})$
- the height $h_{\overline{L}}(Y)$ of a subvariety $Y \subseteq X$ (Gillet-Soulé, Zhang).

Arakelov theory

Algebraic geometry	$\longleftrightarrow$	Arakelov geometry
X/k projective		$X/\mathbb{Q}$ projective
$L \in \mathrm{Pic}(X)$		$\overline{L} \in \widehat{\mathrm{Pic}}(X)$
intersections $L \cdot C$ with curves $C \subset X$		heights $h_{\overline{L}}(x)$ of points $x \in X(\overline{\mathbb{Q}})$
intersection $L^{\dim Y} \cdot Y$ with subvarieties $Y \subseteq X$		height $h_{\overline{L}}(Y)$ of subvarieties $Y \subseteq X$

"And the reader is likely to discover a new and interesting question just by asking for her (his) favorite statement in classical algebraic geometry"

Christophe Soulé, 1995

$X/\mathbb{Q}$ projective variety, $\overline{L} = (L, \|.\|) \in \widehat{\mathrm{Pic}}(X)$.

Definition

Let $s \in H^0(X, L)$ be a non-zero global section. We say that s is small if

$$\|s\|_{v,\sup} \leq 1$$

for every $v \in \mathcal{P} \cup \{\infty\}$, with strict inequality if $v = \infty$.

We denote by $\widehat{\Gamma}(\overline{L}) \subset H^0(X, L)$ the set of small sections.

Definition

We say that $\overline{L}$ is ample if

- L is ample, and
- $H^0(X, L^{\otimes n}) = \mathrm{Span}_{\mathbb{Q}}(\widehat{\Gamma}(\overline{L}^{\otimes n}))$ for every $n \gg 1$.

Theorem (Zhang's arithmetic Nakai-Moishezon criterion)

The following are equivalent:

- $\overline{L}$ *is ample*
- $h_{\overline{L}}(Y) > 0$ *for every subvariety* $Y \subset X_{\overline{\mathbb{Q}}}$
- $\inf_{x \in X(\overline{\mathbb{Q}})} h_{\overline{L}}(x) > 0.$

Let $\zeta_{\mathrm{abs}}(\overline{L}) = \inf_{x \in X(\overline{\mathbb{Q}})} h_{\overline{L}}(x)$. By Zhang's theorem,

$$\zeta_{\mathrm{abs}}(\overline{L}) > 0 \Leftrightarrow \overline{L} \text{ ample.}$$

The **essential minimum** of $\overline{L}$ is the quantity

$$\zeta_{\mathrm{ess}}(\overline{L}) = \inf\{\lambda \in \mathbb{R} \mid \{x \in X(\overline{\mathbb{Q}}) \mid h_{\overline{L}}(x) \leq \lambda\} \text{ is dense in } X\}.$$

Remark

The Bogomolov conjecture can be reformulated as an inequality

$$\zeta_{\mathrm{ess}}(\overline{L}) > 0.$$

Question:

$$\zeta_{\mathrm{ess}}(\overline{L}) > 0 \Leftrightarrow ?$$

Theorem 1 (B., 2021)

We have

$$\zeta_{\mathrm{ess}}(\overline{L}) > 0 \Leftrightarrow \overline{L} \text{ is big.}$$

Definition (Yuan, Moriwaki)

We say that $\overline{L}$ is big if

$$\widehat{\mathrm{vol}}(\overline{L}) := \limsup_{n\to\infty} \frac{\ln \#\widehat{\Gamma}(\overline{L}^{\otimes n})}{n^{\dim X+1}/(\dim X + 1)!} > 0.$$

- In algebraic geometry, toric varieties are special algebraic varieties than can be described combinatorially.
- Let X be a projective toric variety and $L \in \mathrm{Pic}(X)$ a *toric* line bundle

Toric dictionary

$$\text{geometry of } (X, L) \leftrightsquigarrow \text{combinatorics of } \Delta(L)$$

$$\Delta(L) \subset \mathbb{R}^{\dim X} \quad \text{polytope}$$

Arithmetic analogue: Burgos Gil, Philippon and Sombra (2009–2019)

Given a *toric* adelic line bundle $\overline{L}$ on a toric projective variety X over $\mathbb{Q}$, they defined a concave function on $\Delta(L)$ that encodes:

1. the height $h_{\overline{L}}(X)$ and the volume $\widehat{\mathrm{vol}}(\overline{L})$
2. the arithmetic positivity of $\overline{L}$
3. the absolute and essential minima $\zeta_{\mathrm{abs}}(\overline{L})$ and $\zeta_{\mathrm{ess}}(\overline{L})$

Question

What about non-toric varieties?

- Theory of Okounkov bodies (Lazarsfeld and Mustață, Kaveh and Khovanskii)
- X a projective variety of dimension $d \geq 1$, $L \in \mathrm{Pic}(X)$ a big line bundle
- $\nu \colon \mathrm{Rat}(X) \to \mathbb{Z}^d$ a valuation of maximal rank on the field of rational functions $\mathrm{Rat}(X)$

Definition (Lazarsfeld and Mustață, Kaveh and Khovanskii)

The Okounkov body $\Delta_\nu(L) \subset \mathbb{R}^d$ of L with respect to ν is the closure in $\mathbb{R}^d$ of the set

$$\left\{ \frac{\nu(s)}{n} \;\middle|\; 0 \neq s \in H^0(X, L^{\otimes n}),\ n \geq 0 \right\}$$

for the euclidean topology. It is a convex body (convex compact set with non-empty interior).

Theorem (Lazarsfeld and Mustață, Kaveh and Khovanskii)

We have

$$\mathrm{vol}(L) = d!\mu_{\mathbb{R}^d}(\Delta_\nu(L)).$$

Let X be a projective variety over $\mathbb{Q}$ and let $\overline{L} = (L, \|.\|) \in \widehat{\mathrm{Pic}}(X)$. Assume that L is big. Let $d = \dim X$.

Theorem (Boucksom and Chen, 2011)

There exists a concave function $G_{\overline{L},\nu} \colon \Delta_\nu(L) \to \mathbb{R}$ *such that*

$$\widehat{\mathrm{vol}}(\overline{L}) = (d+1)! \int_{\Delta_\nu(L)} \max\{0, G_{\overline{L},\nu}\} d\mu_{\mathbb{R}^d}$$

and

$$h_{\overline{L}}(X) = (d+1)! \int_{\Delta_\nu(L)} G_{\overline{L},\nu} d\mu_{\mathbb{R}^d}.$$

Arakelov theory and convex geometry

In view of the "arithmetic toric dictionary" of Burgos Gil, Philippon and Sombra, it is natural to ask:

Question

Can we characterize

1. arithmetic ampleness
2. absolute minimum $\zeta_{\mathrm{abs}}(\overline{L})$, essential minimum $\zeta_{\mathrm{ess}}(\overline{L})$

in terms of $G_{\overline{L},\nu}$?

Theorem 2 (B., 2022)

The following are equivalent:

1. $\overline{L}$ is ample
2. L is ample and $\inf_{\alpha \in \Delta_\nu(L)} G_{\overline{L},\nu}(\alpha) > 0$.

Key ingredient of the proof: "Adelic Cauchy's inequality".
Let $x \in X(\mathbb{Q})$. If there exists $s \in \widehat{\Gamma}(\overline{L})$ such that $s(x) \neq 0$, then

$$h_{\overline{L}}(x) = -\sum_{v \in \mathcal{P} \cup \{\infty\}} \ln \|s(x)\|_v \geq -\sum_{v \in \mathcal{P} \cup \{\infty\}} \ln \|s\|_{v,\mathrm{sup}} > 0.$$

More generally:

Proposition (B., 2022)

If L is (geometrically) ample, then for any $x \in X(\overline{\mathbb{Q}})$ and for any $\varepsilon > 0$, there exists $\rho(\overline{L}, x, \varepsilon) \in \mathbb{R}$ such that

$$h_{\overline{L}}(x) + \varepsilon \geq -\frac{\mathrm{ord}_x(s)}{n}\rho(\overline{L}, x, \varepsilon)$$

for any $s \in \widehat{\Gamma}(\overline{L}^{\otimes n})$, $n \in \mathbb{N} \setminus \{0\}$.

“Proof” of the proposition:

- Construct a local section s' of $L^{\otimes n}$ with $s'(x) \neq 0$ by applying to s a differential operator of order $\operatorname{ord}_x s$
- For every $v \in \mathcal{P} \cup \{\infty\}$, use Cauchy's inequality on a suitable "disc" around x to have an inequality

$$\|s'(x)\|_v \leq \frac{\|s\|_{v,sup}}{\rho_v^{\operatorname{ord}_x s}} \leq \frac{1}{\rho_v^{\operatorname{ord}_x s}},$$

where ρ_v is the radius of the disc.

$$h_{\overline{L}}(x) = -\frac{1}{n} \sum_{v \in \mathcal{P} \cup \{\infty\}} \ln \|s'(x)\|_v \geq -\frac{\operatorname{ord}_x s}{n} \sum_{v \in \mathcal{P} \cup \{\infty\}} \rho_v .$$

"Proof" of the theorem: assume L is geometrically ample.

- $\overline{L}$ is ample $\Rightarrow$ inf $G_{\overline{L},\nu} > 0$ follows from the definition
- Assume inf $G_{\overline{L},\nu} > 0$

Fact: for every $x \in X(\overline{\mathbb{Q}})$ and $\varepsilon > 0$, there exists $s \in \widehat{\Gamma}(\overline{L}^{\otimes n})$, $n \in \mathbb{N} \setminus \{0\}$ with $\mathrm{ord}_x s \leq n\varepsilon$. By the proposition,

$$h_{\overline{L}}(x) \geq 0,$$

and therefore $\zeta_{\mathrm{abs}}(\overline{L}) \geq 0$.

Refining the argument by slight perturbation of metrics, one even gets $\zeta_{\mathrm{abs}}(\overline{L}) > 0$. By Zhang's arithmetic Nakai-Moishezon theorem, $\overline{L}$ is ample.

Theorem 3 (B.)

We have

$$\zeta_{\mathrm{ess}}(\overline{L}) = \sup_{\alpha \in \Delta_\nu(L)} G_{\overline{L},\nu}(\alpha) \quad \text{and} \quad \zeta_{\mathrm{abs}}(\overline{L}) = \inf_{\alpha \in \Delta_\nu(L)} G_{\overline{L},\nu}(\alpha).$$

"Proof":

1. First equality: consequence of the equivalence

$$\zeta_{\mathrm{ess}}(\overline{L}) > 0 \Leftrightarrow \overline{L} \text{ big}$$

   (Theorem 1).
2. Second equality: direct consequence of Theorem 2

We say that $\overline{L}$ is nef if $\zeta_{\mathrm{abs}}(\overline{L}) = \inf_{x \in X(\overline{\mathbb{Q}})} h_{\overline{L}}(x) \geq 0$.

Theorem (Gillet and Soulé 1992, Moriwaki 2009)

If $\overline{L}$ is nef, then $h_{\overline{L}}(X) = \widehat{\mathrm{vol}}(\overline{L})$.

Corollary 4 (B., 2022)

$\overline{L}$ is nef if and only if $h_{\overline{L}}(X) = \widehat{\mathrm{vol}}(\overline{L})$.

Previously known under additional assumptions:

- $d = 1$ (Moriwaki, 2014)
- $(X, \overline{L})$ is toric (Burgos Gil, Philippon, Moriwaki, Sombra 2016)

By a theorem of Zhang,

$$\zeta_{\mathrm{ess}}(\overline{L}) \geq \widehat{h}_{\overline{L}}(X) := \frac{h_{\overline{L}}(X)}{(d+1)\,\mathrm{vol}(L)}.$$

The case of equality is of particular importance for the study of equidistribution of small points.

Corollary 5 (B., 2022)

Assume that L is big. The following are equivalent.

1. $\zeta_{\mathrm{ess}}(\overline{L}) = \widehat{h}_{\overline{L}}(X)$
2. $\zeta_{\mathrm{abs}}(\overline{L}) = \widehat{h}_{\overline{L}}(X)$
3. the function $G_{\overline{L},\nu} \colon \Delta_\nu(L) \to \mathbb{R}$ is constant.

In that case, we have $G_{\overline{L},\nu}(\alpha) = \zeta_{\mathrm{ess}}(\overline{L})$ for every $\alpha \in \Delta_\nu(L)$.

Thank you!