Asymptotic analysis of the Schrödinger-Lohe system

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• Objective : A system of coupled nonlinear Schrödinger equations (Schrödinger-Lohe model) is studied in three perspectives :

1. Discretized (Semi-discrete SL) [Ha, H., Kim, 2022]
2. Wigner transformed (Wigner-Lohe) [Ha, H., Kim, 2022-2]
3. Semiclassical limit (Vlasov-Lohe) [Ha, H., Kim, submitted]

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Schrödinger-Lohe system

- A quantum synchronization which has been studied extensively is the Schrödinger-Lohe system :

$$\begin{cases} \mathrm{i}\hbar\partial_t\psi_j = -\frac{\hbar^2}{2}\Delta\psi_j + V_j\psi_j + \frac{\mathrm{i}\hbar\kappa}{2N}\sum_{k=1}^{N}\left(\psi_k - \frac{\langle\psi_j,\psi_k\rangle}{\langle\psi_j,\psi_j\rangle}\psi_j\right), \quad t>0, \quad x\in\mathbb{R}^d, \\ \psi_j(0,x) = \psi_j^0(x), \quad j\in\{1,\dots,N\}, \end{cases}$$

where V_j are real valued potential functions and $\hbar$ is the Planck constant.

- Question : What is this model for?

Classical and Quantum Dynamics

Let $H(x,p)$ be a Hamiltonian. x : position, p : momentum(velocity) and $\mathcal{H}(x,\nabla/\mathrm{i})$ be its quantization. Recall that

- Classical dynamics : The equation of motion is as follows

$$\begin{cases} \dot{x} = -\dfrac{\partial H}{\partial p} \\ \dot{p} = \dfrac{\partial H}{\partial x}. \end{cases}$$

- Quantum dynamics : We first consider the Schrödinger equation

$$\mathrm{i}\partial_t \psi = -\frac{1}{2}\Delta\psi + V\psi = \mathcal{H}\psi.$$

Then, we compute the expectation of the physical observable A using ψ, i.e.

$$\mathbb{E}(A) = \langle A\psi, \psi\rangle_{L^2}.$$

Back to the Schrödinger-Lohe system

- Consider the many body wave function

$$\Psi(x_1, \cdots, x_N) := \psi_1(x_1) \cdots \psi_N(x_N).$$

Then, the Schrödinger-Lohe model can be written as follows

$$\mathrm{i}\partial_t \Psi = \mathcal{H}\Psi,$$

where

$$\mathcal{H} = \sum_{k=1}^{N} I \otimes I \otimes \cdots \otimes \mathcal{H}_k \otimes \cdots \otimes I,$$

and

$$\mathcal{H}_k = \left(-\frac{1}{2}\Delta + V_k + \frac{\mathrm{i}\kappa}{2N} \sum_{m=1}^{N} \Big(|\psi_m\rangle\langle\psi_k| - |\psi_k\rangle\langle\psi_m| \Big) \right).$$

What is Schrödinger-Lohe model for?

- Therefore, the Schrödinger-Lohe model can be written as

$$i\partial_t \Psi = \mathcal{H}(\Psi)\Psi,$$

where the Hamiltonian $\mathcal{H}$ depends on the wave function Ψ.

- We can see the S-L model as a feedback control for quantum system.
- We use the S-L model to control quantum particles to synchronize.

Some results on the Schrödinger-Lohe system

1. (Conservation of L^2-norm) : $\|\psi_j^0\|_{L^2} = 1 \Rightarrow \|\psi_j(t)\|_{L^2} = 1$.

2. Let $\mathcal{D}(V) = \max_{i,j} \|V_i - V_j\|_\infty$ and $\mathcal{D}(\psi(t)) = \max_{i,j} \|\psi_i(t) - \psi_j(t)\|_{L^2}$.
   Then, we have

Theorem (Cho-Choi-Ha, '16 & Choi-Ha, '14)

(1) $\kappa > 0, \quad \mathcal{D}(V) = 0, \quad \mathcal{D}(\psi^0) < \frac{1}{2}$ *implies* $\mathcal{D}(\psi(t)) \lesssim e^{-\kappa t}$.

(2) $\kappa > 54\mathcal{D}(V) > 0, \quad \mathcal{D}(\psi^0) < \mathcal{D}_\infty$ *implies* $\lim_{\kappa\to\infty} \limsup_{t\to\infty} \mathcal{D}(\psi(t)) = 0$,

where $\mathcal{D}_\infty$ *is a constant.*

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Semi-discrete SL model

- Semi-discrete Schrödinger-Lohe $\Rightarrow$ Schrödinger-Lohe

1. Stability
2. Continuum limit (from discrete to continuous)

Semi-discrete SL model

For $h \in (0,1]$, we define $\mathbb{Z}_h^d := h\mathbb{Z}^d = \{x = hn : n \in \mathbb{Z}^d\}$, and the discrete Laplacian

$$(\Delta_h f)(x) := \sum_{j=1}^{d} \frac{f(x+he_j) - 2f(x) + f(x-he_j)}{h^2}.$$

Then, for a wave function $\psi_j^h = \psi_j^h(t,x) : \mathbb{R}_+ \times \mathbb{Z}_h^d \to \mathbb{C}$, we define the semi-discrete Schrödinger-Lohe model :

$$\begin{cases} \mathrm{i}\partial_t \psi_j^h = -\frac{1}{2}\Delta_h \psi_j^h + V_j^h \psi_j^h + \frac{\mathrm{i}\kappa}{2N} \sum_{k=1}^{N} \left(\psi_k^h - \frac{\langle \psi_k^h, \psi_j^h \rangle}{\langle \psi_j^h, \psi_j^h \rangle} \psi_j^h \right), \quad j \in [N], \\ \psi_j^h(0,x) = \psi_j^{h,\mathrm{in}}(x), \quad \|\psi_j^{h,\mathrm{in}}\|_{L^2(\mathbb{R}^d)} = 1, \end{cases}$$

where $V_j^h : \mathbb{Z}_h^d \to \mathbb{R}_+$ is an external one-body real-valued potential.

Semi-discrete SL model

- Goal : Semi-discrete SL $\Rightarrow$ SL.
- From continuous to discrete : Given $f \in L^p(\mathbb{R}^d)$, the *discretization* of f, denoted by $f^h : \mathbb{Z}_h^d \to \mathbb{C}$ is defined by

$$x_m := hm \in \mathbb{Z}_h^d, \quad f^h(x_m) := \frac{1}{h^d} \int_{x_m + [0,h)^d} f(y) dy.$$

- From discrete to continuous : Conversely, we define the *linear interpolation operator* p_h which maps a function on $\mathbb{Z}_h^d$ to a function on $\mathbb{R}^d$: for $x \in x_m + [0, h)^d$,

$$(p_h f)(x) := f(x_m) + \sum_{\ell=1}^{d} \frac{f(x_m + he_\ell) - f(x_m)}{h} (x - x_m)_\ell,$$

where $\{e_j\}$ is the standard ONB of $\mathbb{R}^d$.

Notation

- The L^p_h-norm:

$$\|f\|_{L^p_h} := h^{\frac{d}{p}} \|f\|_{\ell^p(\mathbb{Z}^d_h)} = \begin{cases} h^{\frac{d}{p}} \left(\sum_{x \in \mathbb{Z}^d_h} |f(x)|^p \right)^{\frac{1}{p}}, & p \in [1, \infty), \\ \sup_{x \in \mathbb{Z}^d_h} |f(x)|, & p = \infty. \end{cases}$$

- The inner product: $\langle f, g \rangle_{L^2_h} := h^d \sum_{x \in \mathbb{Z}^d_h} \overline{f(x)} g(x)$,
- Sobolev norms on $\mathbb{Z}^d_h$: $\|f\|_{W^{s,p}_h} := \|\langle \nabla_h \rangle^s f\|_{L^p_h} = \|(\langle \xi \rangle^s \hat{f})^\vee\|_{L^p_h}$
- For $p = 2$, $H^s_h := W^{s,2}_h$.

Notation

- The two point correlation function

$$\alpha_{ij} = \langle \psi_i, \psi_j \rangle, \quad \mathcal{A} = (\alpha_{ij})_{i,j}$$

- The functionals are defined as :

$$\mathfrak{M}(\mathcal{A}(t)) := \max_{1 \le i,j \le N} |1 - \alpha_{ij}(t)|, \quad \mathrm{dist}(\mathcal{A}, \tilde{\mathcal{A}}) := \max_{1 \le i,j \le N} |\alpha_{ij} - \tilde{\alpha}_{ij}|$$

- The difference between potentials $V_j(x) = V(x) + \nu_j$

$$\mathcal{D}(\nu) := \max_{1 \le i,j \le N} |\nu_j - \nu_i|$$

- For some technical reason

$$\mathfrak{M}_- := \frac{1}{2}\left(1 - \sqrt{1 - \frac{4\mathcal{D}(\mathcal{V})}{\kappa}}\right), \quad \mathfrak{M}_+ := \frac{1}{2}\left(1 + \sqrt{1 - \frac{4\mathcal{D}(\mathcal{V})}{\kappa}}\right).$$

Asymptotic Stability

- The dynamics of $\mathcal{A} = (\alpha_{ij})$ is (asymptotically) stable.

Theorem (Ha-H.-Kim, '22)

Suppose that potential, coupling strength and initial data satisfy

$$V_j(x) = V(x) + \nu_j, \quad x \in \mathbb{R}^d, \quad j \in [N], \quad \kappa > 4\mathcal{D}(\nu),$$
$$\max\left\{\mathfrak{M}(\mathcal{A}^{\text{in}}), \mathfrak{M}(\tilde{\mathcal{A}}^{\text{in}})\right\} < \mathfrak{M}_+ = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4\mathcal{D}(\mathcal{V})}{\kappa}}\right),$$

and let $\{\psi_j\}$ *and* $\{\tilde{\psi}_j\}$ *be global solutions to SL corresponding to the initial data* $\{\psi_j^{\text{in}}\}$ *and* $\{\tilde{\psi}_j^{\text{in}}\}$*, respectively. Then for* $\mathfrak{M}^\infty \in (\mathfrak{M}_-, \frac{1}{2})$*, there exists* $t^* \in [0, \infty)$ *such that for* $t \geq t^*$*,*

$$\text{dist}(\mathcal{A}(t), \tilde{\mathcal{A}}(t)) \leq \text{dist}(\mathcal{A}(t^*), \tilde{\mathcal{A}}(t^*)) \exp\left[-2\kappa\left|\frac{1}{2} - \mathfrak{M}^\infty\right|(t - t^*)\right].$$

Asymptotic Stability

Step 1 : Calculation

$$\frac{\mathrm{d}\alpha_{ij}}{\mathrm{d}t} = \mathrm{i}(\nu_j - \nu_i)\alpha_{ij} + \frac{\kappa}{2N}\sum_{k=1}^{N}(\alpha_{ki} + \alpha_{jk})(1 - \alpha_{ij}).$$

Step 2 : More calculation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{dist}(\mathcal{A}, \tilde{\mathcal{A}}) \le 2\kappa\left(\mathfrak{M}(\mathcal{A}) - \frac{1}{2}\right)\mathrm{dist}(\mathcal{A}, \tilde{\mathcal{A}}), \quad t > 0.$$

Step 3 : Assumption ⇒ Existence of an absorbing set : there exists a finite time $t^* \in (0, \infty)$ such that

$$\sup_{t_* \le t < \infty} \mathfrak{M}(\mathcal{A}(t)) \le \mathfrak{M}^\infty < \frac{1}{2}.$$

Step 4 : Apply **Step 3** to **2** : $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{dist}(\mathcal{A}, \tilde{\mathcal{A}}) \le -2\kappa\left|\frac{1}{2} - \mathfrak{M}^\infty\right|\mathrm{dist}(\mathcal{A}, \tilde{\mathcal{A}}).$

From discrete to continuous

Theorem (Ha-H.-Kim, '22)

Suppose system parameters and initial data satisfy

$$\kappa > 0, \quad h \in (0,1), \quad \psi_j^{\text{in}} \in H^1(\mathbb{R}^d),$$
$$\max_{1 \le i,j \le N} |\langle \psi_i^{h,\text{in}}, \psi_j^{h,\text{in}} \rangle - \langle \psi_i^{\text{in}}, \psi_j^{\text{in}} \rangle| \lesssim \mathcal{O}(\sqrt{h}),$$

and let $\{\psi_j^h\}$ *and* $\{\psi_j\}$ *be two global smooth solutions to semi-discrete S-L and S-L model, respectively. Then, there exist uniform (with respect to h) constants* G_1 *and* G_2 *such that*

$$\|p_h \psi_j^h(t) - \psi_j(t)\|_{L^2} \le G_1(1 + \|\psi_j^0\|_{H^1}) e^{G_2 t} \sqrt{h}, \quad t > 0.$$

In other words, we see that

$$\|p_h \psi_j^h(t) - \psi_j(t)\|_{L^2} \to 0, \quad \text{as } h \text{ goes to } 0.$$

Main ingredients

1. Discrete vs continuous [Hong, Yang, 2019]

• For $f \in H^1(\mathbb{R}^d)$, let f^h : discretization, $p_h f^h$: linear interpolation.

(1) Discretization vs continous

$$\|f^h\|_{\dot{H}_h^1(\mathbb{Z}_h^d)} \lesssim \|f\|_{\dot{H}^1(\mathbb{R}^d)}, \quad \|p_h f^h\|_{\dot{H}^1(\mathbb{R}^d)} \lesssim \|f^h\|_{\dot{H}_h^1(\mathbb{Z}_h^d)}$$
$$\|p_h f^h - f\|_{L^2(\mathbb{R}^d)} \lesssim h\|f\|_{H^1(\mathbb{R}^d)}.$$

• Let $U(t)$, $U^h(t)$ be the solution operator for the continuous and discrete linear Schrödinger equation.

(2) Discrete flow vs continuous flow

$$\|p_h U^h(t)\psi^{h,\mathrm{in}} - U(t)\psi^{\mathrm{in}}\|_{L^2} \lesssim |t| h^{\frac{1}{2}} \left(\|\psi^{h,\mathrm{in}}\|_{H_h^1} + \|\psi^{\mathrm{in}}\|_{H^1} \right) + \|p_h \psi^{h,\mathrm{in}} - \psi^{\mathrm{in}}\|_{L^2}.$$

Main ingredients

2. H^1-norm bound [Huh, Ha, 2017]
Suppose

$$\psi_j^{\text{in}} \in H^1(\mathbb{R}^d), \quad \|\psi_j^{\text{in}}\|_{L^2} = 1, \quad V_j \in W^{1,\infty}(\mathbb{R}^d).$$

Then, for any $T \in (0, \infty)$, the solution to SL model satisfies

$$\|\psi_j(t)\|_{H^1(\mathbb{R}^d)} \lesssim e^{Ct}, \quad t \in [0, T], \quad j \in [N].$$

3. Stability

$$\max_{1\le i,j\le N} |\alpha_{ij}(t) - \alpha_{ij}^h(t)| \le e^{3\kappa t} \max_{1\le i,j\le N} |\alpha_{ij}^{\text{in}}(t) - \alpha_{ij}^{h,\text{in}}(t)|.$$

Idea of the proof

Step 1 : Duhamel's formula for $p_h\psi_j^h - \psi_j$

$$\begin{aligned} p_h\psi_j^h - \psi_j = p_h U^h(t)\psi_j^{h,\mathrm{in}} - U(t)\psi_j^{\mathrm{in}} \\ + \int_0^t \big(p_h U^h(t-s)(V_j^h\psi_j^h) - U(t-s)(V_j\psi_j)\big)\mathrm{d}s \\ + \int_0^t (p_h U^h(t-s)L_j^h - U(t-s)L_j)\mathrm{d}s. \end{aligned}$$

Step 2 : Estimate each terms using Main ingredients.

$$\max_{1\le j\le N} \|p_h\psi_j^h(t) - \psi_j(t)\|_{L^2} \lesssim h^{\frac{1}{2}} e^{Ct} + \max_{1\le j\le N} \int_0^t \|p_h\psi_j^h(s) - \psi_j(s)\|_{L^2}\mathrm{d}s.$$

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Semiclassical analysis of the S-L model

• From now on, we study the semiclassical analysis of the Schrödinger-Lohe model.

• Write the equation in an approriate form to investigate the effect of the scale $\hbar$.

• This can be done using the Wigner transform and we get Wigner-Lohe system.

• *h* and $\hbar$ are completely different !!

Quantum-Classical Correspondence

- Given a Schrödinger equation

$$\mathrm{i}\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\Delta\psi(x,t) + V(x)\psi(x,t) = H\psi.$$

We want to see what happens when $\hbar \to 0$ (Semiclassical Limit).

- For example, the uncertainty principle $[x,p] = \mathrm{i}\hbar$ breaks down.
- However, doing this directly to the Schrödinger equation is non sense.
- Two approaches : WKB vs Wigner transform

WKB method

- One natural way to see this is via the WKB ansatz,

$$\psi = a(x,t)\exp\left(\frac{\mathrm{i}S(x,t)}{\hbar}\right).$$

- This leads us to

$$\partial_t S = -\frac{1}{2m}|\nabla S|^2 - V,$$
$$\partial_t a = -\frac{1}{m}\nabla S \cdot \nabla a - \frac{1}{2m} a\Delta S.$$

- Weakness : Well-posedness, solution form fixed...

Wigner transform method

- Wigner phase space method

Definition

We define the Wigner transform of $L^2(\mathbb{R}^d)$ functions ψ and ϕ by

$$w^\hbar[\psi,\phi](x,p) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\psi\left(x+\frac{\hbar y}{2}\right)\overline{\phi}\left(x-\frac{\hbar y}{2}\right)e^{\mathrm{i}p\cdot y}\,dy.$$

We call $w^\hbar$, the Wigner function or distribution.

- This also has some weaknesses.

Wigner transform

- Some properties of the Wigner transform ($\hbar = 1$).

Lemma

(1) (Moyal identity)

$$\langle w[\psi,\phi], w[\psi',\phi']\rangle_{L^2(\mathbb{R}^{2d})} = \left(\frac{1}{2\pi\hbar}\right)^d \langle \psi,\psi'\rangle_{L^2}\overline{\langle \phi,\phi'\rangle}_{L^2},$$

(2) $\int_{\mathbb{R}^d} w[\psi,\phi](x,p)dp = \psi(x)\bar{\phi}(x), \quad \int_{\mathbb{R}^d} w[\psi,\psi](x,p)dp = |\psi(x)|^2$

(3) $\int_{\mathbb{R}^d} w[\psi,\phi](x,p)dx = \frac{1}{(2\pi h)^d}\mathcal{F}\psi(\frac{p}{h})\overline{\mathcal{F}\phi}(\frac{p}{h})$

- In particular, the expectation value of an observable G is given by

$$\mathbb{E}(G) = \int w[\psi,\psi](x,p)g(x,p)dxdp.$$

- Remark : Wigner function can take negative values, i.e. not a true probability density. cf) Husimi transform.
- However, as $\hbar \to 0$ the Wigner function becomes more positive.

Wigner transform method

- If ψ satisfies the linear Schrödinger equation, $w^\hbar = w^\hbar[\psi, \psi]$ satisfies the quantum Liouville equation :

$$\partial_t w^\hbar + p \cdot \nabla_x w^\hbar + \frac{\Theta[V](w^\hbar)}{\hbar} = 0,$$

where $\Theta[V]$ is the pseudodifferential operator,

$$\Theta[V](w^\hbar) := -\frac{\mathrm{i}}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \left[V\left(x + \frac{\hbar y}{2}\right) - V\left(x - \frac{\hbar y}{2}\right) \right] w^\hbar(x, p') e^{\mathrm{i}(p-p')\cdot y} dp' dy.$$

- Notice that $\hbar \to 0$ yields (formally)

$$\partial_t w + p \cdot \nabla_x w - \nabla_x V \cdot \nabla_p w = 0.$$

The Wigner-Lohe model

- Schrödinger-Lohe $\Rightarrow$ Wigner-Lohe, via the Wigner transform
- We define the Wigner matrix $W^\hbar = (w_{ij}^\hbar)$ by

$$w_{ij}^\hbar := w^\hbar[\psi_i, \psi_j].$$

- In this part, we set $\hbar = 1$.

The Wigner-Lohe model

- The Wigner-Lohe model for $V_i \equiv V$ is written as

$$\begin{cases} \partial_t w_{ij} + p \cdot \nabla_x w_{ij} + \Theta[V](w_{ij}) \\ \quad = \dfrac{\kappa}{2N} \displaystyle\sum_{k=1}^{N} \left[(w_{ik} + w_{kj}) - \left(\int_{\mathbb{R}^{2d}} (w_{ik} + w_{kj}) \mathrm{d}x \mathrm{d}p \right) w_{ij} \right], \quad (x,p) \in \mathbb{R}^{2d}, \\ w_{ij}(0,x,p) = w_{ij}^0(x,p), \quad i,j \in [N]. \end{cases}$$

where $\Theta[V]$ is the pseudodifferential operator,

$$\Theta[V](w) := -\frac{\mathrm{i}}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \left[V\left(x + \frac{y}{2}\right) - V\left(x - \frac{y}{2}\right) \right] w(x,p') e^{\mathrm{i}(p-p')\cdot y} dp' dy,$$

subject to initial constraints:

$$\int_{\mathbb{R}^{2d}} w_{ii}^0 dxdp = 1, \quad \left| \int_{\mathbb{R}^{2d}} w_{ij}^0 dxdp - 1 \right| < 1, \quad i \neq j \in [N].$$

Complete synchronization

- Our first result is on the uniformization of the Wigner matrix

Theorem (Ha, H., Kim, 2022)

Let w_{ij} be a sufficiently smooth solution to Wigner-Lohe. Then, the complete aggregation emerges asymptotically:

$$\lim_{t\to\infty} \|w_{ik} - w_{jm}\|_{L^2} = 0, \quad i, j, k, m \in [N].$$

- Set

$$z_{ij}(t) := \int_{\mathbb{R}^{2d}} w_{ij}(t, x, p)\mathrm{d}x\mathrm{d}p, \quad i, j \in [N], \quad t > 0,$$

Proof Idea

- Want to derive Grönwall-type inequality for $\|w_{ik} - w_{jk}\|_{L^2}$.
- But, the inequality contains terms like $\|w_{ij}\|_{L^2}$ and $|1 - z_{ij}|$ which are not necessarily bounded.
- Thus, we restrict the initial data to show that $|1 - z_{ij}| \to 0$ and $\|w_{ij}\|_{L^2} \leq C$.
- Then there exist two positive constants C_1 and C_2 such that

$$\frac{d}{dt}\sum_{k=1}^{N} \|w_{ik} - w_{jk}\|_{L^2}^2 \leq -\kappa\Big(1 - C_1 e^{-\kappa t}\Big)\sum_{k=1}^{N} \|w_{ik} - w_{jk}\|_{L^2}^2 + C_2 e^{-\kappa t}, \quad t > 0.$$

Complete synchronization

- Indeed, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|w_{ik} - w_{jk}\|_{L^2}^2 \\
&\qquad \le \frac{\kappa}{N}\sum_{\ell=1}^{N}\int_{\mathbb{R}^{2d}}\Big(|w_{ik} - w_{jk}||w_{i\ell} - w_{j\ell}| - \mathrm{Re}(z_{i\ell} + z_{\ell k})|w_{ik} - w_{jk}|^2 \\
&\qquad\quad + |z_{i\ell} - z_{j\ell}||w_{jk}||w_{ik} - w_{jk}|\Big)\mathrm{d}x\mathrm{d}p \\
&\qquad \le \frac{\kappa}{N}\sum_{\ell=1}^{N}\Big(\|w_{ik} - w_{jk}\|_{L^2}\|w_{i\ell} - w_{j\ell}\|_{L^2} - \mathrm{Re}(z_{i\ell} + z_{\ell k})\|w_{ik} - w_{jk}\|_{L^2}^2 \\
&\qquad\quad + |z_{i\ell} - z_{j\ell}|\|w_{jk}\|_{L^2}\|w_{ik} - w_{jk}\|_{L^2}\Big).
\end{aligned}
$$

Existence theory

- Now we investigate the global solvability.

For this, we define a subset $\mathcal{X}$, a norm and a transport operator:

$$\mathcal{X} := \left\{ f \in L^2(\mathbb{R}^{2d}) : \ \left| \int_{\mathbb{R}^{2d}} f dx dp \right| < \infty \right\},$$

$$\|f\|_{\mathcal{X}} := \|f\|_{L^2} + \left| \int f dx dp \right|, \quad A := -p \cdot \nabla_x .$$

and its N^2 copies:

$$\mathrm{X} := \left\{ F = (F_{ij}) \in (L^2(\mathbb{R}^{2d}))^{\otimes N^2} \quad : \quad \left| \int_{\mathbb{R}^{2d}} F_{ij} dx dp \right| < \infty, \quad i,j \in [N] \right\},$$

$$\|F\|_{\mathrm{X}} := \|F\|_{L^2(\mathbb{R}^{2d})^{\otimes N^2}} + \left| \int_{\mathbb{R}^{2d}} F dx dp \right| := \max_{i,j} \left(\|F_{ij}\| + \left| \int_{\mathbb{R}^{2d}} F_{ij} dx dp \right| \right).$$

Theorem (Ha-H.-Kim, '22)

For $T \in (0, \infty)$, the following assertions hold.

1. *If initial data and the potential satisfy*

$$w_{ij}^0 \in \mathcal{X}, \quad i, j \in [N], \quad \textit{and} \quad V \in L^\infty(\mathbb{R}^d),$$

then there exists a unique mild solution to the Wigner-Lohe:

$$w_{ij} \in C([0, T]; \mathcal{X}), \quad i, j \in [N].$$

2. *If we impose further regularity on initial data and the potential*

$$w_{ij}^0 \in D(A), \quad i, j \in [N], \quad \textit{and} \quad V \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d),$$

then there exists a unique classical solution to the Wigner-Lohe.

$$w_{ij} \in C([0, T]; \mathcal{X}) \cap C^1([0, T]; D(A)), \quad i, j \in [N].$$

Sketch of proof

Idea of the proof :

(1) Mild solution : Fixed point theorem on the space X.

$$\begin{cases} \partial_t W + p \cdot \nabla_x W + \Theta[V](W) \\ \qquad = \dfrac{\kappa}{2N}\left(E_{ij} W C_j + R_i W E_{ij} - W \displaystyle\int_{\mathbb{R}^{2d}} (E_{ij} G C_j + R_i G E_{ij}) dx dp \right), \\ W(0) = W^0. \end{cases}$$

(2) Classical solution : Semigroup theory to the equation of the form

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = f(t, u(t)), \quad t > t_0, \\ u(t_0) = u_0. \end{cases}$$

(2-continued) For $W = (w_{ij})_{(i,j)}$, we introduce an $N \times N$ matrix $F(W)$ whose (i,j)-th component is given as

$$(F(W))_{(i,j)} := \frac{\kappa}{2N} \sum_{k=1}^{N} (z_{ik} + z_{kj}) w_{ij} = \frac{\kappa}{2N} \sum_{k=1}^{N} \left(\int_{\mathbb{R}^{2d}} (w_{ik} + w_{kj}) dxdp \right) w_{ij},$$

Then, we show that F is indeed Lipschitz by using the following :

Lemma (Application of the Gâteaux Mean value theorem)

For $U, V \in \mathrm{X}$, there exists a positive constant $C > 0$ that may depend on time T such that

$$\|F(U) - F(V)\|_{\mathrm{X}} \leq C \|U - V\|_{\mathrm{X}}.$$

Then, the functional derivative, denoted by $\mathrm{D}F$, is continuous. Consequently, F is Lipschitz from a bounded subset of X to X.

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- We study the transition

$$\text{Schrödinger-Lohe} \underbrace{\Rightarrow}_{1} \text{Wigner-Lohe} \underbrace{\Rightarrow}_{2} \text{Vlasov-Lohe}$$

1 : Wigner transform

2 : Semiclassical limit.

- Vlasov-Lohe model

1. Derivation (Formal and Rigorous)
2. Global existence of classical solutions
3. Asymptotic behavior

The semiclassical limit

- Goal : Semiclassical Limit of a quantum synchronization system?

That is, we want the classical counterpart of the Schrödinger-Lohe system: Nonidentical potential + Nonlinearity

$$i\hbar\partial_t\psi_j^\hbar = -\frac{\hbar^2}{2}\Delta\psi_j^\hbar + U_j^\hbar\psi_j^\hbar + \frac{i\hbar\kappa}{2N}\sum_{k=1}^{N}\left(\psi_k^\hbar - \langle\psi_j^\hbar, \psi_k^\hbar\rangle\psi_j^\hbar\right).$$

- But, how to take the Wigner transform?

- We assume that the potentials are of the form $U_j^\hbar(x) = U(x) + \hbar\nu_j$.

$\Rightarrow$ Wigner matrix approach [Gerard-Markowich-Mauser-Poupaud, '97].

$$w^\hbar[\psi_i, \psi_j](x,p) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\psi_i\left(x + \frac{\hbar y}{2}\right)\overline{\psi_j}\left(x - \frac{\hbar y}{2}\right)e^{ip\cdot y}dy.$$

The semiclassical limit

- Question : How to send $\hbar \to 0$?
- Frequently used test function space is defined by

$$\mathcal{A} = \{\phi \in \mathcal{C}_0(\mathbb{R}^d \times \mathbb{R}^d) : \mathcal{F}_{p\to z}\phi(x,z) \in L^1(\mathbb{R}^d_z; \mathcal{C}^0(\mathbb{R}^d_x))\},$$

with the norm given by

$$\|\phi\|_{\mathcal{A}} := \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} |\mathcal{F}_{p\to z}\phi(x,z)| dz,$$

and we denote by $\mathcal{A}^*$ the dual space of $\mathcal{A}$.

- Indeed, we work with N^2-copies of $\mathcal{A}$.

Wigner measures

Proposition (Gerard-Markowich-Mauser-Poupaud, '97)

For a bounded family of $L^2(\mathbb{R}^d)^{\otimes N}$ functions $\{(\psi_j^\hbar)\}$, let $W^\hbar[\Psi^\hbar]$ be a matrix whose elements are generalized Wigner distributions:

$$W^\hbar[\Psi^\hbar]_{i,j} = w^\hbar[\psi_i^\hbar, \psi_j^\hbar].$$

Then, the following assertions hold.

1. *(Boundedness):*

$$\|W^\hbar[\Psi^\hbar]\|_{(\mathcal{A}^{N\times N})^*} \leq \|\Psi^\hbar\|_{L^2(\mathbb{R}^d)^{\otimes N}}.$$

2. *(Weak-$*$ compactness): There exists a subsequence of $W^\hbar[\Psi^\hbar]$ which converges in $(\mathcal{A}^{N\times N})^*$ weak-$*$ to the measure $W = (w_{ij})$. We call W as the Wigner measure associated to the family $\{(\psi_j^\hbar)\}$.*

Formal derivation

- Recall the following Wigner-Lohe sysetm:

$$\partial_t w_{ij}^\hbar + p \cdot \nabla_x w_{ij}^\hbar + \frac{\Theta[V](w_{ij}^\hbar)}{\hbar}$$
$$= \frac{\mathrm{i}}{(2\pi)^d} \int_{\mathbb{R}^{2d}} (\omega_i - \omega_j) w_{ij}^\hbar(x, p') e^{\mathrm{i}(p-p')\cdot y} dp' dy$$
$$+ \frac{\kappa}{2N} \sum_{k=1}^N \left\{ (w_{kj}^\hbar + w_{ik}^\hbar) - \left(\int_{\mathbb{R}^{2d}} (w_{ik}^\hbar + w_{kj}^\hbar) dx dp \right) w_{ij}^\hbar \right\},$$

where the pseudodifferential operator is defined by

$$\Theta[V](w)(x, p; \hbar)$$
$$= -\frac{\mathrm{i}}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \left[V\left(x + \frac{\hbar y}{2}\right) - V\left(x - \frac{\hbar y}{2}\right) \right] w(x, p') e^{\mathrm{i}(p-p')\cdot y} dp' dy.$$

Formal derivation

- Now, sending $\hbar \to 0$, yields the following Vlasov-Lohe equation

$$
\begin{aligned}
&\partial_t w_{ij} + p \cdot \nabla_x w_{ij} - \nabla_x V \cdot \nabla_p w_{ij} \\
&= \mathrm{i}(\nu_i - \nu_j) w_{ij}(x,p) \\
&\quad + \frac{\kappa}{2N} \sum_{k=1}^{N} \left\{ (w_{kj} + w_{ik}) - \left(\int_{\mathbb{R}^{2d}} (w_{ik} + w_{kj}) dxdp \right) w_{ij} \right\}.
\end{aligned}
$$

- In the following, we want to derive

Schrödinger-Lohe $\Rightarrow$ Wigner-Lohe $\Rightarrow$ Vlasov Lohe or

Wave functions $\Rightarrow$ Wigner functions $\Rightarrow$ Wigner measures.

Step 1 : From Schrödinger-Lohe to Wigner-Lohe, i.e. from

$$\begin{cases} i\hbar\partial_t \psi_j^\hbar = -\dfrac{\hbar^2}{2}\Delta\psi_j^\hbar + (V + \hbar\nu_j)\psi_j^\hbar + \dfrac{i\hbar\kappa}{2N}\displaystyle\sum_{k=1}^N (\psi_k^\hbar - \langle \psi_k^\hbar, \psi_j^\hbar\rangle \psi_j^\hbar), \\ \psi_j^\hbar\Big|_{t=0+} = \psi_j^{\hbar,0}, \quad \|\psi_j^{\hbar,0}\|_{L^2} = 1, \quad j \in [N], \end{cases}$$

to

$$\begin{aligned} &\partial_t w_{ij}^\hbar + p\cdot\nabla_x w_{ij}^\hbar \\ &\quad = \frac{i}{\hbar(2\pi)^d}\int_{\mathbb{R}^{2d}} \left[U_i^\hbar\Big(x+\frac{\hbar y}{2}\Big) - U_j^\hbar\Big(x-\frac{\hbar y}{2}\Big)\right] w_{ij}^\hbar(x,p') e^{i(p-p')\cdot y}\, dp'\, dy \\ &\qquad + \frac{\kappa}{2N}\sum_{k=1}^N \left[(w_{kj}^\hbar + w_{ik}^\hbar) - \Big(\int_{\mathbb{R}^{2d}} (w_{ik}^\hbar + w_{kj}^\hbar)\, dx dp\Big) w_{ij}^\hbar\right], \end{aligned}$$

where we set $U_j^\hbar(x) = V(x) + \hbar\nu_j$.

In theorem, we have

Theorem (Ha-H.-Kim, submitted)

Suppose that the potentials $U_j^\hbar := V + \hbar\nu_j$ are in $(\mathcal{C}^1 \cap L^\infty)(\mathbb{R}^d)$, and let $\{\psi_j^\hbar\} \subset \mathcal{C}(\mathbb{R}, H^2(\mathbb{R}^d)) \cap \mathcal{C}^1(\mathbb{R}, L^2(\mathbb{R}^d))$ be a set of solutions to SL. Then, $w_{ij}^\hbar$ satisfies

$$\partial_t w_{ij}^\hbar + p \cdot \nabla_x w_{ij}^\hbar + \frac{\Theta[V](w_{ij}^\hbar)}{\hbar} = \frac{\mathrm{i}(\nu_i - \nu_j)}{(2\pi)^d} \int_{\mathbb{R}^{2d}} w_{ij}^\hbar(x, p') e^{\mathrm{i}(p-p')\cdot y} dp' dy$$

$$+ \frac{\kappa}{2N} \sum_{k=1}^N \left\{ (w_{kj}^\hbar + w_{ik}^\hbar) - \left(\int_{\mathbb{R}^{2d}} (w_{ik}^\hbar + w_{kj}^\hbar) dxdp \right) w_{ij}^\hbar \right\}, \quad i, j \in [N].$$

To show the previous theorem, we prove

Lemma

Suppose that the potentials $U_j^\hbar = V + \hbar\nu_j$ belong to $(\mathcal{C}^1 \cap L^\infty)(\mathbb{R}^d)$, and let $\{\psi_j^\hbar\} \subset \mathcal{C}(\mathbb{R}, H^2(\mathbb{R}^d)) \cap \mathcal{C}^1(\mathbb{R}, L^2(\mathbb{R}^d))$ be a strong solution to SL. Then, one has the following regularity estimates:

1. *The following functions belong to $\mathcal{C}^1(\mathbb{R}; L^\infty(\mathbb{R}^{2d}))$:*

$$w_{ij}^\hbar, \quad \mathrm{i}(\nu_i - \nu_j) w_{ij}^\hbar,$$

$$\frac{\kappa}{2N} \sum_{k=1}^N \left\{ (w_{kj}^\hbar + w_{ik}^\hbar) - \left(\int_{\mathbb{R}^{2d}} (w_{ik}^\hbar + w_{kj}^\hbar) dxdp \right) w_{ij}^\hbar \right\}, \quad i,j \in [N].$$

2. *The following functions belong to $\mathcal{C}(\mathbb{R}; L^\infty(\mathbb{R}^{2d}))$:*

$$\frac{\partial w_{ij}^\hbar}{\partial x_m}, \quad \frac{\partial^2 w_{ij}^\hbar}{\partial x_\ell \partial x_m}, \quad \frac{1}{\hbar}\Theta[V](w_{ij}^\hbar), \quad i,j \in [N], \quad \ell, m \in [d].$$

Step 2 : From Wigner-Lohe to Vlasov-Lohe.

- Starting from

$$\begin{aligned}
&\partial_t w_{ij}^{\hbar} + p \cdot \nabla_x w_{ij}^{\hbar} \\
&\quad = \frac{\mathrm{i}}{\hbar(2\pi)^d} \int_{\mathbb{R}^{2d}} \Big[U_i^{\hbar}\Big(x + \frac{\hbar y}{2}\Big) - U_j^{\hbar}\Big(x - \frac{\hbar y}{2}\Big) \Big] w_{ij}^{\hbar}(x, p') e^{\mathrm{i}(p-p')\cdot y} dp' dy \\
&\qquad + \frac{\kappa}{2N} \sum_{k=1}^{N} \Big[(w_{kj}^{\hbar} + w_{ik}^{\hbar}) - \Big(\int_{\mathbb{R}^{2d}} (w_{ik}^{\hbar} + w_{kj}^{\hbar}) dx dp \Big) w_{ij}^{\hbar} \Big],
\end{aligned}$$

- We arrive at

$$\begin{aligned}
&\partial_t w_{ij} + p \cdot \nabla_x w_{ij} - \nabla_x V \cdot \nabla_p w_{ij} \\
&\quad = \mathrm{i}(\nu_i - \nu_j) w_{ij} + \frac{\kappa}{2N} \sum_{k=1}^{N} \Big\{ (w_{kj} + w_{ik}) - \Big(\int_{\mathbb{R}^{2d}} (w_{ik} + w_{kj}) dx dp \Big) w_{ij} \Big\}.
\end{aligned}$$

Wigner-Lohe to Vlasov-Lohe

• The derivation of Vlasov-Lohe is based on assumption that we have the strong convergence for the initial data

$$\int_{\mathbb{R}^{2d}} w_{ij}^{\hbar}(x,p,0)dxdp \to \int_{\mathbb{R}^{2d}} w_{ij}(dx,dp,0),$$

and applying the weak-$*$ convergence of the Wigner functions to Wigner measures.

Wigner-Lohe to Vlasov-Lohe

- The Wigner measures are weak solutions :

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^{2d}} (\partial_t \phi + p \cdot \nabla_x \phi - \nabla V \cdot \nabla_p \phi) w_{ij}(t, dx, dp) dt \\
&+ \int_{\mathbb{R}^{2d}} \phi(0, x, p) w_{ij}(0, dx, dp) \\
&+ \mathrm{i}(\nu_i - \nu_j) \int_0^\infty \int_{\mathbb{R}^{2d}} \phi w_{ij}(t, dx, dp) dt \\
&+ \frac{\kappa}{2N} \sum_{k=1}^{N} \int_0^\infty \int_{\mathbb{R}^{2d}} \phi (w_{kj} + w_{ik})(t, dx, dp) \\
&- \int_0^\infty \left(\int_{\mathbb{R}^{2d}} (w_{ik} + w_{kj})(t, dx, dp) \right) \left(\int_{\mathbb{R}^{2d}} \phi w_{ij}(t, dx, dp) \right) dt = 0,
\end{aligned}
$$

for any test function $\phi \in \mathcal{C}_c^\infty([0, \infty) \times \mathbb{R}^{2d})$.

Wigner-Lohe to Vlasov-Lohe

Theorem (Ha-H.-Kim, submitted)

Suppose that the potentials and initial data satisfy

$$U_j^\hbar := V + \hbar\nu_j \in (\mathcal{C}^1 \cap L^\infty)(\mathbb{R}^d), \quad j \in [N],$$

and let $\{(\psi_j^\hbar)_j\}_\hbar$ *be a sequence of strong solutions to SL in* $\mathcal{C}(\mathbb{R}; H^2(\mathbb{R}^d)) \cap \mathcal{C}^1(\mathbb{R}; L^2(\mathbb{R}^d))$*. If the corresponding Wigner distributions* $\{w_{ij}^\hbar\}$ *satisfy the assumption on initial data, then the set of corresponding Wigner measures* $\{w_{ij}\}$ *is a global weak solution to VL.*

Global Solvability

- The Vlasov-Lohe model : for $(t, x, p) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$

$$\begin{cases} \partial_t w_{ij} + p \cdot \nabla_x w_{ij} - \nabla_x V \cdot \nabla_p w_{ij} = \mathrm{i}(\nu_i - \nu_j) w_{ij} \\ \quad + \dfrac{\kappa}{2N} \displaystyle\sum_{k=1}^{N} \left\{ (w_{kj} + w_{ik}) - \left(\int_{\mathbb{R}^{2d}} (w_{ik} + w_{kj}) dxdp \right) w_{ij} \right\}, \\ w_{ij} \Big|_{t=0} = w_{ij}^0, \quad i, j \in [N], \end{cases}$$

subject to initial constraints:

$$w_{ij}^0 \in W^{2,\infty}(\mathbb{R}^{2d}), \quad \int_{\mathbb{R}^{2d}} w_{ii}^0 dxdp = 1, \quad \left| \int_{\mathbb{R}^{2d}} w_{ij}^0 dxdp - 1 \right| < 1, \quad i \neq j \in [N].$$

Global Solvability

Proposition (Ha-H.-Kim, submitted)

(Local existence of a classical solution) Suppose that the zeroth order potential and initial datum satisfy

$$V \in \mathcal{C}^3(\mathbb{R}^d) \quad \textit{and} \quad \int_{\mathbb{R}^{2d}} w_{ii}^0 dxdp = 1, \quad \forall\ i \in [N].$$

Then, there exists a time $\tau_* > 0$ such that system has a unique set of solutions $\{w_{ij}\}$ in $\mathcal{C}^1([0, \tau_*) \times \mathbb{R}^{2d})$.

Theorem (Ha-H.-Kim, submitted)

(Global existence of classical solutions) Suppose that the zeroth order potential V lies in $\mathcal{C}^3(\mathbb{R}^d)$ and initial data satisfy the constraints. Then, for any $\tau \in (0, \infty)$, there exists a unique global classical solution in $\mathcal{C}^1([0, \tau) \times \mathbb{R}^{2d})$.

Global Solvability

- Idea of the proof : Method of characteristics.

- Local existence : The nonlocal term is determined by the ODEs : If we set

$$z_{ij}(t) := \int_{\mathbb{R}^{2d}} w_{ij}\, dxdp, \quad i, j \in [N].$$

Then, we integrate the system with respect to (x, p)-variable to find the ODE system for z_{ij}:

$$\frac{dz_{ij}}{dt} = \mathrm{i}(\nu_i - \nu_j) z_{ij} + \frac{\kappa}{2N} \sum_{k=1}^{N} (1 - z_{ij})(z_{ik} + z_{kj}).$$

- Global existence : Using the method of characteristics to estimate the

term

$$\mathcal{E}(t) := \sum_{0 \le |\alpha| + |\beta| \le 2} \sum_{i,j=1}^{N} \|\nabla_x^\alpha \nabla_p^\beta w_{ij}\|_{L^\infty(\mathbb{R}^{2d})}.$$

Uniformization of the Wigner matrix

Let $\{w_{ij}\}$ be a solution to Vlasov-Lohe system. We set

$$\mathcal{D}(\mathcal{Z}) := \max_{1 \le i,j \le N} |1 - z_{ij}|, \quad \mathcal{D}(\mathcal{V}) := \max_{1 \le i,j \le N} |\nu_i - \nu_j|.$$

Theorem Suppose that coupling strength and initial data satisfy

$$\kappa > 0, \quad \mathcal{D}(\mathcal{Z}^0) < \frac{1 + \sqrt{1 - 4(\mathcal{D}(\mathcal{V})/\kappa)}}{2},$$

Then, the following estimates hold.

1. If $\mathcal{D}(\mathcal{V}) = 0$, complete synchronization for $\mathcal{W} = \{w_{ij}\}$ emerges asymptotically:

$$\lim_{t \to \infty} \max_{i,j,\ell,k \in [N]} \|w_{ij}(t) - w_{\ell k}(t)\| = 0.$$

2. If $\mathcal{D}(\mathcal{V}) > 0$ and $\kappa > 4\mathcal{D}(\mathcal{V})$, practical synchronization for $\mathcal{Z} = \{z_{ij}\}$ occurs asymptotically:

$$\lim_{\kappa \to \infty} \limsup_{t \to \infty} \mathcal{D}(\mathcal{Z}(t)) = 0.$$

Summary

(1) Continuum limit : Semi-discrete SL $\Rightarrow$ SL

(2) Semiclassical analysis : Study of the Wigner-Lohe model

(2) Semiclassical limit : SL $\Rightarrow$ Wigner-Lohe $\Rightarrow$ Vlasov-Lohe

Thank you for the listening!