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On calibrated and separating sub-actions

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Abstract. We consider a one-sided transitive subshift of finite type $\sigma\colon \Sigma \to \Sigma$ and a Hölder observable A. In the ergodic optimization model, one is interested in properties of A-minimizing probability measures. If $\bar{A}$ denotes the minimizing ergodic value of A, a sub-action u for A is by definition a continuous function such that $A \geq u \circ \sigma - u + \bar{A}$. We call contact locus of u with respect to A the subset of Σ where $A = u \circ \sigma - u + \bar{A}$. A calibrated sub-action u gives the possibility to construct, for any point $x \in \Sigma$, backward orbits in the contact locus of u. In the opposite direction, a separating sub-action gives the smallest contact locus of A, that we call $\Omega(A)$, the set of non-wandering points with respect to A.
We prove that separating sub-actions are generic among Hölder sub-actions. We also prove that, under certain conditions on $\Omega(A)$, any calibrated sub-action is of the form $u(x) = u(x_i) + h_A(x_i, x)$ for some $x_i \in \Omega(A)$, where $h_A(x, y)$ denotes the Peierls barrier of A. We present the proofs in the holonomic optimization model, a formalism which allows to take into account a two-sided transitive subshift of finite type $(\hat{\Sigma}, \hat{\sigma})$.

Keywords: ergodic optimization, minimizing measures, sub-actions, separating sub-actions, Peierls barrier.

Mathematical subject classification: 37A05, 37A50, 37A60.

1 Introduction

In the *ergodic optimization model* (see, for instance, [2, 3, 5, 9, 15, 16, 18]), given a continuous observable $A\colon X \to \mathbb{R}$, one is interested in understanding which T-invariant Borel probability measure μ of a compact metric space X

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minimizes the average $\int_X A\, d\mu$. Such measures are called *minimizing probability measures*[1].

Minimizing probability measures admit dual objects: the *sub-actions*. A sub-action $u\colon X \to \mathbb{R}$ associated to an observable A enables to replace A by a cohomologous observable whose ergodic minimizing value is actually the absolute minimum. To each sub-action u one associates a compact subset of X called *contact locus* which contains the support of any minimizing probability measure. A sub-action gives therefore important information on T-invariant Borel probability measures that minimize the average of A. It is a relevant problem to investigate the existence of a particular sub-action having the smallest contact locus, that is, the smallest "trapping region" of all minimizing probability measures.

In section 2, we give a simplified version for the ergodic optimization model of the main results, namely, of the theorems 10, 11 and 12. In section 3, we recall the definition of the *holonomic optimization model* and state the main results. We give in section 4 the proof of theorem 10 and in section 5 the proof of theorem 12. We address the reader to [13] for a proof of theorem 11. We will adopt throughout the text the point of view which consists in interpreting ergodic optimization problems as questions of variational dynamics (see, for instance, [9, 13, 18]), similar to Aubry-Mather technics for Lagrangian systems. For an expository introduction to the general theory of ergodic optimization, we refer the reader to the article of O. Jenkinson (see [16]).

We still would like to point out that one of the main conjectures in the theory of ergodic optimization on compact spaces can be roughly formulated in the following way: *in any hyperbolic dynamics, a generic Hölder (or Lipshitz) observable possesses an unique minimizing probability measure, which is supported by a periodic orbit*. Concerning this problem, partial answers were already obtained, among them [3, 9, 15, 18, 20, 21]. Working with a transitive expanding dynamical system, J. Brémont (see [5]) has recently shown how such conjecture might follow from a careful study of the contact loci of typical sub-actions with finitely many connected components. Of course, such result reaffirms the importance of the study of sub-actions as well as of their respective contact loci.

In the same dynamical context, we are in particular interested in finding *separating sub-actions*, that is sub-actions whose contact locus is the smallest one. As mentioned above, these sub-actions give more information on the minimiz-

[1] Maximizing probabilities also appear in the literature. Obviously, replacing the observable A by $-A$, both vocabularies can be interchanged and the rephrased statements will be immediately verified. The maximizing terminology seems more convenient to study the connections with the thermodynamic formalism (see, for example, [9, 17]).

ing measure(s) than does a general sub-action. Our main theorem (namely, theorem 10) states that such sub-actions are actually generic among the set of Hölder sub-actions. An interesting result we also present here and which is independent of the previous considerations is an analysis related to the following situation: it is known that, for each irreducible component of the A-non-wandering set, one can associate via the Peierls barrier a calibrated sub-action. We present in theorem 12 sufficient conditions (by no means necessary) that assure that there exists a dominant one among such calibrated sub-actions.

2 A simplified version of theorems 10, 11 and 12

Let (X, T) be a transitive expanding dynamical system, that is, a continuous covering several-to-one map $T: X \to X$ on a compact metric space X whose inverse branches are uniformly contracting by a factor $0 < \lambda < 1$. We denote by $\mathcal{M}_T$ the set of T-invariant Borel probability measures. Our objective in this section is to summarize the conclusions of theorems 10, 11 and 12 in ergodic optimization theory. We first recall basic definitions from [9] (see also [16]).

Given a continuous observable $A: X \to \mathbb{R}$, we call *ergodic minimizing value* the quantity

$$\bar{A} := \min_{\mu \in \mathcal{M}_T} \int A \, d\mu.$$

We call *A-minimizing probability* a measure $\mu \in \mathcal{M}_T$ which realizes the above minimum.

We say that a continuous function $u: X \to \mathbb{R}$ is a sub-action with respect to the observable A if the following inequality holds everywhere on X

$$A \geq u \circ T - u + \bar{A}.$$

We would like to emphasize that, although the definition of a sub-action can be extended to other regularities (for instance, to the class of bounded measurable functions), we will only consider continuous sub-actions in this paper.

Definition 1. *A sub-action $u: X \to \mathbb{R}$ is said calibrated if*

$$u(x) = \min_{T(y)=x} \left[u(y) + A(y) - \bar{A}\right] \quad \text{for all } \ x \in X.$$

Definition 2. *We call contact locus of a sub-action u the set*

$$\mathbb{M}_A(u) := (A - u \circ T + u)^{-1}(\bar{A}).$$

It is just the subset of X where $A = u \circ T - u + \bar{A}$.

A point $x \in X$ is said to be *non-wandering with respect to A* if, for every $\epsilon > 0$, there exists an integer $k \geq 1$ and a point $y \in X$ such that

$$d(x, y) < \epsilon, \quad d(x, T^k(y)) < \epsilon \text{ and } \left| \sum_{j=0}^{k-1} (A - \bar{A}) \circ T^j(y) \right| < \epsilon.$$

We denote by $\Omega(A)$ the *set of non-wandering points* with respect to the observable $A \in C^0(X)$. When the observable is Hölder, $\Omega(A)$ is a non-empty compact T-invariant set containing the support of all minimizing probability measures. Moreover,

$$\Omega(A) \subset \bigcap \left\{ \mathbb{M}_A(u) \;\middle|\; u \text{ is a continuous sub-action} \right\}.$$

We are interested in finding u so that $\Omega(A) = \mathbb{M}_A(u)$.

Definition 3. *A sub-action $u \in C^0(X)$ is said to be separating (with respect to A) if it satisfies $\mathbb{M}_A(u) = \Omega(A)$.*

The main conclusion of theorem 10 can be stated in the following way. The proof of this particular case will not be given and can be adapted from the one of the general situation (see section 4).

Theorem 4. *Let (X, T) be a transitive expanding dynamical system on a compact metric space and $A \colon X \to \mathbb{R}$ be a θ-Hölder observable. Then there exist a θ-Hölder separating sub-action for A. Furthermore, in the θ-Hölder topology, the subset of θ-Hölder separating sub-actions is generic among all θ-Hölder sub-actions.*

We will present in theorem 6 a result of different nature and independent interest. The item which is totally new on this claim will be item 2.

Contrary to a separating sub-action, a calibrated sub-action u possesses a large contact locus in the sense $T(\mathbb{M}_A(u)) = X$. Calibrated sub-actions are built using a particular sub-action called the *Peierls barrier*. For Hölder observable A, the Peierls barrier of A, $h_A \colon \Omega(A) \times X \to \mathbb{R}$, is a Hölder calibrated sub-action in the second variable defined by

$$h_A(x, y) := \lim_{\epsilon \to 0} \liminf_{k \to +\infty} \inf \Big\{ \sum_{j=0}^{k-1} (A - \bar{A}) \circ T^j(z) \;\Big|\; z \in X,\ d(z, x) < \epsilon \text{ and } d(T^k(z), y) < \epsilon \Big\}.$$

The equivalent theorem to 11 may be stated in the following form.

Theorem 5. *Let (X, T) be a transitive expanding dynamical system on a compact metric space and $A\colon X \to \mathbb{R}$ be a Hölder observable. Then the set of continuous calibrated sub-actions coincides with the set of functions of the form*

$$u(y) = \min_{x\in\Omega(A)} \big[\phi(x) + h_A(x, y)\big], \quad \forall\, y \in X,$$

where $\phi\colon \Omega(A) \to \mathbb{R}$ is any continuous function satisfying

$$\phi(y) - \phi(x) \le h_A(x, y), \quad \forall\, x, y \in \Omega(A).$$

Moreover, u extends ϕ and is thus uniquely characterized by ϕ.

The condition $x \sim y \Leftrightarrow h_A(x, y) + h_A(y, x) = 0$ defines an equivalent relation on $\Omega(A)$. An equivalence class is called an *irreducible component*. It is a closed T-invariant set[2].

In the case $\Omega(A)$ is reduced to a finite number of disjoint irreducible components, the set of calibrated sub-actions is parametrized by a finite number of conditions. More precisely, if $\Omega(A) = \sqcup_{i=1}^{r} C_i$ is equal to a disjoint union of irreducible components and $x_i \in C_i$ are chosen, the *sub-action constraint set* is by definition

$$C_A(x_1, \ldots, x_r) := \big\{(u_1, \ldots, u_r) \in \mathbb{R}^r \mid u_j - u_i \le h_A(x_i, x_j), \quad \forall\, i, j\big\}.$$

Therefore, the analogous result to theorem 12 can be stated as follows.

Theorem 6. *Let (X, T) be a transitive expanding dynamical system on a compact metric space and $A\colon X \to \mathbb{R}$ be a Hölder observable. Assume that $\Omega(A) = \sqcup_{i=1}^{r} C_i$ is equal to a disjoint union of irreducible components.*

1. *There is a one-to-one correspondence between the sub-action constraint set and the set of calibrated sub-actions,*

$$\begin{cases} (u_1, \ldots, u_r) \in C_A(x_1, \ldots, x_r) \\ u(x) = \min\limits_{1\le i\le r} \big[u_i + h_A(x_i, x)\big] \end{cases} \iff \begin{cases} u \text{ is a calibrated sub-action} \\ u_i = u(x_i) \end{cases}.$$

2. *Let $i_0 \in \{1, \ldots, r\}$ and $u_{i_0} \in \mathbb{R}$ fixed. Define $u_i = u_{i_0} + h_A(x_{i_0}, x_i)$ for all i, then $(u_1, \ldots, u_r) \in C_A(x_1, \ldots, x_r)$ and the unique calibrated sub-action u satisfying $u(x_i) = u_i$, for all i, is of the form*

$$u(x) := \min_{1\le i\le r} \big[u_i + h_A(x_i, x)\big] = u_{i_0} + h_A(x_{i_0}, x).$$

[2] We prove these statements in the general setting (see definition-proposition 18 and proposition 19).

When the optimizing probability is unique, the calibrated sub-action is unique (up to additive constants) and generally the proofs of important results are easiest to discuss.

One of the main issues of the thermodynamic formalism at temperature zero is the analysis, in the case there are several ergodic maximizing probabilities for A, which of these probabilities the Gibbs states $\mu_{\beta A}$ accumulates, when the inverse temperature parameter β goes to infinite. It is not clear when there is a unique one in the general Hölder case[3]. In the case of a potential A that depends on finitely many coordinates, this question is addressed in [4, 17].

Let us denote, in our notation, by $C_1, C_2, \ldots, C_r$ the different supports of the ergodic components of the set of maximizing probabilities for A. Then, one can ask: is there a unique one, let us say, with support in C_{i_0} that will be attained as the only limit of Gibbs states $\mu_{\beta A}$ when $\beta \to \infty$?

This question is in some way related to the result of item 2 of theorem 6. Indeed, the dual question can be made for the limits $\frac{1}{\beta}\log\phi_\beta$ when $\beta \to \infty$, where ϕ_β is the normalized eigenfunction for the Ruelle operator associated to the potential βA. It is well-known that any convergent subsequence will determine a calibrated sub-action, but is it not clear if there is only one possible limit.

Hence, which one among the various calibrated sub-actions would be chosen? This is an important question. All functions of the form $h_A(x_i, \cdot)$, with $x_i \in C_i$, are calibrated sub-actions, for any $i = 1, 2, \ldots, r$. Item 2 of theorem 6 gives sufficient conditions to say that a certain $h_A(x_{i_0}, \cdot)$ is preferred in some sense. We believe this fact is related to the important issues described above.

3 Basic concepts and main results

For simplicity, we will restrict the exposition of the *holonomic optimization model* to the symbolic dynamics case. Let (Σ, σ) be a one-sided transitive subshift of finite type given by a $s \times s$ irreducible transition matrix $\mathbf{M}$. More precisely

$$\Sigma := \left\{ \mathbf{x} \in \{1, \ldots, s\}^{\mathbb{N}} \,\middle|\, \mathbf{M}(x_j, x_{j+1}) = 1 \text{ for all } j \geq 0 \right\}$$

and σ is the left shift acting on Σ by $\sigma(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$. Fix $\lambda \in (0, 1)$. We choose a particular metric on Σ defined by $d(\mathbf{x}, \bar{\mathbf{x}}) = \lambda^k$, for any $\mathbf{x}, \bar{\mathbf{x}} \in \Sigma$, $\mathbf{x} = (x_0, x_1, \ldots)$, $\bar{\mathbf{x}} = (\bar{x}_0, \bar{x}_1, \ldots)$ and $k = \min\{j : x_j \neq \bar{x}_j\}$.

[3] Examples of Lipschitz observables on the full shift $\{0, 1\}^{\mathbb{N}}$ for which the zero temperature limit of the associated Gibbs measures does not exist have been recently announced (see [6]).

The holonomic model is a generalization of the ergodic optimization framework. The holonomic model has been introduced first by R. Mañé in an attempt to clarify Aubry-Mather theory for continuous time Lagrangian dynamics (see [8, 19]). In this model, the set of invariant minimizing probability measures is replaced by a broader class of measures called holonomic measures. In Aubry-Mather theory for discrete time Lagrangian dynamics on the n dimensional torus $\mathbb{T}^n$ (see [14]), an holonomic probability measure $\mu(dx, dv)$ is a probability measure on $\mathbb{T}^n \times \mathbb{R}^n$ satisfying

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} f(x+v)\, d\mu(x,v) = \int_{\mathbb{T}^n \times \mathbb{R}^n} f(x)\, d\mu(x,v), \quad \forall\, f \in C^0(\mathbb{T}^n),$$

where the sum $x + v$ is obviously taken modulo $\mathbb{Z}^n$.

One may exploit an interesting analogy with Aubry-Mather theory in symbolic dynamics. Similarly to the previous example of discrete dynamics, Σ will play the role of the "space of positions" (analogous to $\mathbb{T}^n$ in the holonomic model) and the set of inverse branches or possible pasts Σ^* will play the role of the "space of immediately anterior velocities" (analogous to $\mathbb{R}^n$). For a complete exposition and motivation of the holonomic optimization model, see [12, 13].

We call *dual subshift of finite type* the space

$$\Sigma^* := \left\{ \mathbf{y} \in \{1, \ldots, s\}^{\mathbb{N}_*} \,\middle|\, \mathbf{M}(y_{j+1}, y_j) = 1 \text{ for all } j \geq 1 \right\}.$$

We denote by $\mathbf{y} = (\ldots, y_3, y_2, y_1)$ a point of Σ^*. We call *dual shift* the map $\sigma^*(\ldots, y_3, y_2, y_1) := (\ldots, y_3, y_2)$. The *natural extension* of (Σ, σ) will play the role of the "phase space" (analogous to $\mathbb{T}^n \times \mathbb{R}^n$) and will be identified with a subset of $\Sigma^* \times \Sigma$

$$\hat{\Sigma} := \Big\{ (\mathbf{y}, \mathbf{x}) = (\ldots, y_2, y_1 | x_0, x_1, \ldots) \in \Sigma^* \times \Sigma \,\Big|\, \mathbf{x} = (x_0, x_1, \ldots),\ \mathbf{y} = (\ldots, y_2, y_1) \text{ and } \mathbf{M}(y_1, x_0) = 1 \Big\}.$$

Equivalently, one may write $\hat{\Sigma} = \bigcup_{\mathbf{x} \in \Sigma} \Sigma^*_{\mathbf{x}} \times \{\mathbf{x}\}$, where

$$\Sigma^*_{\mathbf{x}} := \left\{ \mathbf{y} = (\ldots, y_2, y_1) \in \Sigma^* \,\middle|\, \mathbf{M}(y_1, x_0) = 1 \right\} \quad \forall\, \mathbf{x} = (x_0, x_1, \ldots) \in \Sigma.$$

The analogue of the "discrete Euler-Lagrange map" is obtained by the usual left shift $\hat{\sigma}$ on the natural extension,

$$\hat{\sigma}(\ldots, y_2, y_1 | x_0, x_1, \ldots) = (\ldots, y_1, x_0 | x_1, x_2, \ldots).$$

Consider then $\tau^*\colon \hat{\Sigma} \to \Sigma^*$ given by

$$\tau^*(\mathbf{y}, \mathbf{x}) := \tau^*_{\mathbf{x}}(\mathbf{y}) := (\ldots, y_2, y_1, x_0).$$

Notice that $\tau^*_{\mathbf{x}}(\mathbf{y}) \in (\sigma^*)^{-1}(\mathbf{y})$. Similarly, inverse branches of $\mathbf{x} \in \Sigma$ with respect to σ are constructed using the map $\tau\colon \hat{\Sigma} \to \Sigma$,

$$\tau(\mathbf{y}, \mathbf{x}) := \tau_{\mathbf{y}}(\mathbf{x}) = (y_1, x_0, x_2, \ldots).$$

Clearly we have $\hat{\sigma}(\mathbf{y}, \mathbf{x}) := (\tau^*_{\mathbf{x}}(\mathbf{y}), \sigma(\mathbf{x}))$ and $\hat{\sigma}^{-1}(\mathbf{y}, \mathbf{x}) = (\sigma^*(\mathbf{y}), \tau_{\mathbf{y}}(\mathbf{x}))$. Note that $\tau = \pi \circ \hat{\sigma}^{-1}$, where $\pi\colon \hat{\Sigma} \to \Sigma$ is the canonical projection onto the $\mathbf{x}$-variable.

Let $\hat{\mathcal{M}}$ be the set of probability measures over the Borel sigma-algebra of $\hat{\Sigma}$. Instead of considering the set of $\hat{\sigma}$-invariant probability measures[4], we introduce the set of *holonomic probability measures*,

$$\hat{\mathcal{M}}_{\text{hol}} := \left\{ \hat{\mu} \in \hat{\mathcal{M}} \,\middle|\, \int_{\hat{\Sigma}} f(\tau_{\mathbf{y}}(\mathbf{x})) d\hat{\mu}(\mathbf{y}, \mathbf{x}) = \int_{\hat{\Sigma}} f(\mathbf{x})\, d\hat{\mu}(\mathbf{y}, \mathbf{x}), \forall f \in C^0(\Sigma) \right\}.$$

It seems important to insist that the holonomic condition demands only the continuous function f to be defined on the one-sided shift of finite type Σ and not on the natural extension $\hat{\Sigma}$ as would be the case for the characterization of $\hat{\sigma}$-invariance. Observe that $\hat{\mu} \in \hat{\mathcal{M}}_{\text{hol}}$ if, and only if, $\pi_*(\hat{\mu}) = \pi_*(\hat{\sigma}_*^{-1}(\hat{\mu}))$ if, and only if, $\sigma_*^{-1}(\hat{\mu})$ projects onto a σ-invariant Borel probability measure. As in section 2, we denote by $\mathcal{M}_\sigma$ the set of σ-invariant Borel probability measures. The triple $(\hat{\Sigma}, \hat{\sigma}, \hat{\mathcal{M}}_{\text{hol}})$ is called the holonomic model. Such a formalism includes the ergodic optimization model discussed in section 2 as we will see.

Let $A \in C^\theta(\hat{\Sigma})$ be a Hölder observable. We would like to emphasize that A is continuous on the natural extension $\hat{\Sigma}$. This is one of the crucial points in the holonomic setting: the possibility of formulating a relevant minimization question for functions defined on the two-sided shift. Then, we call *holonomic minimizing value* of A

$$\begin{aligned} \bar{A} &:= \min \left\{ \int_{\hat{\Sigma}} A(\mathbf{y}, \mathbf{x})\, d\hat{\mu}(\mathbf{y}, \mathbf{x}) \,\middle|\, \hat{\mu} \in \hat{\mathcal{M}}_{\text{hol}} \right\} \\ &= \min \left\{ \int_{\hat{\Sigma}} A \circ \hat{\sigma}(\mathbf{y}, \mathbf{x})\, d\hat{\mu}(\mathbf{y}, \mathbf{x}) \,|\, \pi_*(\hat{\mu}) \in \mathcal{M}_\sigma \right\}. \end{aligned}$$

If $A \circ \hat{\sigma} = B \circ \pi$ depends only on the $\mathbf{x}$-variable, $\bar{A} = \bar{B}$ as in the section 2.

[4] It is well-known that a Hölder observable defined on the two-sided shift is cohomologous to an observable that depends just on future coordinates. So a minimization over $\hat{\sigma}$-invariant probabilities may be reduced to a minimization over σ-invariant probabilities.

The set of *minimizing (holonomic) probability measures* is denoted

$$\hat{\mathcal{M}}_{\mathrm{hol}}(A) := \left\{ \hat{\mu} \in \hat{\mathcal{M}}_{\mathrm{hol}} \;\middle|\; \int_{\hat{\Sigma}} A(\mathbf{y}, \mathbf{x})\, d\hat{\mu}(\mathbf{y}, \mathbf{x}) = \bar{A} \right\}.$$

A continuous function $u\colon \Sigma \to \mathbb{R}$ is called *sub-action* with respect to A if

$$u(\mathbf{x}) - u(\tau_{\mathbf{y}}(\mathbf{x})) \leq A(\mathbf{y}, \mathbf{x}) - \bar{A}, \quad \forall\, (\mathbf{y}, \mathbf{x}) \in \hat{\Sigma},$$

or equivalently $A - \bar{A} \geq u \circ \pi - u \circ \pi \circ \hat{\sigma}^{-1}$. We call *contact locus* of a sub-action u the set

$$\hat{\mathbb{M}}_A(u) := \left(A - u \circ \pi + u \circ \pi \circ \hat{\sigma}^{-1}\right)^{-1}(\bar{A})$$

where the above inequality becomes an equality, that is, a point $(\mathbf{y}, \mathbf{x}) \in \hat{\Sigma}$ belongs to $\hat{\mathbb{M}}_A(u)$ if, and only if, $u(\mathbf{x}) - u(\tau_{\mathbf{y}}(\mathbf{x})) = A(\mathbf{y}, \mathbf{x}) - \bar{A}$. If $A \circ \hat{\sigma} = B \circ \pi$ for some $B\colon \Sigma \to \mathbb{R}$, notice that $\pi \circ \hat{\sigma}^{-1}(\hat{\mathbb{M}}_A(u)) = \mathbb{M}_B(u)$.

A calibrated sub-action is a particular sub-action which possesses a large contact locus in the sense that $\pi(\hat{\mathbb{M}}_A(u)) = \Sigma$.

Definition 7. *A sub-action $u\colon \Sigma \to \mathbb{R}$ is said to be calibrated for A if*

$$u(\mathbf{x}) = \min_{\mathbf{y} \in \Sigma_{\mathbf{x}}^*} \left[u(\tau_{\mathbf{y}}(\mathbf{x})) + A(\mathbf{y}, \mathbf{x}) - \bar{A} \right], \quad \forall\, \mathbf{x} \in \Sigma,$$

where recall that $\Sigma_{\mathbf{x}}^* := \{\mathbf{y} \in \Sigma^* \mid (\mathbf{y}, \mathbf{x}) \in \hat{\Sigma}\}$.

If $\hat{B} := A \circ \hat{\sigma}$ and $B(\mathbf{x}) := \min\{\hat{B}(\mathbf{y}, \mathbf{x}) \mid \mathbf{y} \in \Sigma_{\mathbf{x}}^*\}$, then u is a calibrated sub-action for A if, and only if, u is a calibrated sub-action for B. Indeed,

$$\begin{aligned} u(\mathbf{x}) &= \min_{\sigma(\bar{\mathbf{x}}) = \mathbf{x}} \; \min_{\mathbf{y} \in \Sigma_{\mathbf{x}}^*,\, \tau_{\mathbf{y}}(\mathbf{x}) = \bar{\mathbf{x}}} \left[u(\bar{\mathbf{x}}) + \hat{B}(\sigma^*(\mathbf{y}), \bar{\mathbf{x}}) - \bar{A} \right] \\ &= \min_{\sigma(\bar{\mathbf{x}}) = \mathbf{x}} \left[u(\bar{\mathbf{x}}) + B(\bar{\mathbf{x}}) - \bar{A} \right] = \min_{\sigma(\bar{\mathbf{x}}) = \mathbf{x}} \left[u(\bar{\mathbf{x}}) + B(\bar{\mathbf{x}}) - \bar{B} \right]. \end{aligned}$$

(The definition of B gives $\bar{B} \leq \bar{A}$ and the calibration gives $\bar{B} \geq \bar{A}$.)

A classification theorem for calibrated sub-actions is presented in [13]. A central concept is the set of non-wandering points with respect to A (previously defined in [9, 18] in the ergodic optimization model). We call *path of length k* a sequence $(\mathbf{z}^0, \ldots, \mathbf{z}^k)$ of points of $\hat{\Sigma}$ such that

$$\mathbf{z}^i = (\mathbf{y}^i, \mathbf{x}^i) \;\text{ with }\; \mathbf{x}^i = \tau_{\mathbf{y}^{i+1}}(\mathbf{x}^{i+1}),\; \forall\, i = 0, 1, \ldots, k-1,$$

that is, a sequence $(\mathbf{z}^0, \dots, \mathbf{z}^k)$ where $\mathbf{x}^i = \sigma^i(\mathbf{x}^0)$ for all $i = 0, 1, \dots, k$, $\mathbf{x}^0 = (x_0, x_1, \dots, x_{k-1}, \mathbf{x}^k)$ and

$$\mathbf{z}^0 = \big(\mathbf{y}^0 | x_0, \dots, x_{k-1}, \mathbf{x}^k\big),\ \mathbf{z}^1 = \big(\sigma^*(\mathbf{y}^1), x_0 | x_1, \dots, x_{k-1}, \mathbf{x}^k\big),\ \dots,$$

$$\mathbf{z}^{k-1} = \big(\sigma^*(\mathbf{y}^{k-1}), x_{k-2} | x_{k-1}, \mathbf{x}^k\big),\ \mathbf{z}^k = \big(\sigma^*(\mathbf{y}^k), x_{k-1} | \mathbf{x}^k\big).$$

Note that the point $\mathbf{y}^0$ is free of any restriction except that $\mathbf{M}(y_1^0, x_0) = 1$, more precisely, one just asks that $\mathbf{y}^0 \in \Sigma^*_{\mathbf{x}^0}$ while $\mathbf{y}^j \in \Sigma^*_{\mathbf{x}^j} \cap (\sigma^*)^{-1}(\Sigma^*_{\mathbf{x}^{j-1}})$ for $j = 1, \dots, k$. Equivalently, one could present a path in the following way

$$\mathbf{z}^0 = \big(\mathbf{y}^0, \tau_{\mathbf{y}^1} \circ \tau_{\mathbf{y}^2} \circ \cdots \circ \tau_{\mathbf{y}^k}(\mathbf{x}^k)\big),\ \mathbf{z}^1 = \big(\mathbf{y}^1, \tau_{\mathbf{y}^2} \circ \cdots \circ \tau_{\mathbf{y}^k}(\mathbf{x}^k)\big),\ \dots,$$

$$\mathbf{z}^{k-1} = \big(\mathbf{y}^{k-1}, \tau_{\mathbf{y}^k}(\mathbf{x}^k)\big),\ \mathbf{z}^k = \big(\mathbf{y}^k, \mathbf{x}^k\big).$$

Given $\epsilon > 0$ and $\mathbf{x}, \bar{\mathbf{x}} \in \Sigma$, we say that a path of length k, $(\mathbf{z}^0, \dots, \mathbf{z}^k)$, begins within ϵ of $\mathbf{x}$ and ends within ϵ of $\bar{\mathbf{x}}$ if $d(\mathbf{x}^0, \mathbf{x}) < \epsilon$ and $d(\mathbf{x}^k, \bar{\mathbf{x}}) < \epsilon$. Denote by $\mathcal{P}_k(\mathbf{x}, \bar{\mathbf{x}}, \epsilon)$ the set of such paths. Denote by $\mathcal{P}_k(\mathbf{x})$ the set of paths of length k beginning exactly at $\mathbf{x}$. Notice that a path $(\mathbf{z}^0, \dots, \mathbf{z}^k)$ belongs to $\mathcal{P}_k(\mathbf{x})$ if, and only if, $\pi(\mathbf{z}^i) = \sigma^i(\mathbf{x})$ for all $i = 0, 1, \dots, k$.

A point $\mathbf{x} \in \Sigma$ will be called *non-wandering with respect to* A if, for every $\epsilon > 0$, one can find a path $(\mathbf{z}^0, \dots, \mathbf{z}^k)$ in $\mathcal{P}_k(\mathbf{x}, \mathbf{x}, \epsilon)$, with $k \geq 1$, such that

$$\Big| \sum_{i=1}^{k} (A - \bar{A})(\mathbf{z}^i) \Big| < \epsilon.$$

We will denote by $\Omega(A)$ the set of non-wandering points with respect to A. If $A \circ \hat{\sigma} = B \circ \pi$, notice that $\Omega(A) = \Omega(B)$ as in section 2.

The first two authors have proved in [13] that $\Omega(A)$ is a non-empty compact σ-invariant set and satisfies

$$\Omega(A) \subset \bigcap \Big\{ \pi(\hat{\mathbb{M}}_A(u)) \ \Big|\ u \text{ is a continuous sub-action} \Big\}.$$

Remark 8. The set $\Omega(A)$ is analogous to the projected Aubry set in the continuous time Lagrangian dynamics. One could have introduced the corresponding Aubry set $\hat{\Omega}(A) \subset \hat{\Sigma}$ and proved $\pi(\hat{\Omega}(A)) = \Omega(A)$. Unfortunately, even for Hölder observable A, the graph property is not any more true: $\pi : \hat{\Omega}(A) \to \Omega(A)$ is no more bijective. A counter-example can be found in [13]. It would be interesting to find the right assumptions on $A \in C^\theta(\hat{\Sigma})$ in order to get this property.

Contrary to a calibrated sub-action, a *separating sub-action* is a sub-action with the smallest contact locus. More precisely,

Definition 9. *A sub-action $u \in C^0(\Sigma)$ is said to be separating (with respect to A) if it verifies $\pi(\hat{\mathbb{M}}_A(u)) = \Omega(A)$.*

Our first result is the following one.

Theorem 10. *If $A\colon \hat{\Sigma} \to \mathbb{R}$ is a θ-Hölder observable, then there exists a θ-Hölder separating sub-action. Moreover, in the θ-Hölder topology, the subset of θ-Hölder separating sub-actions is generic among all θ-Hölder sub-actions.*

According to the analogy with continuous time Lagrangian dynamics, sub-actions correspond to viscosity sub-solutions of the stationary Hamilton-Jacobi equation, calibrated sub-actions correspond to the weak KAM solutions introduced by A. Fathi (see [10]) and separating sub-actions correspond to special sub-solutions as described in [11].

By adapting the proof of theorem 10 in [13] and by using definition 14 of the the Peierls barrier h_A, we obtain a structure theorem for calibrated sub-actions. Such characterization corresponds to the one obtained for weak KAM solutions in Lagrangian dynamics (see [7]). The proof of the following theorem will be omitted.

Theorem 11. *Let A be a θ-Hölder observable.*

1. *If u is a continuous calibrated sub-action for A, then*

$$u(\mathbf{x}) = \min_{\bar{\mathbf{x}} \in \Omega(A)} \big[u(\bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x})\big].$$

2. *Conversely, for every continuous application $\phi\colon \Omega(A) \to \mathbb{R}$ satisfying*

$$\phi(\mathbf{x}) - \phi(\bar{\mathbf{x}}) \leq h_A(\bar{\mathbf{x}}, \mathbf{x}), \quad \forall\, \mathbf{x}, \bar{\mathbf{x}} \in \Omega(A),$$

the function $u(\mathbf{x}) := \min_{\bar{\mathbf{x}} \in \Omega(A)}[\phi(\bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x})]$ is a continuous calibrated sub-action extending ϕ on $\Omega(A)$.

In particular, this representation formula for calibrated sub-actions implies immediately that, in order to compare two such functions, we just need to compare their restrictions to $\Omega(A)$. For instance, if two calibrated sub-actions coincide for every non-wandering point with respect to A, then they are the same.

In the case the set of non-wandering points for A is reduced to a finite union of irreducible components $\Omega(A) = C_1 \cup \ldots \cup C_r$, the set of calibrated sub-actions admits a simpler characterization. We first show that the condition $\mathbf{x} \sim \bar{\mathbf{x}} \;\Leftrightarrow\; h_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x}) = 0$ defines an equivalent relation. Each one of its

equivalent classes is called an irreducible component. Let $\bar{\mathbf{x}}^1 \in C_1, \ldots, \bar{\mathbf{x}}^r \in C_r$ fixed. We call *sub-action constraint set* the set

$$C_A(\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^r) = \left\{(u_1, \ldots, u_r) \in \mathbb{R}^r \mid u_j - u_i \leq h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j), \quad \forall\, i, j\right\}.$$

Our second result is the following one.

Theorem 12. *Let A be a Hölder observable. Assume $\Omega(A)$ is a finite union of disjoint irreducible components, namely, $\Omega(A) = \sqcup_{i=1}^r C_i$. Let $\bar{\mathbf{x}}^1 \in C_1, \ldots, \bar{\mathbf{x}}^r \in C_r$ fixed.*

1. *If u is a continuous calibrated sub-action and $u_i := u(\bar{\mathbf{x}}^i)$ for every $i = 1, \ldots, r$, then*

$$(u_1, \ldots, u_r) \in C(\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^r) \quad \text{and} \quad u(\mathbf{x}) = \min_{1 \leq i \leq r} \left[u(\bar{\mathbf{x}}^i) + h_A(\bar{\mathbf{x}}^i, \mathbf{x})\right].$$

2. *If $(u_1, \ldots, u_r) \in C(\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^r)$ and $u(\mathbf{x}) := \min_{1 \leq i \leq r} \left[u_i + h_A(\bar{\mathbf{x}}^i, \mathbf{x})\right]$, then u is a continuous calibrated sub-action satisfying $u(\bar{\mathbf{x}}^i) = u_i$ for all $i = 1, \ldots, r$.*

3. *Take $i_0 \in \{1, \ldots, r\}$ and $(u_1, \ldots, u_r)$ such that $u_i := u_{i_0} + h_A(\bar{\mathbf{x}}^{i_0}, \bar{\mathbf{x}}^i)$ for all $i = 1, \ldots, r$. Then i_0 is unique, $(u_1, \ldots, u_r) \in C(\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^r)$ and the unique calibrated sub-action u satisfying $u(\bar{\mathbf{x}}^i) = u_i$, for all $i = 1, \ldots, r$, is of the form $u(\mathbf{x}) = u_{i_0} + h_A(\bar{\mathbf{x}}^{i_0}, \mathbf{x})$.*

The application we present here has a certain similarity to lemma 6 in [1]. We point out that the local character of viscosity solutions (as in definition 1 of [1]) is not present in our setting.

Application 13. *Let A be a Hölder observable. Consider any continuous sub-action v and a continuous calibrated sub-action u.*

1. *Then $u - v$ is constant on every irreducible component and*

$$\min_{\Sigma}(u - v) = \min_{\Omega(A)}(u - v).$$

2. *Assume $\Omega(A) = \sqcup_{i=1}^r C_i$ is a finite union of disjoint irreducible components. If $\min_\Sigma(u - v)$ is realized on an unique component C_{i_1} and the other components C_i, $i \neq i_1$, are not local minimum for $u - v$, then*

$$u(\mathbf{x}) = u(\bar{\mathbf{x}}_{i_1}) + h_A(\bar{\mathbf{x}}_{i_1}, \mathbf{x}), \quad \forall\, \mathbf{x} \in \Sigma,$$

where $\bar{\mathbf{x}}_{i_1}$ is any point in C_{i_1}.

4 Proof of theorem 10.

We first recall two notions of action potential between two points: the Mañé potential and the Peierls barrier. Given $\epsilon > 0$, $\mathbf{x}, \bar{\mathbf{x}} \in \Sigma$ and $k \geq 1$, we denote

$$S_A^\epsilon(\mathbf{x}, \bar{\mathbf{x}}, k) = \inf \left\{ \sum_{i=1}^{k} (A - \bar{A})(\mathbf{z}^i) \,\Big|\, (\mathbf{z}^0, \ldots, \mathbf{z}^k) \in \mathcal{P}_k(\mathbf{x}, \bar{\mathbf{x}}, \epsilon) \right\}.$$

If $\hat{B} := A \circ \hat{\sigma}$ and $B := \min\{\hat{B}(\mathbf{y}, \mathbf{x}) \,|\, \mathbf{y} \in \Sigma_{\mathbf{x}}^*\}$, notice that

$$S_A^\epsilon(\mathbf{x}, \bar{\mathbf{x}}, k) = \inf \left\{ \sum_{i=0}^{k-1} (B - \bar{B}) \circ \sigma^i(\mathbf{x}^0) \,\Big|\, d(\mathbf{x}^0, \mathbf{x}) < \epsilon,\ d(\sigma^k(\mathbf{x}^0), \bar{\mathbf{x}}) < \epsilon \right\}.$$

Definition 14. *We call Mañé potential the function* $\phi_A \colon \Sigma \times \Sigma \to \mathbb{R} \cup \{+\infty\}$ *defined by*

$$\phi_A(\mathbf{x}, \bar{\mathbf{x}}) = \lim_{\epsilon \to 0} \inf_{k \geq 1} S_A^\epsilon(\mathbf{x}, \bar{\mathbf{x}}, k).$$

We call Peierls barrier the function $h_A \colon \Sigma \times \Sigma \to \mathbb{R} \cup \{+\infty\}$ *defined by*

$$h_A(\mathbf{x}, \bar{\mathbf{x}}) = \lim_{\epsilon \to 0} \liminf_{k \to +\infty} S_A^\epsilon(\mathbf{x}, \bar{\mathbf{x}}, k).$$

Clearly, $\phi_A \leq h_A$ and both functions are lower semi-continuous. We summarize the main properties of these action potentials.

Proposition 15. *Let A be a Hölder observable.*

1. *If u is a continuous sub-action, then $u(\bar{\mathbf{x}}) - u(\mathbf{x}) \leq \phi_A(\mathbf{x}, \bar{\mathbf{x}})$.*
2. *For any points $\mathbf{x}, \bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}} \in \Sigma$, $\phi_A(\mathbf{x}, \bar{\bar{\mathbf{x}}}) \leq \phi_A(\mathbf{x}, \bar{\mathbf{x}}) + \phi_A(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}})$.*
3. *Given a point $\mathbf{x} \in \Sigma$, if there exists a positive integer L such that $0 < L < \min\left\{j > 0 \colon \sigma^j(\mathbf{x}) = \mathbf{x}\right\} \leq +\infty$, then*

$$\phi_A(\mathbf{x}, \mathbf{x}) = \phi_A(\mathbf{x}, \sigma^L(\mathbf{x})) + \phi_A(\sigma^L(\mathbf{x}), \mathbf{x}).$$

Moreover, if $\phi_A(\mathbf{x}, \mathbf{x}) < +\infty$, then there exists a path of length L, $(\bar{\mathbf{z}}^0 = (\bar{\mathbf{y}}^0, \bar{\mathbf{x}}^0), \ldots, \bar{\mathbf{z}}^L = (\bar{\mathbf{y}}^L, \bar{\mathbf{x}}^L))$, beginning at $\mathbf{x}$ ($\bar{\mathbf{x}}^j = \sigma^j(\mathbf{x})$ for all $j = 0, \ldots, L$), such that

$$\phi_A(\mathbf{x}, \sigma^L(\mathbf{x})) = \sum_{j=1}^{L} (A - \bar{A})(\bar{\mathbf{z}}^j).$$

4. *For any points* $\mathbf{x}, \bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}} \in \Sigma$ *and any sequence* $\{\bar{\mathbf{x}}^l\}$ *converging to* $\bar{\mathbf{x}}$,

$$h_A(\mathbf{x}, \bar{\bar{\mathbf{x}}}) \leq \liminf_{l \to +\infty} \phi_A(\mathbf{x}, \bar{\mathbf{x}}^l) + h_A(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}).$$

5. *If* $\mathbf{x} \in \Sigma$, *then* $\mathbf{x} \in \Omega(A) \Leftrightarrow \phi_A(\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow h_A(\mathbf{x}, \mathbf{x}) = 0$.

6. *If* $\mathbf{x} \in \Omega(A)$, *then* $\phi_A(\mathbf{x}, \cdot) = h_A(\mathbf{x}, \cdot)$ *and* $h_A(\mathbf{x}, \cdot)$ *is a Hölder calibrated sub-action with respect to the second variable.*

This proposition shows how to construct Hölder calibrated sub-actions without the use of the Lax-Oleinik fixed point method.

Remark 16. In Lagrangian Aubry-Mather theory on a compact manifold M, it is well known that, for any point $x \in M$, the map $y \in M \mapsto h(x, y) \in \mathbb{R}$ defines a weak KAM solution, where $h\colon M \times M \to \mathbb{R}$ denotes the corresponding Peierls barrier. The analogous result for $h_A(\mathbf{x}, \cdot)$ is however false in the holonomic optimization model. Using item 3, it is not difficulty to built examples where

$$\lim_{L \to +\infty} \phi_A(\mathbf{x}, \sigma^L(\mathbf{x})) = \lim_{L \to +\infty} h_A(\mathbf{x}, \sigma^L(\mathbf{x})) = +\infty,$$

which shows that $h_A(\mathbf{x}, \cdot)$ is not always a continuous function.

Proof of proposition 15. Items 1, 2, 5 and 6 are well known and a demonstration can be found, for instance, in [9, 13]. So let us prove items 3 and 4.

Item 3. We already know from item 2, that

$$\phi_A(\mathbf{x}, \mathbf{x}) \leq \phi_A(\mathbf{x}, \sigma^L(\mathbf{x})) + \phi_A(\sigma^L(\mathbf{x}), \mathbf{x}).$$

Define $\eta = \min\{d(\sigma^i(\mathbf{x}), \sigma^j(\mathbf{x}))\colon 0 \leq i < j \leq L\}$. Fix $\gamma > 0$ and take $\epsilon \in (0, \min\{\lambda, \eta/2\})$ such that $\mathrm{Höld}(A) L \epsilon^\theta < \gamma$. Consider also $\rho \in (0, \epsilon)$ such that $d(\mathbf{x}, \bar{\mathbf{x}}) < \rho$ implies $d(\sigma^j(\mathbf{x}), \sigma^j(\bar{\mathbf{x}})) < \epsilon$ for $1 \leq j \leq L$. Take then a path $(\mathbf{z}^0, \ldots, \mathbf{z}^l) \in \mathcal{P}_l(\mathbf{x}, \mathbf{x}, \rho)$ satisfying

$$\sum_{j=1}^{l} (A - \bar{A})(\mathbf{z}^j) < \inf_{k \geq 1} S_A^\rho(\mathbf{x}, \mathbf{x}, k) + \gamma \leq \phi_A(\mathbf{x}, \mathbf{x}) + \gamma.$$

Let $\mathbf{z}^j = (\mathbf{y}^j, \mathbf{x}^j)$ where $\mathbf{x}^j = \sigma^j(\mathbf{x}^0)$ for all $j = 0, 1, \ldots, l$. We claim that $l > L$. Indeed, ρ has been chosen so that, for each $j \in \{1, 2, \ldots, L\}$,

$$d(\mathbf{x}^j, \mathbf{x}) = d(\sigma^j(\mathbf{x}^0), \mathbf{x}) \geq d(\sigma^j(\mathbf{x}), \mathbf{x}) - d(\sigma^j(\mathbf{x}), \sigma^j(\mathbf{x}^0)) > \eta - \epsilon > \rho.$$

Introduce a new path $(\bar{\mathbf{z}}^0, \dots, \bar{\mathbf{z}}^L) \in \mathcal{P}_L(\mathbf{x}, \mathbf{x}, \epsilon)$ given by $\bar{\mathbf{z}}^j = (\mathbf{y}^j, \sigma^j(\mathbf{x}))$, for all $j = 0, \dots, L$. The definition of ρ guarantees

$$\begin{aligned}\sum_{j=1}^{L}(A - \bar{A})(\bar{\mathbf{z}}^j) &< \sum_{j=1}^{L}(A - \bar{A})(\mathbf{z}^j) + \text{Höld}_\theta(A) L\epsilon^\theta \\ &\leq \sum_{j=1}^{L}(A - \bar{A})(\mathbf{z}^j) + \gamma.\end{aligned}$$

Notice that $(\mathbf{z}^L, \dots, \mathbf{z}^l) \in \mathcal{P}_{l-L}(\sigma^L(\mathbf{x}), \mathbf{x}, \epsilon)$. We finally obtain

$$\begin{aligned}\inf_{k\geq 1} S_A^\epsilon(\mathbf{x}, \sigma^L(\mathbf{x}), k) &+ \inf_{k\geq 1} S_A^\epsilon(\sigma^L(\mathbf{x}), \mathbf{x}, k) \\ &\leq \sum_{j=1}^{L}(A - \bar{A})(\bar{\mathbf{z}}^j) + \sum_{j=L+1}^{l}(A - \bar{A})(\mathbf{z}^j) \\ &\leq \sum_{j=1}^{L}(A - \bar{A})(\mathbf{z}^j) + \sum_{j=L+1}^{l}(A - \bar{A})(\mathbf{z}^j) + \gamma \\ &\leq \inf_{k\geq 1} S_A^\rho(\mathbf{x}, \mathbf{x}, k) + 2\gamma \leq \phi_A(\mathbf{x}, \mathbf{x}) + 2\gamma.\end{aligned}$$

By letting ϵ goes to 0 and γ to 0, we get

$$\phi_A(\mathbf{x}, \sigma^L(\mathbf{x})) + \phi_A(\sigma^L(\mathbf{x}), \mathbf{x}) \leq \phi_A(\mathbf{x}, \mathbf{x}).$$

The first part of item 3 is proved. To prove the second part, the previous computation shows that, for any sufficiently small ϵ, there exists a path $(\bar{\mathbf{z}}_\epsilon^0, \dots, \bar{\mathbf{z}}_\epsilon^L) \in \mathcal{P}_L(\mathbf{x})$ such that

$$\sum_{j=1}^{L}(A - \bar{A})\big(\bar{\mathbf{z}}_\epsilon^j\big) + \inf_{k\geq 1} S_A^\epsilon(\sigma^L(\mathbf{x}), \mathbf{x}, k) \leq \phi_A(\mathbf{x}, \mathbf{x}) + 2\gamma.$$

By taking accumulation points of $\bar{\mathbf{z}}_\epsilon^j$ when $\epsilon \to 0$, we obtain, for any γ, a path $(\bar{\mathbf{z}}^0, \dots, \bar{\mathbf{z}}^L)$ such that

$$\phi_A(\mathbf{x}, \sigma^L(\mathbf{x})) \leq \sum_{j=1}^{L}(A - \bar{A})(\bar{\mathbf{z}}^j) \leq \phi_A(\mathbf{x}, \mathbf{x}) - \phi_A(\sigma^L(\mathbf{x}), \mathbf{x}) + 2\gamma$$

The result follows from item 2 and by taking once more accumulation points of $\bar{\mathbf{z}}^j$ when $\gamma \to 0$.

Item 4. Since ϕ_A is lower semi-continuous, the statement is equivalent to

$$h_A(\mathbf{x}, \bar{\bar{\mathbf{x}}}) \leq \phi_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}), \quad \forall\, \mathbf{x}, \bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}} \in \Sigma.$$

Fix $\gamma > 0$ and $\epsilon \in (0, \lambda/2)$ such that $\text{Höld}(A)(2\epsilon)^\theta/(1-\lambda^\theta) < \gamma$. There exists a path $(\mathbf{z}^0, \ldots, \mathbf{z}^k) \in \mathcal{P}_k(\mathbf{x}, \bar{\mathbf{x}}, \epsilon)$ such that

$$\sum_{j=1}^{k} (A - \bar{A})(\mathbf{z}^j) \;<\; \inf_{n \geq 1} S_A^\epsilon(\mathbf{x}, \bar{\mathbf{x}}, n) + \gamma.$$

For any $N \geq 1$, there exists a path $(\bar{\mathbf{z}}^0, \ldots, \bar{\mathbf{z}}^l) \in \mathcal{P}_l(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}, \epsilon)$ of length $l \geq N$ such that

$$\sum_{j=1}^{l} (A - \bar{A})(\bar{\mathbf{z}}^j) \;<\; \inf_{n \geq N} S_A^\epsilon(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}, n) + \gamma.$$

We define a path $\big(\bar{\bar{\mathbf{z}}}^0, \ldots, \bar{\bar{\mathbf{z}}}^{k+l}\big) \in \mathcal{P}_{k+l}\big(\mathbf{x}, \bar{\bar{\mathbf{x}}}, 3\epsilon\big)$ in the following way

$$\bar{\bar{\mathbf{z}}}^j = \bar{\mathbf{z}}^{j-k},\ \forall\, j = k+1, \ldots, k+l, \qquad \bar{\bar{\mathbf{z}}}^j = (\bar{\bar{\mathbf{y}}}^j, \bar{\bar{\mathbf{x}}}^j),\ \forall\, j = 0, \ldots, k,$$

$$\bar{\bar{\mathbf{y}}}^j = \mathbf{y}^j,\ \forall\, j = 0, \ldots, k, \qquad \bar{\bar{\mathbf{x}}}^k = \bar{\mathbf{x}}^0,\ \bar{\bar{\mathbf{x}}}^{j-1} = \tau_{\mathbf{y}^j}(\bar{\bar{\mathbf{x}}}^j),\ \forall\, j = 1, \ldots, k.$$

We notice that $d\big(\bar{\bar{\mathbf{x}}}^j, \mathbf{x}^j\big) \leq \lambda^{k-j} d\big(\bar{\bar{\mathbf{x}}}^k, \mathbf{x}^k\big)$, for all $j = 0, \ldots, k$. Since

$$d(\bar{\bar{\mathbf{x}}}^k, \mathbf{x}^k) = d(\bar{\mathbf{x}}^0, \mathbf{x}^k) \leq d(\bar{\mathbf{x}}^0, \bar{\mathbf{x}}) + d(\bar{\mathbf{x}}, \mathbf{x}^k) < 2\epsilon,$$

we obtain $d(\bar{\bar{\mathbf{x}}}^0, \mathbf{x}) \leq \lambda^k 2\epsilon + \epsilon < 3\epsilon$. Hence, it follows that

$$\begin{aligned}
\inf_{n \geq N} S_A^{3\epsilon}(\mathbf{x}, \bar{\bar{\mathbf{x}}}, n) &\leq \sum_{j=1}^{k+l} (A - \bar{A})(\bar{\bar{\mathbf{z}}}^j) \\
&\leq \sum_{j=1}^{l} (A - \bar{A})(\bar{\mathbf{z}}^j) + \sum_{j=1}^{k} (A - \bar{A})(\mathbf{z}^j) + \frac{(2\epsilon)^\theta}{1-\lambda^\theta} \text{Höld}_\theta(A) \\
&\leq \inf_{n \geq 1} S_A^\epsilon(\mathbf{x}, \bar{\mathbf{x}}, n) + \inf_{n \geq N} S_A^\epsilon(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}, n) + 3\gamma \\
&\leq \phi_A(\mathbf{x}, \bar{\mathbf{x}}) + \inf_{n \geq N} S_A^\epsilon(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}, n) + 3\gamma.
\end{aligned}$$

By taking first $N \to +\infty$, then $\epsilon \to 0$ and $\gamma \to 0$, we get

$$h_A(\mathbf{x}, \bar{\bar{\mathbf{x}}}) \leq \phi_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}).$$

Other properties of the Mañé potential and the Peierls barrier can be derived from the previous proposition. For instance, item 4 gives us the following inequality

$$h_A(\mathbf{x}, \bar{\bar{\mathbf{x}}}) \leq h_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}), \quad \forall\, \mathbf{x}, \bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}} \in \Sigma.$$

We now begin the proof of theorem 10. It follows immediately from the next lemma.

Lemma 17. *Let $D \subset \Sigma$ be an open set containing $\Omega(A)$. Denote by $\mathcal{D}_A$ the subset of Hölder sub-actions u such that $\pi(\mathbb{M}_A(u)) \subset D$. Then, for the Hölder topology, $\mathcal{D}_A$ is an open dense subset of the Hölder sub-actions.*

We only need a few lines to show that lemma 17 yields theorem 10. As a matter of fact, if one considers, for each positive integer j, the open set $D_j = \{\mathbf{x} \in \Sigma \mid d(\mathbf{x}, \Omega(A)) < 1/j\}$ and the corresponding open dense subset of Hölder sub-actions $\mathcal{D}_{A,j}$, then the set of Hölder separating sub-actions contains the countable intersection $\cap_{j>0}\mathcal{D}_{A,j}$.

Proof of lemma 17. We only discuss the denseness of $\mathcal{D}_A$.

Part 1. Let v be any Hölder sub-action for A. We will show that, for every $\mathbf{x} \notin D$, there exists a Hölder sub-action $v_{\mathbf{x}}$ as close as we want to v in the Hölder topology with a projected contact locus disjoint from $\mathbf{x}$, that is, $\mathbf{x} \notin \pi(\mathbb{M}_A(v_{\mathbf{x}}))$ or

$$v_{\mathbf{x}}(\mathbf{x}) - v_{\mathbf{x}}(\tau_{\mathbf{y}}(\mathbf{x})) < A(\mathbf{y}, \mathbf{x}) - \bar{A}, \quad \forall\, \mathbf{y} \in \Sigma^*_{\mathbf{x}}.$$

Let $\mathbf{x} \notin D$. We discuss two cases.

Case a. We assume there exists an integer $k \geq 0$ such that, for every path of length k beginning at

$$\mathbf{x}, (\mathbf{z}^0 = (\mathbf{y}^0, \mathbf{x}), \ldots, \mathbf{z}^k = (\mathbf{y}^k, \sigma^k(\mathbf{x}))) \in \mathcal{P}_k(\mathbf{x}),$$

the terminal point $\mathbf{z}^k \notin \mathbb{M}_A(v)$. If $k = 0$, we choose $v_{\mathbf{x}} = v$. Assume now $k \geq 1$. Let

$$B := A - \bar{A} - v \circ \pi + v \circ \pi \circ \hat{\sigma}^{-1} \geq 0$$

be the associated normalized observable ($B \geq 0$ and $\bar{B} = 0$). We recall that $\tau_{\mathbf{y}^j}(\sigma^j(\mathbf{x})) = \sigma^{j-1}(\mathbf{x})$, for all $j = 1, \ldots, k$. So by hypothesis

$$B(\mathbf{z}^k) = B(\mathbf{y}^k, \sigma^k(\mathbf{x})) > 0, \quad \forall\, \mathbf{y}^k \in \Sigma^*_{\sigma^k(\mathbf{x})} \text{ s.t. } \sigma^{k-1}(\mathbf{x}) = \tau_{\mathbf{y}^k}(\sigma^k(\mathbf{x})). \qquad \text{(I)}$$

Notice first that, if $(\bar{\mathbf{z}}^0, \ldots, \bar{\mathbf{z}}^k)$ is a path of length k and $\gamma \in (0,1)$ is any constant, as B is non-negative, one has

$$\begin{aligned} B(\bar{\mathbf{z}}^0) &= \sum_{j=0}^{k-1} B(\bar{\mathbf{z}}^j) - \sum_{j=1}^{k} B(\bar{\mathbf{z}}^j) + B(\bar{\mathbf{z}}^k) \\ &\geq \gamma \sum_{j=0}^{k-1} B(\bar{\mathbf{z}}^j) - \gamma \sum_{j=1}^{k} B(\bar{\mathbf{z}}^j) + \gamma B(\bar{\mathbf{z}}^k) \qquad \text{(II)} \\ &\geq \gamma \sum_{j=0}^{k-1} B(\bar{\mathbf{z}}^j) - \gamma \sum_{j=1}^{k} B(\bar{\mathbf{z}}^j). \end{aligned}$$

Let $w_k \colon \Sigma \to \mathbb{R}$ be the function given by

$$w_k(\bar{\mathbf{x}}) := \inf \left\{ \sum_{j=1}^{k} B(\bar{\mathbf{z}}^j) \,\Big|\, (\bar{\mathbf{z}}^0, \ldots, \bar{\mathbf{z}}^k) \in \mathcal{P}_k(\bar{\mathbf{x}}) \right\}, \quad \forall\, \bar{\mathbf{x}} \in \Sigma.$$

Because $\mathcal{P}_k(\bar{\mathbf{x}})$ is a closed subspace of the compact space $\hat{\Sigma}^{k+1}$, the above infimum is effectively a minimum. Moreover, since the application $C(\bar{\mathbf{x}}) := \min\{B \circ \hat{\sigma}(\bar{\mathbf{y}}, \bar{\mathbf{x}}) \mid \bar{\mathbf{y}} \in \Sigma^*_{\bar{\mathbf{x}}}\}$ is Hölder, $w_k = \sum_{j=0}^{k-1} C \circ \sigma^j$ is also Hölder[5].

We first prove that $-\gamma w_k$ is a sub-action. Let $\bar{\mathbf{x}} \in \Sigma$ and $\bar{\mathbf{y}} \in \Sigma^*_{\bar{\mathbf{x}}}$. There exists a path of length k, $(\bar{\mathbf{z}}^0, \ldots, \bar{\mathbf{z}}^k)$, beginning at $\bar{\mathbf{x}}$ and realizing the minimum

$$w_k(\bar{\mathbf{x}}) = \sum_{j=1}^{k} B(\bar{\mathbf{z}}^j).$$

Notice the only constraint on $\bar{\mathbf{y}}^0$ is $\bar{\mathbf{y}}^0 \in \Sigma^*_{\bar{\mathbf{x}}}$, besides $\bar{\mathbf{y}}^0$ does not appear in the previous sum. Choose $\bar{\mathbf{y}}^0 = \bar{\mathbf{y}}$, $\bar{\mathbf{y}}^{-1} \in \Sigma^*_{\bar{\mathbf{x}}^{-1}}$ and call $\bar{\mathbf{x}}^{-1} = \tau_{\bar{\mathbf{y}}}(\bar{\mathbf{x}})$. Then $(\bar{\mathbf{z}}^{-1}, \bar{\mathbf{z}}^0, \ldots, \bar{\mathbf{z}}^{k-1})$ is a path of length k beginning at $\tau_{\bar{\mathbf{y}}}(\bar{\mathbf{x}})$. So denote $\bar{\mathbf{z}} := (\bar{\mathbf{x}}, \bar{\mathbf{y}})$. Thanks to inequality (II)

$$B(\bar{\mathbf{z}}) = B(\bar{\mathbf{z}}^0) \geq \gamma \sum_{j=0}^{k-1} B(\bar{\mathbf{z}}^j) - \gamma \sum_{j=1}^{k} B(\bar{\mathbf{z}}^j) \geq \gamma w_k(\tau_{\bar{\mathbf{y}}}(\bar{\mathbf{x}})) - \gamma w_k(\bar{\mathbf{x}}),$$

which shows $-\gamma w_k$ is a sub-action for B. Moreover, given any $\mathbf{y} \in \Sigma^*_{\mathbf{x}}$, the same computation for $\mathbf{z} := (\mathbf{x}, \mathbf{y})$ instead of $\bar{\mathbf{z}}$ and (I) assure that

$$B(\mathbf{z}) - \gamma w_k(\tau_{\mathbf{y}}(\mathbf{x})) + \gamma w_k(\mathbf{x}) \geq \gamma B(\mathbf{z}^k) > 0.$$

[5] We leave the details to the reader. In particular, one shall note that $w_k = \sum_j C \circ \sigma^j$ means $\min(a+b) = \min a + \min b$, which indicates the importance of what a path is.

We have proved that $\mathbf{x} \notin \pi(\mathbb{M}_B(-\gamma w_k)) = \pi(\mathbb{M}_A(v - \gamma w_k))$.

Since γ can be taken as small as we want, we have shown the existence of a Hölder sub-action $v_{\mathbf{x}} = v - \gamma w_k$ close to v in the Hölder topology satisfying $\mathbf{x} \notin \pi(\mathbb{M}_A(v_{\mathbf{x}}))$.

Case b. We suppose that, for every integer $k \geq 0$, one can find a path of length k, $(\mathbf{z}^0, \ldots, \mathbf{z}^k)$, beginning at $\mathbf{x}$, such that $\mathbf{z}^k \in \mathbb{M}_A(v)$, or equivalently $B(\mathbf{z}^k) = 0$ with B as before. In other words, there exists $\mathbf{y}^0 \in \Sigma^*_{\mathbf{x}}$ with $B(\mathbf{y}^0, \mathbf{x}) = 0$ and, for any $k \geq 1$, there exists $\mathbf{y}^k \in \Sigma^*_{\mathbf{x}^k} \cap (\sigma^*)^{-1}(\Sigma^*_{\mathbf{x}^{k-1}})$ such that $B(\mathbf{y}^k, \mathbf{x}^k) = 0$, where $\mathbf{x}^k = \sigma^k(\mathbf{x})$. Define $\bar{\mathbf{z}}^0 = (\mathbf{y}^0, \mathbf{x})$ and $\bar{\mathbf{z}}^k = (\mathbf{y}^k, \mathbf{x}^k)$ for all $k \geq 1$. Notice that $(\bar{\mathbf{z}}^0, \ldots, \bar{\mathbf{z}}^k)$ is now a path of arbitrary length k, beginning at $\mathbf{x}$, which satisfies $B(\bar{\mathbf{z}}^j) = 0$ for $j = 0, \ldots, k$.

Let $\bar{\mathbf{x}} \in \Omega(A) = \Omega(B)$ be any limit point of $(\mathbf{x}^k)_k$ chosen once for all. Let $w := h_B(\bar{\mathbf{x}}, \cdot)$ be the Hölder sub-action for B given by the corresponding Peierls barrier. Notice that by the definition of the Peierls barrier (see definition 14) we clearly get $h_B \geq 0$, since $B \geq 0$ and $\bar{B} = 0$. Furthermore, we remark that $\phi_B(\mathbf{x}, \sigma^k(\mathbf{x})) = 0$ for all $k \geq 1$ and that

$$w(\mathbf{x}) = h_B(\bar{\mathbf{x}}, \mathbf{x}) = \liminf_{k \to +\infty} \phi_B(\mathbf{x}, \sigma^k(\mathbf{x})) + h_B(\bar{\mathbf{x}}, \mathbf{x}) \geq h_B(\mathbf{x}, \mathbf{x}) > 0.$$

Here we have used item 4 of proposition 15 to obtain the first inequality and item 5 of the same proposition to assure the strict inequality since $\mathbf{x} \notin D \supset \Omega(A) = \Omega(B)$.

Let $\gamma \in (0, 1)$ be any real number as close to 0 as we want. We claim that $\mathbf{x}$ satisfies again the first case, namely, there exists $k \geq 1$ such that, for any path of length k, $(\mathbf{z}^0 = (\mathbf{y}^0, \mathbf{x}^0), \ldots, \mathbf{z}^k = (\mathbf{y}^k, \mathbf{x}^k))$, beginning at $\mathbf{x}$, one has

$$B(\mathbf{z}^k) - \gamma h_B(\bar{\mathbf{x}}, \mathbf{x}^k) + \gamma h_B(\bar{\mathbf{x}}, \mathbf{x}^{k-1}) > 0.$$

(Notice that γw is again a sub-action for B since B is non-negative.) Indeed, by contradiction, for any integer $k \geq 0$, we would have a path of length k, $(\mathbf{z}^0 = (\mathbf{y}^0, \mathbf{x}^0), \ldots, \mathbf{z}^k = (\mathbf{y}^k, \mathbf{x}^k))$, beginning at $\mathbf{x}$, such that $\mathbf{z}^k \in \mathbb{M}_B(\gamma w)$, which would yield

$$0 \leq B(\mathbf{z}^k) = \gamma h_B(\bar{\mathbf{x}}, \mathbf{x}^k) - \gamma h_B(\bar{\mathbf{x}}, \mathbf{x}^{k-1}), \quad \forall k \geq 1.$$

On the one hand, from the inequality $\gamma h_B(\bar{\mathbf{x}}, \mathbf{x}^{k-1}) \leq \gamma h_B(\bar{\mathbf{x}}, \mathbf{x}^k)$, we would obtain $0 < w(\mathbf{x}) = h_B(\bar{\mathbf{x}}, \mathbf{x}) \leq h_B(\bar{\mathbf{x}}, \mathbf{x}^k)$ for all $k \geq 1$. On the other hand, by taking a subsequence of $\{\mathbf{x}^k\} = \{\sigma^k(\mathbf{x})\}$ converging to $\bar{\mathbf{x}}$, $h_B(\bar{\mathbf{x}}, \mathbf{x}^k)$ would converge to $h_B(\bar{\mathbf{x}}, \bar{\mathbf{x}}) = 0$, since $\bar{\mathbf{x}} \in \Omega(B)$. We have thus obtained a contradiction. Hence, case (a) implies that there exists a sub-action $v_{\mathbf{x}}$, close to v in the Hölder topology, satisfying $\mathbf{x} \notin \pi(\mathbb{M}_A(v_{\mathbf{x}}))$.

Part 2. We have just proved that, for any $\mathbf{x} \notin D$, there exists a sub-action $v_{\mathbf{x}}$ close to v and a ball $B(\mathbf{x}, \epsilon_{\mathbf{x}})$ of radius $\epsilon_{\mathbf{x}} > 0$ centered at $\mathbf{x}$ such that

$$\forall\, \bar{\mathbf{x}} \in B(\mathbf{x}, \epsilon_{\mathbf{x}}), \quad \bar{\mathbf{x}} \notin \pi(\mathbb{M}_A(v_{\mathbf{x}})).$$

We can extract from the family of these balls $\{B(\mathbf{x}, \epsilon_{\mathbf{x}})\}_{\mathbf{x}}$ a finite family indexed by $\{\mathbf{x}^j\}_{1 \le j \le K}$ which is still a covering of the compact set $\Sigma \setminus D$. Let

$$u = \frac{1}{K} \sum_{j=1}^{K} v_{\mathbf{x}^j}.$$

Then it is easy to check that u is a Hölder sub-action for A satisfying $\pi(\mathbb{M}_A(u)) \subset D$, namely, $u \in \mathcal{D}_A$. Since each sub-action $v_{\mathbf{x}}$ can be taken as close as we want to v in the Hölder topology, the same is true for u. □

5 Proof of theorem 12.

It was proved in [13] that the projection of the support of a minimizing probability measure $\hat{\mu}$ is included into the A-non-wandering set $\Omega(A)$ when such projection is ergodic. If $\pi_* \hat{\mu}$ is ergodic, $\pi(\mathrm{supp}(\hat{\mu}))$ may be seen as an irreducible component in the sense that any two points can be joined by an ϵ-closed trajectory. We introduce here a more general notion of irreducibility.

Definition-Proposition 18. *Let $A \colon \hat{\Sigma} \to \mathbb{R}$ be a Hölder observable. We say that two points $\mathbf{x}, \bar{\mathbf{x}}$ of $\Omega(A)$ are equivalent and write $\mathbf{x} \sim \bar{\mathbf{x}}$ if*

$$h_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x}) = 0.$$

Then $\sim$ is an equivalent relation. Its equivalent classes are called irreducible components.

Proof. It is obvious that $\sim$ is reflexive ($h_A(\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} \in \Omega(A)$) and symmetric. Let u be a continuous sub-action and $B := A - \bar{A} - u \circ \pi + u \circ \tau$ be the associated normalized observable. Then the definition of the Peierls barrier (see definition 14) implies

$$h_B(\mathbf{x}, \bar{\mathbf{x}}) = h_A(\mathbf{x}, \bar{\mathbf{x}}) - u(\bar{\mathbf{x}}) + u(\mathbf{x}), \quad \forall\, \mathbf{x}, \bar{\mathbf{x}} \in \Sigma.$$

Since $h_B(\mathbf{x}, \bar{\mathbf{x}}) \ge 0$, we see that $\mathbf{x} \sim \bar{\mathbf{x}} \Leftrightarrow h_B(\mathbf{x}, \bar{\mathbf{x}}) = 0$ and $h_B(\bar{\mathbf{x}}, \mathbf{x}) = 0$.

To show the transitivity property, it is enough to prove

$$\mathbf{x} \sim \bar{\mathbf{x}} \text{ and } \bar{\mathbf{x}} \sim \bar{\bar{\mathbf{x}}} \Longrightarrow h_B(\mathbf{x}, \bar{\bar{\mathbf{x}}}) = 0.$$

But proposition 15 guarantees

$$0 \leq h_B(\mathbf{x}, \bar{\bar{\mathbf{x}}}) \leq h_B(\mathbf{x}, \bar{\mathbf{x}}) + h_B(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}) = 0.$$

The transitivity property is proved. □

Proposition 19. *The irreducible components are closed and σ-invariant.*

Proof. Part 1. Let $\mathbf{x} \in \Omega(A)$. Consider $\{\bar{\mathbf{x}}_\epsilon\}_\epsilon$ a sequence of points of $\Omega(A)$ equivalent to $\mathbf{x}$ and within ϵ of $\bar{\mathbf{x}} \in \Omega(A)$. Then on the one hand, $h_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x}) \geq h_A(\mathbf{x}, \mathbf{x}) = 0$, and on the other hand,

$$h_A(\mathbf{x}, \bar{\mathbf{x}}_\epsilon) + h_A(\bar{\mathbf{x}}, \mathbf{x}) \leq h_A(\mathbf{x}, \bar{\mathbf{x}}_\epsilon) + h_A(\bar{\mathbf{x}}, \bar{\mathbf{x}}_\epsilon) + h_A(\bar{\mathbf{x}}_\epsilon, \mathbf{x}) = h_A(\bar{\mathbf{x}}, \bar{\mathbf{x}}_\epsilon).$$

By continuity of $h_A(\mathbf{x}, \cdot)$ and $h_A(\bar{\mathbf{x}}, \cdot)$ with respect to the second variable, the previous inequality gives $h_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x}) \leq 0$. Therefore $\bar{\mathbf{x}} \sim \mathbf{x}$ and the class containing $\mathbf{x}$ is closed.

Part 2. Let $\mathbf{x} \in \Omega(A)$. Either $\sigma(\mathbf{x}) = \mathbf{x}$ and in an obvious way $\sigma(\mathbf{x}) \sim \mathbf{x}$ or $\sigma(\mathbf{x}) \neq \mathbf{x}$ and item 3 of proposition 15 shows $\phi_A(\mathbf{x}, \sigma(\mathbf{x})) + \phi_A(\sigma(\mathbf{x}), \mathbf{x}) = \phi_A(\mathbf{x}, \mathbf{x}) = 0$. Remember that $h_A(\mathbf{y}, \cdot) = \phi_A(\mathbf{y}, \cdot)$ whenever $\mathbf{y} \in \Omega(A)$; note that $\mathbf{x}$ and $\sigma(\mathbf{x})$ belong to the σ-invariant set $\Omega(A)$. Then we get $h_A(\mathbf{x}, \sigma(\mathbf{x})) + h_A(\sigma(\mathbf{x}), \mathbf{x}) = h_A(\mathbf{x}, \mathbf{x}) = 0$ and $\mathbf{x}$ and $\sigma(\mathbf{x})$ belong to the same irreducible class. □

We assume from now on that $\Omega(A)$ is equal to a disjoint union of irreducible components, $\Omega(A) = C_1 \sqcup \ldots \sqcup C_r$. The following proposition shows that the Peierls barrier normalized by a separating sub-action could play the role of a quantized set of levels of energy.

Proposition 20. *Let A be a Hölder observable and assume that $\Omega(A) = \sqcup_{i=1}^{r} C_i$ is equal to a finite union of irreducible components.*

1. *If u is a continuous sub-action, then*

$$(\mathbf{x}^i, \mathbf{x}^j) \mapsto h_A(\mathbf{x}^i, \mathbf{x}^j) - u(\mathbf{x}^j) + u(\mathbf{x}^i) \text{ is constant on } C_i \times C_j.$$

2. *If u is a continuous separating sub-action, then*

$$h_A(\mathbf{x}^i, \mathbf{x}^j) > u(\mathbf{x}^j) - u(\mathbf{x}^i), \quad \forall\, (\mathbf{x}^i, \mathbf{x}^j) \in C_i \times C_j, \quad \forall\, i \neq j.$$

Proof. We first normalize A by taking $B = A - \bar{A} - u \circ \pi + u \circ \tau$ so that $B \geq 0$ and $\bar{B} = 0$.

Part 1. Let $(\mathbf{x}^i, \mathbf{x}^j), (\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j) \in C_i \times C_j$. Then $h_B(\mathbf{x}^i, \bar{\mathbf{x}}^i) = h_B(\mathbf{x}^j, \bar{\mathbf{x}}^j) = 0$ and

$$h_B(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j) \leq h_B(\bar{\mathbf{x}}^i, \mathbf{x}^i) + h_B(\mathbf{x}^i, \mathbf{x}^j) + h_B(\mathbf{x}^j, \bar{\mathbf{x}}^j) \leq h_B(\mathbf{x}^i, \mathbf{x}^j).$$

Conversely $h_B(\mathbf{x}^i, \mathbf{x}^j) \leq h_B(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j)$ and we have proved that $h_B(\cdot, \cdot)$ is constant on $C_i \times C_j$.

Part 2. Let $\{U_i^\eta\}_{\eta > 0}$ be a basis of neighborhoods of C_i. Since $\sigma(C_i) \subset C_i$ is disjoint from each C_j, $j \neq i$, there exists $\eta > 0$ small enough such that $\sigma(U_i^\eta)$ is disjoint from $\cup_{j \neq i} U_j^\eta$. Let $i \neq j$ and $\mathbf{x} \in C_i$, $\bar{\mathbf{x}} \in C_j$. For $\epsilon > 0$ sufficiently small, the ball of radius ϵ centered at $\mathbf{x}$ is included in U_i^η. Let $(\mathbf{z}^0 = (\mathbf{y}^0, \mathbf{x}^0), \ldots, \mathbf{z}^k = (\mathbf{y}^k, \mathbf{x}^k))$ be a path of length k within ϵ of $\mathbf{x}$ and $\bar{\mathbf{x}}$, more precisely, satisfying $d(\mathbf{x}^0, \mathbf{x}) < \epsilon$ and $d(\mathbf{x}^k, \bar{\mathbf{x}}) < \epsilon$. Let $p \geq 1$ be the first time $\sigma^p(\mathbf{x}) \notin U_i^\eta$. Then $\sigma^{p-1}(\mathbf{x}) \in U_i^\eta$ and $\sigma^p(\mathbf{x}) \in \sigma(U_i^\eta) \setminus U_i^\eta$. By the choice of η, $\sigma^p(\mathbf{x}) \notin \cup_{j=1}^r U_j^\eta =: \mathcal{U} \supset \Omega(A)$. Since $\Omega(A) = \pi(\mathbb{M}_A(u))$, let $\hat{\mathcal{U}} := \pi^{-1}(\mathcal{U})$, then $\mathbf{z}^p \notin \hat{\mathcal{U}}$ and

$$\sum_{l=1}^{k} B(\mathbf{z}^l) \geq B(\mathbf{z}^p) \geq \min_{\hat{\Sigma} \setminus \hat{\mathcal{U}}} B =: m > 0.$$

We have proved that $h_B(\mathbf{x}, \bar{\mathbf{x}}) \geq m > 0$. □

We are now in a position to prove our second result.

Proof of theorem 12. We fixed once for all $\bar{\mathbf{x}}^i \in C_i$.

Part 1. We know from theorem 11 that a continuous calibrated sub-action satisfies $u(\mathbf{x}) = \min_{\bar{\mathbf{x}} \in \Omega(A)} [u(\bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x})]$. If $\bar{\mathbf{x}} \in C_i$, then $\bar{\mathbf{x}} \sim \bar{\mathbf{x}}^i$ and $h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \bar{\mathbf{x}}^i) = 0$. Then

$$\begin{aligned} u(\bar{\mathbf{x}}^i) + h_A(\bar{\mathbf{x}}^i, \mathbf{x}) &\leq u(\bar{\mathbf{x}}^i) + h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x}) \\ &= u(\bar{\mathbf{x}}^i) - h_A(\bar{\mathbf{x}}, \bar{\mathbf{x}}^i) + h_A(\bar{\mathbf{x}}, \mathbf{x}) \\ &\leq u(\bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x}). \end{aligned}$$

We have proved that $u(\mathbf{x}) = \min_{1 \leq i \leq r} \left[u(\bar{\mathbf{x}}^i) + h_A(\bar{\mathbf{x}}^i, \mathbf{x}) \right]$. The fact that $(u(\bar{\mathbf{x}}^1), \ldots, u(\bar{\mathbf{x}}^r)) \in C_A(\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^r)$ comes from items 1 and 6 of proposition 15.

Part 2. Let $(u_1, \ldots, u_r) \in C_A(\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^r)$ and define $\phi\colon \Omega(A) \to \mathbb{R}$ by $\phi(\mathbf{x}) := u_i + h_A(\bar{\mathbf{x}}^i, \mathbf{x})$ for all $\mathbf{x} \in C_i$. We notice that ϕ is continuous and we show that $\phi(\bar{\mathbf{x}}) - \phi(\mathbf{x}) \leq h_A(\mathbf{x}, \bar{\mathbf{x}})$ for all $\mathbf{x}, \bar{\mathbf{x}} \in \Omega(A)$. Indeed, if $\mathbf{x} \in C_i$ and $\bar{\mathbf{x}} \in C_j$, then

$$\begin{aligned}\phi(\bar{\mathbf{x}}) - \phi(\mathbf{x}) &= (u_j - u_i) + h_A(\bar{\mathbf{x}}^j, \bar{\mathbf{x}}) - h_A(\bar{\mathbf{x}}^i, \mathbf{x}) \\ &\leq h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j) + h_A(\bar{\mathbf{x}}^j, \bar{\mathbf{x}}) - h_A(\bar{\mathbf{x}}^i, \mathbf{x}) \\ &= h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j) - h_A(\bar{\mathbf{x}}, \bar{\mathbf{x}}^j) - h_A(\bar{\mathbf{x}}^i, \mathbf{x}) \\ &\leq h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}) - h_A(\bar{\mathbf{x}}^i, \mathbf{x}) \leq h_A(\mathbf{x}, \bar{\mathbf{x}}).\end{aligned}$$

(The last but one inequality uses item 1 of proposition 15 and the fact that $h_A(\bar{\mathbf{x}}^i, \cdot)$ is a sub-action.) By theorem 11, we know that the function $u(\mathbf{x}) := \min_{\bar{\mathbf{x}} \in \Omega(A)}[\phi(\bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x})]$ is a continuous calibrated sub-action which extends ϕ on $\Omega(A)$. In particular, $u(\bar{\mathbf{x}}^i) = \phi(\bar{\mathbf{x}}^i) = u_i$ and, thanks to part 1, u coincides with $\min_{1 \leq i \leq r}[u_i + h_A(\bar{\mathbf{x}}^i, \cdot)]$.

Part 3. Let $i_0 \in \{1, \ldots, r\}$. If $(u_1, \ldots, u_r)$ satisfies $u_i = u_{i_0} + h_A(\bar{\mathbf{x}}^{i_0}, \bar{\mathbf{x}}^i)$, then i_0 is unique. Otherwise there would exist $i_1 \neq i_0$ such that $u_i = u_{i_1} + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^i)$. Thus

$$u_{i_1} = u_{i_0} + h_A(\bar{\mathbf{x}}^{i_0}, \bar{\mathbf{x}}^{i_1}) \quad \text{and} \quad u_{i_0} = u_{i_1} + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^{i_0}).$$

We would obtain $h_A(\bar{\mathbf{x}}^{i_0}, \bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^{i_0}) = 0$ contradicting $\bar{\mathbf{x}}^{i_0} \not\sim \bar{\mathbf{x}}^{i_1}$. The fact that $(u_1, \ldots, u_r) \in C_A(\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^r)$ comes from

$$u_j - u_i = h_A(\bar{\mathbf{x}}^{i_0}, \bar{\mathbf{x}}^j) - h_A(\bar{\mathbf{x}}^{i_0}, \bar{\mathbf{x}}^i) \leq h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j).$$

The end of part 3 follows since $u(\mathbf{x}) := u_{i_0} + h_A(\bar{\mathbf{x}}^{i_0}, \mathbf{x})$ already defines a calibrated sub-action satisfying $u(\bar{\mathbf{x}}^i) = u_i$ for all i. □

The proof of application 13 is elementary.

Proof of application 13. Define $B := A - v \circ \pi + v \circ \tau - \bar{A}$, then the null function is a sub-action of B and $v - u$ is a sub-action calibrated to B. Moreover, $h_B(\mathbf{x}, \bar{\mathbf{x}}) = h_A(\mathbf{x}, \bar{\mathbf{x}}) - v(\bar{\mathbf{x}}) + v(\mathbf{x})$ and $\Omega(A) = \Omega(B)$. It is therefore enough to assume A normalized ($A \geq 0$ and $\bar{A} = 0$) and $v = 0$.

Part 1. If $\mathbf{x} \sim \bar{\mathbf{x}}$ are two points of $\Omega(A)$, then $h_A(\mathbf{x}, \bar{\mathbf{x}}) = 0$ and $h_A(\bar{\mathbf{x}}, \mathbf{x}) = 0$. Thanks to items 1 and 6 of proposition 15, we obtain $u(\mathbf{x}) = u(\bar{\mathbf{x}})$. If $\mathbf{x}$ is any point of Σ, by the calibration of u, one can construct an inverse path $\{\mathbf{z}^{-i}\}_{i \geq 0}$ of $\hat{\Sigma}$, with $\pi(\mathbf{z}^0) = \mathbf{x}$, such that $u(\mathbf{x}^{-i}) - u(\mathbf{x}^{-i-1}) =$

$A(\mathbf{z}^{-i})$, $\mathbf{x}^{-i} = \pi(\mathbf{z}^{-i})$, for all i. Let $\bar{\mathbf{x}}$ be an accumulation point of $\{\mathbf{x}^{-i}\}_{i\geq 0}$. Then $\bar{\mathbf{x}} \in \Omega(A)$ and, since $A \geq 0$, the sequence $\{u(\mathbf{x}^{-i})\}_{i\geq 0}$ is decreasing. In particular, $u(\mathbf{x}) \geq u(\bar{\mathbf{x}})$ establishes $\min_{\Sigma} u = \min_{\Omega(A)} u$.

Part 2. Let u_i be the value of u on C_i. Assume we have ordered these values as $u_{i_1} \leq u_{i_2} \leq \ldots \leq u_{i_r}$. Let $\bar{\mathbf{x}}^i \in C_i$ fixed. It suffices to prove $u(\bar{\mathbf{x}}^{i_k}) = u(\bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^{i_k})$ for all $k = 1, \ldots, r$. It is true for $k = 1$. Since $C_{i_{k+1}}$ is not a minimum local of u, one can find a sequence of points $\{\mathbf{x}_\epsilon\}_{\epsilon>0}$ within ϵ of $C_{i_{k+1}}$ such that $u(\mathbf{x}_\epsilon) < u(\bar{\mathbf{x}}^{i_{k+1}})$. From part 1 of theorem 12, there exists an index j such that $u(\mathbf{x}_\epsilon) = u(\bar{\mathbf{x}}^j) + h_A(\bar{\mathbf{x}}^j, \mathbf{x}_\epsilon)$. Since $h_A \geq 0$, $u_j = u(\bar{\mathbf{x}}^j) \leq u(\mathbf{x}_\epsilon) < u_{i_{k+1}}$. So j has to be one of indexes $i_1, \ldots, i_k$. By induction, $u(\bar{\mathbf{x}}^j) = u(\bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^j)$ and

$$u(\mathbf{x}_\epsilon) = u(\bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^j) + h_A(\bar{\mathbf{x}}^j, \mathbf{x}_\epsilon).$$

On the one hand, $h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^j) + h_A(\bar{\mathbf{x}}^j, \mathbf{x}_\epsilon) \geq h_A(\bar{\mathbf{x}}^{i_1}, \mathbf{x}_\epsilon)$ implies

$$u(\mathbf{x}_\epsilon) \geq u(\bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \mathbf{x}_\epsilon).$$

On the other hand, as u is a sub-action, we obtain the reverse inequality and finally

$$u(\mathbf{x}_\epsilon) = u(\bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \mathbf{x}_\epsilon).$$

Letting ϵ go to 0, $\mathbf{x}_\epsilon$ accumulates to $C_{i_{k+1}}$ and

$$u(\bar{\mathbf{x}}^{i_{k+1}}) = u(\bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^{i_{k+1}}).$$

□

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