Conley barriers and their applications: chain-recurrence and Lyapunov functions *

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Abstract

Given a compact metric space X and a continuous map f from X to itself, we construct a barrier function for chain-recurrence. We use it to endow the space of chain-transitive components with a non-trivial ultrametric distance and to construct Lyapunov functions for f. Most of these constructions are then generalized on an arbitrary separable metric space to a continuous compactum-valued map.

1 Introduction

The purpose of this paper is to shed a different light on chain-recurrence for dynamical systems on arbitrary separable metric space. The initial work of Conley [Con78] describes the structure of chain-recurrent points in terms of attractors of f and their basins of attraction. It is in line with the theory of dynamical systems done in the last fifty years, see for example [Shu87]. The work of Conley is surveyed by Hurley [Hur92, Hur98] where it is extended to the settings of arbitrary separable metric space. Moreover, in this work Hurley constructs a type of Lyapunov function which gives a good insight in the structure of chain-recurrent points. Here is a statement.

Theorem 1.1. *Let X be a separable metric space and f be a continuous map from X to itself. Then there exists a continuous function $\phi : X \longrightarrow \mathbb{R}$ such that*

i) *The function ϕ is nonincreasing along orbits of f and is decreasing along orbits of non chain-recurrent points.*

ii) *The function ϕ takes on distinct values on distinct chain-transitive components and sends the set of chain-recurrent points in a subset of the Cantor middle-third set.*

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The point of view taken in this paper is different and is inspired by the recent work of Fathi [Fat] in Weak KAM theory. We will associate a cost to chains in order to construct a barrier function, called a Conley barrier. Here are its main properties.

Theorem 1.2. *Let X be a compact metric space and f be a continuous map from X to itself. Then there exists a continuous function*

$$S : X \times X \longrightarrow \mathbb{R}_+$$

such that

i) *For every $(x, y) \in X^2$, we have $S(x, y) = 0$ if and only if for every $\varepsilon > 0$ there exists an ε-chain from x to y.*

ii) *For every $(x, y, z) \in X^3$, we have $S(x, y) \leq \max(S(x, z), S(z, y))$.*

The existence of such a barrier allows to describe chain-recurrence only in terms of continuous functions. Moreover, the ultrametric inequality satisfied by S will induce a non-trivial ultrametric distance on the set of chain-transitive components. Last, the nonincreasing along orbits of f offers a fundamental starting point towards the construction of Lyapunov functions for f. This will lead to a similar result as Hurley's one, at least in the case of a separable locally compact metric space.

For the sake of clarity, the first part of this paper is devoted to the compact case. Nevertheless, the compactness assumption is not essential to obtain a Conley barrier. This is the object of the second section. Moreover, we will deal with compactum-valued maps since this does not raise any new difficulty. Finally we highlight the link between chain-recurrence for the identity map on X and topological properties of X.

2 The compact case

2.1 Definitions and background

Throughout this section (X, d) will denote a compact metric space and f a continuous map from X to itself.

Definition 2.1. Let $(x, y) \in X^2$ and $\varepsilon > 0$. An *ε-chain* for f from x to y is a finite sequence $(x_0 = x, ..., x_n = y)$, $n \geq 1$, of X such that

$$\forall i \in \{0, .., n-1\},\ d(f(x_i), x_{i+1}) < \varepsilon.$$

A point x in X is called *chain-recurrent* if for every $\varepsilon > 0$ there exists an ε-chain from x to x. We denote by $\mathcal{R}(f)$ the set of *chain-recurrent points* of f. We define an equivalence relation $\backsim$ on the set $\mathcal{R}(f)$ by $x \backsim y$ if and only if for every $\varepsilon > 0$ there are ε-chains from x to y and from y to x. The equivalence classes are called the *chain-transitive components* of f and the associated quotient space is denoted by $\mathcal{R}(f)/\backsim$.

It would be straightforward to verify that these notions are topological and do not depend on the metric d on X. In fact, it will be made clear in section 3. We now describe the main object of this paper.

Definition 2.2. Let X be a compact metric space and f be a continuous map from X to itself. A *Conley barrier* for f is a continuous function

$$S : X \times X \longrightarrow \mathbb{R}_+$$

with the properties that

i) *For every $(x, y) \in X^2$, we have $S(x, y) = 0$ if and only for every $\varepsilon > 0$ there exists an ε-chain from x to y.*

ii) *For every $(x, y, z) \in X^3$, we have $S(x, y) \leq \max(S(x, z), S(z, y))$.*

With respect to property i) any Conley barrier is in fact a barrier for chain-recurrence. The following simple remark will be used many time.

Remark 2.3. For every $x \in X$ and $\varepsilon > 0$, the chain $(x, f(x))$ is always an ε-chain from x to $f(x)$. Thus we have $S(x, f(x)) = 0$ everywhere on X.

As stated in the following theorem, we can always find a Conley barrier for dynamical systems on compact metric space.

Theorem 2.4. *Let X be a compact metric space and f be a continuous map from X to itself. Then there exists a Conley barrier for f.*

Proof. The proof of this theorem will be done in section 2.4. □

Corollary 2.5. *The set $\mathcal{R}(f)$ is a closed subset of X.*

Proof. It follows from property i) that $\mathcal{R}(f) = \{x \in X,\ S(x, x) = 0\}$. Since S is continuous, this set is a closed subset of X. □

Proposition 2.6. *The subset $\mathcal{R}(f)$ and the chain-transitive components are invariant under f.*

Proof. First, we will show that

$$\forall x \in \mathcal{R}(f), S(f(x), x) = 0.$$

Let $x \in \mathcal{R}(f)$. If $f(x) = x$, there is nothing to prove. Therefore, we can assume that $d(f(x), x) > 0$. Let $\varepsilon > 0$ and consider $\eta > 0$ such that $\eta < \min(d(f(x), x), \frac{\varepsilon}{2})$. Since x is chain-recurrent, there exists a η-chain $(x_0 = x, ..., x_m = x)$ from x to x. The condition $\eta < d(f(x), x)$ forces $m \geq 2$. By continuity of f, reducing even more η if necessary, we can also assume that $f(B(f(x), \eta)) \subset B(f^2(x), \frac{\varepsilon}{2})$. The chain $(f(x), x_2, ..., x_m = x)$ is then an ε-chain from $f(x)$ to x. Since ε is arbitrary, it follows that $S(f(x), x) = 0$.

Now if $x \in \mathcal{R}(f)$ then $S(f(x), f(x)) \leq \max(S(f(x), x), S(x, f(x))) = 0$ by remark 2.3. Thus $S(f(x), f(x)) = 0$ and $f(x) \in \mathcal{R}(f)$. Moreover, since $S(x, f(x)) = S(f(x), x) = 0$ the points x and $f(x)$ are in the same chain-transitive component. Thus the subset $\mathcal{R}(f)$ and the chain-transitive components are invariant under f.

□

Before making S explicit, we are going to develop two consequences: an ultrametric distance on the set of chain-transitive components, and the existence of Lyapunov functions for f.

2.2 An ultrametric distance on the space of chain-transitive components

Pseudo-distance In this section, we recall some general facts about pseudo-distances. They will be used to endow the space of chain-transitive components with an ultrametric distance.

Definition 2.7. A *pseudo-distance* on a space E is a function

$$d : E \times E \longrightarrow \mathbb{R}_+$$

such that

i) *For every $x \in E$, we have $d(x, x) = 0$.*

ii) *For every $x, y, z \in E$, we have $d(x, y) \leq d(x, z) + d(z, y)$.*

iii) *For every $x, y \in E$, we have $d(x, y) = d(y, x)$.*

Let d be a pseudo-distance on E. We define an equivalence relation $\mathcal{R}$ on E by

$$x\mathcal{R}y \Longleftrightarrow d(x, y) = 0.$$

We denote by $E/\mathcal{R}$ the set of associated equivalence classes. The following lemma is well-known so we omit its proof.

Lemma 2.8. *The pseudo-distance d induces a distance $\overline{d}$ on the quotient space $E/\mathcal{R}$. Moreover, if the space E is endowed with a topology making d continuous, then the quotient topology is finer than the topology defined by the metric $\overline{d}$.*

Remark 2.9. In the lemma above, if the pseudo-distance d satisfies the stronger *ultrametric* inequality

$$d(x, y) \leq \max(d(x, z), d(z, y))$$

then the distance $\overline{d}$ inherits of the same property and thus defines an *ultrametric distance* on the quotient space $E/\mathcal{R}$.

Ultrametric distance induced by a Conley barrier on the set of chain-transitive components The existence of a Conley barrier leads to the existence of a non-trivial ultrametric distance on the set of chain-transitive components. To see this, let us remark that the equivalence relation $\backsim$ defined on the set of chain-transitive components can be formulated in the following way

$$x \backsim y \Longleftrightarrow \max(S(x,y), S(y,x)) = 0.$$

The quantity

$$\Delta(x,y) := \max(S(x,y), S(y,x))$$

is a symmetric expression in x and y and inherits of the ultrametric inequality satisfied by S. Thus, on the subset $\mathcal{R}(f) = \{x \in X,\ \Delta(x,x) = 0\}$ the function Δ is satisfying all axioms of an ultrametric pseudo-distance. As described in the previous section, it naturally induces an ultrametric distance $\overline{\Delta}$ on the quotient space $\mathcal{R}(f)/\backsim$, i.e. on the space of chain-transitive components.

Corollary 2.10. *Let X be a compact metric space and f be a continuous map from X to itself. Then the set of chain-transitive components with the quotient topology is a compact ultrametric space. We can take as a metric any ultrametric distance induced by a Conley barrier for f. In particular, this set is totally disconnected and Hausdorff.*

Proof. The set of chain-recurrent points is closed in X and hence compact. Since the canonical projection

$$\mathcal{R}(f) \xrightarrow{p} (\mathcal{R}(f)/\backsim, \textit{quotient topology})$$

is continuous, the space $(\mathcal{R}(f)/\backsim, \textit{quotient topology})$ is also compact.

Let $\overline{\Delta}$ introduced above be an ultrametric distance induced by a Conley barrier on the set of chain-transitive components of f. Since Δ is continuous, it follows from lemma 2.8 that the quotient topology is finer than the ultrametric topology induced by $\overline{\Delta}$. Thus, in the following diagram the identity map

$$(\mathcal{R}(f)/\backsim, \textit{quotient topology}) \xrightarrow{Id} \left(\mathcal{R}(f)/\backsim,\ \overline{\Delta}\right)$$

is a continuous bijection. Since the metric space $\left(\mathcal{R}(f)/\backsim, \overline{\Delta}\right)$ is Hausdorff, the same goes for $(\mathcal{R}(f)/\backsim, \textit{quotient topology})$. This set is thus a compact Hausdorff space. The identity map is then an homeomorphism and both topologies are the same. Since for an ultrametric distance every open ball is also closed, the set of chain-transitive components is totally disconnected. □

2.3 Lyapunov functions

Definitions. We can use a Conley barrier to construct different types of Lyapunov functions for f. The following definition is used by Hurley, see [Hur92, Hur98]. For general recalls about Hausdorff dimension, see [HW41]

Definition 2.11. A *strict* Lyapunov function for f is a continuous function $\varphi : X \longrightarrow \mathbb{R}$ such that

i) *For every $x \in X$, we have $\varphi(f(x)) \leq \varphi(x)$.*

ii) *For every $x \in X \setminus \mathcal{R}(f)$, we have $\varphi(f(x)) < \varphi(x)$.*

A strict Lyapunov function is said to be *complete* if it satisfies the following additional property

i') *The function φ is constant on each chain-transitive component, takes on distinct values on distinct chain-transitive components and sends the subset $\mathcal{R}(f)$ into a subset of $\mathbb{R}$ whose Hausdorff dimension is zero.*

Our construction of Lyapunov functions will use a particular kind of functions, called sub-solutions for S. Here is the definition.

Definition 2.12. Let S be a Conley barrier for f. A *sub-solution* for S is a continuous function

$$u : X \longrightarrow \mathbb{R}$$

such that

$$\forall (x, y) \in X^2, \ u(y) - u(x) \leq S(x, y).$$

A sub-solution is said to be *strict* if the inequality is strict as soon as x is not chain-recurrent for f.

Lemma 2.13. *Any sub-solution for S is nonincreasing along orbits of f and any strict sub-solution is decreasing along orbits of non chain-recurrent points. Thus any strict sub-solution for S is a strict Lyapunov function for f.*

Proof. The proof follows from definitions and remark 2.3. □

The following lemma gives a fundamental example of sub-solutions.

Lemma 2.14. *For every z in X, the function*

$$\begin{aligned} S_z : \ & X \longrightarrow \mathbb{R} \\ & x \longmapsto S(z, x) \end{aligned}$$

is a sub-solution for S.

Proof. Since a Conley barrier satisfies an ultrametric inequality, it also satisfies the triangle inequality. Thus for every x, y in X we have

$$S(z, y) \leq S(z, x) + S(x, y)$$

which yields the wanted inequality. □

Strict Lyapunov functions. We now construct a strict Lyapunov function for f. We will see later how sub-solutions of the type S_x can in fact be used to construct a complete Lyapunov function for f.

Theorem 2.15. *Let X be a compact metric space and f be a continuous map from X to itself. There exists a sequence $(x_i)_{i\in\mathbb{N}}$ of points of X and a sequence $(\eta_i)_{i\in\mathbb{N}}$ of positive reals such that the series*

$$\varphi = \sum_{i\in\mathbb{N}} \eta_i S_{x_i}$$

is a strict sub-solution for S, and thus a strict Lyapunov function for f.

Proof. Since the metric space X is compact, it is separable. Let $(x_i)_{i\in\mathbb{N}}$ be a dense sequence in X and $(\eta_i)_{i\in\mathbb{N}}$ be a sequence of positive reals such that $\sum_{i\in\mathbb{N}} \eta_i = 1$. The continuous function S is bounded on the compact set $X \times X$. Thus, the condition $\sum_{i\in\mathbb{N}} \eta_i = 1$ insures that the series $\sum_{i\in\mathbb{N}} \eta_i S_{x_i}$ converges uniformly on X. Hence, it defines a continuous function φ on X. Moreover, the function φ is a sub-solution since a convex combination of sub-solutions is still a sub-solution. Now suppose that $x \in X$ is not chain-recurrent. Then we have $S(x,x) > 0$ and thus $S(x,y)-S(x,x) < S(x,y)$. By density of the $(x_i)_{i\in\mathbb{N}}$ and continuity of S, we can find an integer $j \in \mathbb{N}$ such that $S(x_j,y) - S(x_j,x) < S(x,y)$. Since the functions S_{x_i} are sub-solutions, we always have

$$\forall i \in \mathbb{N},\ S(x_i,y) - S(x_i,x) \le S(x,y)$$

it follows that

$$\begin{aligned} \varphi(y) - \varphi(x) &= \sum_{i\in\mathbb{N}} \eta_i (S(x_i,y) - S(x_i,x)) \\ &< \sum_{i\in\mathbb{N}} \eta_i S(x,y) = S(x,y) \end{aligned}$$

Thus the function φ is a strict sub-solution for S and hence, a strict Lyapunov function for f. □

Complete Lyapunov function The construction of a complete Lyapunov function for f relies on the underlying ultrametric structure of the set of chain-transitive components. It strongly limits values taken by the sub-solutions $S_x, x \in X$ and will lead to functions with images of finite cardinality. The following lemma and corollary are thus fundamental.

Lemma 2.16. *For every $x \in X$, the function S_x is constant in the neighborhood of each point of the set $\mathcal{R}(f) \setminus \{S(x,\cdot) = 0\}$.*

Proof. Let $x \in X$ and $y \in \mathcal{R}(f)$ be such that $S(x,y) > 0$. Consider the open subset $U_{x,y}$ of X

$$U_{x,y} = \{S(y,\cdot) - S(x,y) < 0\} \cap \{S(\cdot,y) - S(x,\cdot) < 0\}.$$

Since $y \in \mathcal{R}(f)$ we have $S(y,y) = 0$ and thus $y \in U_{x,y}$. If $z \in U_{x,y}$ we have

$$S(x,z) \leq \max(S(x,y), S(y,z)) = S(x,y),$$

$$S(x,y) \leq \max(S(x,z), S(z,y)) = S(x,z).$$

Thus $S(x,z) = S(x,y)$ and S_x is constant on $U_{x,y}$. □

Corollary 2.17. *For every $x \in X$, the set $\{S(x,y),\ y \in \mathcal{R}(f)\}$ is countable. Moreover, the only possible accumulation point is zero. In particular, any function of the form $\theta \circ S_x$, where $\theta : \mathbb{R} \to \mathbb{R}$ is constant in a neighbourhood of zero, takes on a finite number of values on $\mathcal{R}(f)$.*

Proof. Let $(x_i)_{i\in\mathbb{N}}$ be a dense sequence in X. Let $x \in X$. At each point of $\mathcal{R}(f)$, the function S_x is either 0 or constant in a neighborhood of that point. Thus, the set $\{S(x,y),\ y \in \mathcal{R}(f)\}$ is included in the set $\{S(x,x_j),\ j \in \mathbb{N}\} \cup \{0\}$ and hence is countable.

Now let α be an accumulation point of the set $\{S(x,y),\ y \in \mathcal{R}(f)\}$. There exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathcal{R}(f)$ such that the sequence $(S(x,y_n))_{n\in\mathbb{N}}$ admits α as a limit with $S(x,y_n) \neq \alpha$, for every $n \in \mathbb{N}$. By compactness of X, we can suppose that y_n admits a limit $y \in X$. Since the set $\mathcal{R}(f)$ is closed, we have $y \in \mathcal{R}(f)$ and the continuity of S implies that $\alpha = S(x,y)$. If α is non zero then S_x would be constant in the neighborhood of y. This would contradicts the fact that for every $n \in \mathbb{N}, S(x,y_n) \neq \alpha$. Thus α is zero.

□

We can now prove the existence of a complete Lyapunov function for f.

Theorem 2.18. *Let X be a compact metric space and f be a continuous map from X to itself. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X, a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals and a sequence $(\theta_n)_{n\in\mathbb{N}}$ of real-valued functions such that the series*

$$\varphi = \sum_{n\in\mathbb{N}} \varepsilon_n \theta_n \circ S_{x_n}$$

defines a complete Lyapunov function for f.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in X. Repeating each x_n infinitely many times, we can suppose without lost of generality that for every $k \in \mathbb{N}$ the sequence $(x_n)_{n\geq k}$ is still dense in X. Now for every $n \in \mathbb{N}$ we set

$$\theta_n(t) := \max(t - \frac{1}{n+1}, 0).$$

Each function θ_n is zero in the neighbourhood of zero. It thus follows from corollary 2.17 that for every $n \in \mathbb{N}$ the function

$$\theta_n \circ S_{x_n} : X \longrightarrow \mathbb{R}$$

takes on a finite number of values on $\mathcal{R}(f)$. It easily follows from the ultrametric inequality satisfied by S and the definition of the relation $\backsim$ on the space of chain-recurrent points

$$x \backsim y \Longleftrightarrow \max(S(x,y), S(y,x)) = 0$$

that the functions S_x for x in X are constant on each chain-transitive components. Thus each function $\theta_n \circ S_{x_n} : X \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, induces a function $\overline{\theta_n \circ S_{x_n}}$ on the set of chain-transitive components with an image of finite cardinality. We will now apply lemma 5.1 of the Appendix to the space $A = \mathcal{R}(f)/\backsim$ together with the family $\left(\overline{\theta_n \circ S_{x_n}}\right)_{n\in\mathbb{N}}$. We just have to prove that this family separates chain-transitive components. If x and y are in distinct chain-transitive components, we have for example $S(x,y) > 0$. Since $S(x,x) = 0$, the continuity of S and the density of the $(x_n)_{n\geq k}$ for every $k \in \mathbb{N}$, implies that we can find an integer $n \in \mathbb{N}$ such that $0 \leq S(x_n, x) < S(x_n, y) - \frac{1}{n+1}$. Hence we have $\overline{\theta_n \circ S_{x_n}}(x) < \overline{\theta_n \circ S_{x_n}}(y)$. We conclude similarly if $S(y,x) > 0$.

Thus, lemma 5.1 furnishes a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals such that the series $\sum_{n\in\mathbb{N}} \varepsilon_n \overline{\theta_n \circ S_{x_n}}$ converges on $\mathcal{R}(f)/\backsim$, separates points of $\mathcal{R}(f)/\backsim$ and has an image in $\mathbb{R}$ whose Hausdorff dimension is zero. Each continuous functions $\theta_n \circ S_{x_n}$ is bounded on the compact set X. Since the positive reals $(\varepsilon_n)_{n\in\mathbb{N}}$ can be chosen arbitrarily small, we can also suppose that the non-negative series

$$\varphi = \sum_{n\in\mathbb{N}} \varepsilon_n \theta_n \circ S_{x_n}$$

converges uniformly on X. The fact that the series $\sum_{n\in\mathbb{N}} \varepsilon_n \overline{\theta_n \circ S_{x_n}}$ separates points of $\mathcal{R}(f)/\backsim$ and has an image in $\mathbb{R}$ whose Hausdorff dimension vanishes precisely means that the function φ takes on distinct values on distinct chain-transitive components and sends $\mathcal{R}(f)$ in a subset of $\mathbb{R}$ whose Hausdorff dimension is zero.

To complete the proof, we just have to show that φ is nonincreasing along orbits of f and decreasing along orbits of non chain-recurrent points. The first part is true since for every $x \in X$ the sub-solution S_x is nonincreasing along orbits of f and each θ_n is monotonic. Now if $x \in X \setminus \mathcal{R}(f)$, we have $S(x,x) > 0$. Since $S(x, f(x)) = 0$ and $(x_n)_{n\geq k}$ is dense for every $k \in \mathbb{N}$, we can find $n \in \mathbb{N}$ such that

$$0 \leq S(x_n, f(x)) < S(x_n, x) - \frac{1}{n+1}.$$

Thus we have $\theta_n \circ S_{x_n}(f(x)) < \theta_n \circ S_{x_n}(x)$ so that $\varphi(f(x)) < \varphi(x)$. $\square$

2.4 Conley barrier

We now come to the construction of a Conley barrier. As a cost for chain, we will consider the maximum of the size of the different jumps. This leads to the following.

Definition 2.19. For every $(x,y) \in X^2$, we set

$$S(x,y) := \inf \left\{ \max_{i\in\{0,..,n-1\}} d(f(x_i), x_{i+1}) \mid n \geq 1,\ x_0 = x, ..., x_n = y \right\}.$$

We now prove that the function S is a Conley barrier for f.

Lemma 2.20. *The function S satisfies the barrier property: for every (x,y) in X^2 we have $S(x,y) = 0$ if and only if for every $\varepsilon > 0$ there exists an ε-chain from x to y.*

Proof. The property becomes clear with the following equivalent definition of S

$$S(x,y) = \inf\{\varepsilon > 0 \mid \text{there exists an } \varepsilon\text{-chain from } x \text{ to } y\}.$$

□

Lemma 2.21. *The function S satisfies the ultrametric inequality*

$$\forall (x,y,z) \in X^3,\ S(x,y) \leq \max(S(x,z), S(z,y)).$$

Proof. Let $x,y,z \in X$ and $(x_0 = x, ..., x_n = z)$, $(z_0 = z, ..., z_m = y)$ be two chains from x to z and from z to y. The concatenated chain provides a chain $(y_0 = x, ..., y_{m+n+1} = y)$ from x to y and thus

$$\begin{aligned} S(x,y) &\leq \max_{j\in\{0,..,m+n\}} d(f(y_j), y_{j+1}) \\ &\leq \max\left(\max_{i\in\{0,..,n-1\}} d(f(x_i), x_{i+1}), \max_{j\in\{0,..,m-1\}} d(f(z_j), z_{j+1}) \right). \end{aligned}$$

The result follows by taking the infimum on chains from x to z and then on chains from z to y. □

Lemma 2.22. *The function S is continuous.*

Proof. Let $x, x', y, y' \in X$. If $(x_0 = x, ..., x_n = y)$ is a chain from x to y, the chain $(\widetilde{x}_0, ..., \widetilde{x}_n)$ obtained by replacing $x_n = y$ by y' is a chain from x to y' such that

$$\begin{aligned} \max_{i\in\{0,..,n-1\}} d(f(\widetilde{x}_i), \widetilde{x}_{i+1}) &\leq \max_{i\in\{0,..,n-1\}} d(f(x_i), x_{i+1}) \\ &\quad + |d(f(x_{n-1}), y) - d(f(x_{n-1}), y')| \\ &\leq \max_{i\in\{0,..,n-1\}} d(f(x_i), x_{i+1}) + d(y, y'). \end{aligned}$$

Hence we get

$$S(x,y') \leq \max_{i\in\{0,..,n-1\}} d(f(\widetilde{x}_i),\widetilde{x}_{i+1}) \leq \max_{i\in\{0,..,n-1\}} d(f(x_i),x_{i+1}) + d(y,y').$$

Taking the infimum on chains $(x_0,...,x_n)$ from x to y we get

$$S(x,y') \leq S(x,y) + d(y,y').$$

Similarly, replacing $x_0 = x$ by x' we have

$$S(x',y) \leq S(x,y) + d(f(x),f(x')).$$

Exchanging role played by x, x' and y, y', we thus get

$$\begin{array}{rcl} |S(x,y') - S(x,y)| & \leq & d(y,y'), \\ |S(x',y) - S(x,y)| & \leq & d(f(x),f(x')). \end{array}$$

It follows that

$$\begin{array}{rcl} |S(x,y) - S(x',y')| & \leq & |S(x,y) - S(x',y)| + |S(x',y) + S(x',y')| \\ & \leq & d(f(x),f(x')) + d(y,y') \end{array}$$

and the continuity of S now follows from the continuity of f. □

Remark 2.23. This last proof shows that every function $S_x = S(x,\cdot)$ is 1-Lipschitzian. It follows that our Lyapunov functions are also Lipschitzian.

3 General construction

We would like to remove the compactness assumption made on X and to cover the case of compactum-valued maps, i.e. maps with values in the set $\Gamma(X)$ of nonempty compact subsets of X. In fact, as we will see, the existence of a Conley barrier only requires the separability of the ambient metric space.

3.1 Hausdorff metric and compactum-valued map

We briefly recall the definition of the Hausdorff topology on $\Gamma(X)$. For more details, see [Nad92].

Definition 3.1. Let (X,d) be a metric space. If K and K' are two compact subsets of X, we define

$$\mathcal{D}_d(K,K') = \inf\{\varepsilon > 0 \mid K' \subset V^d_\varepsilon(K) \text{ and } K \subset V^d_\varepsilon(K')\}$$

where $V^d_\varepsilon(K) = \{x \in X, d(x,K) < \varepsilon\}$.

Proposition 3.2. *The function $\mathcal{D}_d$ is a distance on the set $\Gamma(X)$ of compact subsets of X. The topology it defines does not depend on the metric d used. It is called the Hausdorff topology on $\Gamma(X)$.*

Proof. The fact that the function $\mathcal{D}_d$ is a distance is clear. It does not depends on the metric used since the convergence of a sequence K_n to K can be expressed in a purely topological way. Indeed, the compactness of K implies that $\mathcal{D}_d(K_n, K) \to 0$ as $n \to +\infty$ if and only if

i) For every neighborhood V of K there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $K_n \subset V$.

ii) For every x in K there is a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in K_n$ such that $x_n \to x$ as $n \to +\infty$.

□

Definition 3.3. A *compactum-valued* map is a map from X to $\Gamma(X)$. It is said to be continuous if it is continuous for the Hausdorff topology on $\Gamma(X)$.

3.2 Chain-recurrence on arbitrary separable metric space

In the settings of a noncompact metric space, the notion of chain-recurrence is usually defined using the set $\mathcal{P}$ of continuous functions from X to $\mathbb{R}_+^*$ instead of constants $\varepsilon > 0$. We thus keep topological invariance, see [Hur92]. The notion of $\mathcal{U}$-chain now introduced gives a powerful way to avoid using this set $\mathcal{P}$ and emphasizes the fact that the notion of chain-recurrence is a purely topological one.

Definition 3.4. Let $\mathcal{U}$ be an open covering of X. For $A \subset X$ we set

$$\mathcal{S}t(A,\mathcal{U}) = \bigcup_{\substack{U \in \mathcal{U} \\ A \cap U \neq \emptyset}} U.$$

An open covering $\mathcal{V}$ of X is called an *open refinement* of $\mathcal{U}$ and is denoted by $\mathcal{V} \propto \mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$. An *open barycentric refinement* of $\mathcal{U}$ is an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that

$$\{\mathcal{S}t\left(\{x\},\mathcal{V}\right), x \in X\} \propto \mathcal{U}.$$

Proposition 3.5. *In a metric space X, any open covering of X admits an open barycentric refinement.*

Proof. See for example [Dug78, Chapter VIII, theorem 3.5]. □

Remark 3.6. The notion of barycentric refinement will be used to generalize arguments involving triangular inequalities.

Definition 3.7. Let (X,d) be a metric space and $f : X \longrightarrow \Gamma(X)$ be a compactum-valued map. Given an open covering $\mathcal{U}$ of X and (x,y) in X^2, a $\mathcal{U}$*-chain* from x to y for f is a sequence $(x_0 = x, ..., x_n = y), n \geq 1$, of X such that

$$\forall i \in \{0,..,n-1\},\ x_{i+1} \in \mathcal{S}t(f(x_i),\mathcal{U}).$$

We define similarly the set $\mathcal{R}(f)$ of chain-recurrent points, i.e. of points of X such that for every open covering $\mathcal{U}$ of X there exists a $\mathcal{U}$-chain from x back to x. The chain-transitive components are similarly defined using the equivalence relation $\backsim$ on $\mathcal{R}(f)$ given by $x \backsim y$ if and only if for every open covering $\mathcal{U}$ of X there exists $\mathcal{U}$-chains from x to y and from y to x. Two points x and y in X will be said to be f*-separated* by $\mathcal{U}$ if there exists no $\mathcal{U}$-chain for f from x to y.

Remark 3.8. Any continuous map $f : X \longrightarrow X$ can be seen as a continuous compactum-valued map since singletons are compact. Then, the previous definition just reduces to a sequence $(x_0 = x, ..., x_n = y), n \geq 1$, of X such that

$$\forall i \in \{0,..,n-1\},\ \exists U \in \mathcal{U},\ \begin{cases} f(x_i) \in U, \\ x_{i+1} \in U. \end{cases}$$

3.3 Chain-recurrence adapted distance

From now on, f will denote a continuous compactum-valued map on a separable metric space (X,d). Our purpose is to construct a distance δ on X which allows to define chain-recurrence in the same way as in the compact case. We will follow a scheme given essentially in the work of Hurley, see [Hur92, Hur98].

Definition 3.9. A metric δ on X is said to be *chain-recurrence adapted* for f if it defines the topology of X and if for every x and y in X the following assertions are equivalent:

i) *For every open covering $\mathcal{U}$ of X, there exists a $\mathcal{U}$-chain from x to y.*

ii) *For every number $\varepsilon > 0$, there exists an ε-chain for δ from x to y.*

Remark 3.10. In the compactum-valued case, an ε-chain for δ is defined similarly with $\delta(f(x_i), x_{i+1})$ the distance from the point x_{i+1} to the compact subset $f(x_i)$.

A central point in the construction of a chain-recurrence adapted distance is to show that the elements of the set

$$E = \left\{ (x,y) \in X \times X \mid \begin{array}{l} \textit{there exists an open covering } \mathcal{U} \textit{ of } X \\ \textit{which } f\textit{-separates } x \textit{ and } y \end{array} \right\}$$

can be obtained from a countable family of open coverings of X.

Lemma 3.11. *If the metric space (X,d) is separable then there exists a countable family $(\mathcal{U}_l)_{l\in\mathbb{N}}$ of open coverings of X such that for every $(x,y)\in E$ there exists an open covering $\mathcal{U}_k$ in $(\mathcal{U}_l)_{l\in\mathbb{N}}$ that f-separates x from y.*

Remark 3.12. Such a family will be called a *f-separating* family.

Proof. Let $(x,y)\in E$ and $\mathcal{U}_{x,y}$ be an open covering of X which f-separates x from y. We will show that there are open neighborhoods $W_{x,y}$ of x and $W'_{x,y}$ of y and an open covering $\mathcal{V}_{x,y}$ of X which f-separates every point of $W_{x,y}$ from every point of $W'_{x,y}$.

Let $\widetilde{\mathcal{V}}_{x,y}$ be an open barycentric refinement of the open covering $\mathcal{U}_{x,y}$. The compact subset $f(x)$ is included into the open subset $\mathcal{S}t(f(x),\widetilde{\mathcal{V}}_{x,y})$. Thus by continuity of f, we can find a neighborhood $W_{x,y}$ of x such that

$$\forall x'\in W_{x,y},\ f(x')\subset \mathcal{S}t(f(x),\widetilde{\mathcal{V}}_{x,y}).$$

We first show that the open covering $\widetilde{\mathcal{V}}_{x,y}$ f-separates every point of $W_{x,y}$ from y. Let us suppose that for some $x'\in W_{x,y}$ there exists a $\widetilde{\mathcal{V}}_{x,y}$-chain $(x_0=x',x_1,...,x_n=y)$ from x' to y. Since $x_1\in \mathcal{S}t(f(x'),\widetilde{\mathcal{V}}_{x,y})$ we can find $V_1\in\widetilde{\mathcal{V}}_{x,y}$ such that

$$\begin{cases} V_1\cap f(x')\neq\emptyset,\\ x_1\in V_1.\end{cases}$$

Since $V_1\cap f(x')\neq\emptyset$ and $f(x')\subset \mathcal{S}t(f(x),\widetilde{\mathcal{V}}_{x,y})$, we can find $V_2\in\widetilde{\mathcal{V}}_{x,y}$ such that

$$\begin{cases} V_2\cap f(x)\neq\emptyset,\\ V_1\cap V_2\neq\emptyset.\end{cases}$$

Now, since $V_1,V_2\in\widetilde{\mathcal{V}}_{x,y}$, $V_1\cap V_2\neq\emptyset$ and $\widetilde{\mathcal{V}}_{x,y}$ is an open barycentric refinement of $\mathcal{U}_{x,y}$, we can find $U\in\mathcal{U}_{x,y}$ such that $V_1\cup V_2\subset U$. But then we have $x_1\in U$ and $U\cap f(x)\neq\emptyset$, i.e. $x_1\in\mathcal{S}t(f(x),\mathcal{U}_{x,y})$. Since the open covering $\widetilde{\mathcal{V}}_{x,y}$ is a fortiori an open refinement of $\mathcal{U}_{x,y}$, the chain $(x_0=x,x_1,...,x_n=y)$ is thus a $\mathcal{U}_{x,y}$-chain from x to y, which is absurd. Thus the open covering $\widetilde{\mathcal{V}}_{x,y}$ f-separates every point of $W_{x,y}$ from y.

Now let $\mathcal{V}_{x,y}$ be an open barycentric refinement of $\widetilde{\mathcal{V}}_{x,y}$. Let $W'_{x,y}$ be any open set of $\mathcal{V}_{x,y}$ containing y. Since $\mathcal{V}_{x,y}$ is an open barycentric refinement of $\widetilde{\mathcal{V}}_{x,y}$ and $y\in W'_{x,y}\in\mathcal{V}_{x,y}$, a similar proof shows that if $(x_0,x_1,...,x_n)$ is a $\mathcal{V}_{x,y}$-chain starting in $W_{x,y}$ and ending in $W'_{x,y}$ then the chain $(x_0,...,x_{n-1},y)$ is a $\widetilde{\mathcal{V}}_{x,y}$-chain starting in $W_{x,y}$ and ending at y. Since the open covering $\widetilde{\mathcal{V}}_{x,y}$ f-separates every point of $W_{x,y}$ from y, we conclude that the open covering $\mathcal{V}_{x,y}$ f-separates every point of $W_{x,y}$ from every point of $W'_{x,y}$.

In particular, we have shown that the subset E of $X\times X$ is open. The space X being metric and separable, the same goes for E which thus satisfies the Lindelöf property. We can thus extract from the open covering $\{W_{x,y}\times W'_{x,y},\ (x,y)\in E\}$ of E a countable sub-covering $\left(W_{x_i,y_i}\times W'_{x_i,y_i}\right)_{i\in\mathbb{N}}$.

The family of associated open coverings $(\mathcal{V}_{x_i,y_i})_{i\in\mathbb{N}}$ provides the wanted countable family.

□

We will apply the following well-known lemma to the family $(\mathcal{U}_l)_{l\in\mathbb{N}}$ of open coverings furnished by the previous lemma to obtain the desired chain-recurrence adapted distance.

Lemma 3.13. *Given a countable family $(\mathcal{U}_l)_{l\in\mathbb{N}}$ of open coverings of X, there exists a metric δ on X that defines the topology of X and such that*

$$\forall l \in \mathbb{N},\ \left\{ B_\delta\left(x, \frac{1}{2^l}\right),\ x \in X \right\} \propto \mathcal{U}_l.$$

Proof. Let $l \in \mathbb{N}$. Since any metric space is paracompact, we can find a partition of unity $\left(\varphi_U^l\right)_{U\in\mathcal{U}_l}$ subordinate to $\mathcal{U}_l$ such that the supports of the φ_U^l form a neighborhood finite closed covering of X, see [Dug78, Chapter VIII]. For any open set U in $\mathcal{U}_l$ we set

$$\psi_U^l(x) := \frac{\varphi_U^l(x)}{\sup\limits_{U'\in\mathcal{U}_l} \varphi_{U'}^l(x)}.$$

The function ψ_U^l is well defined since the supports of the $\left(\varphi_U^l\right)_{U\in\mathcal{U}_l}$ form a locally finite family and is continuous since the φ_U^l are. Moreover, we have $0 \leq \psi_U^l \leq 1$ and thus the series $\sum_{l\in\mathbb{N}} \frac{1}{2^l} \max_{U\in\mathcal{U}_l} \left|\psi_U^l(x) - \psi_U^l(y)\right|$ converges uniformly and defines a continuous function on $X \times X$.

We then define

$$\delta(x,y) := d(x,y) + \sum_{l\in\mathbb{N}} \frac{1}{2^l} \max_{U\in\mathcal{U}_l} \left|\psi_U^l(x) - \psi_U^l(y)\right|.$$

The function δ is a distance. Let us show that it induces the topology of X. Since $d \leq \delta$, if $x_n \longrightarrow x$ for δ then $x_n \longrightarrow x$ for d. Conversely, if $x_n \longrightarrow x$ for d then by continuity of the function

$$(x,y) \longmapsto \sum_{l\in\mathbb{N}} \frac{1}{2^l} \max_{U\in\mathcal{U}_l} \left|\psi_U^l(x) - \psi_U^l(y)\right|$$

we have $x_n \longrightarrow x$ for δ.

We now show the refinement property. Let $l \in \mathbb{N}$ and $x \in X$. Since the supports of the $\left(\varphi_U^l\right)_{U\in\mathcal{U}_l}$ form a locally finite family there exists $U_x \in \mathcal{U}_l$ such that $\varphi_{U_x}^l(x) = \sup\limits_{U'\in\mathcal{U}_l} \varphi_U^l(x)$. We then have $\psi_{U_x}^l(x) = 1$. But then, for $y \in B_\delta(x, \frac{1}{2^l})$ we have

$$\frac{1}{2^l}\left|1 - \psi_{U_x}^l(y)\right| \leq \frac{1}{2^l} \max_{U\in\mathcal{U}_l} \left|\psi_U^l(x) - \psi_U^l(y)\right| \leq \delta(x,y) < \frac{1}{2^l}.$$

Thus, we have $\left|1 - \psi^l_{U_x}(y)\right| < 1$ and necessarily $\psi^l_{U_x}(y) > 0$, hence $y \in U_x$. Thus $B_\delta(x, \frac{1}{2^l}) \subset U_x \in \mathcal{U}_l$ and the lemma is proved. $\square$

We can now prove the following theorem.

Theorem 3.14. *Let X be a separable metric space and $f : X \longrightarrow \Gamma(X)$ be a continuous map. Then there exists a chain-recurrence adapted distance for f on X.*

Proof. We apply the previous lemma to the f-separating family $(\mathcal{U}_l)_{l\in\mathbb{N}}$ of lemma 3.11 to obtain a distance δ on X that defines the topology of X. Let us prove that this distance is chain-recurrence adapted. For x, y in X we have to prove that the following assertions are equivalent

i) For every open covering $\mathcal{U}$ of X, there exists a $\mathcal{U}$-chain from x to y.

ii) For every number $\varepsilon > 0$, there exists an ε-chain for δ from x to y.

Let us suppose i). The open coverings $\left\{B_\delta(x', \frac{\varepsilon}{2}),\ x' \in X\right\}$, $\varepsilon > 0$, provides by triangle inequality ε-chains for δ from x to y. Since ε is arbitrary, it shows ii). Conversely, let us suppose ii). For every $l \in \mathbb{N}$ we have

$$\left\{B_\delta(x', \frac{1}{2^l}),\ x' \in X\right\} \propto \mathcal{U}_l.$$

Thus, every $\frac{1}{2^l}$-chain for δ from x to y is in fact a $\mathcal{U}_l$-chain from x to y. Since the family $(\mathcal{U}_l)_{l\in\mathbb{N}}$ is a f-separating one, it shows i). $\square$

3.4 Conley barrier

In the setting of a noncompact metric space, we define what a Conley barrier is using the notion of $\mathcal{U}$-chains.

Definition 3.15. Let X be a metric space and $f : X \longrightarrow \Gamma(X)$ be a continuous map. A *Conley barrier* for f is a continuous function

$$S : X \times X \longrightarrow \mathbb{R}_+$$

with the properties that

i) *For every $(x, y) \in X^2$, $S(x, y) = 0$ if and only if for every open covering $\mathcal{U}$ of X there exists a $\mathcal{U}$-chain for f from x to y.*

ii) *For every $(x, y, z) \in X^3$, we have $S(x, y) \leq \max(S(x, z), S(z, y))$.*

As in the compact case, we will show the following theorem.

Theorem 3.16. *If X is a separable metric space and $f : X \longrightarrow \Gamma(X)$ is a continuous map then there exists a Conley barrier for f.*

Proof. According to theorem 3.14, there exists a chain-recurrence adapted distance δ on X for f. Since chain properties are fully described using the metric δ, it is enough to construct a continuous function S such that

1) For every $(x, y) \in X^2$, we have $S(x, y) = 0$ if and only if for every $\varepsilon > 0$ there exists an ε-chain for δ from x to y.

2) For every $(x, y, z) \in X^3$, we have $S(x, y) \leq \max(S(x, z), S(z, y))$.

The only difference with the compact case is that f is now a compactum-valued map. For every $(x, y) \in X^2$, we thus define similarly S as

$$S(x, y) := \inf \left\{ \max_{i \in \{0,..n-1\}} \delta(f(x_i), x_{i+1}) \mid n \geq 1,\ x_0 = x, ..., x_n = y \right\}.$$

The distance from $f(x_i)$ to x_{i+1} being understood as the distance of the point x_{i+1} to the compact set $f(x_i)$. A similar proof than in the compact case then shows that

$$\left|S(x, y) - S(x', y')\right| \leq \delta(y, y') + \mathcal{D}_\delta(f(x), f(x')).$$

Thus the function S inherits of the continuity of f. The proofs of properties 1) and 2) can now be readily adapted. □

3.5 Ultrametric distance induced on the space of chain-transitive components

The fact that a Conley barrier induces an ultrametric distance on the set of chain-transitive components does not use compactness of X. Thus, the constructions of section 2.2 can be readily adapted. In particular, any Conley barrier furnishes an ultrametric distance on the set of chain-transitive components of f and the induced ultrametric topology is coarser than the quotient topology. Thus, we have the following.

Theorem 3.17. *Let X be a separable metric space and $f : X \longrightarrow \Gamma(X)$ be a continuous map. Then the set of chain-transitive components of f is Hausdorff and totally disconnected.*

Nevertheless, contrary to the compact case, the ultrametric topology induced by a Conley barrier may differ from the quotient topology. A counterexample is given in section 4.2.

3.6 Lyapunov functions

Definitions. In the case of a compactum-valued map, the definitions of Lyapunov functions need to be slightly modified.

Definition 3.18. Given a metric space X and a continuous map $f : X \longrightarrow \Gamma(X)$, a *strict Lyapunov function* for f is a continuous function $\varphi : X \longrightarrow \mathbb{R}$ such that

i) *For every x in X and every y in $f(x)$, we have $\varphi(y) \leq \varphi(x)$.*

ii) *For every x in $X \setminus \mathcal{R}(f)$ and every y in $f(x)$, we have $\varphi(y) < \varphi(x)$.*

A strict Lyapunov function is said to be *complete* if it satisfies the following additional property

i') *The function φ is constant on each chain-transitive component, takes on distinct values on distinct chain-transitive components and sends the subset $\mathcal{R}(f)$ into a subset of $\mathbb{R}$ whose Hausdorff dimension is zero.*

The notion of *sub-solution* for a Conley barrier S is similarly defined. Moreover, proofs of lemma 2.13 and 2.14 are unchanged.

Strict Lyapunov function. Our construction of a strict Lyapunov function for f is still based on sub-solutions of the type S_x for $x \in X$. The existence of a uniform bound for S is there replaced by the following lemma.

Lemma 3.19. *There is a countable open covering $(U_n)_{n\in\mathbb{N}}$ of X such that for every $x \in X$ and for every $n \in \mathbb{N}$, the function S_x is bounded on U_n.*

Proof. Let $x \in X$. By continuity of S_x, there is an open neighborhood U_x of x such that $S(x, \cdot)$ is bounded on U_x. For $x' \in X$ we have

$$\forall y \in U_x,\ S(x', y) \leq \max(S(x', x), S(x, y)).$$

Thus the function $S(x', \cdot)$ is also bounded on U_x. Since the metric space X is separable, it is Lindelöf . Hence, a countable sub-covering of the open covering $\{U_x, x \in X\}$ of X provides the wanted covering. □

Corollary 3.20. *For every sequence $(x_i)_{i\in\mathbb{N}}$ of X, there exists a sequence $(\eta_i)_{i\in\mathbb{N}}$ of positive reals such that the non-negative series $\sum_{i\in\mathbb{N}} \eta_i S_{x_i}$ converges uniformly in the neighborhood of each points of X.*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be an open covering of X furnished by the previous lemma. Each function $S_{x_i}, i \in \mathbb{N}$, is bounded on U_0. Thus, there is a sequence $(\rho_i^0)_{i\in\mathbb{N}}$ of positive reals such that the series $\sum_{i\in\mathbb{N}} \rho_i^0 S_{x_i}$ converges uniformly on U_0. Similarly, there is a sequence $(\rho_i^1)_{i\in\mathbb{N}}$ of positive reals such that the series $\sum_{i\in\mathbb{N}} \rho_i^1 S_{x_i}$ converges uniformly on U_1. Moreover, reducing the ρ_i^1 if necessary, we can also suppose that $\rho_i^1 < \rho_i^0$.

We thus construct using induction sequences $(\rho_i^k)_{i\in\mathbb{N}}$, for k in $\mathbb{N}$, such that $0 < \rho_i^{k+1} < \rho_i^k$ and the series $\sum_{i\in\mathbb{N}} \rho_i^k S_{x_i}$ converges uniformly on U_k. These both conditions then imply that the series $\sum_{i\in\mathbb{N}} \eta_i S_{x_i}$, with $\eta_i = \rho_i^i$, converges uniformly on each $U_k, k \in \mathbb{N}$. The result follows since $(U_n)_{n\in\mathbb{N}}$ is an open covering of X. □

Remark 3.21. If we define instead η_i by $\min(\rho_i^i, \frac{1}{2^{i+1}})$, we can also assume that the series $\sum_{i\geq 1} \eta_i$ converges and belongs to $]0, 1[$. Thus, changing η_0 in $1 - \sum_{i\geq 1} \eta_i$, we can suppose without lost of generality that $\sum_{i\in\mathbb{N}} \eta_i = 1$.

We can now prove the following theorem.

Theorem 3.22. *Let X be a separable metric space and $f : X \longrightarrow \Gamma(X)$ be a continuous map. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ of points of X and a sequence $(\eta_n)_{n\in\mathbb{N}}$ of positive reals such that the series*

$$\varphi = \sum_{n\in\mathbb{N}} \eta_n S_{x_n}$$

is a strict sub-solution for S and thus a strict Lyapunov function for f.

Proof. As in the compact case, let us choose a dense sequence $(x_i)_{i\in\mathbb{N}}$ of X. Let $(\eta_i)_{i\in\mathbb{N}}$ be the associated sequence given by corollary 3.20. Thanks to remark 3.21, we can suppose that $\sum_{i\in\mathbb{N}} \eta_i = 1$. The same proof as in the compact case then shows that the function $\varphi = \sum_{i\in\mathbb{N}} \eta_i S_{x_i}$ is a strict sub-solution for S and thus a strict Lyapunov function for f. □

Complete Lyapunov function. If we had an hypothesis of local compactness, the same tools as in section 2.3 can be used to construct a complete Lyapunov function. In particular, the proof of the following lemma did not use any compactness and is still valid.

Lemma 3.23. *Let X be a separable metric space. For every $x \in X$, the function S_x is constant in the neighborhood of each point of the set $\mathcal{R}(f) \setminus \{S(x, \cdot) = 0\}$.*

Corollary 3.24. *Let X be a separable metric space. For every compact subset K of X and for every x in X, the set $\{S(x, y),\ y \in \mathcal{R}(f) \cap K\}$ is countable and the only possible accumulation point is zero. In particular, any function of the form $\theta \circ S_x$, where $\theta : \mathbb{R} \to \mathbb{R}$ is constant in a neighbourhood of zero, takes on a finite number of values on $\mathcal{R}(f) \cap K$.*

Proof. The proof is the same as proof of corollary 2.17 once the set $\mathcal{R}(f)$ has been replaced by $\mathcal{R}(f) \cap K$. □

Theorem 3.25. *Let X be a locally compact and separable metric space and $f : X \longrightarrow \Gamma(X)$ be a continuous map. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X, a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals and a sequence $(\theta_n)_{n\in\mathbb{N}}$ of real-valued functions such that the series*

$$\varphi = \sum_{n\in\mathbb{N}} \varepsilon_n \theta_n \circ S_{x_n}$$

defines a complete Lyapunov function for f.

Proof. We will denote by p the canonical projection from $\mathcal{R}(f)$ onto $\mathcal{R}(f)/\backsim$. Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in X. Without lost of generality, we can suppose that for every $k \in \mathbb{N}$ the sequence $(x_n)_{n\geq k}$ is still dense in X. Since X is locally compact, metric and separable, there exist a family $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X such that $X = \cup_{n\in\mathbb{N}} K_n$ and for every $n \in \mathbb{N}$, we have $K_n \subset \overset{\circ}{K}_{n+1}$. For every $n \in \mathbb{N}$, we set

$$\theta_n(t) := \max(t - \frac{1}{n+1}, 0).$$

Each function $\theta_k \circ S_{x_k}, k \in \mathbb{N}$, is bounded on the compact set K_n, $n \in \mathbb{N}$. Using a diagonal process, we can find a sequence $(\eta_n)_{n\in\mathbb{N}}$ of positive reals such that the series $\sum_{k\in\mathbb{N}} \eta_k \theta_k \circ S_{x_k}$ converges uniformly on each K_n and thus defines a continuous function on X.

Thanks to corollary 3.24, for every $(k, n) \in \mathbb{N}^2$, the function $\theta_k \circ S_{x_k}$ takes on a finite number of values on $\mathcal{R}(f) \cap K_n$. As in the compact case, each function $\theta_k \circ S_{x_k}$ is constant on the chain-transitive components and induces a function $\overline{\theta_k \circ S_{x_k}}$ on the quotient space $\mathcal{R}(f)/\backsim$. Moreover, this function takes a finite number of values on each compact set $p(K_n \cap \mathcal{R}(f))$, $n \in \mathbb{N}$. We will now use lemma 5.1 with the set $A = \mathcal{R}(f)/\backsim$, the family $(\overline{\theta_n \circ S_{x_n}})_{n\in\mathbb{N}}$ and $A_n = p(K_n \cap \mathcal{R}(f))$. As in the compact case, we easily verify that for every $k \in \mathbb{N}$ the family $(\overline{\theta_n \circ S_{x_n}})_{n\geq k}$ separates points of $\mathcal{R}(f)/\backsim$. Thus lemma 5.1 furnishes a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals such that the series $\sum_{n\in\mathbb{N}} \varepsilon_n \overline{\theta_n \circ S_{x_n}}$ converges on $\mathcal{R}(f)/\backsim$, separates points of $\mathcal{R}(f)/\backsim$ and has an image of zero Hausdorff dimension in $\mathbb{R}$. Since the positive reals $(\varepsilon_n)_{n\in\mathbb{N}}$ can be chosen arbitrarily small, we can also assume that for every $n \in \mathbb{N}$ we have $\varepsilon_n < \eta_n$. Hence, the function

$$\varphi = \sum_{n\in\mathbb{N}} \varepsilon_n \theta_n \circ S_{x_n}$$

converges uniformly on each $K_n, n \in \mathbb{N}$, and thus defines a continuous function on X. It is constant on each chain-transitive component, takes on distinct values on distinct chain-transitive components and sends $\mathcal{R}(f)$ in a subset of $\mathbb{R}$ whose Hausdorff dimension is zero. The rest of the proof is now similar to the compact case. □

4 The case $f = Id_X$

In the particular case $f = Id_X$, a $\mathcal{U}$-chain from x to y just corresponds to a sequence $(U_i)_{0\leq i\leq n}$ of open sets of the open covering $\mathcal{U}$ such that

$$x \in U_0,\ y \in U_n,\ \forall i \in \{0, .., n-1\},\ U_i \cap U_{i+1} \neq \emptyset.$$

In particular, a Conley barrier associated to the identity is symmetric. Chain-recurrence properties are then linked with the topology of X.

4.1 The quasicomponents

Definition 4.1. Let X be a topological space. Two points x and y of X are said to be *separated* in X if the space X can be split into two disjoint open sets U and V containing respectively x and y.

The relation *not being separated* defines an equivalence relation on X. The associated equivalence classes are called the *quasicomponents* of X. Two point x and y lie in the same quasicomponent if and only if every open and closed subset of X containing x or y contains both x and y. Thus, the quasicomponent of a point x coincides with the intersection of open and closed subsets of X that contain x. In particular, the connected component of x is included into the quasicomponent of x.

Remark 4.2. In a compact space, the connected component of a point x coincides with the quasicomponents of x, see [HW41, Chapter II]. Nevertheless, even if the space is locally compact, quasicomponents may be larger than connected components. See for example the counterexample of nested rectangle in [SS95].

The quasicomponents are essentially characterized by a Conley barrier associated to the identity, as shown in the following result.

Lemma 4.3. *Let X be a separable metric space. Then the quasicomponents of X coincide with the chain-transitive components of Id_X.*

Proof. We have to show that two points x and y are separated in X if and only if there exists an open covering $\mathcal{U}$ of X that Id_X-separates x from y. Let us suppose that for every open covering $\mathcal{U}$ of X, there is a $\mathcal{U}$-chain for the identity map from x to y. If x and y where separated in X say by U and V, the open covering $\{U, V\}$ would leads to a contradiction. Conversely, let us suppose that there is an open covering $\mathcal{U}$ of X such that there is no $\mathcal{U}$-chain for the identity map from x to y. Let $U \in \mathcal{U}$ be an open set such that $x \in U$. We consider the set

$$O = \bigcup_{n\in\mathbb{N}} St^n(U,\mathcal{U}) \quad \text{where} \quad St^n(U,\mathcal{U}) = \underbrace{St(...St}_{n\ times}(U,\mathcal{U})..,\mathcal{U}).$$

The set O is open and we claim that the same is true for $X \setminus O$. Indeed, let $z \in X \setminus O$. If we denote by V an element of $\mathcal{U}$ such that $z \in V$, then $V \subset X \setminus O$. Moreover we have $y \in X \setminus O$ since there is no $\mathcal{U}$-chain from x to y for Id_X. The points x and y are thus separated by the open subsets O and $X \setminus O$. $\square$

We then deduce the following corollary.

Corollary 4.4. *Let X be a separable metric space and S be a Conley barrier for the identity map on X. Then two points x and y of X are in the same quasicomponent if and only if $S(x, y) = 0$.*

If the metric space X is compact, the quasicomponents and the connected components of X coincide. We thus obtain the following known result, which follows from corollary 2.10.

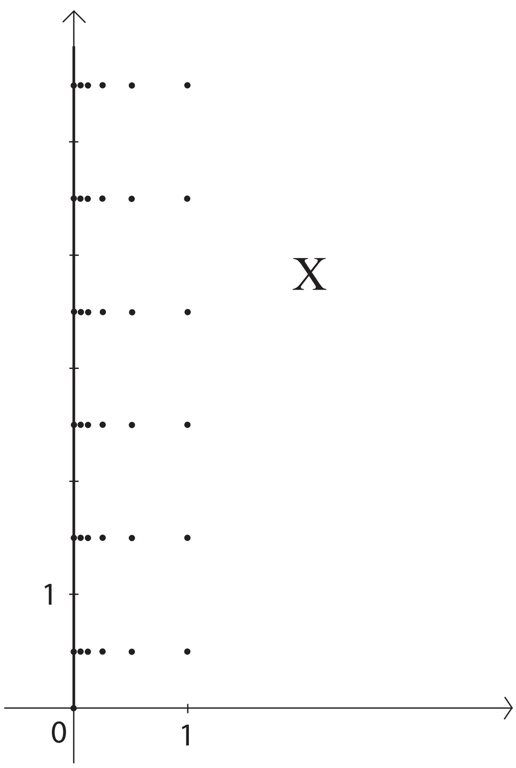
Theorem 4.5. *Let X be a compact metric space. Then the set of connected components of X is an ultrametric space.*

If some quasicomponent fail to be compact, the ultrametric topology induced by a Conley barrier may be strictly coarser than the quotient topology. Such an example is studied in the next section.

4.2 A counterexample

We consider the plane $\mathbb{R}^2$ and for $k \in \mathbb{N}$ we set

$$\mathcal{D} = \{(0, y), y \geq 0\},$$
$$A_k = \left\{ \left(\frac{1}{n}, k + \frac{1}{2} \right), \ n \geq 1 \right\},$$
$$X = \left(\bigcup_{k \in \mathbb{N}} A_k \right) \bigcup \mathcal{D}.$$



We endow the space X with the Euclidean topology inherited from $\mathbb{R}^2$. The space X thus obtained is a closed subset of $\mathbb{R}$ and hence is locally compact.

Lemma 4.6. *For every countable family $(V_i)_{i\in\mathbb{N}}$ of open sets of $\mathbb{R}^2$ containing $\mathcal{D}$, there is an open set V of $\mathbb{R}^2$ containing $\mathcal{D}$ and such that*

$$\forall i \in \mathbb{N},\ X \cap V_i \not\subseteq X \cap V.$$

Proof. We first construct a sequence $(U_k)_{k\in\mathbb{N}}$ of open sets of $\mathbb{R}^2$ such that

i) *For every $k \in \mathbb{N}$, $\{0\} \times [k, k+1] \subset U_k$.*

ii) *For every $k \neq l$, $U_k \cap A_l = \emptyset$.*

iii) *For every $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}^*$ such that $\left(\frac{1}{n_k}, k + \frac{1}{2}\right) \in V_k \setminus U_k$.*

To insure the first two points, it is enough to choose U_k contained in the strip

$$\left\{(x, y), x \in \mathbb{R}, k - \frac{1}{4} < y < k + \frac{5}{4}\right\} \supset \{0\} \times [k, k+1].$$

For the last point, we notice that the point $\left(0, k + \frac{1}{2}\right)$ lies in $V_k \cap \bar{A}_k$. Thus there is an integer $n_k > 0$ such that $\left(\frac{1}{n_k}, k + \frac{1}{2}\right) \in V_k$. We thus set

$$U_k = \left\{(x, y) \in \mathbb{R}^2 \mid x < \frac{1}{n_k},\ k - \frac{1}{4} < y < k + \frac{5}{4}\right\}.$$

From i), the open set $V = \bigcup_{k\in\mathbb{N}} U_k$ contains $\mathcal{D}$. Now let $i \in \mathbb{N}$. By construction we have

$$\left(\frac{1}{n_i}, i + \frac{1}{2}\right) \notin U_i$$

and from ii) we have

$$\forall l \neq i,\ \left(\frac{1}{n_i}, i + \frac{1}{2}\right) \notin U_l$$

Thus $\left(\frac{1}{n_i}, i + \frac{1}{2}\right) \notin X \cap V$ while $\left(\frac{1}{n_i}, i + \frac{1}{2}\right) \in X \cap V_i$. We thus have

$$X \cap V_i \not\subseteq X \cap V$$

as asserted. □

Corollary 4.7. *The set of quasicomponents of the metric space X defined above is not metrizable. Hence, the topology induced by a Conley barrier for Id_X on the set of quasicomponents is strictly coarser than the quotient topology.*

Proof. The quasicomponents of X are the half line $\mathcal{D}$ and the singletons $\left(\frac{1}{n}, k+\frac{1}{2}\right)_{n\geq 1, k>0}$. We will show that $\mathcal{D}$ does not admit any countable basis of open neighborhoods in the quotient topology.

Otherwise, let $(\widetilde{O_i})_{i\in\mathbb{N}}$ be such a basis. The inverse images by the canonical projection p provide a family $(O_i)_{i\in\mathbb{N}}$ of open sets of X that contain $\mathcal{D}$. Thus there is a family $(V_i)_{i\in\mathbb{N}}$ of open set of $\mathbb{R}^2$ containing $\mathcal{D}$ and such that $O_i = V_i \cap X = p^{-1}(\widetilde{O_i})$. According to lemma 4.6, there is an open set V of $\mathbb{R}^2$ containing $\mathcal{D}$ such that for every $i \in \mathbb{N}$, $X \cap V_i \not\subseteq X \cap V$. Since V contains $\mathcal{D}$ and since the quasicomponents of $X \setminus \mathcal{D}$ are reduced to singletons, we have $p^{-1}(p(V)) = V \cap X$. Thus the set $p(V)$ is an open set that contains $\mathcal{D}$. But for every $i \in \mathbb{N}$ the set $p^{-1}(\widetilde{O_i}) = O_i = X \cap V_i$ is not included in $X \cap V$. Thus $\widetilde{O_i} \not\subseteq p(V)$ and this contradicts the fact that $(\widetilde{O_i})_{i\in\mathbb{N}}$ is a basis of open neighborhoods of $\mathcal{D}$ in the quotient. $\square$

4.3 Totally separated space

We can now also answer the following question: under which conditions are chain-transitive components of Id_X reduced to singletons ?

Definition 4.8. A topological space X is said to be

i) *totally disconnected* if connected components of X are reduced to singletons.

ii) *totally separated* if two distinct points of X can always be separated.

iii) *of dimension 0* if every point of X has a basis of open sets with empty boundary.

We always have iii) $\Rightarrow$ ii) $\Rightarrow$ i) and if X is a locally compact space, these notions coincide. In the general setting, they may be different, see [HW41, Chapter II].

Proposition 4.9. *Let X be a separable metric space. Then the chain-transitive components associated to the identity are reduced to singletons if and only if X is totally separated.*

Proof. It is corollary 4.4. $\square$

5 Appendix

5.1 Function series and Hausdorff dimension

In this section, we develop some general facts about the Hausdorff dimension of images of some particular function series. They are used to construct complete Lyapunov functions for f in section 2.3 and 3.6.

Throughout this section, $(f_i)_{i\in\mathbb{N}}$ will denote a family of real valued functions on a set A, such that either

i) *For every $i \in \mathbb{N}$, the set $f_i(A)$ is finite.*

ii) *The family $(f_i)_{i\in\mathbb{N}}$ separates points of A, i.e. for each a, b in A with $a \neq b$, there exists an f_i such that $f_i(a) \neq f_i(b)$.*

or

i) $A = \cup_{n\in\mathbb{N}} A_n$.

ii) *For every $(k, n) \in \mathbb{N}^2$, the set $f_k(A_n)$ is finite.*

iii) *For every $n \in \mathbb{N}$, the family $(f_k)_{k\geq n}$ separates points of A.*

Lemma 5.1. *In both cases, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of arbitrarily small positive reals such that the series $\sum_{n\in\mathbb{N}} \varepsilon_n f_n$ converges on A, separates points of A and has an image of zero Hausdorff dimension in $\mathbb{R}$.*

Proof. We begin with the second case. Considering sets $\tilde{A_n} = \cup_{k\leq n} A_k$ instead of A_n, we can suppose that

$$\forall n \in \mathbb{N}, A_n \subset A_{n+1}.$$

Since for every $(k, n) \in \mathbb{N}^2$ the set $f_k(A_n)$ is finite, we can construct using induction a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals such that

1) $\varepsilon_0 > 0$,

2) $\forall n \in \mathbb{N}, \displaystyle\sum_{k\geq n+1} \varepsilon_k \max_{A_n} |f_k| < \frac{1}{2}\eta_n$,

3) $\forall n \in \mathbb{N}, \displaystyle\sum_{k\geq n+1} \varepsilon_k \max_{A_n} |f_k| < e^{-n\nu_n}$,

where

$$\nu_n = \text{Card}\left(\sum_{k=0}^{n} \varepsilon_k f_k(A_n)\right)$$

and η_n is the minimum of the distance between two distinct points of the finite set $\sum_{k=0}^{n} \varepsilon_k f_k(A_n)$. If this image is reduced to a single point, we just set $\eta_n = 1$. Note that the $(\varepsilon_n)_{n\in\mathbb{N}}$ can be chosen arbitrarily small.

Property 3) implies that the series $\sum_{n\in\mathbb{N}} \varepsilon_n f_n$ converges uniformly on each A_n and thus converges on A. Now, let $a, b \in A$ be two distinct points of A. Since $A = \cup_{n\in\mathbb{N}} A_n$ and $A_n \subset A_{n+1}$, we can choose n large enough so that a and b lie in A_n. If $\sum_{k\leq n} \varepsilon_k f_k(a) \neq \sum_{k\leq n} \varepsilon_k f_k(b)$ then by property 2) we have $\sum_{k\in\mathbb{N}} \varepsilon_k f_k(a) \neq \sum_{k\in\mathbb{N}} \varepsilon_k f_k(b)$. Otherwise, by hypothesis the family $(f_k)_{k\geq n+1}$ separates points of A, thus there is a first $n_0 \geq n+1$ such that $f_{n_0}(a) \neq f_{n_0}(b)$. Hence we have $\sum_{k\leq n_0} \varepsilon_k f_k(a) \neq \sum_{k\leq n_0} \varepsilon_k f_k(b)$. Since $a, b \in A_n \subset A_{n_0}$, we can conclude similarly. Thus the series $\sum_{n\in\mathbb{N}} \varepsilon_n f_n$ separates points of A.

Let us now prove that the set $\sum_{n\in\mathbb{N}} \varepsilon_n f_n(A)$ is a subset of $\mathbb{R}$ whose Hausdorff dimension is zero. Since this property is stable under countable union, it is enough to show that for every $n \in \mathbb{N}$ the set $\sum_{k\in\mathbb{N}} \varepsilon_k f_k(A_n)$ has a zero Hausdorff dimension in $\mathbb{R}$. Let $n \in \mathbb{N}$. We write

$$\sum_{k\in\mathbb{N}} \varepsilon_k f_k(A_n) = \sum_{k\leq n} \varepsilon_k f_k(A_n) + \sum_{k\geq n+1} \varepsilon_k f_k(A_n).$$

By property 3), the subset $\sum_{k\in\mathbb{N}} \varepsilon_k f_k(A_n)$ can be covered by ν_n balls of radius $e^{-n\nu_n}$. Since for every $l \in \mathbb{N}, A_n \subset A_{n+l}$, we conclude that the subset $\sum_{k\in\mathbb{N}} \varepsilon_k f_k(A_n)$ can be covered by ν_{n+l} balls of radius $e^{-(n+l)\nu_{n+l}}$. Since

$$\forall \rho > 0,\ \nu_{n+l}(e^{-(n+l)\nu_{n+l}})^{\rho} \to 0,\ (l \to +\infty)$$

the subset $\sum_{k\in\mathbb{N}} \varepsilon_k f_k(A_n)$ has a zero Hausdorff dimension.

For the first case of the lemma, we take $A_n = A$ for every $n \in \mathbb{N}$ and we construct similarly a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals. If a, b are two distinct points of A then by property 2) there is a first $n \in \mathbb{N}$ such that $\sum_{k\leq n} \varepsilon_k f_k(a) \neq \sum_{k\leq n} \varepsilon_k f_k(b)$. Then, by construction of the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ we have $\sum_{k\in\mathbb{N}} \varepsilon_k f_k(a) \neq \sum_{k\in\mathbb{N}} \varepsilon_k f_k(b)$. The end of the proof is now similar. □

5.2 On the equivalence of chain-recurrence definitions

In this section, we give another definition of chain-recurrence which is used by Hurley in [Hur92] and we prove that it is equivalent to the $\mathcal{U}$-chain approach. Throughout this section, (X, d) will denote a separable metric space and f a continuous map from X to itself. We will denote by $\mathcal{P}$ the set of continuous functions $\varepsilon : X \longrightarrow \mathbb{R}_+^*$. The set $\mathcal{P}$ is introduced by Hurley in [Hur92] in order to keep topological invariance.

Definition 5.2. Let $x, y \in X$ and $\varepsilon \in \mathcal{P}$. An *$\varepsilon$-chain* for f from x to y is a finite sequence $(x_0 = x, ..., x_n = y), n \geq 1$, of X such that

$$\forall i \in \{0, .., n-1\},\ d(f(x_i), x_{i+1}) < \varepsilon(f(x_i)).$$

Remark 5.3. If X is compact, we only need to use constant $\varepsilon > 0$ instead of elements of $\mathcal{P}$ since any continuous function reaches its minimum on X. Definition 5.2 is thus a generalization of the compact case one.

As shown in the following proposition, this definition leads us to the same notion of chain-recurrence than definition 5.2.

Proposition 5.4. *Let $x, y \in X$. The following assertions are equivalent:*

i) *For every $\varepsilon \in \mathcal{P}$, there is an ε-chain from x to y.*

ii) *For every open covering $\mathcal{U}$ of X, there is an $\mathcal{U}$-chain from x to y.*

Proof. Let $\mathcal{U}$ be an open covering of X. A metric space is paracompact so there is a locally finite refinement $\tilde{\mathcal{U}}$ of $\mathcal{U}$. For $U \in \tilde{\mathcal{U}}$ let

$$\varepsilon_U(x) = \frac{d\,(x, X \backslash U)}{2} \quad \text{and} \quad \varepsilon(x) = \max_{U \in \tilde{\mathcal{U}}} \varepsilon_U(x)$$

with the convention that $d(x, \emptyset) = 1$. The function ε is well defined and continuous since the open covering $\tilde{\mathcal{U}}$ is locally finite and each ε_U is continuous. Moreover, this function is positive everywhere on X since $\widetilde{\mathcal{U}}$ is an open covering of X. For an open set $U \in \widetilde{\mathcal{U}}$ that realizes the maximum in the definition of $\varepsilon(x)$, we have $B_d(x, \varepsilon(x)) \subset U$. Thus

$$\{B_d(x, \varepsilon(x)), x \in X)\} \propto \tilde{\mathcal{U}} \propto \mathcal{U}$$

and every ε-chain from x to y provides a $\mathcal{U}$-chain from x to y. It shows i) $\Rightarrow$ ii).

Conversely, let $\varepsilon \in \mathcal{P}$. Then for every $x \in X$, there is an open neighborhood U_x of x such that for every $x' \in U_x$ we have $\varepsilon(x') > \frac{\varepsilon(x)}{2}$. Reducing U_x, we can also suppose that $U_x \subset B_d(x, \varepsilon(x))$. We then consider the open covering $\mathcal{U} = \{U_x,\ x \in X\}$ of X. Let $(x_0 = x, x_1, ..., x_{n-1}, x_n = y)$ be a $\mathcal{U}$-chain from x to y. For every $i \in \{0, ..., n-1\}$ there is $z_i \in X$ such that $f(x_i)$ and x_{i+1} lie in $U_{z_i} \in \mathcal{U}$. Since $U_{z_i} \subset B_d(z_i, \varepsilon(z_i))$ we have $d(f(x_i), x_{i+1}) \leq d(f(x_i), z_i) + d(z_i, x_{i+1}) \leq 2\varepsilon(z_i) < 4\varepsilon(f(x_i))$. The chain $(x, x_1, ..., x_{n-1}, y)$ is thus a 4ε-chain from x to y. It shows ii) $\Rightarrow$ i). $\square$

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