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Zero-temperature phase diagram for double-well type potentials in the summable variation class

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Abstract. We study the zero-temperature limit of the Gibbs measures of a class of long-range potentials on a full shift of two symbols $\{0, 1\}$. These potentials were introduced by Walters as a natural space for the transfer operator. In our case, they are constant on a countable infinity of cylinders and are Lipschitz continuous or, more generally, of summable variation. We assume that there exist exactly two ground states: the fixed points 0^∞ and 1^∞. We fully characterize, in terms of the Peierls barrier between the two ground states, the zero-temperature phase diagram of such potentials, that is, the regions of convergence or divergence of the Gibbs measures as the temperature goes to zero.

1. *Introduction and main results*

We consider the problem of convergence or divergence of Gibbs measures as the absolute temperature goes to zero. By a Gibbs measure, we mean an invariant probability μ_β describing the equilibrium at temperature β^{-1} of one-sided configurations $(x_0, x_1, \ldots) \in \Sigma := \{0, 1\}^{\mathbb{N}}$ interacting according to a potential $H : \Sigma \to \mathbb{R}$, as described in the thermodynamic formalism (see [**3**, **15**, **19**, **20**]). The invariance of the measure is defined with respect to the left shift $\sigma : \Sigma \to \Sigma$, $\sigma(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$. We assume, in the following, that H is non-negative and Lipschitz continuous or, more generally, of summable variation. When $\beta \to +\infty$, the Gibbs measures tend to concentrate on the minima of H. In addition, the limit measure needs to be invariant. We assume that the only invariant ergodic probability measures included in the zero-level set $\{H = 0\}$ are exactly the two Dirac measures δ_{0^∞} and δ_{1^∞}. As the temperature goes to zero ($\beta \to +\infty$), two cases may happen: either the *selection case*, where μ_β converges to a convex combination $c_0\delta_{0^\infty} + c_1\delta_{1^\infty}$, or the *non-selection case*, where, for some subsequence β_k, $\{\mu_{\beta_k}\}$ has two accumulation points, $\mu_{\beta_{2k}} \to \delta_{0^\infty}$ and $\mu_{\beta_{2k+1}} \to \delta_{1^\infty}$. In this work, we consider the smallest class of potentials in which the two cases coexist.

For potentials that depend on a finite number of coordinates, namely, that are constant on a finite number of cylinder sets, the selection case always holds, over both finite alphabets [**6**, **7**, **13**, **17**] and countably infinite alphabets [**11**, **16**]. For potentials that are constant on a countable infinity of cylinders, the selection case has been proved in particular examples (see Baraviera *et al* [**4**], Leplaideur [**18**], Baraviera *et al* [**5**]). The non-selection case has been addressed more recently in [**8**, **10**] and [**9**]. In a seminal paper [**10**], van Enter and Ruszel have produced an example where *chaotic temperature dependence* was observed. However, their alphabet is the unit circle and the construction is only based on properties of the potential and not on the dynamics. Chazottes and Hochman gave, in [**8**], examples of non-selection in any dimension $D \neq 2$ (with respect to an underlying $\mathbb{Z}^D$-action). In one dimension, their potential is equal to the distance to some invariant compact set that has a complex combinatorial construction. In dimension $D \geq 3$, their non-selection examples come from potentials that do depend on a finite number of coordinates. Recently, in [**2**], Aubrun and Sablik extended [**14**], which is the main ingredient in the proof of the multidimensional part of [**8**]. In principle, an analogous proof of the non-selection for $D = 2$ should also work. In [**9**], Coronel and Rivera-Letelier adapted van Enter and Ruszel's ideas for finite alphabets and they ensure the existence of non-selection examples by a perturbative approach combined with entropy arguments, as in [**8**]. Moreover, they were able to verify the non-selection case also for $D = 2$, without using the result of [**2**], but with Lipschitz continuous potentials. Thus, for potentials that depend on a finite number of coordinates in dimension $D = 2$, it is an open question as to whether there exist examples of non-selection.

Our approach is different. We highlight the simplest class of potentials whose zero-temperature phase diagram is completely understood: it contains both the non-selection and the selection cases, with an explicit description of the limit measures in the convergent situation. We show that the criterion for non-selection or selection is whether the Peierls barriers between the two configurations 0^∞ and 1^∞ are both equal to zero or not.

We now detail such a class of potentials. A cylinder of length $n \geq 1$ is a set $C_n := [i_0 i_1 \ldots i_{n-1}]$ of configurations $x \in \Sigma$ such that the first n states $x_0, x_1, \ldots, x_{n-1}$ coincide with $i_0, i_1, \ldots, i_{n-1}$. We say that two points $x, y \in \Sigma$ are n-close, and we write $x \overset{n}{=} y$ if x and y belong to the same cylinder of length n. Let $H : \Sigma \to \mathbb{R}$ be a C^0 non-negative potential. We say that H has summable variation if

$$\sum_{n\geq 1} \mathrm{var}(H, n) < +\infty \quad \text{with } \mathrm{var}(H, n) := \sup\{|H(x) - H(y)| : x \overset{n}{=} y\}. \tag{1.1}$$

We restrict the potential H to a subclass of functions that are constant on a countable infinity of cylinders, as described in the following assumptions. Our subclass is a particular class of Walters potentials with summable variation (see [**21**]).

Definition 1.1. We say that H is a double-well type potential if H is non-negative, has summable variation and is constant on the cylinders $[00^n1]$, $[01^n0]$, $[11^n0]$ and $[10^n1]$. More precisely, there are non-negative sequences $\{a_n^0\}$, $\{a_n^1\}$ and strictly positive sequences $\{b_n^0\}$, $\{b_n^1\}$ such that:

(1) $H(x) = a_n^0 \geq 0$ if $x \in [00^n1]$, $H(x) = a_n^1 \geq 0$ if $x \in [11^n0]$;

(2) $H(x) = b_n^0 > 0$ if $x \in [01^n0]$, $H(x) = b_n^1 > 0$ if $x \in [10^n1]$;

(3) $\sum_{n\geq 1} na_n^0 < +\infty$, $\sum_{n\geq 1} na_n^1 < +\infty$; and

(4) $\sum_{k\geq 1} \sup_{n\geq 0} |b_k^0 - b_{k+n}^0| < +\infty$, $\sum_{k\geq 1} \sup_{n\geq 0} |b_k^1 - b_{k+n}^1| < +\infty$.

Define

$$H_{\min}^0 := \inf_{n\geq 1}\left\{b_n^0 + \sum_{k=1}^{n-1} a_k^1\right\}, \quad H_\infty^0 := \lim_{n\to+\infty} b_n^0 + \sum_{n\geq 1} a_n^1,$$

$$H_{\min}^1 := \inf_{n\geq 1}\left\{b_n^1 + \sum_{k=1}^{n-1} a_k^0\right\}, \quad H_\infty^1 := \lim_{n\to+\infty} b_n^1 + \sum_{n\geq 1} a_n^0.$$

As example of a double-well type potential, consider $H : \Sigma \to [0, +\infty)$, given by $H(0^\infty) = 0 = H(1^\infty)$ and $H(x) = \rho_0^{\theta_0(x)}\rho_1^{\theta_1(x)}$ if x is not a fixed point, where $\rho_0, \rho_1 \in (0, 1)$ and $\theta_0, \theta_1 \geq 1$ are functions such that their restrictions $\theta_0|_{[1]}$, $\theta_1|_{[0]}$, $\theta_0|_{[0^n1]}$ and $\theta_1|_{[1^n0]}$ are identically constant and satisfy $\inf_{n\geq 1}\{\theta_0|_{[0^{n+1}1]} - \theta_0|_{[0^n1]}, \theta_1|_{[1^{n+1}0]} - \theta_1|_{[1^n0]}\} > 0$. For this particular example, Gibbs measures do converge when the system is frozen, as follows from our main result.

Our main theorem describes the zero-temperature phase diagram of double-well type potentials (see Figure 1). The different regions of the diagram are described by a unique parameter, obtained by taking the minimum of three exponents

$$\gamma := \min\left\{\tfrac{1}{2}(H_\infty^1 + H_\infty^0),\, H_{\min}^0 + H_\infty^1,\, H_{\min}^1 + H_\infty^0\right\}. \tag{1.2}$$

Notice that $\gamma = 0$ if and only if $H_\infty^0 = H_\infty^1 = 0$ and if and only if the three exponents coincide. By symmetry, we may assume that $H_\infty^0 \leq H_\infty^1$. We state the theorem in this case. If $\gamma > 0$, one exponent is irrelevant and

$$\gamma = \min\left\{\tfrac{1}{2}(H_\infty^1 + H_\infty^0),\, H_{\min}^1 + H_\infty^0\right\},$$

since $\frac{1}{2}(H_\infty^1 + H_\infty^0) < H_{\min}^0 + H_\infty^1$. We introduce, in that case, the coincidence number κ, which counts how many times the minimum is attained, that is, for $H_n^1 := b_n^1 + \sum_{k=1}^{n-1} a_k^0$,

$$\kappa := \operatorname{card}\left\{n \geq 1 : \tfrac{1}{2}(H_\infty^1 + H_\infty^0) = H_n^1 + H_\infty^0\right\}, \tag{1.3}$$

and a coefficient c, which is the largest solution of the equation $X^2 = \kappa X + 1$,

$$c := \frac{\kappa + \sqrt{\kappa^2 + 4}}{2}. \tag{1.4}$$

Our main theorem is thus stated as follows.

THEOREM 1.2. *Let $H : \Sigma \to \mathbb{R}$ be a double-well type potential. Let μ_β be the Gibbs measure of H at temperature β^{-1}. Assume that $H_\infty^0 \leq H_\infty^1$.*

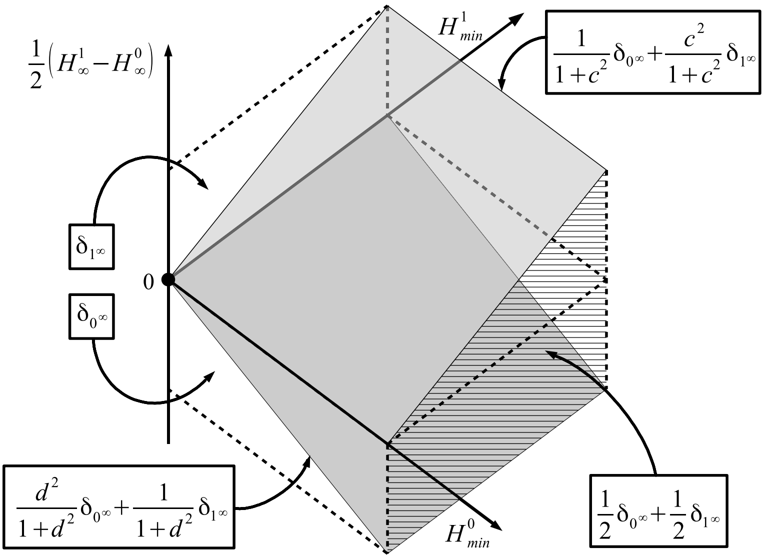

FIGURE 1. Zero-temperature phase diagram. The non-selection case can occur only at the origin. The formulas in the boxes are the limit measures at zero temperature. The two gray planes correspond to the cases of the coincidence of two exponents. Outside these planes, the limit measures are barycenters with rational coefficients. If $H^1_\infty \geq H^0_\infty$, then c is the coefficient given by (1.4). If $H^0_\infty \geq H^1_\infty$, then d is the analogous coefficient.

(1) *If* $\frac{1}{2}(H^1_\infty + H^0_\infty) > H^1_{\min} + H^0_\infty$, *then* $\lim_{\beta\to+\infty} \mu_\beta = \delta_{1^\infty}$.
(2) *If* $H^1_{\min} + H^0_\infty \geq \frac{1}{2}(H^1_\infty + H^0_\infty) > 0$, *then*

$$\lim_{\beta\to+\infty} \mu_\beta = \frac{1}{1+c^2}\delta_{0^\infty} + \frac{c^2}{1+c^2}\delta_{1^\infty}. \tag{1.5}$$

(3) *If* $H^0_\infty = H^1_\infty = 0$, *then there exists a particular choice of* b^0_n, b^1_n *(necessarily* $a^0_n = a^1_n = 0$*) such that* H *is Lipschitz and* μ_β *does not converge. More precisely, there exists a sequence* $\beta_k \to +\infty$ *such that* $\lim_{k\to+\infty} \mu_{\beta_{2k}} = \delta_{0^\infty}$ *and* $\lim_{k\to+\infty} \mu_{\beta_{2k+1}} = \delta_{1^\infty}$.

(Items (1) and (2) correspond to $\gamma > 0$*; item (3) corresponds to* $\gamma = 0$*.)*

In §2, we give general results for potentials of summable variation. In §3, for a double-well type potential H, we compute the measure of every cylinder using two series that capture all the complexity of the limit. In §4, we prove the convergence of Gibbs measures when $\gamma > 0$. Finally, in §5, we provide examples of divergence with $\gamma = 0$. Note that the symmetric case $a^0_n = a^1_n$ and $b^0_n = b^1_n$ gives, in both cases, $\gamma > 0$ or $\gamma = 0$ the convergence to $\frac{1}{2}\delta_{0^\infty} + \frac{1}{2}\delta_{1^\infty}$.

We also show that, in this particular class of potentials, the dichotomy selection/non-selection in Theorem 1.2 can be expressed in terms of the Peierls barrier between the two configurations 0^∞ and 1^∞. The Peierls barrier is defined for any potential with summable variation by

$$h(x, y) := \lim_{p\to+\infty} \lim_{n\to+\infty} S_n^p(x, y) \quad \text{where}$$

$$S_n^p(x, y) := \inf\left\{\sum_{i=0}^{k-1}[H\circ\sigma^i(z) - \bar{H}] : k \geq n,\, z \in \Sigma,\, z \overset{p}{=} x,\, \sigma^n(z) \overset{p}{=} y\right\},$$

$$\bar{H} := \lim_{n\to+\infty} \inf\left\{\frac{1}{n}\sum_{k=0}^{n-1} H\circ\sigma^k(x) : x \in \Sigma\right\}.$$

The Peierls barrier indicates the minimal algebraic cost from x to y using a normalized potential $H - \bar{H}$. In the particular case of double-well type potentials, we have the following result.

COROLLARY 1.3. *Let H be a double-well type potential. Then:*
(1) $\frac{1}{2}(H_\infty^0 + H_\infty^1) = \frac{1}{2}(h(0^\infty, 1^\infty) + h(1^\infty, 0^\infty))$*;*
(2) $H_{\min}^0 + H_\infty^1 = \liminf_{x\to 0^\infty} h(x, 0^\infty)$*;*
(3) $H_{\min}^1 + H_\infty^0 = \liminf_{x\to 1^\infty} h(x, 1^\infty)$*; and*
(4) *the non-selection happens if and only if* $h(0^\infty, 1^\infty) = h(1^\infty, 0^\infty) = 0$.

Note that γ may be seen as the minimum of three energy barriers: $\frac{1}{2}(H_\infty^0 + H_\infty^1)$, the mean energy barrier of a cycle of second order between the two ground states 0^∞ and 1^∞; $H_{\min}^0 + H_\infty^1$, the energy barrier of a cycle of first order at 0^∞; and $H_{\min}^1 + H_\infty^0$, a similar energy barrier at 1^∞.

2. *Basic facts for potentials of summable variation*

In this section, we gather some of the main elements of ergodic optimization theory for potentials of summable variation. Ergodic optimization may be seen as a counterpart at zero temperature of thermodynamic formalism. A useful viewpoint on ergodic optimization is provided by Aubry–Mather theory. For more information, we refer the reader, for instance, to [**12**, **13**] and the references therein.

Definition 2.1. For $H \in C^0(\Sigma)$, a minimizing measure $\mu_{\min}$ is a σ-invariant probability such that

$$\int H\, d\mu_{\min} = \min\left\{\int H\, d\nu : \nu \text{ is a } \sigma\text{-invariant probability measure}\right\}.$$

We call a Mather set of H the invariant compact set

$$\text{Mather}(H) := \bigcup\{\text{supp}(\mu) : \mu \text{ is minimizing}\}.$$

We call a minimizing ergodic value of H the constant

$$\bar{H} := \int H\, d\mu_{\min}.$$

We recall or extend basic results about the Peierls barrier for functions with summable variation.

PROPOSITION 2.2. *If H has summable variation, then*

$$\mathrm{Mather}(H) \subset \{x \in \Sigma : h(x, x) = 0\}. \tag{2.1}$$

The previous proposition follows from Atkinson's theorem [**1**] and from the existence of a continuous calibrated sub-action.

Definition 2.3. We call the Lax–Oleinik operator the nonlinear operator acting on continuous functions $V \in C^0(\Sigma)$ defined by

$$T[V](y) := \min\{V(x) + H(x) : x \in \Sigma, \sigma(x) = y\} \quad \text{for all } y \in \Sigma.$$

We call a calibrated sub-action any continuous function V solution of the equation $T[V] = V + \bar{H}$.

Clearly, $V \circ \sigma - V \leq H - \bar{H}$ when V is a calibrated sub-action, which, in particular, ensures that $h(x, x) \geq 0$ for all $x \in \Sigma$. Atkinson's theorem provides the opposite inequality if $x \in \mathrm{Mather}(H)$. These are the main ingredients of the proof of Proposition 2.2. To obtain a calibrated sub-action, we will introduce a stronger notion of regularity on $C^0(\Sigma)$. Consider thus

$$\mathbb{K} := \left\{ V \in C^0(\Sigma) : \forall n \geq 1,\ \mathrm{var}(V, n) \leq \sum_{k \geq n+1} \mathrm{var}(H, k) \right\}.$$

We also recall that the transfer operator is defined on the space $C^0(\Sigma)$ by

$$\mathcal{L}_\beta[\Phi](x) = e^{-\beta H(0x)}\Phi(0x) + e^{-\beta H(1x)}\Phi(1x) \quad \text{for all } x \in \Sigma.$$

The next theorem contains a version of Ruelle–Perron–Frobenius theorem and provides a calibrated sub-action in the context of potentials with summable variation, making explicit well-known connections between thermodynamic formalism and ergodic theory.

THEOREM 2.4. *Let $H : \Sigma \to \mathbb{R}$ be a potential with summable variation.*

(1) *The transfer operator admits a unique positive and continuous eigenfunction Φ_β satisfying* $\max \Phi_\beta = 1$, *which is associated with a positive eigenvalue λ_β.*
(2) *If $V_\beta := -(1/\beta) \ln \Phi_\beta$, then $V_\beta \in \mathbb{K}$ and* $\min V_\beta = 0$.
(3) *The dual operator $\mathcal{L}_\beta^*$ admits a unique eigenprobability ν_β. The corresponding eigenvalue is equal to λ_β, $\mathcal{L}_\beta^*[\nu_\beta] = \lambda_\beta \nu_\beta$.*
(4) *Define $\mu_\beta := \Phi_\beta \nu_\beta / \int \Phi_\beta \, d\nu_\beta$. Then μ_β is a σ-invariant probability measure, and any weak* accumulation point of μ_β as $\beta \to +\infty$ is a minimizing measure.*
(5) *There exists a sequence $\beta_k \to +\infty$ such that (in the sup-norm topology) $\{V_{\beta_k}\}$ converges to a function $V_\infty \in \mathbb{K}$ with* $\min V_\infty = 0$. *Moreover, any accumulation function V_∞ of $\{V_\beta\}$ as $\beta \to +\infty$ is a calibrated sub-action for H.*

Proof. The proof of these results are standard (see [**13**, **19**, **20**]), and hence we focus on the part leading to the existence of calibrated sub-actions. We define a nonlinear operator T_β by

$$T_\beta[u] := -\frac{1}{\beta} \ln(\mathcal{L}_\beta[\exp(-\beta u)]).$$

Fix $x_0 \in \Sigma$ and define $\mathbb{K}_0 := \{U \in \mathbb{K} : U(x_0) = 0\}$. The set $\mathbb{K}_0$ is closed in the $C^0(\Sigma)$ topology and bounded. By the uniform continuity of $\mathbb{K}$ and Arzelà–Ascoli theorem, the set $\mathbb{K}_0$ is compact. In addition, $\mathbb{K}_0$ is convex.

If $x \stackrel{n}{=} y$, then

$$T_\beta[u](x) - T_\beta[u](y) \le \mathrm{var}(H, n+1) + \mathrm{var}(u, n+1).$$

In particular, $\mathrm{var}(T_\beta[u], n) \le \mathrm{var}(H, n+1) + \mathrm{var}(u, n+1)$ and the map

$$\tilde{T}_\beta[u] := T_\beta[u] - T_\beta[u](x_0)$$

preserves $\mathbb{K}_0$. By the Schauder theorem, $\tilde{T}_\beta$ admits a fixed point or, in an equivalent way, T_β admits an additive eigenfunction $T_\beta[U_\beta] = U_\beta + \bar{H}_\beta$, which yields

$$\mathcal{L}_\beta[\Phi_\beta] = \lambda_\beta \Phi_\beta \quad \text{with } \Phi_\beta := e^{-\beta(U_\beta - \min U_\beta)},\ \lambda_\beta = e^{-\beta \bar{H}_\beta}.$$

Let $\tilde{\Phi}$ be another positive and continuous eigenfunction associated with some positive eigenvalue $\tilde{\lambda}$. We choose $s, t > 0$ such that $s\Phi_\beta \le \tilde{\Phi} \le t\Phi_\beta$. By iterating $\mathcal{L}_\beta$, we obtain $s\lambda_\beta^n \Phi_\beta \le \tilde{\lambda}^n \tilde{\Phi} \le t\lambda_\beta^n \Phi_\beta$. Then $\tilde{\lambda} = \lambda_\beta$. Let s be such that $\min(\tilde{\Phi} - s\Phi_\beta) = 0$. Then the identity

$$\mathcal{L}_\beta[\tilde{\Phi} - s\Phi_\beta] = \lambda_\beta(\tilde{\Phi} - s\Phi_\beta)$$

implies that the set $\arg\min_x (\tilde{\Phi} - s\Phi_\beta)(x)$ is invariant by σ^{-1} and therefore $\tilde{\Phi} = s\Phi_\beta$. The uniqueness of the eigenfunction is proved.

Note that the family $\{V_\beta = -(1/\beta)\ln \Phi_\beta\}_{\beta > 0}$ belongs to the compact subset $\{V \in \mathbb{K} : \min V = 0\}$. Passing to the limit with respect to a suitable sequence $\beta_k \to +\infty$, we see that $T[V_\infty] = V_\infty + c$ for $c = \lim \bar{H}_{\beta_k}$. From min-plus algebra, it is well known that the only additive eigenvalue is $c = \bar{H}$. □

The following proposition shows how calibrated sub-actions are related to the Peierls barrier.

PROPOSITION 2.5. *If H has summable variation, then the following items hold.*

(1) *For every $x \in \mathrm{Mather}(H)$, as a function of its second variable, $h(x, \cdot)$ belongs to $\mathbb{K}$ and is a calibrated sub-action.*

(2) *If $V \in C^0(\Sigma)$ is a calibrated sub-action, then $V \in \mathbb{K}$ and V admits a representation formula*†

$$V(y) = \min\{V(x) + h(x, y) : x \in \mathrm{Mather}(H)\} \quad \textit{for all } y \in \Sigma. \tag{2.2}$$

Proof. For the Lipschitz class, these results may be found in the literature (see, for instance, [**12**, **13**] and the references therein). All proofs may be easily extended by just adapting the arguments to the regularity here considered. For the convenience of the reader, we outline the proofs of items (1) and (2).

Item (2). Suppose $y \stackrel{n}{=} z$. Denoting $y_0 = y$, since V is a calibrated sub-action, there exists a sequence $\{y_k\} \subset \Sigma$ such that

$$V(y_0) = V(y_k) + \sum_{i=1}^{k-1} [H \circ \sigma^i(y_k) - \bar{H}], \quad \sigma(y_k) = y_{k-1} \quad \text{for all } k \ge 1. \tag{2.3}$$

† This representation is usually stated using the Aubry set instead of the Mather set.

For $z_0 = z$, we thus consider a sequence $\{z_k\}$, with $\sigma(z_k) = z_{k-1}$, such that $z_k \overset{n+k}{=} y_k$ for all k. Note that

$$V(z_0) \leq V(z_k) + \sum_{i=1}^{k-1} [H \circ \sigma^i(z_k) - \bar{H}] \quad \text{for all } k \geq 1. \tag{2.4}$$

From (2.3) and (2.4), $\mathrm{var}(V, n) \leq \sum_{k \geq n+1} \mathrm{var}(H, k)$: that is, $V \in \mathbb{K}$.

From the inequality $V \circ \sigma - V \leq H - \bar{H}$, given any $y \in \Sigma$, $V(y) \leq \min\{V(x) + h(x, y) : x \in \mathrm{Mather}(H)\}$. For $y_0 = y$, we consider again (2.3). Since $V(y_k) = V(y_{k+p}) + \sum_{i=1}^{p-1} [H \circ \sigma^i(y_{k+p}) - \bar{H}]$ for all $k, p \geq 0$, one may deduce that a limit $\bar{x} \in \Sigma$ of subsequence $\{y_{k_j}\}$ satisfies $h(\bar{x}, \bar{x}) = 0$. By passing to the limit in $V(y_0) = V(y_{k_j}) + \sum_{i=1}^{k_j-1} [H \circ \sigma^i(y_{k_j}) - \bar{H}]$, we see that $V(y) = V(\bar{x}) + h(\bar{x}, y)$. For all x in the same irreducible class as $\bar{x}$ (see [**12**, Definition 18]), we may extend the equality $V(y) = V(x) + h(x, y)$. As in [**12**, Proposition 19], and also for the summable variation case, each irreducible class is compact and invariant, so that it contains the support of at least one minimizing measure.

Item (1). It suffices to explain how to show that $h(x, \cdot)$, $x \in \mathrm{Mather}(H)$ is a calibrated sub-action. The argument is standard. For $x \in \mathrm{Mather}(H)$, one may use Atkinson's theorem [**1**] to obtain that, as a function of the second variable, $h(x, \cdot)$ is finite everywhere on Σ. Then the calibration property follows from the very definition of the Peierls barrier. For details, see [**12**, **13**] and the references therein. □

3. *Explicit formulas for double-well type potentials*

From now on, we assume that H is a double-well type potential (see Definition 1.1). We show, in Lemma 3.2, that we can reduce the complexity of the notation by taking a suitable coboundary, which is constant on a countable infinity of cylinders. As the issue of selection or non-selection is independent of the cohomological class of the potential, this lemma will enable us to simplify the proof by using the following reduced assumptions.

Definition 3.1. Let H be a double-well type potential. We say that H is reduced if $H = 0$ on $[00] \cup [11]$. More precisely, for every $n \geq 0$:

(1) $H(x) = 0$ if $x \in [00] \cup [11]$;

(2) $H(x) = H_n^0 > 0$ if $x \in [01^n0]$, $H(x) = H_n^1 > 0$ if $x \in [10^n1]$; and

(3) $\sum_{k \geq 1} \sup_{n \geq 0} |H_k^0 - H_{k+n}^0| < +\infty$, $\sum_{k \geq 1} \sup_{n \geq 0} |H_k^1 - H_{k+n}^1| < +\infty$.

Define

$$H_\infty^0 := \lim_{n \to +\infty} H_n^0, \quad H_\infty^1 := \lim_{n \to +\infty} H_n^1,$$
$$H_{\min}^0 := \inf_{n \geq 1} H_n^0, \quad H_{\min}^1 := \inf_{n \geq 1} H_n^1.$$

LEMMA 3.2. *If H is a double-well type potential, then there exists a function $V : \Sigma \to \mathbb{R}$, which is constant on a countable infinity of cylinders, such that $\tilde{H} := H - (V \circ \sigma - V)$ is reduced.*

Proof. Let

$$V(x) := \sum_{k=n}^{+\infty} a_k^0 + \sum_{k\geq 1} a_k^1 \quad \text{if } x \in [0^n 1] \text{ and } n \geq 1,$$

$$V(x) := \sum_{k=n}^{+\infty} a_n^1 + \sum_{k\geq 1} a_k^0 \quad \text{if } x \in [1^n 0] \text{ and } n \geq 1.$$

Then

$$V \circ \sigma - V = \begin{cases} \displaystyle\sum_{k\geq n} a_k^0 - \sum_{k\geq n+1} a_k^0 = a_n^0 & \text{on } [00^n 1], \\ \displaystyle\sum_{k\geq n} a_k^1 - \sum_{k\geq n+1} a_k^1 = a_n^1 & \text{on } [11^n 0], \\ \displaystyle\Big(\sum_{k\geq n} a_k^0 + \sum_{k\geq 1} a_k^1\Big) - \Big(\sum_{k\geq 1} a_k^1 + \sum_{k\geq 1} a_k^0\Big) & \text{on } [10^n 1], \\ \displaystyle\Big(\sum_{k\geq n} a_k^1 + \sum_{k\geq 1} a_k^0\Big) - \Big(\sum_{k\geq 1} a_k^0 + \sum_{k\geq 1} a_k^1\Big) & \text{on } [01^n 0]. \end{cases}$$

The new double-well type potential $\tilde{H} := H - (V \circ \sigma - V)$ becomes

$$\tilde{H}(x) = 0 \quad \text{if } x \in [00] \cup [11],$$

$$\tilde{H}(x) = H_n^0 := b_n^0 + \sum_{k=1}^{n-1} a_k^1 \quad \text{if } x \in [01^n 0],$$

$$\tilde{H}(x) = H_n^1 := b_n^1 + \sum_{k=1}^{n-1} a_k^0 \quad \text{if } x \in [10^n 1]. \qquad \square$$

From now on, H is supposed to be a reduced double-well type potential. We follow the same methods as in [**4**] and [**18**]. Our main goal is to find the characteristic equation of the eigenvalue λ_β and the measures $\mu_\beta([0])$ and $\mu_\beta([1])$. We also want to identify the criterion of divergence in terms of the Peierls barrier.

Since H is non-negative and $H(0^\infty) = H(1^\infty) = 0$, H has null ergodic minimizing value: $\bar{H} = 0$. Since $\{0^\infty, 1^\infty\}$ is the only invariant set included in $\{H = 0\} \subset [00] \cup [11] \cup \{01^\infty, 10^\infty\}$, the Mather set is reduced to the two fixed points, namely, $\text{Mather}(H) = \{0^\infty, 1^\infty\}$.

The next proposition gives a complete description of the Peierls barrier.

PROPOSITION 3.3. *If H is a reduced double-well type potential, then:*

(1) $h(0^\infty, x) = 0$ *for all* $x \in [0]$, *(in particular* $h(0^\infty, 0^\infty) = 0$*);*

(2) $h(0^\infty, x) = \inf_{k\geq n} H_k^0$ *for all* $x \in [1^n 0]$ *(in particular,* $h(0^\infty, 1^\infty) = H_\infty^0$*);*

(3) $\liminf_{x\to 0^\infty} h(x, 0^\infty) = H_{\min}^0 + H_\infty^1$*;*

(4) $h(1^\infty, x) = 0$ *for all* $x \in [1]$ *(in particular,* $h(1^\infty, 1^\infty) = 0$*);*

(5) $h(1^\infty, x) = \inf_{k\geq n} H_k^1$ *for all* $x \in [0^n 1]$ *(in particular,* $h(1^\infty, 0^\infty) = H_\infty^1$*); and*

(6) $\liminf_{x\to 1^\infty} h(x, 1^\infty) = H_{\min}^1 + H_\infty^0$.

Proof.

Item (1) Clearly, $h(0^\infty, x) = 0$ for all $x \in [0]$, since $H \geq 0$ and $H = 0$ on $[00]$.

Item (2) Let $x \in [1^n 0]$ and $p \geq 1$. Every $z \in \Sigma$ satisfying $z \overset{p}{=} 0^\infty$ and $\sigma^k(z) \overset{p}{=} x$ belongs to $[0^{m_1} 1^{n_1} \ldots 0^{m_r} 1^{n_r} 0]$, with $m_1 \geq p$, $n_r \geq n$ and $k = m_1 + n_1 + \cdots + n_r - n$. The corresponding sum $\sum_{i=0}^{k-1} [H \circ \sigma^i(z) - \bar{H}]$ is $H^0_{n_1} + H^1_{m_2} + \cdots + H^0_{n_r}$, which gives (for every $m \geq p$)

$$S^p_m(0^\infty, x) = \inf_{k \geq n} H^0_k, \quad h(0^\infty, x) = \inf_{k \geq n} H^0_k.$$

By continuity of $x \mapsto h(0^\infty, x)$ (see Proposition 2.5), $h(0^\infty, 1^\infty) = H^0_\infty$.

Item (3) On the one hand, if $x \in [0]$, $x \neq 0^\infty$ and $p \geq 1$, then every z satisfying $z \overset{p}{=} x$ and $\sigma^k(z) \overset{p}{=} 0^\infty$ has the form $z = 0^{m_1} 1^{n_1} \cdots 0^{m_r} 1^{n_r} 0^p \cdots$ with $m_i \geq 1$, $n_i \geq 1$ and $k = m_1 + n_1 + \cdots + n_r$. The corresponding sum $\sum_{i=0}^{k-1} [H \circ \sigma^i(z) - \bar{H}]$ is bounded from below by $H^0_{\min} + \inf_{q \geq p} H^1_q$ and we obtain $h(x, 0^\infty) \geq H^0_{\min} + H^1_\infty$. On the other hand, for every $m, n \geq 1$ and $k \geq p \geq m + n$, $S^p_k(0^m 1^n 0^\infty, 0^\infty) = H^0_n + H^1_\infty$. These facts together imply that

$$\liminf_{x \to 0^\infty} h(x, 0^\infty) = H^0_{\min} + H^1_\infty.$$

The other expressions are similarly obtained by permuting zero and one. □

We recall the notion of a Jacobian J of a probability measure ν that is not necessarily invariant by the shift σ. It is a non-negative Borel function $J : \Sigma \to \mathbb{R}^+$ such that, for every bounded Borel test function $f : \Sigma \to \mathbb{R}$,

$$\int_{[0]} f \circ \sigma(x) J(x) \, d\nu(x) = \int_{[1]} f \circ \sigma(x) J(x) \, d\nu(x) = \int_\Sigma f(x) \, d\nu(x).$$

Note that, if such a Jacobian exists, it is unique.

From now on, whenever a function $f : \Sigma \to \mathbb{R}$ is constant on a cylinder $[i_0 i_1 \ldots i_{n-1}]$, we denote by $f(i_0 i_1 \ldots i_{n-1})$ the constant value $f|_{[i_0 i_1 \ldots i_{n-1}]}$.

PROPOSITION 3.4. *Let H be a reduced double-well type potential. Let Φ_β, ν_β and λ_β be the solutions of the Perron–Frobenius equation, as defined in Theorem 2.4. Then Φ_β is constant on every cylinder $[0^n 1]$ or $[1^n 0]$, $n \geq 1$ and ν_β has constant Jacobian J_β on the cylinders $[0^2]$, $[1^2]$, $[01^n 0]$ and $[10^n 1]$, $n \geq 1$. More precisely:*

(1) $$\Phi_\beta(0^n 1) = \sum_{k \geq n} \frac{\exp(-\beta H^1_k)}{\lambda_\beta^{k-n+1}} \Phi_\beta(10), \quad \Phi_\beta(0^\infty) = \frac{\exp(-\beta H^1_\infty)}{\lambda_\beta - 1} \Phi_\beta(10);$$

(2) $$\Phi_\beta(1^n 0) = \sum_{k \geq n} \frac{\exp(-\beta H^0_k)}{\lambda_\beta^{k-n+1}} \Phi_\beta(01), \quad \Phi_\beta(1^\infty) = \exp(-\beta H^0_\infty)/\lambda_\beta - 1 \Phi_\beta(01);$$

(3) *if* $H^0_\infty = H^1_\infty = 0$, *then* $\max \Phi_\beta = \max\{\Phi_\beta(0^\infty), \Phi_\beta(1^\infty)\} = 1$;

(4) $$\nu_\beta[1^n 0] = \frac{1}{\lambda_\beta^{n-1}} \nu_\beta[10] \quad \text{or} \quad J_\beta(x) = \lambda_\beta \quad \text{for all } x \in [1^2];$$

(5) $$\nu_\beta[0^n 1] = \frac{1}{\lambda_\beta^{n-1}} \nu_\beta[01] \quad \text{or} \quad J_\beta(x) = \lambda_\beta \quad \text{for all } x \in [0^2];$$

(6) $\quad \nu_\beta[01^n0] = \dfrac{\exp(-\beta H_n^0)}{\lambda_\beta^n}\nu_\beta[10] \quad$ *or* $\quad J_\beta(x) = \dfrac{\lambda_\beta}{\exp(-\beta H_n^0)} \quad$ *for all* $x \in [01^n0]$*;*

and

(7) $\quad \nu_\beta[10^n1] = \dfrac{\exp(-\beta H_n^1)}{\lambda_\beta^n}\nu_\beta[01] \quad$ *or* $\quad J_\beta(x) = \dfrac{\lambda_\beta}{\exp(-\beta H_n^1)} \quad$ *for all* $x \in [10^n1]$.

Proof.

Part 1. The equation $\mathcal{L}_\beta[\Phi_\beta] = \lambda_\beta\Phi_\beta$ implies that

$$\begin{aligned}\Phi_\beta(0^n1) &= \frac{1}{\lambda_\beta}\Phi_\beta(0^{n+1}1) + \frac{1}{\lambda_\beta}\exp(-\beta H_n^1)\Phi_\beta(10)\\ &= \frac{1}{\lambda_\beta^2}\Phi_\beta(0^{n+2}1) + \left[\frac{1}{\lambda_\beta}\exp(-\beta H_n^1) + \frac{1}{\lambda_\beta^2}\exp(-\beta H_{n+1}^1)\right]\Phi_\beta(10)\\ &= \cdots = \left[\frac{1}{\lambda_\beta}\exp(-\beta H_n^1) + \frac{1}{\lambda_\beta^2}\exp(-\beta H_{n+1}^1) + \cdots\right]\Phi_\beta(10).\end{aligned}$$

A similar computation is carried out for $\Phi_\beta(1^n0)$.

Part 2. For every bounded Borel function $f : \Sigma \to \mathbb{R}$,

$$\int \mathbb{1}_{[0]} f\circ\sigma \frac{\lambda_\beta}{\exp(-\beta H)}\,d\nu_\beta = \int \frac{\mathcal{L}_\beta}{\lambda_\beta}\left[\mathbb{1}_{[0]} f\circ\sigma \frac{\lambda_\beta}{\exp(-\beta H)}\right]d\nu_\beta = \int f\,d\nu_\beta.$$

A similar computation is carried out for $\mathbb{1}_{[1]}$. We thus obtain

$$J_\beta(x) = \frac{\lambda_\beta}{\exp(-\beta H(x))} \quad \text{for all } x \in \Sigma.$$

In particular, $J_\beta(x) = \lambda_\beta$ for $x \in [0^2] \cup [1^2]$, $J_\beta(x) = \lambda_\beta/\exp(-\beta H_n^0)$ for $x \in [01^n0]$ and $J_\beta(x) = \lambda_\beta/\exp(-\beta H_n^1)$ for $x \in [10^n1]$.

Part 3. With respect to the eigenmeasure, we discuss items (4) and (6); the others are similarly proved. Hence, by applying the Jacobian, just note that

$$\begin{aligned}\nu_\beta[10] &= \lambda_\beta\nu_\beta[1^20] = \lambda_\beta^2\nu_\beta[1^30] = \cdots = \lambda_\beta^{n-1}\nu_\beta[1^n0]\\ &= \frac{\lambda_\beta^n}{\exp(-\beta H_n^0)}\nu_\beta[01^n0].\end{aligned}$$

□

For every reduced double-well type potential, we define the following analytic functions that will play a fundamental role in the dichotomy

$$F_\beta^0(\lambda) := \sum_{k\geq 1}\frac{1}{\lambda^k}\exp(-\beta H_k^0), \qquad F_\beta^1(\lambda) := \sum_{k\geq 1}\frac{1}{\lambda^k}\exp(-\beta H_k^1), \tag{3.1}$$

$$\tilde{F}_\beta^0(\lambda) := \sum_{k\geq 1}\frac{k}{\lambda^k}\exp(-\beta H_k^0), \qquad \tilde{F}_\beta^1(\lambda) := \sum_{k\geq 1}\frac{k}{\lambda^k}\exp(-\beta H_k^1). \tag{3.2}$$

We will also keep in mind the following equalities

$$\text{for all } N \geq 0, \qquad \sum_{k\geq N+1}\frac{1}{\lambda^k} = \frac{1}{\lambda^N(\lambda-1)}, \qquad \sum_{k\geq N+1}\frac{k}{\lambda^k} = \frac{N(\lambda-1)+\lambda}{\lambda^N(\lambda-1)^2}. \tag{3.3}$$

COROLLARY 3.5. *Let H be a reduced double-well type potential. Then:*

(1) $F^0_\beta(\lambda_\beta)F^1_\beta(\lambda_\beta) = 1$ *(the characteristic equation);*

(2) $\Phi_\beta(01) = F^1_\beta(\lambda_\beta)\Phi_\beta(10)$, $\Phi_\beta(10) = F^0_\beta(\lambda_\beta)\Phi_\beta(01)$*; and*

(3) $\nu_\beta[01] = F^0_\beta(\lambda_\beta)\nu_\beta[10]$, $\nu_\beta[10] = F^1_\beta(\lambda_\beta)\nu_\beta[01]$.

Proof. Item (1) of Proposition 3.4 implies, by taking $n = 1$, that

$$\Phi_\beta(01) = F^1_\beta(\lambda_\beta)\Phi_\beta(10) \quad \text{and} \quad \Phi_\beta(10) = F^0_\beta(\lambda_\beta)\Phi_\beta(01).$$

By multiplying term to term, we obtain $F^0_\beta(\lambda_\beta)F^1_\beta(\lambda_\beta) = 1$. Also

$$\nu_\beta[01] = \sum_{n\geq 1} \nu_\beta[01^n0] = \sum_{n\geq 1} \frac{1}{\lambda^n_\beta} \exp(-\beta H^0_n)\nu_\beta[10] = F^0_\beta(\lambda_\beta)\nu_\beta[10]. \qquad \square$$

COROLLARY 3.6. *Let H be a reduced double-well type potential. Then:*

(1) $\mu_\beta[01] = \mu_\beta[10]$*;*

(2) $$\frac{\mu_\beta[0^n1]}{\mu_\beta[01]} = \bigg[\sum_{k\geq n} \frac{1}{\lambda^k_\beta} \exp(-\beta H^1_k)\bigg] F^0_\beta(\lambda_\beta), \qquad \frac{\mu_\beta[0]}{\mu_\beta[01]} = \frac{\tilde F^1_\beta(\lambda_\beta)}{F^1_\beta(\lambda_\beta)};$$

(3) $$\frac{\mu_\beta[1^n0]}{\mu_\beta[10]} = \bigg[\sum_{k\geq n} \frac{1}{\lambda^k_\beta} \exp(-\beta H^0_k)\bigg] F^1_\beta(\lambda_\beta), \qquad \frac{\mu_\beta[1]}{\mu_\beta[10]} = \frac{\tilde F^0_\beta(\lambda_\beta)}{F^0_\beta(\lambda_\beta)};$$

(4) $$\frac{\mu_\beta[01^n0]}{\mu_\beta[10]} = \frac{\exp(-\beta H^0_n)F^1_\beta(\lambda_\beta)}{\lambda^n_\beta}, \qquad \frac{\mu_\beta[10^n1]}{\mu_\beta[01]} = \frac{\exp(-\beta H^1_n)F^0_\beta(\lambda_\beta)}{\lambda^n_\beta}; \text{ and}$$

(5) $$\frac{\mu_\beta[0]}{\mu_\beta[1]} = \frac{F^0_\beta(\lambda_\beta)}{F^1_\beta(\lambda_\beta)} \frac{\tilde F^1_\beta(\lambda_\beta)}{\tilde F^0_\beta(\lambda_\beta)}.$$

We know that $\lambda_\beta \to 1$ as $\beta \to +\infty$. In order to understand the behavior of μ_β, it is fundamental to have a better Puiseux series expansion of λ_β, as is the case for potentials that depend on finite number of coordinates (see [**13**]). The log-scale limit, the limit of $-(1/\beta)\ln(\lambda_\beta - 1)$, is usually easy to obtain using a min-plus technique. This may be sufficient to show the convergence of μ_β when there is no coincidence of exponents, as happens in [**5**]. Usually, the limit is then a periodic measure. In general, the log-scale limit is not sufficient and an expansion of the form $\lambda_\beta = 1 + ce^{-\beta\gamma} + o(e^{-\beta\gamma})$ needs to be founded, as in [**4**, **18**]. A barycenter of periodic measures with irrational coefficients may be the limit in this case. Let us recall, from equation (1.2), the definition of the key parameter γ, which, from now on, we call the Puiseux exponent

$$\gamma := \min\big\{\tfrac{1}{2}(H^1_\infty + H^0_\infty),\, H^0_{\min} + H^1_\infty,\, H^1_{\min} + H^0_\infty\big\}.$$

The coincidence of exponents is understood in the sense that the minimum γ may be attained several times. The following proposition gives the log-scale limit of the main quantities that appear in the dichotomy. We will give better estimates in the next section.

PROPOSITION 3.7. *Let H be a reduced double-well type potential. Then:*

(1) $\lim_{\beta\to+\infty} -\frac{1}{\beta}\ln(\lambda_\beta - 1) = \gamma$;

(2) $\lim_{\beta\to+\infty} -\frac{1}{\beta}\ln F_\beta^{0,1}(\lambda_\beta) = \min_{n\geq 1}\{H_n^{0,1},\ H_\infty^{0,1} - \gamma\}$; and

(3) $\lim_{\beta\to+\infty} -\frac{1}{\beta}\ln \tilde{F}_\beta^{0,1}(\lambda_\beta) = \min_{n\geq 1}\{H_n^{0,1},\ H_\infty^{0,1} - 2\gamma\}$.

Proof.

Part 1. We claim that any limit point of $-(1/\beta)\ln(\lambda_\beta - 1)$ is finite. Recall that H is non-negative and $\max \Phi_\beta = 1$. Hence, given $x_\beta^{\max} \in \arg\max \Phi_\beta$, we see that $\lambda_\beta = \mathcal{L}_\beta[\Phi_\beta](x_\beta^{\max}) \leq 2$. Since $\lambda_\beta \Phi_\beta(0^\infty) = \mathcal{L}_\beta[\Phi_\beta](0^\infty)$ yields $\lambda_\beta = 1 + \exp(-\beta H_\infty^1)\Phi_\beta(10^\infty)/\Phi_\beta(0^\infty) \geq 1$, we have the *a priori* estimate $1 \leq \lambda_\beta \leq 2$. Furthermore, from

$$\frac{\exp(-\beta \max_k H_k^0)}{\lambda_\beta - 1} \leq F_\beta^0(\lambda_\beta) = \frac{1}{F_\beta^1(\lambda_\beta)} \leq \frac{\lambda_\beta - 1}{\exp(-\beta \max_k H_k^1)},$$

we conclude that $\exp(-\beta(\max H_k^0 + \max H_k^1)/2) \leq \lambda_\beta - 1 \leq 1$.

Part 2. For some subsequence $\beta \to +\infty$, assume that $-(1/\beta)\ln(\lambda_\beta - 1) \to \bar{\gamma}$. We claim that $-(1/\beta)\ln F_\beta^0(\lambda_\beta) \to \min_{n\geq 1}(H_n^0,\ H_\infty^0 - \bar{\gamma})$ for the same subsequence. Indeed, let $\epsilon > 0$. We choose $N \geq 1$ such that $|H_n^0 - H_\infty^0| < \epsilon$ for all $n \geq N$. We split the series (3.1) into two terms. For the first term, for β large enough,

$$\exp\Big(-\beta\Big(\min_{1\leq k\leq N} H_k^0 + \epsilon\Big)\Big) \leq \sum_{k=1}^{N} \frac{1}{\lambda_\beta^k}\exp(-\beta H_k^0) \leq \exp\Big(-\beta\Big(\min_{1\leq k\leq N} H_k^0 - \epsilon\Big)\Big).$$

For the second term, using the estimates (3.3), for β large enough

$$\exp(-\beta(\bar{\gamma} + \epsilon)) \leq \lambda_\beta^N(\lambda_\beta - 1) \leq \exp(-\beta(\bar{\gamma} - \epsilon)),$$

$$\frac{\exp(-\beta(H_\infty^0 + \epsilon))}{\lambda_\beta^N(\lambda_\beta - 1)} \leq \sum_{k>N} \frac{1}{\lambda_\beta^k}\exp(-\beta H_k^0) \leq \frac{\exp(-\beta(H_\infty^0 - \epsilon))}{\lambda_\beta^N(\lambda_\beta - 1)},$$

$$\exp(-\beta(H_\infty^0 - \bar{\gamma} + 2\epsilon)) \leq \sum_{k>N} \frac{1}{\lambda_\beta^k}\exp(-\beta H_k^0) \leq \exp(-\beta(H_\infty^0 - \bar{\gamma} - 2\epsilon)).$$

The claim is proved by adding the two terms, changing the scale and passing to the limits as $\beta \to +\infty$ and $\epsilon \to 0$.

Part 3. We show there is a unique limit point $\bar{\gamma}$ by showing that it is the unique solution of a min-plus equation. Indeed, from the characteristic equation $1 = F_\beta^0(\lambda_\beta)F_\beta^1(\lambda_\beta)$, we obtain

$$0 = \min_{n\geq 1}\{H_n^0,\ H_\infty^0 - \bar{\gamma}\} + \min_{n\geq 1}\{H_n^1,\ H_\infty^1 - \bar{\gamma}\}.$$

This equation is equivalent to

$$\min_{n\geq 1} H_n^0 + H_\infty^1 - \bar{\gamma} = 0 \quad \text{or} \quad \min_{n\geq 1} H_n^1 + H_\infty^0 - \bar{\gamma} = 0 \quad \text{or} \quad H_\infty^0 + H_\infty^1 - 2\bar{\gamma} = 0.$$

We have shown that $\bar{\gamma}$ is the Puiseux exponent γ.

Part 4. We prove item (3) similarly as in part 2. We choose $\epsilon > 0$ and $N \geq 1$ as before. The first part of the series (3.2) satisfies

$$\lim_{\beta \to +\infty} -\frac{1}{\beta} \ln \sum_{k=1}^{N} \frac{k}{\lambda_\beta^k} \exp(-\beta H_k^0) = \min_{1 \leq k \leq N} H_k^0.$$

Using again the estimate (3.3), for β large enough, the remaining part gives

$$\exp(-\beta(2\gamma + \epsilon)) \leq \frac{\lambda_\beta^N (\lambda_\beta - 1)^2}{N(\lambda_\beta - 1) + \lambda_\beta} \leq \exp(-\beta(2\gamma - \epsilon)),$$

$$\exp(-\beta(H_\infty^0 - 2\gamma + 2\epsilon)) \leq \sum_{k>N} \frac{k}{\lambda_\beta^k} \exp(-\beta H_k^0) \leq \exp(-\beta(H_\infty^0 - 2\gamma - 2\epsilon)). \quad \square$$

COROLLARY 3.8. *Let H be a reduced double-well type potential and V be a calibrated sub-action. Then V is constant on every cylinder of the form $[0^n 1]$ and $[1^n 0]$, where $n \geq 1$. More precisely:*

(1) $V(x) = \min\Big\{V(0^\infty),\ V(1^\infty) + \inf_{k \geq n} H_k^1\Big\}$ *for all $x \in [0^n 1]$; and*

(2) $V(x) = \min\Big\{V(1^\infty),\ V(0^\infty) + \inf_{k \geq n} H_k^0\Big\}$ *for all $x \in [1^n 0]$.*

In particular, $\min V = \min\{V(0^\infty), V(1^\infty)\}$. *With respect to $\Phi_\beta = e^{-\beta V_\beta}$, which is the eigenfunction used in Theorem 2.4 to ensure the existence of calibrated sub-actions, we have the following complementary information.*

(3) *If $\gamma > 0$ and $H_\infty^1 \geq H_\infty^0$, then $\{V_\beta\}$ converges uniformly to the calibrated sub-action V_∞ characterized by*

$$V_\infty(x) = \min\Big\{H_\infty^1 - \gamma,\ \inf_{k \geq n} H_k^1\Big\} \quad \textit{for all } x \in [0^n 1],\ \textit{for all } n \geq 1,$$

$$V_\infty(x) = 0 \quad \textit{for all } x \in [1].$$

(4) *If $\gamma = 0$, then $\{V_\beta\}$ converges uniformly to 0, which is the unique calibrated sub-action satisfying* $\min V = 0$.

Proof.

Part 1. Items (1)–(2) are consequences of the representation formula (2.2).

Part 2. If $H_\infty^1 \geq H_\infty^0$, then $H_\infty^1 + H_\infty^0 - 2\gamma \geq 0 \geq H_\infty^0 - \gamma$. Item (1) of Proposition 3.4, item (2) of Corollary 3.5 and items (1) and (2) of Proposition 3.7 imply that

$$\lim_{\beta \to +\infty} [V_\beta(0^\infty) - V_\beta(01)] = H_\infty^1 + H_\infty^0 - 2\gamma \geq 0.$$

From item (2) of Proposition 3.4 and item (1) of Proposition 3.7,

$$\lim_{\beta \to +\infty} [V_\beta(1^\infty) - V_\beta(01)] = H_\infty^0 - \gamma \leq 0.$$

Therefore, we obtain

$$\lim_{\beta \to +\infty} [V_\beta(0^\infty) - V_\beta(1^\infty)] = H_\infty^1 - \gamma \geq 0.$$

Let V_∞ be any accumulation function of $\{V_\beta\}$. Then V_∞ is a calibrated sub-action and, in particular, satisfies items (1) and (2), which have already been proved. Thus, since $\min V_\infty = 0$, necessarily $V_\infty(1^\infty) = 0$ and $V_\infty(0^\infty) = H^1_\infty - \gamma$, so that the characterization given in item (3) is proved. As this is the uniquely defined limit function, we have actually shown that $V_\beta \to V_\infty$ uniformly.

Part 3. If $\gamma = 0$, then $H^0_\infty = H^1_\infty = 0$. Let V_∞ be any accumulation function of $\{V_\beta\}$. Then V_∞ is a calibrated sub-action. By passing to the limit as $n \to +\infty$ in items (1) and (2), we obtain $V_\infty(0^\infty) = V_\infty(1^\infty)$. Since $\min V_\infty = 0$, V_∞ is necessarily the null function. By uniqueness of the accumulation function, we have proved that $V_\beta \to V_\infty$ uniformly. □

4. *The selection case*

We assume that H is reduced and that $\gamma > 0$, which is equivalent to $\max\{H^0_\infty, H^1_\infty\} > 0$. We also suppose that $H^0_\infty \le H^1_\infty$ (the opposite case is similar). In particular, $H^1_\infty > 0$. We know that the only accumulation points of μ_β are barycenters $c_0\delta_{0^\infty} + c_1\delta_{1^\infty}$. Our goal is to find an equivalent of $\mu_\beta[0]/\mu_\beta[1]$ as $\beta \to +\infty$ and therefore to prove the convergence of μ_β.

Proof of item (1) of Theorem 1.2. Assume that $\frac{1}{2}(H^1_\infty + H^0_\infty) > H^1_{\min} + H^0_\infty$. Then $\gamma = H^1_{\min} + H^0_\infty > 0$ since $H^1_{\min} = 0 \Leftrightarrow H^1_\infty = 0$. We will see that it is enough to estimate the quotient of the measures at the log-scale. Proposition 3.7 implies that

$$\lim_{\beta\to+\infty} -\frac{1}{\beta} \ln F^0_\beta(\lambda_\beta) = \min\{H^0_{\min}, H^0_\infty - \gamma\} = H^0_\infty - \gamma,$$
$$\lim_{\beta\to+\infty} -\frac{1}{\beta} \ln \tilde{F}^0_\beta(\lambda_\beta) = \min\{H^0_{\min}, H^0_\infty - 2\gamma\} = H^0_\infty - 2\gamma,$$
$$\lim_{\beta\to+\infty} -\frac{1}{\beta} \ln \tilde{F}^1_\beta(\lambda_\beta) = \min\{H^1_{\min}, H^1_\infty - 2\gamma\}.$$

The estimate for F^1_β is obtained from the characteristic equation. Thus

$$\begin{aligned}\lim_{\beta\to+\infty} -\frac{1}{\beta} \ln\left(\frac{\mu_\beta[0]}{\mu_\beta[1]}\right) &= \lim_{\beta\to+\infty} -\frac{1}{\beta} \ln\left(\frac{F^0_\beta(\lambda_\beta)}{F^1_\beta(\lambda_\beta)} \frac{\tilde{F}^1_\beta(\lambda_\beta)}{\tilde{F}^0_\beta(\lambda_\beta)}\right) \\ &= H^0_\infty + \min\{H^1_{\min}, H^1_\infty - 2\gamma\} > 0.\end{aligned}$$

We have proved that $\mu_\beta[0]/\mu_\beta[1] \to 0$ or $\mu_\beta \to \delta_{1^\infty}$. □

For the proof of item (2) of Theorem 1.2, the previous log-scale estimate is not enough. We need to develop an analytical technique that gives equivalents of the quantities $F^{0,1}_\beta(\lambda_\beta)$, $\tilde{F}^{0,1}_\beta(\lambda_\beta)$, and $\lambda_\beta - 1$.

We first need the following lemma on sequences.

LEMMA 4.1. *Let $\{H_n\}_{n\ge0}$ be a converging sequence satisfying*

$$\sum_{n\ge0} \sup_{k\ge0} |H_n - H_{n+k}| < +\infty.$$

Then $\lim_{n\to+\infty}(H_n - H_\infty)\ln(n) = 0$, *where* $H_\infty = \lim_{n\to+\infty} H_n$.

Proof. Define $K_n := \sup_{k\geq 0} |H_n - H_{n+k}|$ for all $n \geq 0$. Note then that $|H_n - H_\infty| \leq K_n$ and that $\{K_n\}_{n\geq 0}$ is a non-increasing sequence converging to zero such that $\sum_{n\geq 0} K_n < +\infty$. Assume, by contradiction, that there exist $\epsilon > 0$ and a subsequence $N_i \to +\infty$ such that $K_{N_i} \ln(N_i) \geq \epsilon$. Thanks to the non-increasing property,

$$\sum_{i\geq 1} \frac{N_{i+1} - N_i}{\ln(N_{i+1})} \leq \frac{1}{\epsilon} \sum_{i\geq 1} \sum_{N_i \leq n < N_{i+1}} K_n < +\infty.$$

We thus observe that

$$\frac{1 - N_i/N_{i+1}}{\ln(N_{i+1})/N_{i+1}} \to 0 \quad \Longrightarrow \quad \frac{N_i}{N_{i+1}} \to 1,$$

which implies that, for every i sufficiently large,

$$\frac{N_{i+1} - N_i}{\ln(N_{i+1})} = \frac{N_i}{\ln(N_{i+1})}\bigg(\frac{N_{i+1}}{N_i} - 1\bigg) \geq \frac{N_{i+1}}{N_i} - 1 \geq \ln\bigg(\frac{N_{i+1}}{N_i}\bigg).$$

But then $\sum_{i\geq 1}[\ln(N_{i+1}) - \ln(N_i)] < +\infty$ contradicts $N_i \to +\infty$. □

From now on, we write $f(\beta) \sim g(\beta)$ to indicate that the positive functions f and g are equivalent as $\beta \to +\infty$. Also, as usual $f(\beta) \ll g(\beta)$ means that f is negligible with respect to g as $\beta \to +\infty$.

Proof of item (2) of Theorem 1.2. Assume that $0 < \frac{1}{2}(H^1_\infty + H^0_\infty) \leq H^1_{\min} + H^0_\infty$. Then $\gamma = \frac{1}{2}(H^0_\infty + H^1_\infty)$. We recall that the coincidence number κ has been defined in (1.3) and the coefficient c in (1.4). We will prove that

$$\begin{aligned}
\lambda_\beta &= 1 + c\exp(-\beta\gamma) + o(\exp(-\beta\gamma)),\\
F^0_\beta(\lambda_\beta) &\sim \frac{\exp(-\beta H^0_\infty)}{\lambda_\beta - 1} \sim \frac{1}{c}\exp\bigg(\beta\frac{H^1_\infty - H^0_\infty}{2}\bigg),\\
\tilde{F}^0_\beta(\lambda_\beta) &\sim \frac{\exp(-\beta H^0_\infty)}{(\lambda_\beta - 1)^2} \sim \frac{1}{c^2}\exp(\beta H^1_\infty),\\
F^1_\beta(\lambda_\beta) &\sim c\exp\bigg(-\beta\frac{H^1_\infty - H^0_\infty}{2}\bigg),\\
\tilde{F}^1_\beta(\lambda_\beta) &\sim \frac{\exp(-\beta H^1_\infty)}{(\lambda_\beta - 1)^2} \sim \frac{1}{c^2}\exp(\beta H^0_\infty).
\end{aligned} \tag{4.1}$$

Using item (5) of Corollary 3.6, we will obtain $\mu_\beta[0]/\mu_\beta[1] \to 1/c^2$ and the convergence of the Gibbs measure, as in (1.5).

Part 1. We determine an equivalent of $F^0_\beta(\lambda_\beta)$. If H^0_k is constant and equal to H^0_∞, then the proof is complete and

$$F^0_\beta(\lambda_\beta) = \frac{\exp(-\beta H^0_\infty)}{\lambda_\beta - 1} \quad \text{and} \quad \tilde{F}^0_\beta(\lambda_\beta) = \frac{\exp(-\beta H^0_\infty)}{(\lambda_\beta - 1)^2}.$$

We may now assume that H^0_k is not constant. Let $\epsilon > 0$. For β large enough, there exists a smallest positive integer N_β such that

$$\beta|H^0_{N_\beta} - H^0_\infty| \geq \epsilon \quad \text{and} \quad \beta|H^0_k - H^0_\infty| \leq \epsilon \quad \text{for all } k \geq N_\beta + 1.$$

Lemma 4.1 implies that $|H_n^0 - H_\infty^0| \ln(n) \to 0$. Since $|H_{N_\beta}^0 - H_\infty^0| \geq \epsilon/\beta$, we obtain (even in the case when N_β is bounded with respect to β)

$$\lim_{\beta\to+\infty} \frac{1}{\beta} \ln N_\beta = 0. \tag{4.2}$$

Hence, we may show that

$$N_\beta(\lambda_\beta - 1) \exp(-\beta H_{\min}^0) \ll \exp(-\beta H_\infty^0) \quad \text{and} \quad \lambda_\beta^{N_\beta} \to 1. \tag{4.3}$$

For the first estimate, by taking $-(1/\beta)\ln$ on both terms and using item (1) of Proposition 3.7, one has $\gamma + H_{\min}^0 > H_\infty^0$ (according to the two cases: if $H_\infty^1 > H_\infty^0$, then $\gamma > H_\infty^0$; if $H_\infty^1 = H_\infty^0$, then $H_{\min}^0 > 0$). For the above limit, note that

$$\frac{\lambda_\beta - 1}{\exp(-\beta H_{\min}^1)} \leq \frac{1}{F_\beta^1(\lambda_\beta)} = F_\beta^0(\lambda_\beta) \leq \frac{1}{\lambda_\beta - 1},$$
$$\lambda_\beta \leq 1 + \exp(-\beta H_{\min}^1/2), \quad \lambda_\beta^{N_\beta} \leq \exp(N_\beta \exp(-\beta H_{\min}^1/2)).$$

As $H_{\min}^1 > 0$, using (4.2), $N_\beta \ll \exp(\beta H_{\min}^1/2)$ and $\lambda_\beta^{N_\beta} \to 1$.

We are now able to compute an equivalent of $F_\beta^0(\lambda_\beta)$. We split the series $F_\beta^0(\lambda_\beta)$ into two parts and use (4.3) to obtain, for β sufficiently large,

$$\frac{\exp(-\beta H_\infty^0 - \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)} \leq F_\beta^0(\lambda_\beta) \leq N_\beta \exp(-\beta H_{\min}^0) + \frac{\exp(-\beta H_\infty^0 + \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)},$$
$$\frac{\exp(-\beta H_\infty^0 - 2\epsilon)}{\lambda_\beta - 1} \leq F_\beta^0(\lambda_\beta) \leq \frac{\exp(-\beta H_\infty^0 + 2\epsilon)}{\lambda_\beta - 1}.$$

By taking $\epsilon \to 0$, we have just proved that

$$F_\beta^0(\lambda_\beta) \sim \frac{\exp(-\beta H_\infty^0)}{\lambda_\beta - 1}. \tag{4.4}$$

Part 2. We determine an equivalent of $\tilde{F}_\beta^0(\lambda_\beta)$. We use the same definition of N_β as before and similarly prove the estimates

$$N_\beta(\lambda_\beta - 1) \ll 1, \quad N_\beta^2(\lambda_\beta - 1)^2 \exp(-\beta H_{\min}^0) \ll \exp(-\beta H_\infty^0). \tag{4.5}$$

We split the series $\tilde{F}_\beta^0(\lambda_\beta)$ and use the computation (3.3) to obtain

$$\frac{(N_\beta(\lambda_\beta - 1) + \lambda_\beta) \exp(-\beta H_\infty^0 - \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)^2} \leq \tilde{F}_\beta^0(\lambda_\beta)$$
$$\tilde{F}_\beta^0(\lambda_\beta) \leq N_\beta^2 \exp(-\beta H_{\min}^0) + \frac{(N_\beta(\lambda_\beta - 1) + \lambda_\beta) \exp(-\beta H_\infty^0 + \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)^2}.$$

Using the estimates (4.5), for β sufficiently large,

$$\frac{\exp(-\beta H_\infty^0 - 2\epsilon)}{(\lambda_\beta - 1)^2} \leq \tilde{F}_\beta^0(\lambda_\beta) \leq \frac{\exp(-\beta H_\infty^0 + 2\epsilon)}{(\lambda_\beta - 1)^2}.$$

Letting $\epsilon \to 0$, we have just proved that

$$\tilde{F}^0_\beta(\lambda_\beta) \sim \frac{\exp(-\beta H^0_\infty)}{(\lambda_\beta - 1)^2}. \tag{4.6}$$

Part 3. We determine an equivalent of $F^1_\beta(\lambda_\beta)$. As before, we discuss two cases. If H^1_k is constant and equal to H^1_∞, then the coincidence number (1.3) is $\kappa = 0$ and the coefficient (1.4) is $c = 1$. We immediately obtain

$$F^1_\beta(\lambda_\beta) = \frac{\exp(-\beta H^1_\infty)}{\lambda_\beta - 1} \quad \text{and} \quad \tilde{F}^1_\beta(\lambda_\beta) = \frac{\exp(-\beta H^1_\infty)}{(\lambda_\beta - 1)^2}.$$

We may assume that H^1_k is not constant. For β large enough, we redefine N_β as the smallest positive integer such that

$$\beta|H^1_{N_\beta} - H^1_\infty| \geq \epsilon \quad \text{and} \quad \beta|H^1_k - H^1_\infty| \leq \epsilon \quad \text{for all } k \geq N_\beta + 1.$$

As before, $(1/\beta)\ln N_\beta \ll 1$. Recall now that $H^1_{\min} \geq \frac{1}{2}(H^1_\infty - H^0_\infty)$. In the case when $\kappa > 0$, $H^1_{\min} < H^1_\infty$ and we introduce another exponent

$$H^{1*}_{\min} := \min\{H^1_k : k \text{ s.t. } H^1_k + H^0_\infty \neq \tfrac{1}{2}(H^1_\infty + H^0_\infty)\} > H^1_{\min}.$$

In the case when $\kappa = 0$, by convention, $H^{1*}_{\min} = H^1_{\min}$. We show the first estimate

$$N_\beta(\lambda_\beta - 1)\exp(-\beta H^{1*}_{\min}) \ll \exp(-\beta H^1_\infty). \tag{4.7}$$

Indeed, by taking $-(1/\beta)\ln$, it is enough to argue that $\gamma + H^{1*}_{\min} > H^1_\infty$. In the case when $\kappa > 0$, $H^1_{\min} + H^0_\infty = \frac{1}{2}(H^1_\infty + H^0_\infty) = \gamma$ and

$$\gamma + H^{1*}_{\min} > \gamma + H^1_{\min} = H^1_\infty.$$

In the case when $\kappa = 0$, $H^1_{\min} + H^0_\infty > \frac{1}{2}(H^1_\infty + H^0_\infty) = \gamma$ and

$$\gamma + H^{1*}_{\min} = \gamma + H^1_{\min} > H^1_\infty.$$

The limit $\lambda_\beta^{N_\beta} \to 1$ is similarly proved. We are now able to compute an equivalent of $F^1_\beta(\lambda_\beta)$. As before, we split the series into two parts: in the finite sum, we keep the indices corresponding to the incidences and the exponents $H^1_{\min}$; the rest of the indices have a larger exponent $H^{1*}_{\min}$ (unless $\kappa = 0$, where we only use one exponent $H^1_{\min}$). Thus, for β large enough,

$$(e^{-\epsilon}\kappa)\exp(-\beta H^1_{\min}) + \frac{\exp(-\beta H^1_\infty - \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)} \leq F^1_\beta(\lambda_\beta),$$

$$F^1_\beta(\lambda_\beta) \leq \kappa\exp(-\beta H^1_{\min}) + N_\beta\exp(-\beta H^{1*}_{\min}) + \frac{\exp(-\beta H^1_\infty + \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)}.$$

Taking into account the estimate (4.7), for β sufficiently large,

$$\left[\kappa\exp(-\beta H^1_{\min}) + \frac{\exp(-\beta H^1_\infty)}{\lambda_\beta - 1}\right]e^{-2\epsilon} \leq F^1_\beta(\lambda_\beta)$$

$$F^1_\beta(\lambda_\beta) \leq \left[\kappa\exp(-\beta H^1_{\min}) + \frac{\exp(-\beta H^1_\infty)}{\lambda_\beta - 1}\right]e^{2\epsilon}.$$

Letting $\epsilon \to 0$, we have proved (in both cases, $\kappa > 0$ or $\kappa = 0$) that

$$F^1_\beta(\lambda_\beta) \sim \kappa \exp(-\beta H^1_{\min}) + \frac{\exp(-\beta H^1_\infty)}{\lambda_\beta - 1}. \tag{4.8}$$

Part 4. We show an equivalent of $\lambda_\beta - 1$. The characteristic equation (item (1) of Corollary 3.5), the equivalents (4.4) and (4.8) give

$$(\lambda_\beta - 1)^2 \exp(\beta(H^1_\infty + H^0_\infty)) \sim \kappa(\lambda_\beta - 1)\exp(\beta(H^1_\infty + H^0_\infty)/2) + 1.$$

(In the case when $\kappa > 0$, we use the equality $H^1_{\min} + H^0_\infty = \frac{1}{2}(H^1_\infty + H^0_\infty)$.) Let $X_\beta = (\lambda_\beta - 1)\exp(\beta(H^1_\infty + H^0_\infty)/2)$. Then $X^2_\beta \sim \kappa X_\beta + 1$. Necessarily, X_β is bounded with respect to β, it is non-negative and any accumulation point c satisfies $c^2 = \kappa c + 1$. We have just proved that

$$\lambda_\beta - 1 \sim c \exp\big(-\beta \tfrac{1}{2}(H^1_\infty + H^0_\infty)\big). \tag{4.9}$$

Using the previous equivalents (4.4) and (4.6) as well as the characteristic equation, one obtains the equivalents of $F^0_\beta(\lambda_\beta)$, $\tilde{F}^0_\beta(\lambda_\beta)$ and $F^1_\beta(\lambda_\beta)$. For the equivalent of $\tilde{F}^1_\beta(\lambda_\beta)$, since $2\gamma + H^1_{\min} = H^1_\infty + H^1_{\min} + H^0_\infty > H^1_\infty$, one first notices that

$$N^2_\beta(\lambda_\beta - 1)^2 \exp(-\beta H^1_{\min}) \ll \exp(-\beta H^1_\infty). \tag{4.10}$$

The series $\tilde{F}^1_\beta(\lambda_\beta)$ is then split in a more crude way

$$\frac{(N_\beta(\lambda_\beta - 1) + \lambda_\beta)\exp(-\beta H^1_\infty - \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)^2} \le \tilde{F}^1_\beta(\lambda_\beta),$$

$$\tilde{F}^1_\beta(\lambda_\beta) \le N^2_\beta \exp(-\beta H^1_{\min}) + \frac{(N_\beta(\lambda_\beta - 1) + \lambda_\beta)\exp(-\beta H^1_\infty + \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)^2},$$

and, therefore,

$$\tilde{F}^1_\beta(\lambda_\beta) \sim \frac{\exp(-\beta H^1_\infty)}{(\lambda_\beta - 1)^2} \sim \frac{1}{c^2}\exp(\beta H^0_\infty). \tag{4.11}$$

The proof of all the equivalents (4.1) is now complete. □

5. *The non-selection case*

We construct an example of a Lipschitz double-well type potential satisfying $H^0_\infty = H^1_\infty = 0$ that produces a non-convergent family of Gibbs measure as the temperature goes to zero. Notice that any symmetric example, $H^0_n = H^1_n$, for all $n \ge 1$, provides a family of symmetric Gibbs measures $\{\mu_\beta\}$ that converge to $\frac{1}{2}\delta_{0^\infty} + \frac{1}{2}\delta_{1^\infty}$. We show that the subclass of double-well type potentials is rich enough to break the symmetry in an alternating way. Notice also that H is necessarily reduced in order to obtain the non-selection case.

The two fixed points 0^∞, 1^∞ are connected by two heteroclinic orbits, $\{0^n 1^\infty\}_{n\ge1}$ and $\{1^n 0^\infty\}_{n\ge1}$. The oscillation between the two minimizing measures δ_{0^∞} and δ_{1^∞} are obtained by choosing a symmetric potential H, where both $\{H^0_n\}_{n\ge1}$ and $\{H^1_n\}_{n\ge1}$ are non-increasing and converge to zero. The level sets of H alternate, as in Figure 2, and are chosen according to the following rules that are similar to the rules in [**10**].

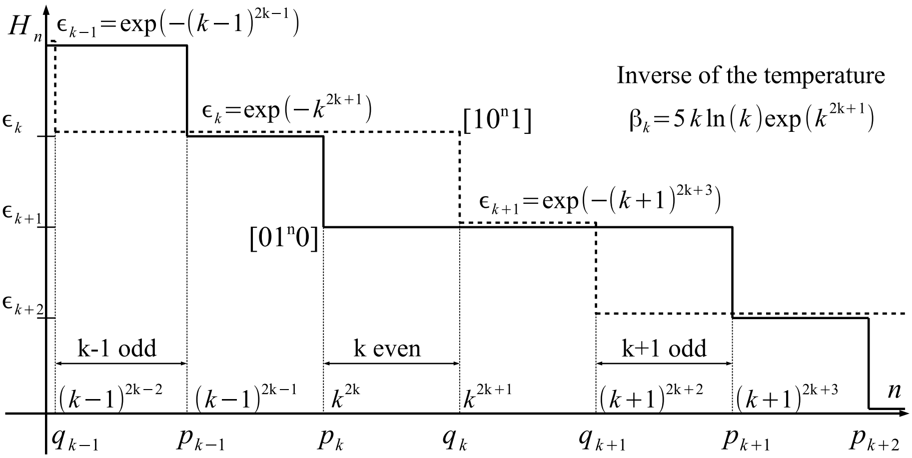

FIGURE 2. The non-selection case for a Lipschitz example. The level sets satisfy $H = \epsilon_k = \exp(-k^{2k+1})$ on $[01^n 0]$ for every $p_{k-1} < n \le p_k$ and on $[10^n 1]$ for every $q_{k-1} < n \le q_k$. If k is even, $p_k = k^{2k}$ and $q_k = k^{2k+1}$. If k is odd, $p_k = k^{2k+1}$ and $q_k = k^{2k}$.

- *Rule 1.* We choose two increasing sequences $\{p_k\}_{k\ge 0}$ and $\{q_k\}_{k\ge 0}$, which alternate according to the parity of the index k: that is,

$$1 \le p_0 < q_0 < q_1 < p_1 < p_2 < q_2 < q_3 < p_3 < \cdots,$$
$$p_{2l} < q_{2l} < q_{2l+1} < p_{2l+1} < p_{2l+2} < q_{2l+2} < \cdots.$$

- *Rule 2.* We choose a decreasing sequence $\{\epsilon_k\}_{k\ge 0}$ of positive numbers which goes to zero. We choose H so that a level set of H corresponds to a union of cylinders $[01^n 0]$ (respectively, $[10^n 1]$) over $n \in \{p_{k-1}+1, \ldots, p_k\}$ (respectively, over $n \in \{q_{k-1}+1, \ldots, q_k\}$). By convention, $p_{-1} = q_{-1} = 0$ and

$$H_n^0 := \epsilon_k \quad \text{for all } p_{k-1} < n \le p_k, \qquad H_n^1 := \epsilon_k \quad \text{for all } q_{k-1} < n \le q_k.$$

The contribution of the potential H_n^0 (respectively, H_n^1) exhibits a large drop at the level p_k (respectively, q_k): that is,

$$\begin{aligned} &\text{for all } n \le p_k, \quad H_n^0 \ge \epsilon_k, \quad \text{for all } n \ge p_k+1, \quad H_n^0 \le \epsilon_{k+1},\\ &\text{for all } n \le q_k, \quad H_n^1 \ge \epsilon_k, \quad \text{for all } n \ge q_k+1, \quad H_n^1 \le \epsilon_{k+1}.\end{aligned}$$

- *Rule 3.* We choose a decreasing sequence of temperatures $\beta_k^{-1} \to 0$ which forces the Gibbs measure to give larger mass to either [0] for an even index or [1] for an odd index. The only constraints on $\{p_k\}$, $\{q_k\}$, $\{\epsilon_k\}$ and $\{\beta_k\}$ that we use are

$$\lim_{k\to+\infty} p_k^2 \exp(-\beta_k\epsilon_k) = 0, \qquad \lim_{k\to+\infty} q_k^2 \exp(-\beta_k\epsilon_k) = 0,$$
$$\lim_{k\to+\infty} \beta_k\epsilon_{k+1} = 0, \qquad \lim_{k\to+\infty} \frac{q_{2k}}{p_{2k}} = +\infty, \qquad \lim_{k\to+\infty} \frac{p_{2k+1}}{q_{2k+1}} = +\infty,$$
$$\sum_{k\ge 1} (p_k - p_{k-1}) \exp(-\epsilon_k) < +\infty, \qquad \sum_{k\ge 1} (q_k - q_{k-1}) \exp(-\epsilon_k) < +\infty.$$

The last two conditions ensure the summability of the variation.

The three previous rules enable us to say that, at the temperature β_k^{-1}, for k even or odd, the system is mainly governed by a system having a potential $\tilde{H}$ equal to zero on $[00] \cup [01^{p_k+1}] \cup [11] \cup [10^{q_k+1}]$ (thanks to $\epsilon_{k+1} \ll \epsilon_k$) and positive elsewhere.

Proof of item (3) of Theorem 1.2. Let k be even. The other case is similar. To simplify the notation, we write $p = p_k$, $q = q_k$ and $\lambda = \lambda_{\beta_k}$. Remember the *a priori* estimate $\lambda \le 2$.

Part 1. We rewrite $F_\beta^0(\lambda)$ as if the energy H_n^0 were negligible for $n > p$. Then

$$F_\beta^0(\lambda) = \frac{1}{\lambda^p(\lambda-1)}(\alpha_0 + \lambda^p(\lambda-1)\theta_0), \tag{5.1}$$

where

$$\alpha_0 := \lambda^p(\lambda-1)\sum_{n\ge p+1}\frac{1}{\lambda^n}\exp(-\beta_k H_n^0) \quad \text{and} \quad \theta_0 := \sum_{n=1}^{p}\frac{1}{\lambda^n}\exp(-\beta_k H_n^0).$$

As $H_n^0 \le \epsilon_{k+1}$ for $n \ge p+1$ and $H_n^0 \ge \epsilon_k$ for $n \le p$, we obtain

$$\exp(-\beta_k\epsilon_{k+1}) \le \alpha_0 \le 1, \qquad \theta_0 \le p\exp(-\beta_k\epsilon_k).$$

Rule 3 implies that $\alpha_0 \to 1$ and $\theta_0 \to 0$ as $k \to +\infty$. Similarly,

$$F_\beta^1(\lambda) = \frac{1}{\lambda^q(\lambda-1)}(\alpha_1 + \lambda^q(\lambda-1)\theta_1), \tag{5.2}$$

with

$$\alpha_1 := \lambda^q(\lambda-1)\sum_{n\ge q+1}\frac{1}{\lambda^n}\exp(-\beta_k H_n^1) \quad \text{and} \quad \theta_1 := \sum_{n=1}^{q}\frac{1}{\lambda^n}\exp(-\beta_k H_n^1).$$

As $H_n^1 \le \epsilon_{k+1}$ for $n \ge q+1$ and $H_n^1 \ge \epsilon_k$ for $n \le q$, the third rule also implies that $\alpha_1 \to 1$ and $\theta_1 \to 0$ as $k \to +\infty$. As $F_\beta^0(\lambda)F_\beta^1(\lambda) = 1$,

$$\lambda^{p+q}(\lambda-1)^2 = [\alpha_0 + \lambda^p(\lambda-1)\theta_0][\alpha_1 + \lambda^q(\lambda-1)\theta_1] := \delta^2.$$

Part 2. We show that $\delta \to 1$ as $k \to +\infty$. Let $N := (p+q)/2$. We first observe that, for k large enough, $\lambda^N \ge e$. If not,

$$\lambda - 1 \ge \delta e^{-1} \ge e^{-1}\sqrt{\alpha_0\alpha_1}. \tag{5.3}$$

On the one hand $\lambda - 1 \to 0$ and on the other hand $\alpha_0\alpha_1 \to 1$, so we get a contradiction. We next observe that $\lambda - 1 \ge 1/N$. Indeed,

$$\lambda = 1 + \frac{\delta}{\lambda^N}, \quad \ln(\lambda) \le \frac{\delta}{\lambda^N}, \quad 1 \le N\ln(\lambda) \le \frac{N\delta}{\lambda^N}, \quad \lambda^N \le N\delta, \tag{5.4}$$

and, from the equation $\lambda^N(\lambda-1) = \delta$, we finally obtain $\lambda - 1 \ge 1/N$. We rewrite the two terms $\lambda^p(\lambda-1)$ and $\lambda^q(\lambda-1)$ as

$$\begin{aligned}
\lambda^p(\lambda-1) &= (\lambda^N)^{p/N}(\lambda-1) = [\lambda^N(\lambda-1)]^{p/N}(\lambda-1)^{1-p/N} \\
&= \delta^{p/N}(\lambda-1)^{(q-p)/(q+p)} \le \delta^{p/N}, \\
\lambda^q(\lambda-1) &= (\lambda^N)^{q/N}(\lambda-1) = [\lambda^N(\lambda-1)]^{q/N}(\lambda-1)^{1-q/N} \\
&= \delta^{q/N}(\lambda-1)^{-(q-p)/(q+p)} \le \delta^{q/N}(\lambda-1)^{-1} \le q\delta^{q/N}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\delta^2 &\le [\alpha_0 + \delta^{p/N}\theta_0][\alpha_1 + q\delta^{q/N}\theta_1]\\ &= \alpha_0\alpha_1 + \alpha_0\theta_1 q\delta^{q/N} + \alpha_1\theta_0\delta^{p/N} + \theta_0\theta_1 q\delta^2.\end{aligned}$$

Using $\delta^{p/N} \le 1+\delta^2$ and $\delta^{q/N} \le 1+\delta^2$,

$$\alpha_0\alpha_1 \le \delta^2 \le \frac{\alpha_0\alpha_1 + (\alpha_0 q\theta_1 + \alpha_1\theta_0)}{1-(\alpha_0 q\theta_1 + \alpha_1\theta_0 + \theta_0 q\theta_1)}.$$

Since $q\theta_1 \le q^2\exp(-\beta_k\epsilon_k) \to 0$ and $\theta_0 \to 0$ as $k \to +\infty$, we obtain $\delta \to 1$.

Part 3. We first prove that $q(\lambda-1) \to +\infty$. Since $N < q$, it is enough to show that $N(\lambda-1) \to +\infty$. Indeed, for every $C \ge 1$ and for k sufficiently large, $\lambda^N \ge \exp(C)$, as in (5.3). Using the same estimates as in (5.4),

$$C\lambda^N \le N\delta \quad\text{and}\quad N(\lambda-1) \ge C.$$

Therefore, from the estimates of part 2, we see that

$$\begin{aligned}\frac{\lambda^p(\lambda-1)^2}{p(\lambda-1)+\lambda} &\le \frac{\lambda^p(\lambda-1)}{p} \le \frac{\delta^{p/N}}{p} \le \frac{1+\delta^2}{p} \to 0,\\ \frac{\lambda^q(\lambda-1)^2}{q(\lambda-1)+\lambda} &\le \frac{\lambda^q(\lambda-1)}{q} \le \frac{\delta^{q/N}}{q(\lambda-1)} \le \frac{1+\delta^2}{q(\lambda-1)} \to 0.\end{aligned}$$

Part 4. We decompose $\tilde F^0_\beta(\lambda)$ as before

$$\tilde F^0_\beta(\lambda) = \frac{p(\lambda-1)+\lambda}{\lambda^p(\lambda-1)^2}\left(\tilde\alpha_0 + \frac{\lambda^p(\lambda-1)^2}{p(\lambda-1)+\lambda}\tilde\theta_0\right), \tag{5.5}$$

where

$$\exp(-\beta_k\epsilon_{k+1}) \le \tilde\alpha_0 := \frac{\lambda^p(\lambda-1)^2}{p(\lambda-1)+\lambda}\sum_{n\ge p+1}\frac{n}{\lambda^n}\exp(-\beta_k H^0_n) \le 1$$

$$\text{and}\quad \tilde\theta_0 := \sum_{n=1}^{p}\frac{n}{\lambda^n}\exp(-\beta_k H^0_n) \le p^2\exp(-\beta_k\epsilon_k).$$

Then $\tilde\alpha_0 \to 1$ and $\tilde\theta_0 \to 0$. Similar estimates are obtained for $\tilde F^1_\beta(\lambda)$.

Part 5. We may now conclude the proof. Since $\lambda^p(\lambda-1)/p \to 0$, $\lambda^q(\lambda-1)/q \to 0$, $p\theta_0 \to 0$ and $q\theta_1 \to 0$, equations (5.1) and (5.2) imply that

$$F^0_\beta(\lambda) \sim \frac{1}{\lambda^p(\lambda-1)} \quad\text{and}\quad F^1_\beta(\lambda) \sim \frac{1}{\lambda^q(\lambda-1)}.$$

As $\lambda^p(\lambda-1)^2/(p(\lambda-1)+\lambda) \to 0$ and $\lambda^q(\lambda-1)^2/(q(\lambda-1)+\lambda) \to 0$, equation (5.5) and a similar expression for $\tilde F^1_\beta(\lambda)$ provide

$$\tilde F^0_\beta(\lambda) \sim \frac{p(\lambda-1)+\lambda}{\lambda^p(\lambda-1)^2} \quad\text{and}\quad \tilde F^1_\beta(\lambda) \sim \frac{q(\lambda-1)+\lambda}{\lambda^q(\lambda-1)^2}.$$

Item (5) of Corollary 3.6 thus gives

$$\frac{\mu_\beta[0]}{\mu_\beta[1]} = \frac{F^0_\beta(\lambda)}{F^1_\beta(\lambda)}\frac{\tilde F^1_\beta(\lambda)}{\tilde F^0_\beta(\lambda)} \sim \frac{q(\lambda-1)+\lambda}{p(\lambda-1)+\lambda} \ge \min\left\{\frac{q}{2p}, \frac{q(\lambda-1)}{2\lambda}\right\} \to +\infty.$$

As a matter of fact, rule 3 requires $\lim_{l\to+\infty}(q_{2l}/p_{2l}) = +\infty$. Hence $\mu_{\beta_{2l}} \to \delta_{0^\infty}$. □

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