Globally minimizing parabolic motions in the Newtonian N-body Problem

E. MADERNA, A. VENTURELLI

Abstract

We consider the N-body problem in $\mathbb{R}^d$ with the newtonian potential $1/r$. We prove that for every initial configuration x_i and for every minimizing normalized central configuration x_0, there exists a collision-free parabolic solution starting from x_i and asymptotic to x_0. This solution is a minimizer in every time interval. The proof exploits the variational structure of the problem, and it consists in finding a convergent subsequence in a family of minimizing trajectories. The hardest part is to show that this solution is parabolic and asymptotic to x_0.

1. Introduction

In this paper we consider N positive masses in an euclidean space $\mathbb{R}^d$, submitted to a gravitational interaction. We find some interesting solutions with a given asymptotic behaviour. The equation of motion of the N-body problem is written

$$\ddot{\boldsymbol{r}}_i = - \sum_{j=1,\dots,N,\ j\neq i} \frac{m_j(\boldsymbol{r}_i - \boldsymbol{r}_j)}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3}. \tag{1}$$

where m_i is the mass and $\boldsymbol{r}_i \in \mathbb{R}^d$ the position of the i-th body. Since these equations are invariant by translation, we can assume that the center of mass is at the origin.

These equations are Euler-Lagrange equations of the Lagrangian action functional (we will define it precisely in the next section), therefore solutions of (1) are critical points of the action in a set of paths with fixed ends. The simplest kind of critical points are minima, so it is natural to search for minimizers of the lagrangian action joining two given configurations in a fixed time. The potential of the N-body problem being singular at collision configurations, a main difficult involved in this approach is to show that minimizers are collision-free. The following theorem, essentially due to C. Marchal, is a major advanced in this subject.

Theorem 1 *Given two N-body configurations $x_i = (\boldsymbol{r}_1, ..., \boldsymbol{r}_N) \in (\mathbb{R}^d)^N$, $x_f = (\boldsymbol{s}_1, ..., \boldsymbol{s}_N) \in (\mathbb{R}^d)^N$ and a time $T > 0$, an action minimizing path joining x_i to x_f in time T is collision-free for $t \in (0, T)$.*

See [14] and [8] for a claim and a proof of this theorem for $d = 2$ and $d = 3$. See [11] for a proof in any dimension. This theorem, together with the lower semicontinuity of the action (see Section 2), implies in particular that there always exists a collision-free minimizing solution joining two given collision-free N-body configuration in a given time.

There is an extensive recent literature on the search of action minimizing periodic and quasiperiodic solutions in the *N*-body problem (see for instance [2,3,5, 6,8–11,14,17,18]). A natural extension of Marchal's theorem is to search solutions defined on an infinite interval $[0, +\infty)$, starting from a given configuration at $t = 0$ and having a given asymptotic behaviour for $t \to +\infty$. The classification of all possible asymptotic behaviour of solutions in the N-body problem has been investigated since the beginning of the last century. The main results in this direction are due to J. Chazy. In [4] it is shown that there are only seven possible final evolutions in the three-body problem. Among these seven possibilities there are the so-called *parabolic motions*. A solution $t \mapsto (\boldsymbol{r}_1, ..., \boldsymbol{r}_N)(t)$ of the N-body problem is said to be parabolic if the velocity of every body tends to zero as $t \to +\infty$. We introduce the functions

$$I(x) = \sum_{i=1}^{N} m_i |\boldsymbol{r}_i|^2, \qquad U(x) = \sum_{1 \le i < j \le N} \frac{m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}, \qquad x = (\boldsymbol{r}_1, ..., \boldsymbol{r}_N), \quad (2)$$

respectively equal to the moment of inertia with respect to the center of mass and to the Newtonian potential.

Notation *Given a configuration x, we denote by $\tilde{x} = I(x)^{-1/2}x$ the associated normalized configuration.*

It is well known (see for instance [12] and [7]) that if $t \mapsto x(t)$ is a parabolic solution, the normalized trajectory $\tilde{x}(t)$ is asymptotic to the set of *central configurations* (i.e. critical points of $\tilde{U} = I^{1/2}U$). Given a central configuration x_0 with $I(x_0) = 1$, we say that a parabolic solution $t \mapsto x(t)$ is asymptotic to x_0 if $\tilde{x}(t) \to x_0$ as $t \to +\infty$. A central configuration x_0 is said to be minimizing if it is an absolute minimum of $\tilde{U}$. We can now state the main result of this paper.

Main Theorem *Given any initial configuration x_i and any minimizing normalized central configuration x_0, there exists a parabolic solution $\gamma : [0, +\infty) \to (\mathbb{R}^d)^N$ starting from x_i at $t = 0$ and asymptotic to x_0 for $t \to +\infty$. This solution is a minimizer of the lagrangian action with fixed ends in every compact interval contained in $[0, +\infty)$ and it is collision-free for $t > 0$.*

We do not require any hypothesis of nondegeneracy of the central configuration x_0.

The parabolic solution γ is constructed as limit of a sequence $\gamma_n : [0, t_n] \to (\mathbb{R}^d)^N$ of minimizers connecting x_i with a configuration homothetic to x_0 in time t_n, and $t_n \to +\infty$. In Section 3 we construct the sequence γ_n and we prove that it is uniformly convergent on every compact subset of $\mathbb{R}$. In Sections 4 and 5 we show that

γ is parabolic and asymptotic to x_0. The proof of this last property is achieved by comparing the action of the N-body problem with the action of a Kepler problem, and using Lambert's Theorem to estimates the action. In the Appendix we state and prove some technical estimates concerning the Kepler problem on the line that we need to construct γ and to prove its parabolicity. The authors believe that these minimizing parabolic solutions are in fact calibrated curves of some weak KAM solutions of the N-body problems, whose existence has been proved in [13] by one of the authors. Our Main Theorem has a natural interpretation in terms of McGehee vector field and collision manifold. Indeed, in [7, 15, 16] it is shown that if x_0 is a central configuration with $I(x_0) = 1$, the state $(x_0, v_0 x_0)$ with $v_0 = (2U(x_0))^{1/2}$ is a critical point of the McGehee vector field in the collision manifold, and its stable set corresponds to parabolic solutions asymptotic to x_0 as $t \to +\infty$. Thus, we can formulate the Main Theorem by saying that the stable set of $(x_0, v_0 x_0)$ (for the McGehee vector field) projects on the whole configuration space, provided x_0 is a minimizing central configuration.

We think that variational methods could be used to study some important features on the global dynamics of N-body problem. In particular, it should be interesting to study hyperbolic solutions using variational methods. We recall that a solution $\gamma \colon [0, +\infty) \to (\mathbb{R}^d)^N$ is said to be hyperbolic if there exists a (collision-free) configuration x_0 such that

$$\gamma(t) = x_0 t + o(t), \qquad t \to +\infty \qquad (3)$$

A hyperbolic solution has necessarily positive energy, and replacing x_0 by a normalized configuration, (3) is equivalent to $\gamma(t) = \sqrt{2h}x_0 t + o(t)$ as $t \to +\infty$ (see [4]), where h is the energy of the solution. In this case we will say that $\gamma(t)$ is hyperbolic for $t \to +\infty$ and asymptotic to x_0. Since there is no constraint to the limit configuration x_0 of a hyperbolic solution (see again [4]), it is natural to ask the following two questions. The second one has been asked by R. Montgomery.

Question 1. Given an initial configuration x_i and a normalized non-collision configuration x_0, does there exist a hyperbolic motion starting from x_i at $t = 0$ and asymptotic to x_0 for $t \to +\infty$?

Question 2. For which couple of normalized non-collision configurations x_0 and x_0' does there exist a solution that is hyperbolic both for $t \to +\infty$ and for $t \to -\infty$ and is asymptotic to x_0 for $t \to +\infty$ and to x_0' for $t \to -\infty$?

We hope that it will be possible to answer these questions using variational methods similar to those developed in this paper.

2. Variational setting

Since equations (1) are invariant by translation, we fix the origin of our inertial frame at the center of mass of the system. We define the *configuration space* of the

system as

$$\mathscr{X} = \left\{ x = (\boldsymbol{r}_1, ..., \boldsymbol{r}_N) \in (\mathbb{R}^d)^N, \quad \sum_{i=1}^{N} m_i \boldsymbol{r}_i = 0 \right\},$$

and we endow $\mathscr{X}$ with the *mass scalar product* :

$$x \cdot y = \sum_{i=1}^{N} m_i < \boldsymbol{r}_i, \boldsymbol{s}_i >$$
$$x = (\boldsymbol{r}_1, ..., \boldsymbol{r}_N) \in \mathscr{X}, \quad y = (\boldsymbol{s}_1, ..., \boldsymbol{s}_N) \in \mathscr{X},$$

where $< , >$ is the usual euclidean product in $\mathbb{R}^d$. We denote by $\| \|$ the euclidean norm on $\mathscr{X}$ associated to the mass scalar product. A configuration $x = (\boldsymbol{r}_1, ..., \boldsymbol{r}_N) \in \mathscr{X}$ is said to be a *collision configuration* if $\boldsymbol{r}_i = \boldsymbol{r}_j$ for some $i \neq j$. We denote by *Coll* the set of collision configurations and by $\widehat{\mathscr{X}} = \mathscr{X} \setminus Coll$ the set of collisions-free configurations. Equations (1) can be written in a more compact form as a second order differential equation on $\widehat{\mathscr{X}}$

$$\ddot{x} = \nabla U(x), \tag{4}$$

where U is the newtonian potential already defined in (2), the gradient is calculated with respect to the mass scalar product. Since $\widehat{\mathscr{X}}$ is an open subset of $\mathscr{X}$, the tangent space of $\widehat{\mathscr{X}}$ is identified with $\widehat{\mathscr{X}} \times \mathscr{X}$. The following functions defined on $\widehat{\mathscr{X}} \times \mathscr{X}$

$$K = y \cdot y, \quad L = \frac{K}{2} + U, \quad H = \frac{K}{2} - U,$$

are respectively equal to twice the kinetic energy, to the lagrangian and to the energy first integral.

Given an absolutely continuous path $\gamma : [a, b] \to \mathscr{X}$, we define its *Lagrange action* by :

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt,$$

where L is naturally extended to a function defined over $\mathscr{X} \times \mathscr{X}$ by $L(x, y) = +\infty$ if $x \in Coll$. It is well known that collision-free extremals of A_L are solutions of equations (4).

Definition 2 *We say that an absolutely continuous path $\gamma : [a, b] \to \mathscr{X}$ is a minimizer if $A_L(\sigma) \geq A_L(\gamma)$ for every absolutely continuous path $\sigma : [a, b] \to \mathscr{X}$ having the same extremities. If $I \subset \mathbb{R}$ is any interval, we say that $\gamma : I \to \mathscr{X}$ is a minimizer if for every compact interval $[a, b]$ contained in I, the path $\gamma|_{[a,b]}$ is a minimizer.*

Given a positive real number T and two configurations x_i and x_f, let $\Sigma(x_i, x_f; T)$ be the set of absolutely continuous paths defined in the interval $[0, T]$ and joining x_i to x_f in time T. The following proposition is well known.

Proposition 3 *For every $x_i, x_f \in \mathscr{X}$ and for every $T > 0$ there exists a minimizer $\gamma : [0, T] \to \mathscr{X}$ joining x_i to x_f.*

In [19] and [11] one can find a proof of this proposition when the functional A_L is defined over H^1 paths (i.e. absolutely continuous paths with derivative in L^2.) joining x_i to x_f. An absolutely continuous path having a finite action is necessarily in H^1, therefore minimizers among H^1 paths are also minimizers among absolutely continuous paths.

The proposition above do not ensure that γ is collision-free, but by the already cited Marchal's theorem, if $d \geq 2$, minimizers are collision-free for $t \in (0, T)$.

3. Construction of the solution

In this section we construct the solution $\gamma : [0, +\infty) \to \mathscr{X}$ of the main theorem as limit of minimizers. We will show in Sections 4 and 5 that γ is parabolic and asymptotic to x_0.

Before stating the main result of this section, we recall a classical result concerning parabolic solutions (see [7] or [12]) for a proof).

Proposition 4 *If $\gamma : [0, +\infty) \to \mathscr{X}$ is a parabolic solution of the N-body problem, the energy of γ is necessarily zero, moreover we have*

$$I(t) = \alpha^2 t^{\frac{4}{3}} + o(t^{\frac{4}{3}}), \qquad \nabla \tilde{U}(\tilde{\gamma}(t)) \to 0, \qquad \tilde{U}(\tilde{\gamma}(t)) \to U_0$$

as $t \to +\infty$, where

$$\alpha = (9U_0/2)^{1/3} \tag{5}$$

In particular, the ω-limit of $\tilde{\gamma}(t)$ is contained in the set of normalized central configuration.

Since there are always infinitely many normalized central configurations for a given critical level of $\tilde{U}$, (the orthogonal group acts on $\widehat{\mathscr{X}}$ leaving invariant $\tilde{U}$), we cannot say *a priori* that the ω-limit of $\tilde{\gamma}(t)$ is a given configuration. If $\gamma(t)$ is a parabolic solution asymptotic to normalized central configuration x_0 (i.e. $\tilde{\gamma}(t)$ converges to x_0), by Proposition 4 we have the asymptotic estimates

$$\gamma(t) = \alpha x_0 t^{\frac{2}{3}} + o(t^{\frac{2}{3}}), \quad \text{as} \quad t \to +\infty \tag{6}$$

The following Lemma is a converse of Proposition 4.

Lemma 5 *Let x_0 be a normalized central configuration, $U_0 = \tilde{U}(x_0)$ and α the constant defined in (5). A solution $\gamma : [0, +\infty) \to \mathscr{X}$ satisfying the asymptotic estimates (6) is parabolic and asymptotic to x_0.*

Proof. We just need to prove that γ is parabolic. Replacing (6) in the equation of motion we find $\ddot{\gamma}(t) = \mathcal{O}(t^{-\frac{4}{3}})$, as $t \to +\infty$. Therefore, the velocity $\dot{\gamma}(t)$ has a limit for $t \to +\infty$ that we denote $\dot{\gamma}_\infty$. Moreover we have

$$\dot{\gamma}(t) = \dot{\gamma}_\infty + \mathcal{O}(t^{-\frac{1}{3}}), \qquad t \to +\infty.$$

Integrating this expression we find

$$\gamma(t) = \dot{\gamma}_\infty t + \mathcal{O}(t^{\frac{2}{3}}), \qquad t \to +\infty,$$

thus $\dot{\gamma}_\infty = 0$ and $\gamma(t)$ is parabolic.

By the way, if x_0 is a normalized central configuration, the path

$$\gamma_0 : [0, +\infty) \to \mathscr{X}, \qquad \gamma_0(t) = \alpha x_0 t^{\frac{2}{3}}, \tag{7}$$

is a solution of the *N*-body problem. γ_0 is called *homothetic-parabolic solution asymptotic to* x_0.

We state now the main result of this section. We recall that x_i is the initial configuration of the Main Theorem, x_0 is a normalized minimizing central configuration, U_0 and α are as before, $\gamma_0(t)$ is given by (7).

Theorem 6 *There exists a minimizing solution* $\gamma : [0, +\infty) \to \mathscr{X}$ *starting from* x_i, *a sequence of positive numbers* $t_n \to +\infty$ *and a sequence of minimizers* $\gamma_n \in \Sigma(x_i, \gamma_0(t_n); t_n)$ *such that* γ_n *converges uniformly to* γ *on every compact interval contained in* $[0, +\infty)$. *Moreover* $\gamma(t)$ *is collision-free for* $t > 0$.

We prove this Theorem in several steps. At Proposition 9 we show that if T and t/T are sufficiently great, for every minimizer $\overline{\gamma} \in \Sigma(x_i, \gamma_0(t); t)$ the action $A_L(\overline{\gamma}|_{[0,T]})$ has a uniform bound (independent of t). Successively, using Ascoli's theorem and a diagonal trick, we find the sequence $(\gamma_n)_{n=1}^{+\infty}$. We start with some preliminary definitions and remarks. Given two configurations x and x' and a time T, we denote by $\mathscr{A}(x, x'; T)$ the action of a minimizing path joining x to x' in time T (the same function is denoted $\phi(x, x', T)$ in [13]). In a similar way, given two positive real numbers a and b and a time T, we denote by $S(a, b; T)$ the action (for the one dimensional keplerian problem with lagrangian $\frac{\dot{r}^2}{2} + \frac{U_0}{r}$) of a minimizing path joining a to b in time T.

By the homogeneity of U, if $\varpi : [0, T] \to \mathscr{X}$ is a solution of (4) and $\lambda > 0$, the path

$$\varpi^\lambda : [0, \lambda T] \to \mathscr{X}, \qquad \varpi^\lambda(t) = \lambda^{\frac{2}{3}} \varpi(t/\lambda)$$

is still a solution of (4). Moreover, if ϖ is a minimizer, ϖ^λ is still a minimizer. A similar property holds for solutions and minimizers of a one dimensional Kepler problem. Therefore we have

$$\mathscr{A}(\lambda^{\frac{2}{3}}x, \lambda^{\frac{2}{3}}x'; \lambda T) = \lambda^{\frac{1}{3}}\mathscr{A}(x, x'; T), \qquad S(\lambda^{\frac{2}{3}}a, \lambda^{\frac{2}{3}}b; \lambda T) = \lambda^{\frac{1}{3}}S(a, b; T).$$

Lemma 7 *We have*

$$\mathscr{A}(x, x'; T) \geq S(\|x\|, \|x'\|; T)$$

with equality if and only if x *and* x' *are on the half-line starting from zero generated by* $\hat{x}$, *where* $\hat{x}$ *is a normalized minimizing configuration (i.e.* $\|\hat{x}\| = 1$ *and* $\tilde{U}(\hat{x}) = U_0$).

Proof. Let $\varpi : [0,T] \to \mathscr{X}$ be a minimizer joining x to x' in time T and let $r(s) = \|\varpi(s)\|$. By Sundman inequality we have

$$\|\dot{\varpi}(s)\|^2 \geq \dot{r}^2(s),$$

with equality if and only if $\dot{\varpi}(s)$ is parallel to $\varpi(s)$. Since U_0 is the minimum of $\tilde{U}$ we have also

$$U(\varpi(s)) \geq \frac{U_0}{r(s)},$$

with equality if and only if $\tilde{U}(\varpi(s)) = U_0$. Therefore

$$\mathscr{A}(x,x';T) = \int_0^T \left(\frac{\|\dot{\varpi}(s)\|^2}{2} + U(\varpi(s)) \right) ds$$

$$\geq \int_0^T \left(\frac{\dot{r}^2(s)}{2} + \frac{U_0}{r(s)} \right) ds$$

$$\geq S(\|x\|, \|x'\|; T)$$

with equality if and only if $\varpi(s) = \mu(s)\hat{x}$, where $\hat{x}$ is a minimizing normalized configuration and $s \mapsto \mu(s) \in \mathbb{R}_+$ is a minimizer (for the one-dimensional Kepler problem) joining $\|x\|$ to $\|x'\|$ in time T. This proves the Lemma.

In order to simplify the exposition we introduce the following notation. If $x, x' \in \mathscr{X}$ are two configurations and $0 \leq \tau < T < t$ we term

$$\mathscr{M}(x,x';\tau,T,t) = \mathscr{A}(0,x;T+\tau) + \mathscr{A}(x,x';t-T) - \mathscr{A}(0,x';t-\tau). \quad (8)$$

In a similar way, if $r, r' \in [0, +\infty)$ and $0 \leq \tau < T < t$ we term

$$\mathscr{N}(r,r';\tau,T,t) = S(0,r;T+\tau) + S(r,r';t-T) - S(0,r';t-\tau). \quad (9)$$

Lemma 8 *Let $0 < \tau < T < t$ be real numbers. If $\xi \in \Sigma(0,x_i;\tau)$ and $\overline{\gamma} \in \Sigma(x_i,\gamma_0(t);t)$ are two minimizers we have*

$$\mathscr{M}(\overline{\gamma}(T),\gamma_0(t);\tau,T,t) \leq 2A_L(\xi) \quad and \quad \mathscr{N}(\|\overline{\gamma}(T)\|, \alpha t^{\frac{2}{3}};\tau,T,t) \leq 2A_L(\xi).$$

Proof. In order to prove the first inequality, let $\eta \in \Sigma(0,\overline{\gamma}(T);T+\tau)$ and $\zeta \in \Sigma(0,\gamma_0(t);t-\tau)$ be two minimizers. The path ζ is nothing but a reparametrization of $\gamma_0 \big|_{[0,t]}$. Since η and $\overline{\gamma}$ are minimizers, we have the triangular inequalities

$$A_L(\eta) \leq A_L(\xi) + A_L(\overline{\gamma}\big|_{[0,T]})$$
$$A_L(\overline{\gamma}) \leq A_L(\xi) + A_L(\zeta),$$

therefore

$$A_L(\eta) + A_L(\overline{\gamma}\big|_{[T,t]}) \leq A_L(\xi) + A_L(\overline{\gamma}) \leq 2A_L(\xi) + A_L(\zeta).$$

This gives the first inequality.

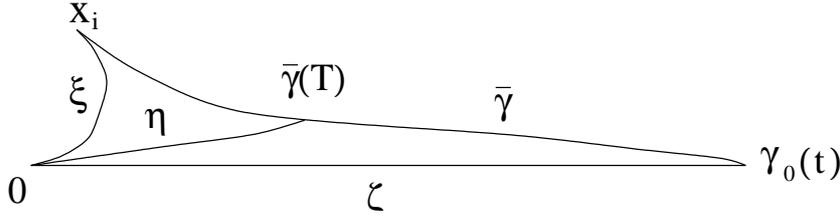The second inequality is a direct consequence of the first one and of Lemma 7.

Fig. 1. The paths $\overline{\gamma}$, η, ζ and ξ in the configuration space.

Proposition 9 *There exist three constants $K > 0$, $\overline{T} > 0$ and $\overline{s} > 1$ such that for every $T \geq \overline{T}$, for every $t \geq \overline{s}T$ and for every minimizer $\overline{\gamma} \in \Sigma(x_i, \gamma_0(t); t)$ we have*

$$\|\overline{\gamma}(T)\| \leq KT^{\frac{2}{3}}.$$

Proof. Suppose, for the sake of a contradiction, that there exist three sequences of positive real numbers $(K_n)_{n=0}^{+\infty}$, $(T_n)_{n=0}^{+\infty}$ and $(t_n)_{n=0}^{+\infty}$ satisfying

$$K_n \to +\infty, \qquad T_n \to +\infty, \qquad \frac{t_n}{T_n} \to +\infty,$$

and a sequence of minimizers $\overline{\gamma}_n \in \Sigma(x_i, \gamma_0(t_n); t_n)$ such that for every $n \in \mathbb{N}$:

$$\|\overline{\gamma}_n(T_n)\| \geq K_n T_n^{\frac{2}{3}}.$$

Let $\tau > 0$ and $\xi : [0, \tau] \to \mathcal{X}$ be a minimizer connecting 0 to x_i in time τ. Without loss of generality we can assume $0 < \tau < T_n < t_n$. By homothety invariance and by the second inequality of Lemma 8 we have

$$T_n^{\frac{1}{3}} \mathcal{N}\left(\frac{\|\overline{\gamma}_n(T_n)\|}{T_n^{\frac{2}{3}}}, \alpha\left(\frac{t_n}{T_n}\right)^{\frac{2}{3}}; \frac{\tau}{T_n}, 1, \frac{t_n}{T_n} \right) \leq 2A_L(\xi). \qquad (10)$$

Since $\frac{\|\overline{\gamma}_n(T_n)\|}{T_n^{\frac{2}{3}}} \to +\infty$ and $\frac{t_n}{T_n} \to +\infty$, by Proposition 21 of the Appendix we have

$$\mathcal{N}\left(\frac{\|\overline{\gamma}_n(T_n)\|}{T_n^{\frac{2}{3}}}, \alpha\left(\frac{t_n}{T_n}\right)^{\frac{2}{3}}; \frac{\tau}{T_n}, 1, \frac{t_n}{T_n} \right) \to +\infty$$

as $n \to +\infty$. This contradicts inequality (10).

We need now an estimates of the minimal action $\mathcal{A}(x, x'; T)$ when $\|x\|$ and $\|x'\|$ are less then a given size.

Proposition 10 *There exist two positive constants C_1 and C_2 such that if $R > 0$ and $T > 0$, if $x \in \mathcal{X}$ and $x' \in \mathcal{X}$ are two configurations satisfying $\|x\| \leq R$ and $\|x'\| \leq R$, we can find an absolutely continuous path $\gamma_{xx'} : [0, T] \to \mathcal{X}$ joining x to x' in time T such that the following inequality holds*

$$A_L(\gamma_{xx'}) \leq C_1 \frac{R^2}{T} + C_2 \frac{T}{R}. \qquad (11)$$

In particular we have

$$\mathscr{A}(x,x';T) \le C_1 \frac{R^2}{T} + C_2 \frac{T}{R}. \tag{12}$$

Proof. An alternative proof of this Proposition can be found in [13].

Let $x_0' \in \widehat{\mathscr{X}}$ be any normalized collision-free configuration. We construct an absolutely continuous path $\gamma_x : [0,T/2] \to \mathscr{X}$ joining x to Rx_0' and verifying

$$A_L(\gamma_x) \le A_1 \frac{R^2}{T} + A_2 \frac{T}{R} \tag{13}$$

where A_1 and A_2 are two positive constants independent on R, T and x. An analogous path $\gamma_{x'} : [0,T/2] \to \mathscr{X}$ joining Rx_0' to x' can be constructed in exactly the same way. Pasting γ_x and $\gamma_{x'}$ together and choosing $C_1 = 2A_1$ and $C_2 = 2A_2$ we get a path $\gamma_{xx'}$ verifying (11). Inequality (12) is an obvious consequence of (11).

Let $x_0' = (\boldsymbol{c}_1,...,\boldsymbol{c}_N)$. We term $\boldsymbol{c}_{ij} = \boldsymbol{c}_j - \boldsymbol{c}_i$ and $c_{ij} = |\boldsymbol{c}_{ij}|$ for $1 \le i < j \le N$. In a similar way, given $x = (\boldsymbol{r}_1,...,\boldsymbol{r}_N) \in \mathscr{X}$ with $\|x\| \le R$, we term $\boldsymbol{r}_{ij} = \boldsymbol{r}_j - \boldsymbol{r}_i$ and $r_{ij} = |\boldsymbol{r}_{ij}|$. Let λ_{ij} be the coefficients

$$\lambda_{ij} = \frac{r_{ij}}{Rc_{ij} + r_{ij}} \in [0,1),$$

and let h be the cardinality of the set $\{\lambda_{ij}\}_{1 \le i < j \le N}$. The inequality $1 \le h \le N(N-1)/2$ holds. Let us denote

$$0 \le \mu_1 < ... < \mu_h < 1$$

the elements of the set $\{\lambda_{ij}\}_{1 \le i < j \le N}$ ordered increasingly. We define $\mu_0 = 0$ and $\mu_{h+1} = 1$. For every $i = 0,...,h$ we term

$$\tau_i = \frac{T(\mu_{i+1} - \mu_i)^{\frac{3}{2}}}{2 \sum\limits_{k=0}^{h} (\mu_{k+1} - \mu_k)^{\frac{3}{2}}}.$$

We observe that $\tau_0 \ge 0$ and $\tau_i > 0$ if $i \ge 1$, moreover $\sum\limits_{i=0}^{h} \tau_i = T/2$. Defining

$$\sigma_0 = 0, \qquad \sigma_i = \tau_0 + ... + \tau_{i-1}, \qquad i = 1,...,h+1.$$

we have $\sigma_{h+1} = T/2$. Let $\lambda : [0,T/2] \to [0,1]$ be the path defined by

$$\lambda(t) = \begin{cases} \mu_1 \left(1 - \left(\frac{\tau_0 - t}{\tau_0}\right)^{\frac{2}{3}}\right), & \text{if } t \in [0,\sigma_1] \\[2ex] \mu_i + \left(\frac{t - \sigma_i}{\tau_i/2}\right)^{\frac{2}{3}} \frac{\mu_{i+1} - \mu_i}{2}, & \text{if } t \in [\sigma_i, \sigma_i + \frac{\tau_i}{2}], \quad i = 1,...,h-1 \\[2ex] \mu_{i+1} - \left(\frac{\sigma_{i+1} - t}{\tau_i/2}\right)^{\frac{2}{3}} \frac{\mu_{i+1} - \mu_i}{2}, & \text{if } t \in [\sigma_i + \frac{\tau_i}{2}, \sigma_{i+1}], i = 1,...,h-1 \\[2ex] \mu_h + (1 - \mu_h)\left(\frac{t - \sigma_h}{T/2 - \sigma_h}\right)^{\frac{2}{3}}, & \text{if } t \in [\sigma_h, T/2]. \end{cases}$$

The definition of $\lambda(t)$ in the interval $[\sigma_0, \sigma_1]$ has some meaning only if $\sigma_0 < \sigma_1$ (i.e. if $\tau_0 > 0$). The path

$$\gamma_x(t) = (1 - \lambda(t))x + \lambda(t)Rx_0', \qquad t \in [0, T/2]$$

connects x to Rx_0' in the time $T/2$. If $1 \le i \le h - 1$, the action of the restriction $\gamma_x\big|_{[\sigma_i, \sigma_i + \tau_i/2]}$ is given by

$$A_L(\gamma_x\big|_{[\sigma_i, \sigma_i + \frac{\tau_i}{2}]}) = \frac{\|Rx_0' - x\|^2}{2} \int_{\sigma_i}^{\sigma_i + \frac{\tau_i}{2}} \lambda(t)^2 \, dt$$

$$+ \sum_{1 \le j < k \le N} m_j m_k \int_{\sigma_i}^{\sigma_i + \frac{\tau_i}{2}} \frac{dt}{|(1 - \lambda(t))\boldsymbol{r}_{jk} + \lambda(t)R\boldsymbol{c}_{jk}|}.$$

As $t \in [\sigma_i, \sigma_i + \tau_i/2]$ the path $\lambda(t)$ increases from μ_i to $(\mu_i + \mu_{i+1})/2$, hence the coefficient λ_{jk} that is closest to $\lambda(t)$ is exactly μ_i. Using the triangular inequality we find

$$|(1 - \lambda(t))\boldsymbol{r}_{jk} + \lambda(t)R\boldsymbol{c}_{jk}| \ge |r_{jk} - \lambda(t)(r_{jk} + Rc_{jk})|$$

$$= (r_{jk} + Rc_{jk})|\lambda_{jk} - \lambda(t)|$$

$$\ge Rc_{jk}|\lambda_{jk} - \lambda(t)|$$

$$\ge Rc_{jk}(\lambda(t) - \mu_i),$$

for $t \in [\sigma_i, \sigma_i + \tau_i/2]$ and for every $1 \le j < k \le N$. Therefore, since $\|x\| \le R$ and $\|x_0'\| = 1$

$$A_L(\gamma_x\big|_{[\sigma_i, \sigma_i + \frac{\tau_i}{2}]}) \le 2R^2 \left(\frac{\mu_{i+1} - \mu_i}{2}\right)^2 \left(\frac{2}{\tau_i}\right)^{\frac{4}{3}} \int_{\sigma_i}^{\sigma_i + \frac{\tau_i}{2}} \frac{4}{9}(t - \sigma_i)^{-\frac{2}{3}} \, dt$$

$$+ \sum_{1 \le j < k \le N} \frac{m_j m_k}{Rc_{jk}} \int_{\sigma_i}^{\sigma_i + \frac{\tau_i}{2}} \frac{dt}{\lambda(t) - \mu_i}$$

$$= \frac{4R^2}{3} \frac{(\mu_{i+1} - \mu_i)^2}{\tau_i} + \frac{U(x_0')}{R} \frac{2}{\mu_{i+1} - \mu_i} \left(\frac{\tau_i}{2}\right)^{\frac{2}{3}} \int_{\sigma_i}^{\sigma_i + \frac{\tau_i}{2}} \frac{dt}{(t - \sigma_i)^{\frac{2}{3}}}$$

$$= \frac{4R^2}{3} \frac{(\mu_{i+1} - \mu_i)^2}{\tau_i} + \frac{3U(x_0')}{R} \frac{\tau_i}{\mu_{i+1} - \mu_i}.$$

In a similar way we find

$$A_L(\gamma_x\big|_{[\sigma_i + \frac{\tau_i}{2}, \sigma_{i+1}]}) \le \frac{4R^2}{3} \frac{(\mu_{i+1} - \mu_i)^2}{\tau_i} + \frac{3U(x_0')}{R} \frac{\tau_i}{\mu_{i+1} - \mu_i}, \qquad i = 1, ..., h - 1$$

$$A_L(\gamma_x\big|_{[\sigma_0, \sigma_1]}) \le \frac{8R^2}{3} \frac{\mu_1^2}{\tau_0} + \frac{3U(x_0')}{R} \frac{\tau_0}{\mu_1},$$

$$A_L(\gamma_x\big|_{[\sigma_h, \sigma_{h+1}]}) \le \frac{8R^2}{3} \frac{(1 - \mu_h)^2}{\tau_h} + \frac{3U(x_0')}{R} \frac{\tau_h}{1 - \mu_h}.$$

That gives

$$A_L(\gamma_x\big|_{[\sigma_i,\sigma_{i+1}]}) \leq \frac{8R^2}{3} \frac{(\mu_{i+1}-\mu_i)^2}{\tau_i} + \frac{6U(x_0')}{R} \frac{\tau_i}{\mu_{i+1}-\mu_i}, \qquad i = 0,...,h,$$

and by definition of τ_i

$$A_L(\gamma_x) \leq \frac{16R^2}{3T} \left(\sum_{i=0}^{h} (\mu_{i+1}-\mu_i)^{\frac{3}{2}} \right) \left(\sum_{i=0}^{h} (\mu_{i+1}-\mu_i)^{\frac{1}{2}} \right)$$
$$+ \frac{3U(x_0')T}{R} \frac{\sum_{i=0}^{h} (\mu_{i+1}-\mu_i)^{\frac{1}{2}}}{\sum_{i=0}^{h} (\mu_{i+1}-\mu_i)^{\frac{3}{2}}}. \tag{14}$$

By definition of μ_i we have

$$\mu_{i+1} - \mu_i \geq 0, \qquad \sum_{i=0}^{h} (\mu_{i+1}-\mu_i) = 1.$$

Let us introduce now the functions

$$f_1 : \mathbb{R}_+^{h+1} \to \mathbb{R}, \qquad f_1(z) = \sum_{i=0}^{h} z_i^{\frac{3}{2}},$$
$$f_2 : \mathbb{R}_+^{h+1} \to \mathbb{R}, \qquad f_2(z) = \sum_{i=0}^{h} z_i^{\frac{1}{2}},$$
$$g : \mathbb{R}_+^{h+1} \to \mathbb{R}, \qquad g(z) = \sum_{i=0}^{h} z_i,$$

and study minima and maxima of f_1 and f_2 with the constraint $g(z) = 1$. We show by induction on h that

$$\min_{g(z)=1} f_1(z) = \frac{1}{(h+1)^{\frac{1}{2}}}, \qquad \max_{g(z)=1} f_1(z) = 1. \tag{15}$$

If $h = 0$, condition $g(z) = 1$ implies $z_0 = 1$, thus

$$\min_{g(z)=1} f_1(z) = 1, \qquad \max_{g(z)=1} f_1(z) = 1$$

Assuming now the statement is true up to order $h-1$, let us prove it is true at order h. By Lagrange multiplier theorem, the unique interior critical point of f_1 under the condition $g(z) = 1$ is given by the equations

$$\frac{\partial f_1}{\partial z_i}(z) = \lambda \frac{\partial g}{\partial z_i}(z), \quad i = 0,...,h, \qquad \lambda \in \mathbb{R}, \qquad g(z) = 1,$$

this gives

$$z_i = \frac{1}{h+1}, \quad i = 0,...,h, \qquad f_1(z) = \frac{1}{(h+1)^{\frac{1}{2}}}.$$

The boundary of the simplex $g(z) = 1$ is the set of $z = (z_0, ..., z_h)$ such that $\sum_{i=0}^{h} z_i = 1$ and $z_i = 0$ for at least one indices i. By inductive hypothesis, the minimum of $f_1(z)$ on the boundary of $g(z) = 1$ is $1/h^{1/2}$ and the maximum is 1. Comparing with the value of f_1 on the unique interior critical point of f_1 we find (15). In a similar way one prove

$$\min_{g(z)=1} f_2(z) = 1, \qquad \max_{g(z)=1} f_2(z) = (h+1)^{\frac{1}{2}}. \tag{16}$$

Replacing these estimates in (14) we find

$$A_L(\gamma_x) \leq \frac{16R^2}{3T}(h+1)^{\frac{1}{2}} + \frac{3U(x_0')T}{R}(h+1),$$

since $h \leq \frac{N(N-1)}{2}$, inequality (13) is proved.

We give now the proof of Theorem 6.

Proof of Theorem 6. By Propositions 9 and 10, there exist three constants $a > 0$, $\overline{T} > 0$ and $\overline{s} > 1$ such that for every $T \geq \overline{T}$, for every $t \geq \overline{s}T$ and for every minimizer $\overline{\gamma} \in \Sigma(x_i, \gamma_0(t); t)$ we have

$$A_L(\overline{\gamma}|_{[0,T]}) \leq aT^{\frac{1}{3}} \tag{17}$$

Let us prove the equicontinuity of the family

$$\left\{ \overline{\gamma}|_{[0,T]} \right\}, \tag{18}$$

where $\overline{\gamma}$ is any minimizer joining x_i to $\gamma_0(t)$ in time t and $t \geq \overline{s}T$. By (17) we have

$$\int_0^T \|\dot{\overline{\gamma}}(s)\|^2 ds \leq 2aT^{\frac{1}{3}},$$

hence, by Cauchy-Schwarz inequality, for every $0 \leq s < s' \leq T$ we have

$$|\overline{\gamma}(s') - \overline{\gamma}(s)| \leq \int_s^{s'} \|\dot{\overline{\gamma}}(u)\| du \leq \sqrt{s - s'} \left(\int_s^{s'} \|\dot{\overline{\gamma}}(u)\|^2 du \right)^{\frac{1}{2}}$$
$$\leq (2aT^{\frac{1}{3}})^{\frac{1}{2}} \sqrt{s - s'}.$$

This gives the equicontinuity of the family (18). By the way, since $\overline{\gamma}(0) = x_i$, the family is also equibounded. By Ascoli theorem we can find a divergent sequence $(t_n)_{n=1}^{+\infty}$ satisfying $t_n \geq \overline{s}T$ and a sequence of minimizers $\gamma_n \in \Sigma(x_i, \gamma_0(t_n); t_n)$ such that the restriction $(\gamma_n|_{[0,T]})_{n=1}^{+\infty}$ converges uniformly. Applying this argument on an increasing and divergent sequence $(T_k)_{k=1}^{+\infty}$, by a diagonal trick we can find an increasing and divergent sequence of times $(t_n)_{n=1}^{+\infty}$, a sequence of minimizers $\gamma_n \in \Sigma(x_i, \gamma_0(t_n); t_n)$ and a path $\gamma : [0, +\infty) \to \mathcal{X}$ such that $(\gamma_n)_{n=1}^{+\infty}$ converges uniformly to γ on every compact interval. Moreover, by lower semi-continuity of the action we have

$$A_L(\gamma|_{[0,T]}) \leq \liminf_{n \to +\infty} A_L(\gamma_n|_{[0,T]}) \leq aT^{\frac{1}{3}} \tag{19}$$

for every $T > 0$, proving in particular that $A_L(\gamma|_{[0,T]})$ is finite. Therefore, $\gamma(T)$ is a non-collision configuration for almost all $T > 0$. We prove now that γ is a
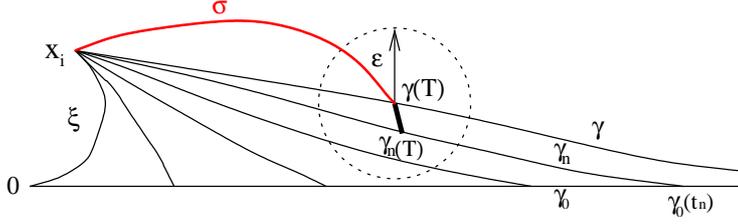
Fig. 2. $\sigma_{\varepsilon,n}$ is obtained by pasting σ (reparametrized) with the straight line joining $\gamma(T)$ to $\gamma_n(T)$

minimizing path. Since we want to show that $\gamma\big|_{[0,T]}$ is a minimizer for every $T > 0$, it is sufficient to prove that $\gamma\big|_{[0,T]}$ is a minimizer for T arbitrary great. We can assume, without loss of generality, that $\gamma(T)$ is a non-collision configuration. Assuming, for the sake of a contradiction, that $\gamma\big|_{[0,T]}$ is not a minimizer, there would exists an absolutely continuous path $\sigma : [0,T] \to \mathscr{X}$ joining x_i to $\gamma(T)$ such that

$$A_L(\sigma) < A_L(\gamma\big|_{[0,T]}). \tag{20}$$

Moreover, there exists $M > 0$ and $\overline{\varepsilon} > 0$ such that

$$\forall x \in \overline{B}(\gamma(T), \overline{\varepsilon}) \Rightarrow U(x) \le M,$$

where $\overline{B}(\gamma(T), \overline{\varepsilon})$ is the closed ball centered in $\gamma(T)$ with radius $\overline{\varepsilon}$. Since the sequence $\gamma_n\big|_{[0,T]}$ converges uniformly to $\gamma\big|_{[0,T]}$, given $0 < \varepsilon < \overline{\varepsilon}$ there exists a positive integer $N_{T,\varepsilon}$ such that for every $n \ge N_{T,\varepsilon}$ we have $\gamma_n(T) \in \overline{B}(\gamma(T), \varepsilon)$. Let $\sigma_{\varepsilon,n} : [0,T] \to \mathscr{X}$ be the path defined by

$$\sigma_{\varepsilon,n}(t) = \begin{cases} \sigma\left(\frac{T}{T-\varepsilon}t\right) & \text{if } t \in [0, T-\varepsilon] \\ \frac{T-t}{\varepsilon}\gamma(T) + \frac{t-T+\varepsilon}{\varepsilon}\gamma_n(T) & \text{if } t \in [T-\varepsilon, T], \end{cases}$$

where $n \ge N_{T,\varepsilon}$. By construction $\sigma_{\varepsilon,n}$ joins x_i to $\gamma_n(T)$ in time T (see Figure 2). Moreover, if $t \in [T-\varepsilon, T]$, the configuration $\sigma_{\varepsilon,n}(t)$ is contained in the ball $\overline{B}(\gamma(T), \varepsilon)$. Computing the action of $\sigma_{\varepsilon,n}$ we get

$$A_L(\sigma_{\varepsilon,n}) \le \tfrac{1}{2}\left(\tfrac{T}{T-\varepsilon}\right)^2 \int_0^{T-\varepsilon} \left\|\dot{\sigma}\left(\frac{Tt}{T-\varepsilon}\right)\right\|^2 dt + \int_0^{T-\varepsilon} U\left(\sigma\left(\frac{Tt}{T-\varepsilon}\right)\right) dt$$

$$+ \left(M + \tfrac{1}{2}\right)\varepsilon$$

$$= \frac{T}{T-\varepsilon}\int_0^T \frac{1}{2}\|\dot{\sigma}(t)\|^2 dt + \frac{T-\varepsilon}{T}\int_0^T U(\sigma(t))dt + \left(M + \frac{1}{2}\right)\varepsilon$$

$$= A_L(\sigma) + \mathcal{O}(\varepsilon).$$

Inequalities (19) and (20) imply

$$A_L(\sigma_{\varepsilon,n}) < A_L(\gamma_n\big|_{[0,T]})$$

if ε is sufficiently small and n sufficiently great. This contradicts the minimizing property of γ_n and proves that γ is a minimizer. By Marchal theorem, γ is collision-free (and in particular it is a real solution of the N-body problem) for $t > 0$. This complete the proof of Theorem 6.

4. Parabolicity of the solution

To complete the proof of the main theorem we still have to show that the limit solution $\gamma(t)$ is parabolic and asymptotic to x_0. By Lemma 5 we just need to verify the asymptotic estimates (6). We introduce now the following

Notation *Given the functions $f(r,x_1,...,x_n)$ and $g(r,x_1,...,x_n) \neq 0$, we write $f(r,x_1,...,x_n) = o_r(g(r,x_1,...,x_n))$ as $r \to r_0$ if the quotient $\frac{f(r,x_1,...,x_n)}{g(r,x_1,...,x_n)}$ is infinitesimal as $r \to r_0$, uniformly on $(x_1,...,x_n)$. In a similar way, we write $f(r,x_1,...,x_n) = \mathscr{O}_r(g(r,x_1,...,x_n))$ if the quotient $\frac{f(r,x_1,...,x_n)}{g(r,x_1,...,x_n)}$ is locally bounded for r close to r_0, uniformly on the variables $(x_1,...,x_n)$.*

Let us give now a refinement of Lemma 8.

Lemma 11 *Let $\tau > 0$ and $\xi \in \Sigma(0,x_i;\tau)$ be a minimizer. There exist two constants $\overline{T} > \tau$ and $\overline{s} > 1$ such that for every $T \geq \overline{T}$, for every $t \geq \overline{s}T$ and for every minimizer $\overline{\gamma} \in \Sigma(x_i,\gamma_0(t);t)$ we have*

$$\mathscr{M}(\overline{\gamma}(T),\gamma_0(t);0,T,t) \leq 2A_L(\xi) + \mathscr{O}_T(T^{-\frac{2}{3}})$$

$$\mathscr{N}(\|\overline{\gamma}(T)\|,\alpha t^{\frac{2}{3}};0,T,t) \leq 2A_L(\xi) + \mathscr{O}_T(T^{-\frac{2}{3}}).$$

as $T \to +\infty$.

Proof. The second inequality is a direct consequence of the first one and of Lemma 7. Let us prove the first inequality. We consider τ as a fixed constant, while T and t are variables. Let $\overline{T} > 0$, $\overline{s} > 1$ and $K > 0$ be like in Proposition 9. Without loss of generality we can assume $\overline{T} > \tau$. Let $T \geq \overline{T}$ and $t \geq \overline{s}T$. Let $\eta_{T+\tau} \in \Sigma(0,\overline{\gamma}(T);T+\tau)$ be a minimizer. The path

$$\overline{\eta}_T : [0,T] \to \mathscr{X}, \qquad \overline{\eta}_T(s) = \eta_{T+\tau}\left(\frac{T+\tau}{T}s\right).$$

is a reparametrization of $\eta_{T+\tau}$ and it joins 0 to $\overline{\gamma}(T)$ in time T, thus

$$\mathscr{A}(0,\overline{\gamma}(T);T) \leq A_L(\overline{\eta}_T).$$

A computation of the action of $\overline{\eta}_T$ gives

$$A_L(\overline{\eta}_T) = (1 + \mathscr{O}(1/T))A_L(\eta_{T+\tau}), \qquad T \to +\infty. \tag{21}$$

Since $\eta_{T+\tau}$ is a minimizer joining 0 to $\overline{\gamma}(T)$ in time $T + \tau$, by Propositions 9 and 10 we obtain

$$A_L(\eta_{T+\tau}) \leq C_1\frac{K^2T^{\frac{4}{3}}}{T+\tau} + C_2\frac{T+\tau}{KT^{\frac{2}{3}}} = \mathscr{O}(T^{\frac{1}{3}}). \tag{22}$$

Combining inequalities (21) and (22), by definition of $\eta_{T+\tau}$ and $\overline{\eta}_T$ we get

$$\mathscr{A}(0,\overline{\gamma}(T);T) - \mathscr{A}(0,\overline{\gamma}(T);T+\tau) \leq \mathscr{O}(T^{-\frac{2}{3}}). \quad T \to +\infty \qquad (23)$$

In a similar way, let us consider a minimizer $\eta_T \in \Sigma(0,\overline{\gamma}(T);T)$ The path

$$\overline{\eta}_{T+\tau} : [0,T+\tau] \to \mathscr{X}, \qquad \overline{\eta}_{T+\tau}(s) = \eta_T\left(\left(\frac{T}{T+\tau}\right)s\right).$$

is a reparametrization of η_T, and it joins 0 to $\overline{\gamma}(T)$ in time $T+\tau$, hence

$$\mathscr{A}(0,\overline{\gamma}(T);T+\tau) \leq A_L(\overline{\eta}_{T+\tau}).$$

Arguing as before we get the estimates

$$\mathscr{A}(0,\overline{\gamma}(T);T+\tau) - \mathscr{A}(0,\overline{\gamma}(T);T) \leq \mathscr{O}(T^{-\frac{2}{3}}), \quad T \to +\infty. \qquad (24)$$

Combining inequalities (23) with (24) we obtain

$$\mathscr{A}(0,\overline{\gamma}(T);T+\tau) = \mathscr{A}(0,\overline{\gamma}(T);T) + \mathscr{O}_T(T^{-\frac{2}{3}}), \quad T \to +\infty, \qquad (25)$$

uniformly on $t \geq \overline{s}T$ and $\overline{\gamma} \in \Sigma(0,\gamma_0(t);t)$. With the same argument we find the following estimates

$$\mathscr{A}(0,\gamma_0(t);t-\tau) - \mathscr{A}(0,\gamma_0(t);t) = \mathscr{O}_t(t^{-\frac{2}{3}}), \quad t \to +\infty. \qquad (26)$$

Replacing (25) and (26) into the first inequality of Lemma 8, since we assume $t \geq \overline{s}T$ and $\overline{s} > 1$, we obtain the first inequality of this Lemma. This ends the proof.

To simplify the notations we introduce now the functions

$$\mathscr{F} : \mathscr{X} \times (1,+\infty) \to \mathbb{R}_+$$
$$\mathscr{F}(x,s) = \mathscr{M}(x,\gamma_0(s);0,1,s) = \mathscr{A}(0,x;1) + \mathscr{A}(x,\gamma_0(s);s-1) - \mathscr{A}(0,\gamma_0(s);s),$$

and

$$\mathscr{G} : \mathbb{R}_+ \times (1,+\infty) \to \mathbb{R}_+$$
$$\mathscr{G}(r,s) = \mathscr{N}(r,\alpha s^{\frac{2}{3}};0,1,s) = S(0,r;1) + S(r,\alpha s^{\frac{2}{3}};s-1) - S(0,\alpha s^{\frac{2}{3}};s).$$

Lemma 12 *Given $s > 1$ and $x \in \mathscr{X}$ we have*

$$\mathscr{F}(x,s) \geq 0$$

with equality if and only if $x = \alpha x_0$.

Proof. By Lemma 7 we have

$$\mathscr{F}(x,s) \geq \mathscr{G}(\|x\|,s),$$

with equality if and only if $x = \|x\|x_0$. Since $u \mapsto \alpha u^{\frac{2}{3}}$ is the unique solution of the one-dimensional Kepler problem joining 0 to $\alpha s^{\frac{2}{3}}$ in time s (see Lemma 16 in the Appendix), it is also a minimizer, therefore

$$\mathscr{G}(r,s) \geq 0,$$

with equality if and only if $r = \alpha$. This proves the Lemma.

By homothety invariance, the conclusion of Lemmas 11 and 12 can be written in the more compact form

$$0 \leq T^{1/3}\mathscr{F}\left(\frac{\overline{\gamma}(T)}{T^{2/3}}, \frac{t}{T}\right) \leq 2A_L(\xi) + \mathscr{O}_T(T^{-2/3})$$

$$0 \leq T^{1/3}\mathscr{G}\left(\frac{\|\overline{\gamma}(T)\|}{T^{2/3}}, \frac{t}{T}\right) \leq 2A_L(\xi) + \mathscr{O}_T(T^{-2/3}). \tag{27}$$

as $T \to +\infty$, uniformly on $t \geq \overline{s}T$ and $\overline{\gamma} \in \Sigma(0, \gamma_0(t); t)$ minimizer.

The following Theorem is a main tool in the proof of the Main Theorem. It shows that if $\mathscr{F}(x,s)$ is sufficiently small and s is sufficiently great, the configuration x is close to αx_0.

Theorem 13 *There exist $\overline{\varepsilon} > 0$ and a function $\delta : (0, \overline{\varepsilon}] \to \mathbb{R}_+$ satisfying $\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0^+$, such that for every $\varepsilon \in (0, \overline{\varepsilon}]$, there exists $\overline{s}_\varepsilon > 1$, such that for every $s \geq \overline{s}_\varepsilon$, the set of configurations $x \in \mathscr{X}$ satisfying the inequality*

$$\mathscr{F}(x,s) \leq \varepsilon \tag{28}$$

is contained in the ball $\overline{B}(\alpha x_0, \delta(\varepsilon))$.

Before giving the proof of Theorem 13, we show that this theorem achieve the proof of the Main Theorem.

Proof of the Main Theorem. Let $\gamma : [0, +\infty) \to \mathscr{X}$ be the limit solution constructed in Theorem 6 and let $\gamma_n \in \Sigma(x_i, \gamma_0(t_n); t_n)$ be the sequence of minimizers uniformly convergent to γ on every compact interval. Let $\overline{\varepsilon}$ be as in Theorem 13, let $\overline{T}$ and $\overline{s}$ be as in Lemma 11 and $0 < \varepsilon < \overline{\varepsilon}$. An immediate consequence of inequalities (27) is the existence of $\overline{T}_\varepsilon \geq \overline{T}$ such that if $T \geq \overline{T}_\varepsilon$ and $t_n \geq \overline{s}T$ we have

$$\mathscr{F}\left(\frac{\gamma_n(T)}{T^{\frac{2}{3}}}, \frac{t_n}{T}\right) \leq \varepsilon$$

and by Theorem 13

$$\left\|\frac{\gamma_n(T)}{T^{2/3}} - \alpha x_0\right\| \leq \delta(\varepsilon),$$

for t_n sufficiently great. The sequence $\gamma_n\big|_{[0,T]}$ converges uniformly to $\gamma\big|_{[0,T]}$ as $n \to +\infty$, hence

$$\left\|\frac{\gamma(T)}{T^{2/3}} - \alpha x_0\right\| \leq \delta(\varepsilon),$$

for every $T \geq \overline{T}_\varepsilon$. Since $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, we have proved that

$$\frac{\gamma(T)}{T^{2/3}} \to \alpha x_0, \quad \text{as} \quad T \to +\infty,$$

that is to say, γ is parabolic and asymptotic to x_0. This achieves the proof of the Main Theorem.

The next section is devoted to prove Theorem 13.

5. Proof of Theorem 13

In order to achieve the proof of Theorem 13 we compare the *N*-body problem with a Kepler problem on the configuration space with a lagrangian given by

$$L_0(x, \dot{x}) = \frac{\|\dot{x}\|^2}{2} + \frac{U_0}{\|x\|}, \qquad (x, \dot{x}) \in \mathscr{X} \times \mathscr{X}.$$

Let $A_{L_0}(\varpi)$ denote the action (for the lagrangian L_0) of an absolutely continuous path ϖ and $\mathscr{A}_0(x_1, x_2; s)$ the infimum of $A_{L_0}(\varpi)$ over all absolutely continuous paths ϖ joining x_1 to x_2 in time s. We have the inequality

$$\mathscr{A}(x_1, x_2; s) \geq \mathscr{A}_0(x_1, x_2; s) \geq S(\|x_1\|, \|x_2\|; s),$$

with $\mathscr{A}(x_1, x_2; s) = \mathscr{A}_0(x_1, x_2; s)$ if and only if there exists a minimizing path (for the lagrangian *L*) $\varpi : [0, s] \to \mathscr{X}$ joining x_1 with x_2 such that $\tilde{U}(\varpi(u)) = U_0$ for every $u \in [0, s]$, and $\mathscr{A}_0(x_1, x_2; s) = S(\|x_1\|, \|x_2\|; s)$ if and only if x_1 and x_2 are on a same half-line starting from the origin. The function

$$\mathscr{F}_0 : \mathscr{X} \times (1, +\infty) \to \mathbb{R}_+,$$
$$\mathscr{F}_0(x, s) = \mathscr{A}_0(0, x; 1) + \mathscr{A}_0(x, \gamma_0(s); s - 1) - \mathscr{A}_0(0, \gamma_0(s); s),$$

verifies the inequality

$$\mathscr{F}(x, s) \geq \mathscr{F}_0(x, s) \geq \mathscr{G}(\|x\|, s) \geq 0. \tag{29}$$

Roughly speaking, to achieve the proof of Theorem 13, we replace $\mathscr{F}(x, s)$ with $\mathscr{F}_0(x, s)$ and we show that if ε is small and s great, the inequality $\mathscr{F}_0(x, s) \leq \varepsilon$ can be satisfied only if x is in a small ball centered in αx_0.

This goal will be achieved in two steps. In Proposition 14 we prove that if s is sufficiently great, the set of $r \in \mathbb{R}_+$ verifying $\mathscr{G}(r, s) \leq \varepsilon$ is contained in a small interval centered in α. Hence, by inequality (29), the set of configuration x verifying $\mathscr{F}(x, s) \leq \varepsilon$ is contained in a thin hollow sphere with inner and outer radious close to α. In Proposition 15 we show that the set of configurations x verifying $\mathscr{F}_0(x, s) \leq \varepsilon$ is a small neighborhood of αx_0.

Proposition 14 *There exist $\varepsilon_1 > 0$ and a function $\delta_1 : (0, \overline{\varepsilon}_1] \to \mathbb{R}_+$ satisfying $\delta_1(\varepsilon) = o(1)$ as $\varepsilon \to 0^+$, such that for every $\varepsilon \in (0, \overline{\varepsilon}_1]$ there exists $\overline{s}_\varepsilon^1 > 1$, such that for every $s \geq \overline{s}_\varepsilon^1$, the set of $r \in \mathbb{R}_+$ satisfying the inequality*

$$\mathscr{G}(r, s) \leq \varepsilon$$

is contained in the interval $[\alpha - \delta_1(\varepsilon), \alpha + \delta_1(\varepsilon)]$.

Proof. By Proposition 21 of the Appendix there exists $\overline{r} > 0$ and $\overline{s} > 0$ such that for every $r \geq \overline{r}$ and for every $s \geq \overline{s}$ we have $\mathscr{G}(r, s) > 1$. Without loss of generality we will assume $\alpha < \overline{r} < \overline{s}^{1/3}$. By Proposition 20 of the Appendix we have

$$\mathscr{G}(r, s) = S(0, r; 1) - \beta_0 r^{\frac{1}{2}} + g(r, s),$$

where $g(r, s) = o_s(1)$ as $s \to +\infty$, uniformly on $0 \leq r \leq s^{\frac{1}{3}}$, and where $\beta_0 = (8U_0)^{\frac{1}{2}}$. Let us introduce now the function

$$G(r) = S(0, r; 1) - \beta_0 r^{\frac{1}{2}}.$$

By Lemma 16 the solution joining 0 with r in time 1 is monotonic for $r \geq \beta$, where $\beta = 2 \left(\frac{U_0}{\pi^2} \right)^{1/3}$. We remark that $\beta < \alpha$. The energy $h(0, r; 1)$ of this solution is negative if and only if $0 \leq r < \alpha$, moreover $h(0, \beta; 1) = -U_0/\beta$. Let us term $h = h(0, r; 1)$. The action $S(0, r; 1)$ is given by

$$S(0, r; 1) = \begin{cases} \displaystyle\int_0^{-\frac{U_0}{h}} \sqrt{2 \left(h + \frac{U_0}{u} \right)} \, du + \int_r^{-\frac{U_0}{h}} \sqrt{2 \left(h + \frac{U_0}{u} \right)} \, du - h \text{ if } & r < \beta \\[4mm] \displaystyle\int_0^r \sqrt{2 \left(h + \frac{U_0}{u} \right)} \, du - h & \text{if } \quad r \geq \beta, \end{cases}$$

hence by Lemma 18, the functions $r \mapsto S(0, r; 1)$ and $r \mapsto G(r)$ are of class $\mathscr{C}^1$ on $(0, +\infty)$, moreover we have

$$G'(r) = \begin{cases} -\sqrt{2 \left(h(0, r; 1) + \frac{U_0}{r} \right)} - \sqrt{\frac{2U_0}{r}} \text{ if } & 0 < r < \beta \\[4mm] \sqrt{2 \left(h(0, r; 1) + \frac{U_0}{r} \right)} - \sqrt{\frac{2U_0}{r}} & \text{if } \quad r \geq \beta, \end{cases}$$

proving that $G(r)$ is in fact of class $\mathscr{C}^2$ on $(0, \beta) \cup (\beta, +\infty)$. Since the function $r \mapsto h(0, r; 1)$ is increasing and $h(0, \alpha; 1) = 0$, the function $G(r)$ is decreasing for $r \in (0, \alpha)$ and it is increasing for $r \in (\alpha, +\infty)$. The absolute minimum of $G(r)$ is achieved at $r = \alpha$, and we have

$$G(\alpha) = \int_0^\alpha \sqrt{\frac{2U_0}{u}} \, du - \sqrt{8U_0 \alpha} = 0.$$

By Lemma 18, a direct computation of the second derivative of G at α gives

$$G''(\alpha) = \frac{5U_0^{\frac{1}{2}}}{2^{\frac{1}{2}}\alpha^{\frac{3}{2}}},$$

hence, since $G(\alpha) = G'(\alpha) = 0$, there exists $\overline{\delta} > 0$ and $C_1 > 0$ such that

$$\forall r \in [\alpha - \overline{\delta}, \alpha + \overline{\delta}], \qquad G(r) \geq C_1(r - \alpha)^2.$$

Without loss of generality we shall assume $\alpha - \overline{\delta} > \beta$ and $\alpha + \overline{\delta} < \overline{r}$. Let $\overline{\varepsilon}_1 = \min\{\frac{C_1\overline{\delta}^2}{2}, 1\}$ and let us define the function

$$\delta_1 : (0, \overline{\varepsilon}_1] \to \mathbb{R}_+, \qquad \delta_1(\varepsilon) = \sqrt{\frac{2\varepsilon}{C_1}}.$$

Since $G(r)$ is decreasing for $r \leq \alpha$ and increasing for $r \geq \alpha$, for every $\varepsilon \in (0, \overline{\varepsilon}_1]$ we have

$$\forall r \in (0, \alpha - \delta_1(\varepsilon)) \cup (\alpha + \delta_1(\varepsilon), +\infty), \qquad G(r) > C_1\delta_1(\varepsilon)^2 = 2\varepsilon. \qquad (30)$$

We come back now to the function $\mathscr{G}(r,s) = G(r) + g(r,s)$. Since $g(r,s)$ is infinitesimal for $s \to +\infty$ and $0 \leq r \leq s^{\frac{1}{3}}$, for every $\varepsilon \in (0, \overline{\varepsilon}_1]$ there exists $\overline{s}_\varepsilon^1 > \overline{s}$ such that for every $s \geq \overline{s}_\varepsilon^1$ and for every r verifying $0 \leq r \leq s^{\frac{1}{3}}$ we have $|g(r,s)| \leq \varepsilon$. If $s \geq \overline{s}_\varepsilon^1$ and $r \geq \overline{r}$ we have

$$\mathscr{G}(r,s) > 1 \geq \overline{\varepsilon}_1 \geq \varepsilon.$$

If $s \geq \overline{s}_\varepsilon^1$ and $r \in (0, \overline{r})$, by (30), for every $r \in (0, \alpha - \delta_1(\varepsilon)) \cup (\alpha + \delta_1(\varepsilon), \overline{r})$ we have

$$\mathscr{G}(r,s) = G(r) + g(r,s) > 2\varepsilon - \varepsilon = \varepsilon.$$

This ends the proof of the Proposition.

We introduce the following notation : given two configurations x_1 and x_2, the angle between x_1 and x_2 is denoted by the symbol $\angle(x_1, x_2)$. We always have $0 \leq \angle(x_1, x_2) \leq \pi$.

Proposition 15 *If $\overline{\varepsilon}_1$ and $\delta_1 : (0, \overline{\varepsilon}_1] \to \mathbb{R}_+$ are like in Proposition 14, there exist $\overline{\varepsilon} \in (0, \overline{\varepsilon}_1]$ and $C_2 > 0$ such that given the function*

$$\delta_2 : (0, \overline{\varepsilon}] \to \mathbb{R}_+, \qquad \delta_2(\varepsilon) = (C_2\varepsilon)^{\frac{1}{2}}, \qquad (31)$$

for every $\varepsilon \in (0, \overline{\varepsilon}]$, there exists $\overline{s}_\varepsilon^2 > 1$ such that for every $s \geq \overline{s}_\varepsilon^2$ and for every configuration $x \in \mathscr{X}$ satisfying

$$|\|x\| - \alpha| \leq \delta_1(\varepsilon), \qquad \angle(x, x_0) > \delta_2(\varepsilon) \qquad (32)$$

we have

$$\mathscr{F}_0(x,s) > \varepsilon.$$

Proof. The basic tool of this proof is Lambert's Theorem. Our reference is [1]. Let $C_2 > 0$ and $\overline{\varepsilon} \in (0, \overline{\varepsilon}_1]$. Let $\delta_2 : (0, \overline{\varepsilon}] \to \mathbb{R}_+$ be the function defined in (31). In the following we will ask more precise conditions on C_2 and $\overline{\varepsilon}$. Let $0 < \varepsilon \leq \overline{\varepsilon}$, let x be a configuration verifying (32) and $s > 1$. The minimizer (for L_0) $\sigma : [0, s-1] \to \mathscr{X}$ joining x to $\gamma_0(s)$ in time $s-1$ is a collision-free Keplerian arc, hence it is contained in the plane generated by 0, x and $\gamma_0(s)$. Introducing a system of polar coordinates in this plane, we can identify x with $re^{\iota\theta}$ and $\gamma_0(s)$ with $\alpha s^{\frac{2}{3}} \in \mathbb{R} \subset \mathbb{C}$ where

$$|r - \alpha| \leq \delta_1(\varepsilon), \qquad \delta_2(\varepsilon) < |\theta| \leq \pi.$$

Moreover, the path σ can be written in polar coordinates by

$$\sigma(u) = \rho(u)e^{\iota\phi(u)}, \quad u \in [0, s-1],$$

where

$$\rho(0) = r \quad \phi(0) = \theta$$
$$\rho(s-1) = \alpha s^{\frac{2}{3}} \quad \phi(s-1) \in 2\pi\mathbb{Z}.$$

Since σ is collision-free, $\rho(u) > 0$ for all $u \in [0, s-1]$. By definition of $\mathscr{F}_0$ and using the properties of $\mathscr{A}_0$ we have

$$\mathscr{F}_0(re^{\iota\theta}, s) = \mathscr{A}_0(0, re^{\iota\theta}; 1) + \mathscr{A}_0(re^{\iota\theta}, \gamma_0(s); s-1) - \mathscr{A}_0(0, \gamma_0(s); s)$$
$$= S(0, r; 1) + \mathscr{A}_0(re^{\iota\theta}, \gamma_0(s); s-1) - S(0, \alpha s^{\frac{2}{3}}; s).$$

We prove now that σ is a *direct path*, that is to say, the total variation of the polar angle ϕ is less than or equal to π. Assume, for the sake of contradiction, that $|\phi(s-1) - \phi(0)| > \pi$. Eventually changing the orientation of the plane, we can assume without loss of generality $\phi(s-1) - \phi(0) > \pi$, hence there exists a unique integer $k \geq 1$ and a unique real number $\alpha \in (-\pi, \pi]$ such that

$$\phi(s-1) - \phi(0) = 2k\pi + \alpha.$$

The path $\overline{\rho}e^{\iota\overline{\phi}}$ defined by

$$\overline{\rho}(u) = \rho(u), \quad \overline{\phi}(u) = \phi(0) + \frac{\alpha}{2k\pi + \alpha}(\phi(u) - \phi(0)),$$

has the same ends as the original one, moreover

$$A_{L_0}(\overline{\rho}e^{\iota\overline{\phi}}) - A_{L_0}(\rho e^{\iota\phi}) = \frac{1}{2}\left[\left(\frac{\alpha}{2k\pi + \alpha}\right)^2 - 1\right]\int_0^{s-1}(\rho^2\dot{\phi}^2)(u)du < 0,$$

and we get a contradiction. Lambert's Theorem state that if x_1 and x_2 are two configurations and $\tau > 0$, the action $\mathscr{A}_0(x_1, x_2; \tau)$ of the direct Keplerian arc joining x_1 to x_2 in time τ is a function of three parameters only : the time τ, the distance $\|x_1 - x_2\|$ between the two ends and the sum of the distances between the ends and the origin (i.e. $\|x_1\| + \|x_2\|$). Comparing now σ with a direct collinear arc, by Lambert's Theorem we find

$$\mathscr{A}_0(re^{\iota\theta}, \gamma_0(s); s-1) = S(d_1(r, \theta, s), d_2(r, \theta, s); s-1),$$

where

$$d_1(r,\theta,s) = \frac{r + \alpha s^{\frac{2}{3}} - |re^{\imath\theta} - \alpha s^{\frac{2}{3}}|}{2},$$
$$d_2(r,\theta,s) = \frac{r + \alpha s^{\frac{2}{3}} + |re^{\imath\theta} - \alpha s^{\frac{2}{3}}|}{2}.$$

Moreover

$$|re^{\imath\theta} - \alpha s^{\frac{2}{3}}| = \alpha s^{\frac{2}{3}} - r\cos\theta + l(r,\theta,s),$$

where

$$l(r,\theta,s) = \mathscr{O}_s(s^{-2/3}), \quad s \to +\infty$$

uniformly on $\delta_2(\varepsilon) < |\theta| \le \pi$ and $|r - \alpha| \le \delta_1(\varepsilon)$. Therefore we get

$$d_1(r,\theta,s) = r\left(\frac{1+\cos\theta}{2}\right) - \frac{l(r,\theta,s)}{2}$$
$$d_2(r,\theta,s) = \alpha s^{\frac{2}{3}} + r\left(\frac{1-\cos\theta}{2}\right) + \frac{l(r,\theta,s)}{2}.$$

Since $S(0, \alpha s^{\frac{2}{3}}; s) = \alpha_0 s^{\frac{1}{3}}$, applying Proposition 20 of the Appendix to $S(d_1(r,\theta,s), d_2(r,\theta,s); s-1)$ we find

$$\mathscr{F}_0(re^{\imath\theta},s) = G(r) + \beta_0 r^{\frac{1}{2}} \left[1 - \left(\frac{1+\cos\theta}{2} - \frac{l(r,\theta,s)}{2r} \right)^{\frac{1}{2}} \right] + g(r,\theta,s),$$

where $g(r,\theta,s)$ is infinitesimal as $s \to +\infty$, uniformly on r and θ. In Proposition 14 we showed that $G(r) \ge 0$ for all $r > 0$. Let $\bar{s}_\varepsilon^2 > 0$ such that for every $s \ge \bar{s}_\varepsilon^2$, for every θ satisfying $|\theta| \in (\delta_2(\varepsilon),\pi]$ and for every $r \in [\alpha - \delta_1(\varepsilon), \alpha + \delta_1(\varepsilon)]$ we have

$$|g(r,\theta,s)| \le \varepsilon, \qquad \left| \frac{l(r,\theta,s)}{2r} \right| \le \varepsilon.$$

Since the function $x \mapsto \cos x$ is decreasing in $[0,\pi]$, chosing $C_2 > 4$ and using the classical expansions of $\cos x$ and $(1+x)^{\frac{1}{2}}$ we find

$$\mathscr{F}_0(re^{\imath\theta},s) \ge \beta_0(\alpha - \delta_1(\varepsilon))^{\frac{1}{2}} \left[1 - \left(\frac{1+\cos\delta_2(\varepsilon)}{2} + \varepsilon \right)^{\frac{1}{2}} \right] - \varepsilon$$
$$= \varepsilon \left[\beta_0 \alpha^{\frac{1}{2}} \left(1 - \frac{\delta_1(\varepsilon)}{\alpha} \right)^{\frac{1}{2}} \left(\frac{C_2 - 4}{8} + \mu(\varepsilon) \right) - 1 \right],$$

where $\mu(\varepsilon) = o(1)$ as $\varepsilon \to 0$. Chosing $0 < \overline{\varepsilon} < \overline{\varepsilon}_1$ such that

$$\forall \varepsilon \in (0,\overline{\varepsilon}], \qquad |\mu(\varepsilon)| < \frac{1}{8}, \quad \text{and} \quad |\delta_1(\varepsilon)| < \frac{\alpha}{2},$$

and chosing C_2 in such a way

$$C_2 > 5 + \frac{16\sqrt{2}}{\beta_0 \alpha^{\frac{1}{2}}}$$

we find

$$\mathscr{F}_0(re^{\imath\theta},s) > \varepsilon,$$

for every $r \in [\alpha - \delta_1(\varepsilon), \alpha + \delta_1(\varepsilon)]$, for every θ such that $|\theta| \in (\delta_2(\varepsilon), \pi]$ and for every $s \geq \overline{s}_\varepsilon^2$. This proves the Proposition.

The proof of Theorem 13 is essentially the juxtaposition of the two previous Propositions.

Proof of Theorem 13.

We use the same notations of the previous two Propositions. Given $\varepsilon \in (0, \overline{\varepsilon}]$, let $\overline{s}_\varepsilon = \max\{\overline{s}_\varepsilon^1, \overline{s}_\varepsilon^2\}$. By Proposition 14 and 15 and by inequality (29), if $s \geq \overline{s}_\varepsilon$ and x is a configuration verifying $\mathscr{F}(x,s) \leq \varepsilon$ we have

$$\big| \|x\| - \alpha \big| \leq \delta_1(\varepsilon) \quad \text{and} \quad \angle(x, x_0) \leq \delta_2(\varepsilon). \tag{33}$$

Let δ be the function

$$\delta : (0, \overline{\varepsilon}] \to \mathbb{R}_+, \qquad \delta(\varepsilon) = \big[2\alpha \, (\alpha + \delta_1(\varepsilon)) \, (1 - \cos \delta_2(\varepsilon)) + \delta_1(\varepsilon)^2 \big]^{\frac{1}{2}},$$

an easy computation show that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and the set of configurations verifying (33) is contained in the ball $\overline{B}(\alpha x_0, \delta(\varepsilon))$. The Theorem is proved.

Appendix : Some estimates for the one-dimensional Kepler Problem

The Kepler problem on the half-line $\mathbb{R}_+$ is defined by the equation

$$\ddot{r} = -\frac{U_0}{r^2}, \tag{34}$$

where $U_0 > 0$ is the gravitational constant. The Lagrangian function of the problem and the energy are written

$$l = \frac{\dot{r}^2}{2} + \frac{U_0}{r}, \qquad h = \frac{\dot{r}^2}{2} - \frac{U_0}{r}.$$

A parabolic solution of the Kepler problem is nothing but a solution with zero energy. There is a unique increasing parabolic solution, namely $r(s) = \alpha s^{2/3}$ where $\alpha = (9U_0/2)^{1/3}$. Given $0 \leq a \leq b$, the energy of a solution connecting a to b is necessarily greater or equal to $-U_0/b$. Moreover, if $0 \leq a < b$, for $h \geq 0$ or $h = -U_0/b$ there is a unique segment of solution of energy h joining a to b, this solution increases from a to b. If $-U_0/b < h < 0$ there are exactly two segments of solutions of energy h joining a to b, a monotonic one, that increases from a to b, and a non-monotonic one, that increases from a to $-U_0/h$ and decreases from $-U_0/h$ to b. Let $\overline{s}(a,b)$ be the time employed by the solution of energy $-U_0/b$ to connect a to b. We have the following lemma, whose proof is left to the reader.

Lemma 16 *Given $0 \leq a \leq b$, and $s > 0$, there exists a unique segment of solution joining a to b in time s, moreover, the solution is monotonic if and only if $0 < s \leq \overline{s}(a,b)$.*

Definition 17 *Given $0 \leq a \leq b$ and $s > 0$, we denote by $h(a,b;s)$ the energy of the unique segment of solution joining a to b in time s, and we denote by $S(a,b;s)$ the Lagrangian action of this solution.*

Since the solution joining a to b in time s is unique, $S(a,b;s)$ is also the minimum of the action of absolutely continuous paths joining a to b in time s.

We shall study the behaviour of the function $r \mapsto h(0,r;s)$ for fixed $s > 0$.

Lemma 18 *Given $s > 0$, the function $r \mapsto h(0,r;s)$ is $\mathscr{C}^1$ in $(0,+\infty)$ with a strictly positive derivative. Moreover*

$$\frac{\partial h}{\partial r}(0,\alpha s^{\frac{2}{3}};s) = \frac{5U_0}{\alpha^2 s^{\frac{4}{3}}}.$$

The proof is left to the reader. We shall also need the following two Propositions

Proposition 19 *Let $\overline{\varepsilon} > 0$. We have*

$$S(0,r;1+\varepsilon) = \frac{r^2}{2(1+\varepsilon)} + o_r(r^2) \tag{35}$$

as $r \to +\infty$, uniformly for $\varepsilon \in [0,\overline{\varepsilon}]$.

Proof. The parabolic solution $u \mapsto \alpha u^{\frac{2}{3}}$ has zero energy, hence $h(0,\alpha(1+\varepsilon)^{\frac{2}{3}};1+\varepsilon) = 0$. Since we are interested at what happens when $r \to +\infty$, we assume $r > \alpha(1+\overline{\varepsilon})^{\frac{2}{3}}$. By Lemma 18 the energy $h(0,r;1+\varepsilon)$ is positive and the solution joining 0 to r in time $1+\varepsilon$ is monotonic. The function $h = h(0,r;1+\varepsilon)$ verifies the identity

$$1+\varepsilon = \int_0^r \frac{du}{\sqrt{2\left(h+\frac{U_0}{u}\right)}} = \frac{U_0}{2^{\frac{1}{2}}h^{\frac{3}{2}}}E\left(\frac{hr}{U_0}\right), \tag{36}$$

where $E : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$E(x) = \int_0^x \sqrt{\frac{s}{1+s}}\,ds, \qquad x \in \mathbb{R}_+,$$

and it verifies the estimates

$$\begin{aligned} E(x) &= \tfrac{2}{3}x^{\frac{3}{2}} + o(x^{\frac{3}{2}}) \quad \text{as} \quad x \to 0^+ \\ E(x) &= x + o(x), \quad \text{as} \quad x \to +\infty. \end{aligned} \tag{37}$$

Let us prove now that

$$h(0,r;1+\varepsilon) \to +\infty, \qquad \text{as} \quad r \to +\infty \tag{38}$$

uniformly on $\varepsilon \in [0,\overline{\varepsilon}]$. Assuming, for the sake of contradiction, that (38) is false, there would exist two sequence $r_n \to +\infty$ and $\varepsilon_n \in [0,\overline{\varepsilon}]$ such that $h(0,r_n;1+\varepsilon_n)$ is bounded. To simplify notations let us denote $h_n = h(0,r_n;1+\varepsilon_n)$. By identities (36) and (37), the sequence $h_n r_n$ is bounded too. This implies that $h_n \to 0$ as $n \to +\infty$. Since $E(x)$ is continuous and strictly increasing, identity (36) gives $h_n r_n \to 0$ as $n \to +\infty$. Applying again (36) and the first of (37) we obtain

$$\lim_{n \to +\infty} 1+\varepsilon_n = \lim_{n \to +\infty} \frac{U_0}{\sqrt{2}}\left(\frac{r_n}{U_0}\right)^{\frac{3}{2}}\left(\frac{U_0}{h_n r_n}\right)^{\frac{3}{2}}E\left(\frac{h_n r_n}{U_0}\right) = +\infty$$

that gives a contradiction and proves (38). Writing now (36) as

$$1 + \varepsilon = \frac{r}{\sqrt{2h}} \left(\frac{U_0}{hr} \right) E \left(\frac{hr}{U_0} \right),$$

using the second of (37) we obtain the following estimates

$$h = h(0, r; 1 + \varepsilon) = \frac{1}{2} \left(\frac{r}{1 + \varepsilon} \right)^2 + o_r(r^2) \tag{39}$$

as $r \to +\infty$, uniformly on $\varepsilon \in [0, \overline{\varepsilon}]$. Let us consider now the action $S(0, r; 1 + \varepsilon)$. Let $t \mapsto u(t)$ be the solution joining 0 with r in time $1 + \varepsilon$. We have

$$\begin{aligned} S(0, r; 1 + \varepsilon) &= \int_0^{1+\varepsilon} \left(\frac{\dot{u}^2}{2} + \frac{U_0}{u} \right) dt = \int_0^r \frac{h + \frac{2U_0}{u}}{\sqrt{2 \left(h + \frac{U_0}{u} \right)}} du \\ &= \sqrt{2h} \int_0^r \sqrt{1 + \frac{U_0}{hu}} du - \sqrt{\frac{h}{2}} \int_0^r \frac{du}{\sqrt{1 + \frac{U_0}{hu}}} \\ &= \frac{U_0}{\sqrt{h}} \left(\sqrt{2} F \left(\frac{hr}{U_0} \right) - \frac{1}{\sqrt{2}} E \left(\frac{hr}{U_0} \right) \right), \end{aligned} \tag{40}$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$F(x) = \int_0^x \sqrt{\frac{s+1}{s}} ds, \qquad x \geq 0.$$

The function F verifies the asymptotic estimates

$$F(x) = x + o(x), \qquad x \to +\infty. \tag{41}$$

Replacing (39) in (40) we find (35).

Proposition 20 *Let $A > 0$ and $B > 0$ be two constants. If we set*

$$\alpha_0 = (8U_0 \alpha)^{\frac{1}{2}} \quad \text{and} \quad \beta_0 = (8U_0)^{\frac{1}{2}}$$

then we have

$$S(r, \alpha(s^{\frac{2}{3}} + \xi); s + \eta) = \alpha_0 s^{\frac{2}{3}} - \beta_0 r^{\frac{1}{2}} + o_s(1) \tag{42}$$

as $s \to +\infty$, uniformly on $r \in [0, s^{1/3}]$, $|\xi| \leq A$ and $|\eta| \leq B$.

Proof. We first prove that the (unique) solution joining r to $\alpha(s^{2/3} + \xi)$ in time $s + \eta$ is monotonic. In order to simplify the exposition let us term $\lambda(\xi, s) = \alpha(s^{2/3} + \xi)$. We shall compare $s + \eta$ with the time employed by the solution of energy

$-\frac{U_0}{\lambda(\xi,s)}$ to connect r to $\lambda(\xi,s)$. As usual we denote $\overline{s}(r,\lambda(\xi,s))$ this time. By definition of α we have

$$
\begin{aligned}
\overline{s}(r,\lambda(\xi,s)) &= \int_r^{\lambda(\xi,s)} \frac{du}{\sqrt{2\left(-\frac{U_0}{\lambda(\xi,s)} + \frac{U_0}{u}\right)}} \\
&= \frac{\lambda(\xi,s)^{3/2}}{(2U_0)^{1/2}} \int_{\frac{r}{\lambda(\xi,s)}}^1 \frac{dv}{\sqrt{\frac{1}{v}-1}} \\
&= \frac{3s}{2}\left(1 + \frac{\xi}{s^{2/3}}\right)^{3/2}\left(\frac{\pi}{2} - H\left(\frac{r}{\lambda(\xi,s)}\right)\right),
\end{aligned}
$$

where we define

$$
H : \mathbb{R}_+ \to \mathbb{R}, \qquad H(x) = \int_0^x \sqrt{\frac{v}{1-v}}\,dv.
$$

Since we assume

$$
0 \le r \le s^{1/3} \qquad \text{and} \qquad |\xi| \le A
$$

we have

$$
\frac{r}{\lambda(\xi,s)} \to 0 \qquad \text{as} \qquad s \to +\infty.
$$

An easy computation shows that

$$
H(x) = \frac{2}{3}x^{3/2} + \mathcal{O}(x^{5/2}), \qquad x \to 0,
$$

hence we get the estimates

$$
\overline{s}(r,\lambda(\xi,s)) = \frac{3\pi}{4}s\left(1 + \mathcal{O}_s(s^{-1/2})\right).
$$

Since $\frac{3\pi}{4} > 1$, we have $\overline{s}(r,\lambda(\xi,s)) > s + \eta$ for s sufficiently great, and by Lemma 16 the solution joining r to $\lambda(\xi,s)$ in time $s + \eta$ is monotonic.

Let $h = h(r,\lambda(\xi,s);s+\eta)$ be the energy of the solution joining r to $\lambda(\xi,s)$ in time $s + \eta$. We prove that $h = o_s(1/s)$ for $s \to +\infty$, uniformly on $0 \le r \le s^{1/3}$, $|\xi| \le A$ and $|\eta| \le B$. The energy h satisfies the identity

$$
\begin{aligned}
s + \eta &= \int_r^{\lambda(\xi,s)} \frac{du}{\sqrt{2\left(h + \frac{U_0}{u}\right)}} \\
&= \frac{\lambda(\xi,s)^{3/2}}{(2U_0)^{1/2}} \int_{\frac{r}{\lambda(\xi,s)}}^1 \frac{dv}{\sqrt{\frac{\lambda(\xi,s)}{U_0}h + \frac{1}{v}}}
\end{aligned}
\tag{43}
$$

Introducing the functions

$$x(r,s,\xi) = \left(\frac{r}{\lambda(\xi,s)}\right)^{1/2}, \qquad\qquad y(s,\xi) = \frac{\xi}{s^{2/3}},$$

(44)

$$k(r,s,\xi,\eta) = \frac{\lambda(\xi,s)}{U_0}h(r,\lambda(\xi,s);s+\eta), \quad z(s,\eta) = \frac{\eta}{s},$$

and using the definition of α, the relation (43) becomes

$$F(x(r,s,\xi),y(s,\xi),z(s,\eta),k(r,s,\xi,\eta)) = 0,$$

(45)

where $F(x,y,z,k)$ is defined by

$$F(x,y,z,k) = \int_{x^2}^{1} \left(\frac{v}{1+kv}\right)^{1/2} dv - \frac{2}{3}(1+z)(1+y)^{-3/2}.$$

We think now at (x,y,z,k) as independent variables. Using the implicit function theorem we show that the equation

$$F(x,y,z,k) = 0$$

(46)

defines a unique $\mathscr{C}^2$ function $k = k(x,y,z)$ for (x,y,z) close to $(0,0,0)$. We observe that $F(x,y,z,k)$ is of class $\mathscr{C}^2$ with respect to the variables y and z. Moreover F is derivable with respect to x and

$$\frac{\partial F}{\partial x}(x,y,z,k) = -\frac{2x|x|}{(1+kx^2)^{1/2}}, \qquad \frac{\partial F}{\partial x}(0,0,0,0) = 0.$$

$\frac{\partial F}{\partial x}$ is derivable with respect to x and k, and we have

$$\frac{\partial^2 F}{\partial x^2}(x,y,z,k) = -\frac{2|x|(2+kx^2)}{(1+kx^2)^{3/2}}$$

(47)

$$\frac{\partial^2 F}{\partial k\partial x}(x,y,z,k) = \frac{x^3|x|}{(1+kx^2)^{3/2}},$$

showing that $\frac{\partial F}{\partial x}$ is of class $\mathscr{C}^1$ in a neighborhood of $(0,0,0,0)$. In particular

$$\frac{\partial^2 F}{\partial x^2}(0,0,0,0) = 0, \qquad \frac{\partial^2 F}{\partial k\partial x}(0,0,0,0) = 0.$$

By the theorem of differentiation under the integral sign, $\frac{\partial F}{\partial k}$, $\frac{\partial^2 F}{\partial k^2}$ and $\frac{\partial^2 F}{\partial x\partial k}$ are well defined, moreover

$$\frac{\partial F}{\partial k}(x,y,z,k) = -\frac{1}{2}\int_{x^2}^{1}\left(\frac{v}{1+kv}\right)^{3/2} dv, \quad \frac{\partial F}{\partial k}(0,0,0,0) = -\frac{1}{5},$$

$$\frac{\partial^2 F}{\partial k^2}(x,y,z,k) = \frac{3}{4}\int_{x^2}^{1}\left(\frac{v}{1+kv}\right)^{5/2} dv, \quad \frac{\partial^2 F}{\partial k^2}(0,0,0,0) = \frac{3}{14},$$

$$\frac{\partial^2 F}{\partial x\partial k}(x,y,z,k) = \frac{x^3|x|}{(1+kx^2)^{3/2}}, \qquad\qquad \frac{\partial^2 F}{\partial x\partial k}(0,0,0,0) = 0.$$

By the way, we have also

$$\frac{\partial^2 F}{\partial k \partial y}(x,y,z,k) = \frac{\partial^2 F}{\partial k \partial z}(x,y,z,k) = 0.$$

These computations show that F is of class $\mathscr{C}^2$ in a neighborhood of $(0,0,0,0)$. Moreover

$$F(0,0,0,0) = \int_0^1 \sqrt{v}\,dv - \frac{2}{3} = 0.$$

By the implicit function theorem, equation (46) defines a $\mathscr{C}^2$ function $k = g(x,y,z)$ in a neighborhood of $(0,0,0)$ such that $g(0,0,0) = 0$ and

$$\frac{\partial g}{\partial x}(0,0,0) = \frac{\partial^2 g}{\partial x^2}(0,0,0) = \frac{\partial^2 g}{\partial x \partial y}(0,0,0) = \frac{\partial^2 g}{\partial x \partial z}(0,0,0) = 0,$$

that is to say

$$g(x,y,z) = \mathscr{O}(|y| + |z|) + o(x^2 + y^2 + z^2). \tag{48}$$

Coming back to original variables, identity (48) gives

$$h(r, \lambda(\xi,s); s+\eta) = \frac{U_0}{\lambda(\xi,s)} g\left(\left(\frac{r}{\lambda(\xi,s)} \right)^{1/2}, \frac{\xi}{s^{2/3}}, \frac{\eta}{s} \right)$$

$$= o_s(1/s), \tag{49}$$

as $s \to +\infty$, uniformly on $0 \le r \le s^{\frac{1}{3}}$, $|\xi| \le A$, and $|\eta| \le B$. We compute now the action $S(r, \lambda(\xi,s); s+\eta)$. Since the solution joining r to $\lambda(\xi,s)$ in time $s+\eta$ (denoted here $t \mapsto u(t)$) is monotonic, we have

$$S(r, \lambda(\xi,s); s+\eta) = \int_0^{s+\eta} \left(\frac{\dot{u}^2(t)}{2} + \frac{U_0}{u(t)} \right) dt$$

$$= \int_r^{\lambda(\xi,s)} \frac{h + \frac{2U_0}{u}}{\sqrt{2\left(h + \frac{U_0}{u}\right)}} du$$

$$= \int_r^{\lambda(\xi,s)} \sqrt{2\left(h + \frac{U_0}{u}\right)} \, du - (s+\eta)h.$$

Introducing the integration variable $v = \frac{u}{\lambda(\xi,s)}$, by (49) we find

$$S(r, \lambda(\xi,s); s+\eta) = (2U_0 \lambda(\xi,s))^{\frac{1}{2}} A(x,k) + o_s(1), \tag{50}$$

where $x = x(r,s,\xi)$ and $k = k(r,s,\xi,\eta)$ are the functions defined like in (44) and

$$A(x,k) = \int_{x^2}^1 \sqrt{k + \frac{1}{v}} \, dv = A_0(k) - B(x,k),$$

where

$$A_0(k) = \int_0^1 \sqrt{k + \frac{1}{v}}\, dv, \qquad B(x,k) = \int_0^{x^2} \sqrt{k + \frac{1}{v}}\, dv.$$

Once again, we think at x and k as independent variables and we give an asymptotic expansion of $A(x,k)$ for x and k close to 0. By the classical theorem of differentiation under the integral sign, $A_0(k)$ is derivable in 0 and

$$A_0(k) = 2 + \frac{k}{3} + o(k).$$

Moreover we have the following estimates for $B(x,k)$

$$B(x,k) = \int_0^{x^2} \frac{dv}{\sqrt{v}} + \int_0^{x^2} \left(\sqrt{k + \frac{1}{v}} - \sqrt{\frac{1}{v}} \right) dv$$

$$= 2|x| + k \int_0^{x^2} \frac{\sqrt{v}}{\sqrt{1 + kv} + 1}\, dv$$

$$= 2|x| + \mathcal{O}(k|x|^3),$$

hence

$$A(x,k) = 2 + \frac{k}{3} - 2|x| + \mathcal{O}(k|x|^3) + o(k),$$

as $x \to 0$ and $k \to 0$. Replacing in (50) and using (49) we find the final estimates (42).

The two previous Propositions imply the following one.

Proposition 21 *Given $\overline{\varepsilon} > 0$, we have*

$$\lim_{\substack{s \to +\infty \\ r \to +\infty}} \mathcal{N}(r, \alpha s^{\frac{2}{3}}; \varepsilon, 1, s) = +\infty,$$

uniformly on $\varepsilon \in [0, \overline{\varepsilon}]$, where $\mathcal{N}$ is the function defined in (9).

Proof. If $0 \leq \varepsilon \leq \overline{\varepsilon}$ and $0 \leq r \leq s^{\frac{1}{3}}$, from Propositions (19) and (20) we have :

$$\mathcal{N}(r, \alpha s^{\frac{2}{3}}; \varepsilon, 1, s) = \frac{r^2}{2(1+\varepsilon)}(1 + o_r(1)) - \beta_0 r^{\frac{1}{2}} + o_s(1),$$

therefore

$$\lim_{\substack{r \to +\infty \\ 0 \leq r \leq s^{\frac{1}{3}}}} \mathcal{N}(r, \alpha s^{\frac{2}{3}}; \varepsilon, 1, s) = +\infty, \qquad (51)$$

uniformly on $\varepsilon \in [0, \overline{\varepsilon}]$. Let us consider now the case $r \geq s^{\frac{1}{3}}$. Forgetting the term $S(r, \alpha s^{\frac{2}{3}}; s - 1)$ in $\mathcal{N}(r, \alpha s^{\frac{2}{3}}; \varepsilon, 1, s)$ and applying again Propositions (19) and (20) we find

$$\mathcal{N}(r, \alpha s^{\frac{2}{3}}; \varepsilon, 1, s) \geq \frac{r^2}{2(1+\varepsilon)}(1 + o_r(1)) - \alpha_0 s^{\frac{1}{3}} + o_s(1)$$

$$\geq \frac{s^{\frac{2}{3}}}{2(1+\varepsilon)}(1 + o_s(1)) - \alpha_0 s^{\frac{1}{3}} + o_s(1).$$

This estimates implies the limit

$$\lim_{\substack{s \to +\infty \\ r \geq s^{\frac{1}{3}}}} \mathcal{N}(r, \alpha s^{\frac{2}{3}}; \varepsilon, 1, s) = +\infty, \tag{52}$$

uniformly on $\varepsilon \in [0, \overline{\varepsilon}]$.

The two limits (51) and (52) achieve a proof of the Proposition.

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Instituto de Matemática y Estadística "Prof. Rafael Laguardia",
Universidad de la República,
Herrera y Reissig 565, 12000 Montevideo,
Uruguay.
e-mail: eze@fing.edu.uy

and

Laboratoire d'Analyse non linéaire et Géometrie,
Université d'Avignon et des pays de Vaucluse,
33, Rue Louis Pasteur - 84000, Avignon,
France.
e-mail: Andrea.Venturelli@univ-avignon.fr