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Calibrated Configurations for Frenkel–Kontorova Type Models in Almost Periodic Environments

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Abstract. The Frenkel–Kontorova model describes how an infinite chain of atoms minimizes the total energy of the system when the energy takes into account the interaction of nearest neighbors as well as the interaction with an exterior environment. An almost periodic environment leads to consider a family of interaction energies which is stationary with respect to a minimal topological dynamical system. We focus, in this context, on the existence of calibrated configurations (a notion stronger than the standard minimizing condition). In any dimension and for any continuous superlinear interaction energies, we exhibit a set, called projected Mather set, formed of environments that admit calibrated configurations. In the one-dimensional setting, we then give sufficient conditions on the family of interaction energies that guarantee the existence of calibrated configurations for every environment. The main mathematical tools for this study are developed in the frameworks of discrete weak KAM theory, Aubry–Mather theory and spaces of Delone sets.

1. Introduction

The original Frenkel–Kontorova model [11–13] describes a one-dimensional chain of classical coupled particles which are subjected to an environment via an *interaction energy* $E : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. Given a finite configuration $(x_m, x_{m+1}, \dots, x_n)$ of points in $\mathbb{R}^d$, define

$$E(x_m, x_{m+1}, \dots, x_n) := \sum_{k=m}^{n-1} E(x_k, x_{k+1}).$$

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A *minimizing configuration* $(x_k)_{k\in\mathbb{Z}}$ for the interaction energy E is an infinite chain of points in $\mathbb{R}^d$ arranged so that the energy of each finite segment $(x_m, x_{m+1}, \dots, x_n)$ cannot be lowered by changing the configuration inside the segment while fixing the two boundary points, i.e., for all $m < n$, for all $y_m, y_{m+1}, \dots, y_n \in \mathbb{R}^d$ satisfying $y_m = x_m$ and $y_n = x_n$, one has

$$E(x_m, x_{m+1}, \dots, x_n) \le E(y_m, y_{m+1}, \dots, y_n). \tag{1}$$

In the periodic setting, that is, if the interaction energy is C^0, *coercive* and *translation periodic*,

$$\lim_{R\to+\infty} \inf_{\|y-x\|\ge R} E(x,y) = +\infty \quad \text{and} \tag{2}$$

$$\forall\, t \in \mathbb{Z}^d,\ \forall\, x, y \in \mathbb{R}^d, \quad E(x+t, y+t) = E(x,y), \tag{3}$$

it is easy to show (see [2] for $d = 1$ and [15] for any dimension) that minimizing configurations do exist. The proof in Aubry and Le Dearon [2] makes heavy use of the fact that $d = 1$ and the assumption that E is C^2 and *twist* in the following *strong* sense

$$\frac{\partial^2 E}{\partial x \partial y} \le -\alpha < 0. \tag{4}$$

We will relax slightly the twist condition allowing us anharmonic interactions, by using, for example, $E(x,y) = \frac{1}{4}|y-x-\lambda|^4 + V(x)$ instead of the harmonic interaction $E(x,y) = \frac{1}{2}|y-x-\lambda|^2 + V(x)$.

For environments which are aperiodic, namely when the energy E is not translation periodic, few results are known (see, for instance, [8,14,27]). For $d = 1$, Gambaudo, Guiraud and Petite [14] showed that minimizing configurations do exist for a family of aperiodic C^2 twist energies. They also proved that every minimizing configuration has a rotation number and any nonnegative real number is the rotation number of a minimizing configuration.

A notion stronger than the usual minimizing condition is provided by the concept of calibration. A *calibrated configuration* (at the level $c \in \mathbb{R}$) is a sequence $(x_n)_{n\in\mathbb{Z}}$ such that, for every $m < n$,

$$E(x_m, \dots, x_n) - (n-m)c \ \le\ \inf_{\ell\ge1} \inf_{y_0=x_m,\dots,y_\ell=x_n} \big[E(y_0,\dots,y_\ell) - \ell c\big]. \tag{5}$$

Notice that the number of sites on the right-hand side is arbitrary.

This paper mainly concerns the existence of calibrated configuration in the aperiodic context. A calibrated configuration is obviously minimizing, but the converse is false in general.

In the periodic setting and for $d \ge 1$, an argument using the notion of weak KAM solutions as in [10,15,16] shows that there exist calibrated configurations at a level $\bar{E}$ depending only on the energy E. Conversely, if $d = 1$ and E is twist translation periodic, every minimizing configuration is calibrated for some modified energy $E_\lambda(x,y) = E(x,y) - \lambda(y-x)$, $\lambda \in \mathbb{R}$, at a level $\bar{E}_\lambda$ (see [2]).

Even if $d = 1$, in the aperiodic context it is not known in general whether calibrated configurations exist. In order to give conditions to ensure the existence of calibrated configurations, we will consider in this paper an interaction energy which is *almost periodic* in a sense that will include the periodic case. This will lead to look at a family of interaction energies parameterized by a minimal topological dynamical system (a weak form of homogeneity). Such an approach is similar to studies for the Hamilton–Jacobi equation (see, for instance, [5–7,17,18,22]), where a stationary ergodic setting has been taken into account.

We will assume there exists a family of interaction energies $\{E_\omega\}_\omega$ depending on an environment ω. Let Ω denote the collection of all possible environments. We assume that every chain of atoms $(x_k + t)_{k\in\mathbb{Z}}$, translated in the direction $t \in \mathbb{R}^d$ and interacting with the environment ω, has the same local energy that $(x_k)_{k\in\mathbb{Z}}$ interacting with the shifted environment $\tau_t(\omega)$ for some bijective transformation $\tau_t \colon \Omega \to \Omega$. More precisely, each environment ω defines an interaction $E_\omega(x, y)$ which is assumed to be *topologically stationary* in the following sense

$$\forall\, \omega \in \Omega,\ \forall\, t \in \mathbb{R}^d,\ \forall\, x, y \in \mathbb{R}^d, \quad E_\omega(x+t, y+t) = E_{\tau_t(\omega)}(x, y). \tag{6}$$

In order to ensure the topological stationarity, the interaction energy will be supposed to have a *Lagrangian form*. Formally, we will use the following definition.

Definition 1. Let Ω be a compact metric space.

1. A minimal $\mathbb{R}^d$-action is a couple $\big(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}\big)$, where $\{\tau_t\}_{t\in\mathbb{R}^d}$ is a family of homeomorphisms $\tau_t : \Omega \to \Omega$ satisfying
   - $\tau_s \circ \tau_t = \tau_{s+t}$ for all $s, t \in \mathbb{R}^d$ (the group property),
   - $\tau_t(\omega)$ is jointly continuous with respect to (t, ω),
   - $\forall\, \omega \in \Omega,\ \{\tau_t(\omega)\}_{t\in\mathbb{R}^d}$ is dense in Ω (the minimality property).
2. A family of interaction energies $\{E_\omega\}_{\omega\in\Omega}$ is said to derive from a Lagrangian if there exists a continuous function $L : \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that
$$\forall\, \omega \in \Omega,\ \forall\, x, y \in \mathbb{R}^d, \quad E_\omega(x, y) := L(\tau_x(\omega), y - x). \tag{7}$$
3. An almost periodic interaction model is the set of data $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$, where $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d})$ is a minimal $\mathbb{R}^d$-action and L is a continuous function on $\Omega \times \mathbb{R}^d$.

Notice that the expression "almost periodic" shall not be understood in the sense of H. Bohr. The almost periodicity according to Bohr is canonically relied to the uniform convergence. See [3] for a discussion on the different concepts of almost periodicity in conformity with the uniform topology or with the compact open topology.

Because of the particular form (7) of $E_\omega(x, y)$, these energies are *translation bounded* and *translation uniformly continuous* in the sense that for all $R > 0$, $\sup_{\|y-x\|\le R} E_\omega(x, y) < +\infty$ and $E_\omega(x, y)$ is uniformly continuous in $\|y - x\| \le R$. We make precise the notions of *coercivity* and *superlinearity* for the Lagrangian form.

Definition 2. Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ be an almost periodic interaction model.

1. L is said to be coercive if $\lim_{R\to+\infty} \inf_{\omega\in\Omega} \inf_{\|t\|\geq R} L(\omega,t) = +\infty$.
2. L is said to be superlinear if $\lim_{R\to+\infty} \inf_{\omega\in\Omega} \inf_{\|t\|\geq R} \frac{L(\omega,t)}{\|t\|} = +\infty$.

Let us illustrate our abstract notions by three typical examples.

Example 3. The one-dimensional periodic Frenkel–Kontorova model [11–13]. The interaction energies are given by $E_\omega(x,y) = W(y-x) + V_\omega(x)$, with $\omega \in \mathbb{R}/\mathbb{Z}$, written in Lagrangian form as

$$L(\omega,t) = W(t) + V(\omega) = \frac{1}{2}|t-\lambda|^2 + \frac{K}{(2\pi)^2}\big(1-\cos 2\pi\omega\big), \tag{8}$$

where λ and K are constants. Here $\Omega = \mathbb{R}/\mathbb{Z}$ and $\tau_t : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is given by $\tau_t(\omega) = \omega + t$. We observe that $\{\tau_t\}_t$ is minimal.

Example 4. The one-dimensional almost crystalline model based on [14]. For $\alpha \in (0,1)\backslash\mathbb{Q}$, consider the aperiodic subset of $\mathbb{R}$ defined by

$$\omega(\alpha) := \{k \in \mathbb{Z} : \lfloor k\alpha \rfloor - \lfloor (k-1)\alpha \rfloor = 1\},$$

where $\lfloor\cdot\rfloor$ denotes the integer part. Represented as an ordered subset $\omega(\alpha) = \{\omega_n\}_{n\in\mathbb{Z}}$ possesses the property that the distance between two consecutive points is either $\lfloor\frac{1}{\alpha}\rfloor$ or $\lfloor\frac{1}{\alpha}\rfloor+1$. We choose two smooth functions $U_0, U_1 : \mathbb{R}\to\mathbb{R}$ with supports, respectively, in $(0, \lfloor\frac{1}{\alpha}\rfloor)$ and $(0, \lfloor\frac{1}{\alpha}\rfloor+1)$. We then construct a potential $V_{\omega(\alpha)} : \mathbb{R}\to\mathbb{R}$ and an interaction energy in the following way

$$\forall\, \omega_n \leq x < \omega_{n+1}, \quad V_{\omega(\alpha)}(x) = U_{\omega_{n+1}-\omega_n-\lfloor\frac{1}{\alpha}\rfloor}(x-\omega_n),$$
$$\forall\, x,y \in \mathbb{R}, \quad E_{\omega(\alpha)}(x,y) = \frac{1}{2}|x-y-\lambda|^2 + V_{\omega(\alpha)}(x).$$

More generally, one may similarly define a potential $V_\omega(x)$ and an interaction energy $E_\omega(x,y)$ for any subset $\omega \in \mathbb{R}$ having the property that the distance between two consecutive points belongs to $\{\lfloor\frac{1}{\alpha}\rfloor, \lfloor\frac{1}{\alpha}\rfloor+1\}$. Let Ω' be the set of all such subsets ω. Then, for any $x,t \in \mathbb{R}$, $V_\omega(x+t) = V_{\omega-t}(x)$, where $\omega - t := \{p-t : p \in \omega\}$. Let $\Omega \subset \Omega'$ be the hull of the $\omega(\alpha)$ as explained in Sect. 4. Then, Ω is compact, the group of translations $\tau_t(\omega) := \omega - t$ acts minimally, and $E_\omega(x,y)$ derives from the Lagrangian

$$L(\omega,t) := \frac{1}{2}|t-\lambda|^2 + V_\omega(t). \tag{9}$$

We will extend in Sect. 4 the construction given in Example 4 to any *quasicrystal* ω of $\mathbb{R}$. The associated almost periodic interaction model will be of *almost crystalline type* as we will describe below. Our third example illustrates an almost periodic interaction model on $\mathbb{R}$ which is not almost crystalline.

Example 5. The one-dimensional almost periodic Frenkel–Kontorova model. The underlying minimal flow is given by the irrational flow $\tau_t(\omega) = \omega + t(1,\sqrt{2})$ acting on $\Omega = \mathbb{R}^2/\mathbb{Z}^2$. The family of interaction energies E_ω derives from the Lagrangian

$$L(\omega,t) := \frac{1}{2}|t-\lambda|^2 + \frac{K_1}{(2\pi)^2}\big(1-\cos 2\pi\omega_1\big) + \frac{K_2}{(2\pi)^2}\big(1-\cos 2\pi\omega_2\big), \tag{10}$$

where $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2/\mathbb{Z}^2$.

We will consider calibrated configurations at a specific level.

Definition 6. We call *ground energy* of a family of interactions $\{E_\omega\}_{\omega\in\Omega}$ of Lagrangian form $L : \Omega \times \mathbb{R}^d \to \mathbb{R}$ the quantity

$$\bar{E} := \lim_{n\to+\infty} \inf_{\omega\in\Omega} \inf_{x_0,\ldots,x_n\in\mathbb{R}^d} \frac{1}{n} E_\omega(x_0, \ldots, x_n).$$

Since the sequence $(\inf_{\omega\in\Omega} \inf_{x_0,\ldots,x_n\in\mathbb{R}^d} E_\omega(x_0, \ldots, x_n))_n$ is superadditive, the above limit is actually a supremum by Fekete's Lemma, which is finite if L is assumed to be coercive. Besides, we clearly have *a priori* bounds

$$\inf_{\omega\in\Omega} \inf_{x,y\in\mathbb{R}^d} E_\omega(x, y) \le \bar{E} \le \inf_{\omega\in\Omega} \inf_{x\in\mathbb{R}^d} E_\omega(x, x). \tag{11}$$

In the same way, we may define the *ground energy* $\bar{E}_\omega$ *in the environment* ω as

$$\bar{E}_\omega := \lim_{n\to+\infty} \inf_{x_0,\ldots,x_n\in\mathbb{R}^d} \frac{1}{n} E_\omega(x_0, \ldots, x_n). \tag{12}$$

The ground energy $\bar{E}_\omega$ measures the lowest mean energy per site among all infinite configurations in the environment ω. We will see (Proposition 13) that the minimality of the group action $\{\tau_t\}_t$ implies that $\bar{E} = \bar{E}_\omega$ for all $\omega \in \Omega$.

In this context, let us precise the definition of calibrated configuration. For an environment ω, we say that a configuration $(x_k)_{k\in\mathbb{Z}}$ is *calibrated for* E_ω (at the level $\bar{E}$) if, for all $m < n$,

$$E_\omega(x_m, \ldots, x_n) - (n-m)\bar{E} \;=\; \inf_{\ell\ge1} \inf_{y_0=x_m,\ldots,y_\ell=x_n} \left[E_\omega(y_0, \ldots, y_\ell) - \ell\bar{E}\right]. \tag{13}$$

We show two results that give sufficient conditions for the existence of calibrated configurations. The first one applies to almost periodic interaction models in any dimension. We describe a set, called *projected Mather set*, consisting of environments that allow the existence of calibrated configurations. The second result is more restrictive and holds only for one-dimensional *almost crystalline interaction model*. We then show that a calibrated configuration exists for every environment.

The following definition is basic in our analysis. The vocabulary is borrowed from the weak KAM theory (see [9, 10, 23, 24]).

Definition 7. Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ be an almost periodic interaction model.

1. A measure μ on $\Omega \times \mathbb{R}^d$ is said to be holonomic if it is a probability and
$$\forall\, f \in C^0(\Omega), \quad \int f(\omega)\,\mu(d\omega, dt) = \int f(\tau_t(\omega))\,\mu(d\omega, dt).$$
Let $\mathbb{M}_{hol}$ denote the set of holonomic measures.
2. A measure μ is said to be minimizing if it is holonomic and $\bar{E} = \int L\, d\mu$.
3. We call *Mather set* of L the subset of $\Omega \times \mathbb{R}^d$ defined by
$$\mathrm{Mather}(L) := \cup_{\mu\in\mathbb{M}_{min}(L)} \mathrm{supp}(\mu),$$
where $\mathbb{M}_{min}(L)$ denotes the set of minimizing measures.

The projected Mather set is the projection $pr(\text{Mather}(L))$ of the Mather set into Ω by the canonical projection $pr\colon \Omega \times \mathbb{R}^d \to \Omega$.

Holonomic measures have been defined in [23] in the context of Lagrangian flows on tangent spaces. The $\mathbb{R}^d$-action introduced in Definition 1 plays the role in the case $d = 1$ of the projection of the Lagrangian flow on position space.

Note that $\mathbb{M}_{hol}$ is nonempty as it contains $\delta_{(\omega,0)}$, $\omega \in \Omega$. It can be shown that the Mather set is a nonempty compact set for any superlinear Lagrangian (Proposition 13 and Lemma 21).

Our first result applies to an almost periodic interaction model in every dimension and extends the classical periodic Aubry–Mather theory.

Theorem 8. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ be an almost periodic interaction model. Assume L is superlinear. Then, for all $\omega \in pr(\mathit{Mather}(L))$, there exists a calibrated configuration $(x_k)_{k\in\mathbb{Z}}$ for E_ω at the level $\bar{E}$ such that $x_0 = 0$ and $\sup_{k\in\mathbb{Z}} \|x_{k+1} - x_k\| < +\infty$.*

Let us recall that, by the stationarity hypothesis (6), a configuration $(x_k)_{k\in\mathbb{Z}}$ is calibrated for E_ω if, and only if, for all $t \in \mathbb{R}^d$, the configuration $(x_k - t)_{k\in\mathbb{Z}}$ is calibrated for $E_{\tau_t(\omega)}$. So, by Theorem 8, each environment in the $\{\tau_t\}_{t\in\mathbb{R}^d}$-orbit of the projected Mather set admits a calibrated configuration.

However, it may happen that the orbit of the projected Mather set is a small set. Indeed, in the one-dimensional almost periodic Frenkel–Kontorova model described in Example 5, for $\lambda = 0$, it is easy to check that $\bar{E} = 0$, the Mather set is reduced to the point $(0_{\mathbb{T}^2}, 0_{\mathbb{R}})$, and $x_k = 0$, $k \in \mathbb{Z}$, defines a calibrated configuration. We conjecture that there does not exist a calibrated configuration for $\omega \notin \{(t, t\sqrt{2}) : t \in \mathbb{R}\}$ when $\lambda = 0$. A similar case occurs when there is no exact corrector for the homogenization problem in Hamilton–Jacobi equations in the stationary ergodic setting [5,22].

Our second result applies to a specialized one-dimensional almost periodic interaction model called almost crystalline.

Definition 9. Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be a minimal $\mathbb{R}$-action.

1. An open set $U \subset \Omega$ is said to be a *flow box* of size $R > 0$ if there exists a compact subset $\Xi \subset \Omega$, called *transverse section*, such that
   (a) the induced topology on Ξ admits a basis of closed and open subsets, called clopen subsets,
   (b) the map $(t, \omega) \in B_R \times \Xi \mapsto \tau(t, \omega) = \tau_t(\omega) \in \Omega$ is a homeomorphism onto U, where $B_R = B(0, R)$ denotes the open ball of radius R and center 0.
2. Two flow boxes $U_i = \tau(B_{R_i} \times \Xi_i)$ and $U_j = \tau(B_{R_j} \times \Xi_j)$ are said to be admissible if, whenever $U_i \cap U_j \neq \emptyset$, there exists $a_{i,j} \in \mathbb{R}$ such that
$$\tau_{(j)}^{-1} \circ \tau(t, \omega) = (t - a_{i,j}, \tau_{a_{i,j}}(\omega)), \quad \forall\, (t, \omega) \in \tau_{(i)}^{-1}(U_i \cap U_j),$$
   where $\tau_{(i)}^{-1} : U_i \to B_R \times \Xi$ denotes the inverse map.
3. A flow box decomposition $\{U_i\}_{i\in I}$ is a cover of Ω by admissible flow boxes.

4. A flow box $\tau(B_R \times \Xi)$ is said to be compatible with respect to a flow box decomposition $\{U_i\}_{i\in I}$, where $U_i = \tau(B_{R_i} \times \Xi_i)$, if for every $|t| < R$, there exist $i \in I$, $|t_i| < R_i$ and a clopen subset $\tilde{\Xi}_i$ of Ξ_i such that $\tau_t(\Xi) = \tau_{t_i}(\tilde{\Xi}_i)$.

Of course, the circle has a flow box decomposition. Less trivially, a typical example is a suspension of a minimal homeomorphism on a Cantor set with a locally constant ceiling function. But in general, a minimal $\mathbb{R}$-action does not possess a transverse section. We will describe in Sect. 4 how such a decomposition is obtained for the hull of a quasicrystal (Example 4 is a prototype of a quasicrystal). Yet, our notion is more general than this one because it also includes, for instance, nonexpansive $\mathbb{R}$ actions. The next definition is central in our second main result.

Definition 10. Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be an almost periodic interaction model admitting a flow box decomposition $\{U_i\}_{i\in I}$. L is said to be *locally transversally constant* with respect to $\{U_i\}_{i\in I}$ if, for every compatible flow box $\tau(B_R \times \Xi)$,

$$\forall\, \omega, \omega' \in \Xi,\ \forall\, |x|, |y| < R, \quad E_{\omega'}(x, y) = E_\omega(x, y).$$

We will show in Sect. 4 that the Lagrangians in Examples 3 and 4 are locally transversally constant.

The standard one-dimensional Aubry–Mather theory assumes that the interaction energy $E(x, y)$ is strongly twist as in (4). An energy of the form $E(x, y) = \frac{1}{4}|t - \lambda|^4 + V(x)$ is not strongly twist. We extend slightly this definition: $E(x, y)$ is said to be *weakly twist* if E is a C^2 function and satisfies

$$\forall\, x, y \in \mathbb{R}, \quad \frac{\partial^2 E}{\partial x \partial y}(x, \cdot) < 0 \quad \text{and} \quad \frac{\partial^2 E}{\partial x \partial y}(\cdot, y) < 0 \quad \text{a.e.} \tag{14}$$

Definition 11. Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be a one-dimensional almost periodic interaction model. The interaction model $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ is said to be *almost crystalline* if

1. $\{\tau_t\}_{t\in\mathbb{R}}$ is uniquely ergodic (with unique invariant probability measure λ),
2. L is superlinear and weakly twist (for every $\omega \in \Omega$, E_ω is weakly twist),
3. L is locally transversally constant with respect to a flow box decomposition.

Our second result states that calibrated configurations exist for every environment of an almost crystalline interaction model.

Theorem 12. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be an almost crystalline interaction model. Then, for every $\omega \in \Omega$, there exists a configuration $(x_{k,\omega})_{k\in\mathbb{Z}}$ which is calibrated for E_ω, with bounded jumps and at a bounded distance from the origin uniformly in ω, i.e.,*

$$\sup_{\omega\in\Omega}\ \sup_{k\in\mathbb{Z}}\ |x_{k+1,\omega} - x_{k,\omega}| < +\infty, \quad \sup_{\omega\in\Omega}\ |x_{0,\omega}| < +\infty.$$

Actually, to show this result it is enough, by Theorem 8, to prove that the projected Mather set intersects every $\{\tau_t\}_{t\in\mathbb{R}}$-orbit.

The paper is organized as follows. Section 2 is dedicated to the Proof of Theorem 8, whose strategy takes advantage of a fundamental characterization of the ground energy via a *sup-inf formula*. We give in "Appendix" another proof of this formula. In Sect. 3, we improve classical results about the rearranging of the atoms of a minimizing configuration for weakly twist Lagrangians. We especially show that no coincidence may happen. In Sect. 4, by extending Example 4, we explain how to construct almost crystalline interaction models using quasicrystals and strongly equivariant functions. In particular, Corollary 32 describes an explicit family of almost crystalline interaction models. Section 5 is devoted to the Proof of Theorem 12.

2. Almost Periodic Interaction Models

This section is devoted to the proof of the existence of calibrated configurations for almost periodic interaction models in any dimension. In the periodic setting, the proof is done using calibrated sub-actions as in [15]. We do not know how to extend this tool in the aperiodic case. We use instead a new tool: the *Mañé subadditive cocycle*. We start showing different ways of computing the ground energy. The ground energy computed using the *sup-inf formula* is fundamental for the construction of the Mañé subadditive cocycle. In the second subsection, we use this cocycle to build a calibrated configuration when the environment belongs to the projected Mather set. The Proof of Theorem 8 is given at the end of this section. In all these sections, we will consider an almost periodic interaction model (Definition 1). Most of the results hold for coercive Lagragians.

2.1. Ground Energy and Mather Set

Let $\omega \in \Omega$ be a fixed environment. The ground energy $\bar{E}_\omega$ (Eq. (12)) is computed by taking the limit of the minimum $\frac{1}{n}E_\omega(x_0,\dots,x_n)$ over all finite configurations. We will identify this number with quantities defined globally on the phase space $\Omega \times \mathbb{R}^d$ so that its computation will be interpreted in the framework of ergodic optimization.

To roughly explain this relation, observe that

$$\frac{1}{n}E_\omega(x_0,\dots,x_n) = \int L(\omega,t)\,\mu_{n,\omega}(d\omega,dt),$$

where $\mu_{n,\omega} := \frac{1}{n}\sum_{k=0}^{n-1}\delta_{(\tau_{x_k}(\omega),\,x_{k+1}-x_k)}$. We then check that, for every $f \in C^0(\Omega)$,

$$\int f(\omega)\,\mu_{n,\omega}(d\omega,dt) - \int f(\tau_t(\omega))\,\mu_{n,\omega}(d\omega,dt) = \frac{1}{n}\Big(f\circ\tau_{x_n}(\omega) - f\circ\tau_{x_0}(\omega)\Big).$$

If $(\mu_{n,\omega})_{n\geq 1}$ where tight, we could extract a subsequence converging to a probability measure μ for the weak* topology which would be holonomic as in Definition 7. But the tightness or the fact that $|x_k - x_{k-1}|$ is uniformly bounded whenever $(x_k)_{k=0}^n$ minimizes $\frac{1}{n}E_\omega(x_0,\dots,x_n)$ is *a priori* unclear.

We give several equivalent definitions of the ground energy in the next proposition. Let us recall that $\mathbb{M}_{hol}$ and $\mathbb{M}_{min}(L)$, respectively, denote the sets of holonomic probabilities and of minimizing measures (see Definition 7).

Proposition 13. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ be an almost periodic interaction model. Assume L is coercive. Then*

1. **(the ergodic formula)**
   $\bar{E} = \inf\big\{\int L\,d\mu : \mu \in \mathbb{M}_{hol}\big\}$, *and* $\mathbb{M}_{min}(L) \neq \emptyset$,
2. **(the sup-inf formula)**
   $\bar{E} = \sup_{u\in C^0(\Omega)} \inf\big\{L(\omega,t) + u(\omega) - u\circ\tau_t(\omega) : \omega\in\Omega,\ t\in\mathbb{R}^d\big\}$,
3. **(the ground energy per environment)**
   $\forall\,\omega\in\Omega, \quad \bar{E} = \lim_{n\to+\infty} \inf_{x_0,\ldots,x_n\in\mathbb{R}^d} \frac{1}{n}E_\omega(x_0,\ldots,x_n)$.

We remark that, in Aubry–Mather and weak KAM theories, the central constant known as *Mather's minimal average action/energy* or *Mañé's critical value* is equal minus the corresponding ergodic formula. We prefer to use the opposite sign because this convention allows us to match in harmony the previous definitions.

We also remark that the ground energy per environment actually comes from the minimality of the action. Observe the sup-inf and ergodic formulas are dual to each other as in convex analysis. Although the supremum in the sup-inf formula is achieved for periodic models, we are unable to prove it for general almost periodic interaction models. We note temporarily

$$\bar{E}_\omega = \lim_{n\to+\infty} \inf_{x_0,\ldots,x_n\in\mathbb{R}^d} \frac{1}{n}E_\omega(x_0,\ldots,x_n), \quad \bar{L} := \inf\left\{\int L\,d\mu : \mu\in\mathbb{M}_{hol}\right\},$$
$$\text{and} \quad \bar{K} := \sup_{u\in C^0(\Omega)} \inf_{\omega\in\Omega,\ t\in\mathbb{R}^d} \big[L(\omega,t) + u(\omega) - u\circ\tau_t(\omega)\big].$$

We first prove the equality $\bar{E}_\omega = \bar{E}$. We next show that $\bar{E} \geq \bar{K} \geq \bar{L} \geq \bar{E}$. We will use Birkhoff ergodic theorem for the Markov extension of a holonomic measure. For the convenience of the reader, we recall this construction.

Let $\hat{\Omega} := \Omega\times(\mathbb{R}^d)^{\mathbb{N}}$. Let us recall that every probability measure μ on $\Omega\times\mathbb{R}^d$ admits a unique disintegration along the first projection $pr : \Omega\times\mathbb{R}^d \to \Omega$,

$$\mu(d\omega, dt) := pr_*(\mu)(d\omega)P(\omega, dt),$$

where $\{P(\omega, dt)\}_{\omega\in\Omega}$ is a measurable family of probability measures on $\mathbb{R}^d$.

Definition 14. We call Markov extension of μ the probability measure $\hat{\mu}$ defined on $\hat{\Omega}$ by the Markov construction with initial distribution $pr_*(\mu)$ and transition probabilities $P(\omega, dt)$,

$$\hat{\mu}(d\omega, d\underline{t}) = pr_*(d\omega)P(\omega, dt_0)P(\tau_{t_0}(\omega), dt_1)\cdots P(\tau_{t_0+\cdots+t_{n-1}}(\omega), dt_n).$$

The following lemma shows that the Markov extension of an holonomic measure is invariant for a map. Since its proof is straightforward, we omit it.

Lemma 15. *If μ is holonomic, then $\hat{\mu}$ is invariant with respect to the shift map*

$$\hat{\tau} : (\omega, t_0, t_1, \ldots) \in \hat{\Omega} \mapsto (\tau_{t_0}(\omega), t_1, t_2, \ldots) \in \hat{\Omega}.$$

Conversely, the projection of every $\hat\tau$-invariant probability measure $\tilde\mu$ on $\Omega\times\mathbb{R}^d$ is holonomic. Moreover, if $\hat L(\omega,\underline{t}) := L(\omega,t_0)$ is the natural extension of L on $\hat\Omega$, then $\bar L = \inf\big\{\int \hat L\,d\tilde\mu : \tilde\mu$ is a $\hat\tau$-invariant probability measure$\big\}$.

Proof of Proposition 13. Step $\bar E_\omega = \bar E$. By stationarity of E_ω and minimality of τ_t, we have

$$\begin{aligned}\inf_{x_0,\dots,x_n\in\mathbb{R}^d} E_\omega(x_0,\dots,x_n) &= \inf_{x_0,\dots,x_n\in\mathbb{R}^d}\ \inf_{t\in\mathbb{R}^d}\ E_\omega(x_0+t,\dots,x_n+t)\\ &= \inf_{x_0,\dots,x_n\in\mathbb{R}^d}\ \inf_{t\in\mathbb{R}^d}\ E_{\tau_t(\omega)}(x_0,\dots,x_n)\\ &= \inf_{x_0,\dots,x_n\in\mathbb{R}^d}\ \inf_{\omega\in\Omega}\ E_\omega(x_0,\dots,x_n),\end{aligned}$$

which clearly yields $\bar E_\omega = \bar E$ for every $\omega\in\Omega$.

Step $\bar E\ge\bar K$. Given $c<\bar K$, there exists $u\in C^0(\mathbb{R}^d)$ such that, for every $\omega\in\Omega$ and any $t\in\mathbb{R}^d$, $u(\tau_t(\omega))-u(\omega)\le L(\omega,t)-c$. Let $u_\omega(x)=u(\tau_x(\omega))$. Then,

$$\forall\, x,y\in\mathbb{R}^d,\quad u_\omega(y)-u_\omega(x)\le E_\omega(x,y)-c,$$

which implies $\bar E\ge c$ for every $c<\bar K$, and therefore, $\bar E\ge\bar K$.

Step $\bar K\ge\bar L$. This part is the core of the proof of $\bar E=\bar K$. We give another proof in "Appendix" A.

Let $X:=C^0_b(\Omega\times\mathbb{R}^d)$. A coboundary is a function f of the form $f(\omega,t)=u\circ\tau_t(\omega)-u(\omega)$ for some $u\in C^0(\Omega)$. Consider

$$\begin{aligned}&A:=\{(f,s)\in X\times\mathbb{R} : f \text{ is a coboundary and } s\ge\bar K\}\quad\text{and}\\ &B:=\{(f,s)\in X\times\mathbb{R} : \inf_{\omega\in\Omega,\ t\in\mathbb{R}^d}(L-f)(\omega,t)>s\}.\end{aligned}$$

Then, A and B are nonempty convex subsets of $X\times\mathbb{R}$. They are disjoint by the definition of $\bar K$ and B is open because L is coercive. By Hahn–Banach theorem, there exists a nonzero continuous linear form Λ on $X\times\mathbb{R}$ which separates A and B. The linear form Λ is given by $\lambda\otimes\alpha$, where λ is a continuous linear form on X and $\alpha\in\mathbb{R}$. The linear form λ is, in particular, continuous on $C^0_0(\Omega\times\mathbb{R}^d)$ and, by Riesz–Markov theorem,

$$\forall\, f\in C^0_0(\Omega\times\mathbb{R}^d),\quad \lambda(f)=\int f\,d\mu,$$

for some signed measure μ. By separation, we have

$$\lambda(f)+\alpha s\le\lambda(u-u\circ\tau)+\alpha s',$$

for $u\in C^0(\Omega)$, $f\in X$ and $s,s'\in\mathbb{R}$ such that $\inf_{\Omega\times\mathbb{R}^d}(L-f)>s$ and $s'\ge\bar K$. By multiplying u by an arbitrary constant, one obtains

$$\forall\, u\in C^0(\Omega),\quad \lambda(u-u\circ\tau)=0.$$

The case $\alpha=0$ is not admissible, since otherwise $\lambda(f)\le 0$ for every $f\in X$ and λ would be the null form, which is not possible. The case $\alpha<0$ is not admissible either, since otherwise one would obtain a contradiction by taking

$f = 0$ and $s \to -\infty$. By dividing by $\alpha > 0$ and changing λ/α to λ (as well as μ/α to μ), one obtains

$$\forall f \in X, \quad \lambda(f) + \inf_{\Omega\times\mathbb{R}^d}(L - f) \leq \bar{K}.$$

By taking $f = c\mathbb{1}$, one obtains $c(\lambda(\mathbb{1})-1) \leq \bar{K} - \inf_{\Omega\times\mathbb{R}^d} L$ for every $c \in \mathbb{R}$, and thus, $\lambda(\mathbb{1}) = 1$. By taking $-f$ instead of f, one obtains $\lambda(f) \geq \inf_{\Omega\times\mathbb{R}^d} L - \bar{K}$ for every $f \geq 0$, which (again arguing by contradiction) yields $\lambda(f) \geq 0$. In particular, μ is a probability measure. We claim that

$$\forall u \in C^0(\Omega), \quad \int (u - u\circ\tau)\, d\mu = 0.$$

Indeed, given $R > 0$, consider a continuous function $0 \leq \phi_R \leq 1$, with compact support on $\Omega \times B_{R+1}(0)$, such that $\phi_R \equiv 1$ on $\Omega \times B_R(0)$. Then

$$u - u\circ\tau \geq (u - u\circ\tau)\phi_R + \min_{\Omega\times\mathbb{R}^d}(u - u\circ\tau)(1 - \phi_R).$$

Since λ and μ coincide on $C_0^0(\Omega\times\mathbb{R}^d) + \mathbb{R}\mathbb{1}$, one obtains

$$0 = \lambda(u - u\circ\tau) \geq \int (u - u\circ\tau)\phi_R\, d\mu + \min_{\Omega\times\mathbb{R}^d}(u - u\circ\tau)\int(1 - \phi_R)\, d\mu.$$

By letting $R \to +\infty$, it follows that $\int (u - u\circ\tau)\, d\mu \leq 0$ and the claim is proved by changing u to $-u$. In particular, μ is holonomic. We claim that

$$\forall f \in X, \quad \int f\, d\mu + \inf_{\Omega\times\mathbb{R}^d}(L - f) \leq \bar{K}.$$

Indeed, we first notice that the left-hand side does not change by adding a constant to f. Moreover, if $f \geq 0$ and $0 \leq f_R \leq f$ is any continuous function with compact support on $\Omega \times B_{R+1}(0)$ which is identical to f on $\Omega \times B_R(0)$, the claim follows by letting $R \to +\infty$ in

$$\int f_R\, d\mu + \inf_{\Omega\times\mathbb{R}^d}(L - f) \leq \lambda(f_R) + \inf_{\Omega\times\mathbb{R}^d}(L - f_R) \leq \bar{K}.$$

We finally prove the inequality $\bar{L} \leq \bar{K}$. Given $R > 0$, denote $L_R = \min(L, R)$. Since L is coercive, $L_R \in X$. Then $L - L_R \geq 0$ and $\int L_R\, d\mu \leq \bar{K}$. By letting $R \to +\infty$, one obtains $\int L\, d\mu \leq \bar{K}$ for some holonomic measure μ.

Step $\bar{L} \geq \bar{E}$. We claim the infimum is attained in $\bar{L} := \inf\{\int L\, d\mu : \mu \in \mathbb{M}_{hol}\}$. Indeed, let

$$C := \sup_{\omega\in\Omega} L(\omega, 0) \geq \bar{L} \quad \text{and} \quad \mathbb{M}_{hol,C} := \Big\{\mu \in \mathbb{M}_{hol} : \int L\, d\mu \leq C\Big\}.$$

We equip the set of probability measures on $\Omega \times \mathbb{R}^d$ with the weak topology (convergence of sequence of measures by integration against compactly supported continuous test functions). By coercivity, for every $\epsilon > 0$ and $M > \inf L$ such that $\epsilon > (C - \inf L)/(M - \inf L)$, there exists $R(\epsilon) > 0$ with $\inf_{\omega\in\Omega, \|t\|\geq R(\epsilon)} L(\omega, t) \geq M$. By integrating $L - \inf L$, we get

$$\forall \mu \in \mathbb{M}_{hol,C}, \quad \mu\big(\Omega \times \{t : \|t\| \geq R(\epsilon)\}\big) \leq \int \frac{L - \inf L}{M - \inf L}\, d\mu \leq \frac{C - \inf L}{M - \inf L} < \epsilon.$$

We have just proved that the set $\mathbb{M}_{hol,C}$ is tight. Let $(\mu_n)_{n\geq 0} \subset \mathbb{M}_{hol,C}$ be a sequence of holonomic measures such that $\int L\, d\mu_n \to \bar{L}$. By tightness, we may assume that $\mu_n \to \mu_\infty$ with respect to the strong topology (convergence of sequence of measures by integration against bounded continuous test functions). In particular, μ_∞ is holonomic. Moreover, for every $\phi \in C^0(\Omega, [0,1])$, with compact support,

$$0 \leq \int (L - \bar{L})\phi\, d\mu_\infty = \lim_{n\to+\infty} \int (L - \bar{L})\phi\, d\mu_n \leq \liminf_{n\to+\infty} \int (L - \bar{L})\, d\mu_n = 0.$$

Therefore, μ_∞ is minimizing.

We now prove that $\bar{L} \geq \bar{E}$. Let μ be a minimizing holonomic measure with Markov extension $\hat{\mu}$ (see Definition 14 and Lemma 15). If $(\omega, \underline{t}) \in \hat{\Omega}$, then

$$\sum_{k=0}^{n-1} \hat{L} \circ \hat{\tau}^k(\omega, \underline{t}) = E_\omega(x_0, \ldots, x_n) \quad \text{with} \quad x_0 = 0 \text{ and } x_k = t_0 + \cdots + t_{k-1},$$

and, by Birkhoff ergodic theorem,

$$\bar{E} \leq \int \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{L} \circ \hat{\tau}^k\, d\hat{\mu} = \int L\, d\mu = \bar{L}.$$

□

2.2. Mañé Subadditive Cocycle

As in weak KAM theory, we will make use of the notion of Mañé potential.

Definition 16. We call *Mañé potential* in the environment ω the function on $\mathbb{R}^d \times \mathbb{R}^d$ given by

$$S_\omega(x, y) := \inf_{n\geq 1} \inf_{x=x_0,\ldots,x_n=y} \big[E_\omega(x_0, \ldots, x_n) - n\bar{E}\big].$$

Observe that a calibrated configuration $(x_k)_{k\in\mathbb{Z}}$ for E_ω (Eq. (13)) satisfies, for all $m < n$,

$$E_\omega(x_m, \ldots, x_n) - (n - m)\bar{E} = S_\omega(x_m, x_n). \tag{15}$$

We will see in this section that the Mañé potential is always finite and shares the same properties as a pseudometric. A calibrated configuration may be seen as a geodesic for an "algebraic distance" $E_\omega(x, y) - \bar{E}$.

Since the interaction energy $E_\omega(x, y)$ derives from a Lagrangian $L(\omega, t)$, the Mañé potential $S_\omega(x, y)$ can be lifted to $\Omega \times \mathbb{R}^d$ to a function $\Phi(\omega, t)$ that we call *Mañé subadditive cocycle*.

Definition 17. Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ be an almost periodic interaction model. We call Mañé subadditive cocycle associated with L the function defined on $\Omega \times \mathbb{R}^d$ by

$$\Phi(\omega, t) := \inf_{n\geq 1} \inf_{0=x_0,x_1,\ldots,x_n=t} \sum_{k=0}^{n-1} \big[L(\tau_{x_k}(\omega), x_{k+1} - x_k) - \bar{E}\big].$$

Note that $S_\omega(x, y) = \Phi(\tau_x(\omega), y - x)$.

A function $U : \Omega \times \mathbb{R}^d \to [-\infty, +\infty)$ is said to be *a subadditive cocycle* if

$$\forall\, \omega \in \Omega,\ \forall\, s, t \in \mathbb{R}^d, \quad U(\omega, s+t) \le U(\omega, s) + U(\tau_s(\omega), t). \tag{16}$$

The very definitions of Φ and $\bar{E}$ show that Φ is a subadditive cocycle. In addition, Φ does not take infinite values and satisfies, for every $\omega \in \Omega$ and $s, t \in \mathbb{R}^d$,

$$0 \le \Phi(\omega, 0) \quad \text{and} \quad \bar{E} - L(\tau_t(\omega), -t) \le \Phi(\omega, t) \le L(\omega, t) - \bar{E}. \tag{17}$$

Inequality $0 \le \Phi(\omega, 0)$ is proved using the fact that, for a fixed ω, the sequence

$$\bar{E}_n(\omega, 0) := \inf_{x_1, \ldots, x_{n-1}} E_\omega(0, x_1, \ldots, x_{n-1}, 0)$$

is subadditive in n and $\bar{E} \le \lim_{n\to\infty} \frac{1}{n} \bar{E}_n(\omega, 0) = \inf_{n\ge1} \frac{1}{n} \bar{E}_n(\omega, 0)$. The inequality $\Phi(\omega, t) \le L(\omega, t) - \bar{E}$ comes from the definition of Φ. These two inequalities together with the subadditivity lead to

$$0 \le \Phi(\omega, 0) \le \Phi(\omega, t) + \Phi(\tau_t(\omega), -t) \le \Phi(\omega, t) + L(\tau_t(\omega), -t) - \bar{E},$$

showing the remaining inequality in (17).

Note that calibrated configurations are configurations realizing the infimum in Definition 17. We first weaken the notion of calibration in the way described below. As usual, L is supposed to be coercive.

Definition 18. A measurable subadditive cocycle $U : \Omega \times \mathbb{R}^d \to [-\infty, +\infty)$ is said to be calibrated (with respect to L) when

1. $\forall\, \omega \in \Omega,\ \forall\, s, t \in \mathbb{R}^d, \quad U(\omega, t) \le L(\omega, t) - \bar{L} \quad \text{and} \quad U(\omega, 0) \ge 0,$
2. for every $\mu \in \mathbb{M}_{hol}$ and $\hat{\mu}$ its Markov extension, if $\int L\, d\mu < +\infty$, then, for every $n \ge 1$, $\int U(\omega, \sum_{k=0}^{n-1} t_k)\, \hat{\mu}(d\omega, d\underline{t}) \ge 0$.

The existence of a calibrated subadditive cocycle enables us to easily construct calibrated configurations.

Lemma 19. *If U is a calibrated subadditive cocycle U, then U grows sublinearly, $\sup_{\omega\in\Omega, t\in\mathbb{R}^d} |U(\omega, t)|/(1 + \|t\|) < +\infty$, in particular it is finite everywhere. Besides, for every $\mu \in \mathbb{M}_{min}(L)$ and $\hat{\mu}$ its Markov extension,*

$$\forall\, n \ge 1, \quad U\left(\omega, \sum_{k=0}^{n-1} t_k\right) = \sum_{k=0}^{n-1} [\hat{L} - \bar{L}] \circ \hat{\tau}^k(\omega, \underline{t}), \quad \hat{\mu}(d\omega, d\underline{t}) \ a.e.$$

Proof. Part 1. We show that U is sublinear. Let $K := \sup_{\omega\in\Omega,\ \|t\|\le1} [L(\omega, t) - \bar{L}]$. Given $t \in \mathbb{R}^d$, let $n = \lfloor \|t\| + 1 \rfloor$ and $t_k = \frac{k}{n} t$ for $k = 0, \ldots, n-1$. Then, the subadditive cocycle property implies, on the one hand,

$$\forall\, \omega \in \Omega,\ \forall\, t \in \mathbb{R}^d, \quad U(\omega, t) \le \sum_{k=0}^{n-1} U(\tau_{t_k}(\omega), t_{k+1} - t_k) \le nK \le (1 + \|t\|)K.$$

On the other hand, thanks to the hypothesis $U(\omega, 0) \ge 0$, we obtain

$$\forall\, \omega \in \Omega,\ \forall\, t \in \mathbb{R}^d, \quad U(\omega, t) \ge U(\omega, 0) - U(\tau_t(\omega), -t) \ge -(1 + \|t\|)K.$$

Part 2. Suppose μ is minimizing. Since

$$\forall\, \omega \in \Omega,\ \forall\, t_0, \ldots, t_{n-1} \in \mathbb{R}^d, \quad \sum_{k=0}^{n-1} \big[\hat{L} - \bar{L}\big] \circ \hat{\tau}^k(\omega, \underline{t}) \geq U\left(\omega, \sum_{k=0}^{n-1} t_k\right),$$

by integrating with respect to $\hat{\mu}$, the left-hand side has a null integral, whereas the right-hand side has a nonnegative integral. The previous inequality is thus an equality that holds almost everywhere. □

Proposition 20. *Assume that L is coercive. Then, Φ is upper semi-continuous and calibrated. More precisely, for every $\mu \in \mathbb{M}_{min}(L)$ and $\hat{\mu}$ its Markov extension, for every $(\omega, \underline{t}) \in supp(\hat{\mu})$, $i < j$, $x_0 = 0$ and $x_{k+1} = x_k + t_k$, $(x_k)_{k\geq 0}$ is a one-sided calibrated configuration for E_ω,*

$$\Phi\big(\tau_{x_i}(\omega), x_j - x_i\big) = \sum_{k=i}^{j-1} \big[L - \bar{L}\big] \circ \hat{\tau}^k(\omega, \underline{t}) = E_\omega(x_i, x_{i+1}, \ldots, x_j) - (j-i)\bar{E}.$$

Proof. Part 1. We first show the existence of a particular measurable calibrated subadditive cocycle $U(\omega, t)$. From the sup-inf formula (Proposition 13), for every $p \geq 1$, there exists $u_p \in C^0(\Omega)$ such that

$$\forall\, \omega \in \Omega,\ \forall\, t \in \mathbb{R}^d, \quad u_p \circ \tau_t(\omega) - u_p(\omega) \leq L(\omega, t) - \bar{L} + 1/p.$$

Let $U_p(\omega, t) := u_p \circ \tau_t(\omega) - u_p(\omega)$ and $U := \limsup_{p\to+\infty} U_p$. Then, U is clearly a subadditive cocycle and satisfies $U(\omega, 0) = 0$. Besides, U is finite everywhere, since $0 = U(\omega, 0) \leq U(\omega, t) + U(\tau_t(\omega), -t)$ and $U(\omega, t) \leq L(\omega, t) - \bar{L}$. We just check the second property in Definition 18. Let $\mu \in \mathbb{M}_{hol}$ be such that $\int L\, d\mu < +\infty$. Define, for every $n \geq 1$,

$$\hat{S}_{n,p}(\omega, \underline{t}) := \sum_{k=0}^{n-1} \left[\hat{L} - \bar{L} + \frac{1}{p}\right] \circ \hat{\tau}^k(\omega, \underline{t}) - U_p\left(\omega, \sum_{k=0}^{n-1} t_k\right) \geq 0.$$

Since

$$U_p\left(\omega, \sum_{k=0}^{n-1} t_k\right) = \sum_{k=0}^{n-1} \hat{U}_p \circ \hat{\tau}^k(\omega, \underline{t}), \quad \hat{U}_p(\omega, \underline{t}) := U_p(\omega, t_0),$$

by integrating with respect to $\hat{\mu}$, we obtain

$$0 \leq \int \inf_{p\geq q} \hat{S}_{n,p}\, d\hat{\mu} \leq \inf_{p\geq q} \int \hat{S}_{n,p}(\omega, \underline{t})\, d\hat{\mu} \leq n \int \left[L - \bar{L} + \frac{1}{q}\right] d\mu.$$

By Lebesgue's monotone convergence theorem, as $q \to +\infty$, we have

$$\int \left[n(\hat{L} - \bar{L}) - U\left(\omega, \sum_{k=0}^{n-1} t_k\right)\right] d\hat{\mu} \leq \int n[L - \bar{L}]\, d\mu \quad \text{and}$$

$$\int U\left(\omega, \sum_{k=0}^{n-1} t_k\right) \hat{\mu}(d\omega, d\underline{t}) \geq 0.$$

Part 2. We next show that Φ is calibrated. We have already noticed that Φ satisfies the subadditive cocycle property, $\Phi \le L - \bar{L}$, $\Phi(\omega, 0) \ge 0$, and $\Phi(\omega, t)$ is finite everywhere. Moreover, $\Phi(\omega, t) \ge U(\omega, t)$ and the second property of Definition 18 follows from part 1.

Part 3. We show that Φ is upper semi-continuous. Define

$$\forall\, \omega \in \Omega, \quad \forall\, n \ge 1, \quad \Phi_n(\omega, t) := \inf\{E_\omega(x_0, \ldots, x_n) : x_0 = 0,\ x_n = t\}.$$

Then, $\Phi = \inf_{n\ge1}(\Phi_n - n\bar{E})$ is upper semi-continuous if we prove that Φ_n is continuous. Let $D > 0$, $c_0 := \inf_{\omega,x,y} E_\omega(x, y)$ and $K_D := \sup_{\omega\in\Omega,\ \|t\|\le D} E_\omega(0, \ldots, 0, t)$. By coercivity, there exists $R_D > 0$ such that

$$\forall\, x, y \in \mathbb{R}^d, \quad \|y - x\| > R_D \Rightarrow\ \forall\, \omega \in \Omega,\ E_\omega(x, y) > K_D - (n-1)c_0.$$

Choose $\omega, x_0, \ldots, x_n$ such that $E_\omega(x_0, \ldots, x_n) \le K_D$. Then, for every $0 \le k < n$,

$$K_D \ge E_\omega(x_0, \ldots, x_n) \ge (n-1)c_0 + E_\omega(x_k, x_{k+1})\ \Rightarrow \|x_{k+1} - x_k\| \le R_D.$$

We have proved that the infimum in the definition of $\Phi_n(\omega, t)$, when $\omega \in \Omega$ and $\|t\| \le D$, can be realized over $\|x_k\| \le kR_D$, $\forall\, 0 \le k \le n$. By the uniform continuity of $E_\omega(x_0, \ldots, x_n)$ on the product space $\Omega \times \Pi_k\{\|x_k\| \le kR\}$, we obtain that Φ_n is continuous on $\Omega \times \{\|t\| \le D\}$.

Part 4. Let μ be a minimizing measure with Markov extension $\hat{\mu}$. We show that every $(\omega, \underline{t})$ in the support of $\hat{\mu}$ is calibrated. Let

$$\hat{\Sigma} := \left\{ (\omega, \underline{t}) \in \Omega \times (\mathbb{R}^d)^{\mathbb{N}} : \forall\, n \ge 1,\ \Phi\left(\omega, \sum_{k=0}^{n-1} t_k\right) \ge \sum_{k=0}^{n-1} \left[L - \bar{L}\right] \circ \hat{\tau}^k(\omega, \underline{t}) \right\}.$$

The set $\hat{\Sigma}$ is closed, since Φ is upper semi-continuous. By Lemma 19, $\hat{\Sigma}$ has full $\hat{\mu}$-measure and therefore contains $\mathrm{supp}(\hat{\mu})$. Hence, the proposition is proved thanks to the subadditive cocycle property of Φ and the $\hat{\tau}$-invariance of $\mathrm{supp}(\hat{\mu})$. □

Lemma 21. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ be an almost periodic interaction model.*

1. *If L is coercive, then $\mathbb{M}_{min}(L) \ne \emptyset$ and Mather(L) = supp(μ) for some $\mu \in \mathbb{M}_{min}(L)$. In particular, the Mather set is closed.*
2. *If L is superlinear, the Mather set is compact.*

Proof of Lemma 21. Item 1. The existence of minimizing measures is actually part of item 1 of Proposition 13. Thus, let $\{V_i\}_{i\in\mathbb{N}}$ be a countable basis of the topology of $\Omega \times \mathbb{R}^d$ and let

$$I := \{i \in \mathbb{N} : V_i \cap \mathrm{supp}(\nu) \ne \emptyset \text{ for some } \nu \in \mathbb{M}_{min}(L)\}.$$

We reindex $I = \{i_1, i_2, \ldots\}$ and choose for every $k \ge 1$ a minimizing measure μ_k so that $V_{i_k} \cap \mathrm{supp}(\mu_k) \ne \emptyset$ or equivalently $\mu_k(V_{i_k}) > 0$. Let $\mu := \sum_{k\ge1} \frac{1}{2^k}\mu_k$. Then, μ is minimizing. Suppose some V_i is disjoint from the support of μ. Then, $\mu(V_i) = 0$ and, for every $k \ge 1$, $\mu_k(V_i) = 0$. Suppose by contradiction that $V_i \cap \mathrm{supp}(\nu) \ne \emptyset$ for some $\nu \in \mathbb{M}_{min}(L)$, then $i = i_k$ for some $k \ge 1$ and, by the choice of μ_k, $\mu_k(V_i) > 0$, which is not possible. Therefore, V_i is disjoint from the Mather set and we have just proved $\mathrm{Mather}(L) \subseteq \mathrm{supp}(\mu)$ or $\mathrm{Mather}(L) = \mathrm{supp}(\mu)$.

Item 2. We now assume that L is superlinear. From Lemma 19, the Mañé subadditive cocycle is sublinear. There exists $R > 0$ such that

$$\forall\, \omega \in \Omega,\ \forall\, t \in \mathbb{R}^d, \quad |\Phi(\omega, t)| \leq R(1 + \|t\|).$$

By superlinearity, there exists $B > 0$ such that

$$\forall\, \omega \in \Omega,\ \forall\, t \in \mathbb{R}^d, \quad L(\omega, t) \geq 2R\|t\| - B.$$

Let μ be a minimizing measure. Since $\Phi = L - \bar{L}$ μ a.e. (Lemma 19), we obtain

$$\|t\| \leq (R + B + |\bar{L}|)/R, \quad \mu(d\omega, dt) \text{ a.e.}$$

We have proved that the support of every minimizing measure is compact. In particular, the Mather set is compact. □

Proof of Theorem 8. We show that, for every environment ω in the projected Mather set, there exists a calibrated configuration for E_ω passing through the origin. Let μ be a minimizing measure such that $\mathrm{supp}(\mu) = \mathrm{Mather}(L)$. Let $\hat{\mu}$ denote its Markov extension. For $n \geq 1$, consider

$$\hat{\Omega}_n := \left\{ (\omega, \underline{t}) \in \Omega \times (\mathbb{R}^d)^{\mathbb{N}} : \Phi\left(\omega, \sum_{k=0}^{2n-1} t_k\right) \geq \sum_{k=0}^{2n-1} \left[L - \bar{L}\right] \circ \hat{\tau}^k(\omega, \underline{t}) \right\}.$$

From Proposition 20, $\mathrm{supp}(\hat{\mu}) \subseteq \hat{\Omega}_n$. From the upper semi-continuity of Φ, $\hat{\Omega}_n$ is closed. To simplify the notations, for every $\underline{t}$, we define a configuration $(x_0, x_1, \ldots)$ by

$$x_0 = 0,\ x_{k+1} = x_k + t_k \quad \text{so that} \quad \hat{\tau}^k(\omega, \underline{t}) = (\tau_{x_k}(\omega), (t_k, t_{k+1}, \ldots)).$$

Notice that, if $(\omega, \underline{t}) \in \hat{\Omega}_n$, thanks to the subadditive cocycle property of Φ and the fact that $\Phi \leq L - \bar{L}$, the finite configuration $(x_0, \ldots, x_{2n})$ is calibrated in the environment ω, that is,

$$\forall\, 0 \leq i < j \leq 2n, \quad \Phi\left(\tau_{x_i}(\omega), \sum_{k=i}^{j-1} t_k\right) = \sum_{k=i}^{j-1} \left[L - \bar{L}\right] \circ \hat{\tau}^k(\omega, \underline{t}),$$

or written using the family of interaction energies E_ω,

$$\forall\, 0 \leq i < j \leq 2n, \quad S_\omega(x_i, x_j) = E_\omega(x_i, \ldots, x_j) - (j - i)\bar{E}.$$

Thanks to the sublinearity of S_ω, there exists a constant $R > 0$ such that, uniformly in $\omega \in \Omega$ and $x, y \in \mathbb{R}^d$, we have $|S_\omega(x, y)| \leq R(1 + \|y - x\|)$. Besides, thanks to the superlinearity of E_ω, there exists a constant $B > 0$ such that $E_\omega(x, y) \geq 2R\|y - x\| - B$. Since $S_\omega(x_k, x_{k+1}) = E_\omega(x_k, x_{k+1}) - \bar{E}$, we thus obtain a uniform upper bound $D := (R + B + |\bar{E}|)/R$ on the jumps of calibrated configurations:

$$\forall\, (\omega, \underline{t}) \in \hat{\Omega}_n, \quad \forall\, 0 \leq k < 2n, \quad \|x_{k+1} - x_k\| \leq D.$$

Let $\hat{\Omega}'_n = \hat{\tau}^n(\hat{\Omega}_n)$. Thanks to the uniform bounds on the jumps, $\hat{\Omega}'_n$ is again closed. Since $\hat{\mu}(\hat{\Omega}_n) = 1$, $\hat{\mu}(\hat{\Omega}'_n) = 1$ by invariance of $\hat{\tau}$. Let $\nu := pr_*(\mu)$ be the projected measure on Ω. Then, $\mathrm{supp}(\nu) = pr(\mathrm{Mather}(L))$. By the definition of $\hat{\Omega}'_n$, we have

$$\begin{aligned} &\hat{pr}(\hat{\Omega}'_n) = \{\omega \in \Omega : \exists\, (x_{-n}, \ldots, x_n) \in \mathbb{R}^d \ \text{ s.t. } \ x_0 = 0 \ \text{ and } \\ &S_\omega(x_{-n}, x_n) \geq E_\omega(x_{-n}, \ldots, x_n) - 2n\bar{E}\}. \end{aligned}$$

Again by the uniform boundness of the jumps, $\hat{pr}(\hat{\Omega}'_n)$ is closed and has full ν-measure. Thus, $\hat{pr}(\hat{\Omega}'_n) \supseteq pr(\mathrm{Mather}(L))$. By a diagonal extraction procedure, we obtain, for every $\omega \in \mathrm{Mather}(L)$, a bi-infinite calibrated configuration with uniformly bounded jumps passing through the origin. □

3. Aubry Theory for Weakly Twist Interactions

The one-dimensional Aubry theory is based of the strong form of the twist condition (4). The main consequence of this condition is that the set of infinite two-sided minimizing configurations is well ordered. The weak form of the twist condition (14) allows us to use anharmonic interactions. We extend in this section some proofs of the Aubry theory for weakly twist Lagrangians. We show that minimizing finite configurations are strictly well ordered. The fact that there is no superposition of atoms is new and more delicate to prove. We will use these results for the Proof of Theorem 12.

From now on, we consider almost periodic interaction models where L is supposed to be weakly twist. The following lemma extends Aubry crossing lemma. It enables to give restrictions on the combinatorics of minimizing configurations. In particular, we will obtain they are ordered.

Lemma 22 (Aubry crossing lemma). *Given $\omega \in \Omega$, if $x_0, x_1, y_0, y_1 \in \mathbb{R}$ satisfy $(y_0 - x_0)(y_1 - x_1) < 0$, then*

$$\begin{aligned} &\big[E_\omega(x_0, x_1) + E_\omega(y_0, y_1)\big] - \big[E_\omega(x_0, y_1) + E_\omega(y_0, x_1)\big] \\ &\quad = \alpha(y_0 - x_0)(y_1 - x_1) > 0, \end{aligned}$$

where $\alpha = \frac{1}{(y_0-x_0)(y_1-x_1)} \int_{x_0}^{y_0} \int_{x_1}^{y_1} \frac{\partial^2 E_\omega}{\partial x \partial y}(x, y)\, dydx < 0$.

The proof is similar to the standard Aubry crossing lemma [2] and is left to the reader. We start showing that strictly monotone finite configurations minimize the energy.

Lemma 23. *Let $\omega \in \Omega$. For $n \geq 2$, let $x_0, \ldots, x_n \in \mathbb{R}$ be a nonmonotone sequence (that is, a sequence which does not satisfy $x_0 \leq \ldots \leq x_n$ nor $x_0 \geq \ldots \geq x_n$).*

1. *If $x_0 = x_n$, then $E_\omega(x_0, \ldots, x_n) > \sum_{i=0}^{n-1} E_\omega(x_i, x_i)$.*
2. *If $x_0 \neq x_n$, then there exists a subset $\{i_0, i_1, \ldots, i_r\}$ of $\{0, \ldots, n\}$, with $i_0 = 0$ and $i_r = n$, such that $(x_{i_0}, x_{i_1}, \ldots, x_{i_r})$ is strictly monotone and*

$$E_\omega(x_0, \ldots, x_n) > E_\omega(x_{i_0}, \ldots, x_{i_r}) + \sum \_i \notin \{i_0, \ldots, i_r\} E_\omega(x_i, x_i).$$

(Note that it may happen that $x_i = x_j$ for $i \notin \{i_0, \ldots, i_r\}$ and $j \in \{i_0, \ldots, i_r\}$.)

Proof. We prove the lemma by induction.

Let $x_0, x_1, x_2 \in \mathbb{R}$ be a nonmonotone sequence. If $x_0 = x_2$, then $E_\omega(x_0, x_1, x_2) > E_\omega(x_0, x_0) + E_\omega(x_1, x_1)$. If $x_0 \neq x_2$, then x_0, x_1, x_2 are three distinct points. Thus, $x_0 < x_1$ implies $x_2 < x_1$ and $x_1 < x_0$ implies $x_1 < x_2$. In both cases, Lemma 22 tells us that

$$E_\omega(x_0, x_1) + E_\omega(x_1, x_2) > E_\omega(x_0, x_2) + E_\omega(x_1, x_1).$$

Let $(x_0, \ldots, x_{n+1})$ be a nonmonotone sequence. We have two cases: either $x_0 \leq x_n$ or $x_0 \geq x_n$. We shall only give the proof for the case $x_0 \leq x_n$.

Case $x_0 = x_n$. Then, $(x_0, \ldots, x_n)$ is nonmonotone and by induction

$$\begin{aligned} E_\omega(x_0, \ldots, x_{n+1}) &> E_\omega(x_n, x_{n+1}) + \sum_{i=0}^{n-1} E_\omega(x_i, x_i) \\ &= E_\omega(x_0, x_{n+1}) + \sum_{i=1}^{n} E_\omega(x_i, x_i). \end{aligned}$$

The conclusion holds whether $x_{n+1} = x_0$ or not.

Case $x_0 < x_n$. Whether $(x_0, \ldots, x_n)$ is monotone or not, we may choose a subset of indices $\{i_0, \ldots, i_r\}$ such that $i_0 = 0$, $i_r = n$, $x_{i_0} < x_{i_1} < \ldots < x_{i_r}$ and

$$E_\omega(x_0, \ldots, x_{n+1}) \geq \left(E_\omega(x_{i_0}, \ldots, x_{i_r}) + \sum_{i \notin \{i_0, \ldots, i_r\}} E_\omega(x_i, x_i) \right) + E_\omega(x_n, x_{n+1}).$$

If $x_n \leq x_{n+1}$, then $(x_0, \ldots, x_n)$ is necessarily nonmonotone and the previous inequality is strict. If $x_n = x_{n+1}$, the lemma is proved by modifying $i_r = n + 1$. If $x_n < x_{n+1}$, the lemma is proved by choosing $r + 1$ indices and $i_{r+1} = n + 1$.

If $x_{n+1} < x_n = x_{i_r}$, by applying Lemma 22, one obtains

$$\begin{aligned} E_\omega(x_{i_{r-1}}, x_{i_r}) + E_\omega(x_n, x_{n+1}) &> E_\omega(x_n, x_{i_r}) + E_\omega(x_{i_{r-1}}, x_{n+1}), \\ E_\omega(x_0, \ldots, x_{n+1}) &> E_\omega(x_{i_0}, \ldots, x_{i_{r-1}}, x_{n+1}) \\ &\quad + \left[\sum_{i \notin \{i_0, \ldots, i_r\}} E_\omega(x_i, x_i) \right] + E_\omega(x_n, x_n). \end{aligned}$$

If $x_{i_{r-1}} < x_{n+1}$, the lemma is proved by changing $i_r = n$ to $i_r = n + 1$. If $x_{i_{r-1}} = x_{n+1}$, the lemma is proved by choosing $r - 1$ indices and $i_{r-1} = n + 1$. If $x_{n+1} < x_{i_{r-1}}$, we apply again Lemma 22 until there exists a largest $s \in \{0, \ldots, r\}$ such that $x_s < x_{n+1}$ or $x_{n+1} \leq x_0$. In the former case, the lemma is proved by choosing $s + 1$ indices and by modifying $i_{s+1} = n + 1$. In the latter case, namely, when $x_{n+1} \leq x_0 < x_n$, we have

$$E_\omega(x_0, \ldots, x_{n+1}) > E_\omega(x_0, x_{n+1}) + \sum_{i=1}^{n} E_\omega(x_i, x_i)$$

and the lemma is proved whether $x_{n+1} = x_0$ or $x_{n+1} < x_0$. $\square$

As a consequence, note that it is enough to minimize over strictly monotone configurations, unless $t = 0$, in Definition 17 of the Mañé subadditive cocycle $\Phi(\omega, t)$.

Proposition 24. *The Mañé subadditive cocycle $\Phi(\omega, t)$ satisfies, for every $\omega \in \Omega$,*

- *if $t = 0$, $\Phi(\omega, 0) = E_\omega(0, 0) - \bar{E}$,*
- *if $t > 0$, $\Phi(\omega, t) = \inf_{n \geq 1} \inf_{0 = x_0 < x_1 < \ldots < x_n = t} [E_\omega(x_0, \ldots, x_n) - n\bar{E}]$,*
- *if $t < 0$, $\Phi(\omega, t) = \inf_{n \geq 1} \inf_{0 = x_0 > x_1 > \ldots > x_n = t} [E_\omega(x_0, \ldots, x_n) - n\bar{E}]$.*

Proof. Lemma 23 tells us that we can minimize the energy of $E_\omega(x_0, \ldots, x_n) - n\bar{E}$ by the sum of two terms:

- either $x_n = x_0$, then
$$E_\omega(x_0, \ldots, x_n) - n\bar{E} \geq \big[E_\omega(x_0, x_0) - \bar{E}\big] + \sum_{i \notin \{0, n\}} \big[E_\omega(x_i, x_i) - \bar{E}\big];$$
- or $x_n \neq x_0$, then for some $(x_{i_0}, \ldots, x_{i_r})$ strictly monotone, with $i_0 = 0$ and $i_r = n$,
$$E_\omega(x_0, \ldots, x_n) - n\bar{E} \geq \big[E_\omega(x_{i_0}, \ldots, x_{i_r}) - r\bar{E}\big] + \sum_{i \notin \{i_0, \ldots, i_r\}} \big[E_\omega(x_i, x_i) - \bar{E}\big].$$

We conclude the proof by noticing that $\bar{E} \leq \inf_{x \in \mathbb{R}} E_\omega(x, x)$.

□

The next lemma shows that minimizing finite configurations are strictly ordered.

Proposition 25. *Let $\omega \in \Omega$. If $(x_0, \ldots, x_n)$ is a minimizing configuration for E_ω such that x_i is strictly between x_0 and x_n for every $0 < i < n-1$, then $(x_0, \ldots, x_n)$ is strictly monotone.*

Proof. Let $(x_0, \ldots, x_n)$ be such a minimizing sequence. We show, in part 1, it is monotone, and, in part 2, it is strictly monotone.

Part 1. Assume by contradiction that $(x_0, \ldots, x_n)$ is not monotone. According to Lemma 23, one can find a subset of indices $\{i_0, \ldots, i_r\}$ of $\{0, \ldots, n\}$, with $i_0 = 0$ and $i_r = n$, such that $(x_{i_0}, \ldots, x_{i_r})$ is strictly monotone and
$$E_\omega(x_0, \ldots, x_n) > E_\omega(x_{i_0}, \ldots, x_{i_r}) + \sum_{i \notin \{i_0, \ldots, i_r\}} E_\omega(x_i, x_i).$$
We choose the largest integer r with the above property. Since $(x_0, \ldots, x_n)$ is not monotone, we have necessarily $r < n$. Since $(x_0, \ldots, x_n)$ is minimizing, one can find $i \notin \{i_0, \ldots, i_r\}$ such that $x_i \notin \{x_{i_0}, \ldots, x_{i_r}\}$. Let s be one of the indices of $\{0, \ldots, r\}$ such that x_i is between x_{i_s} and $x_{i_{s+1}}$. Then, by Lemma 22,
$$E_\omega(x_{i_s}, x_{i_{s+1}}) + E_\omega(x_i, x_i) > E_\omega(x_{i_s}, x_i) + E_\omega(x_i, x_{i_{s+1}}).$$
We have just contradicted the maximality of r. Therefore, $(x_0, \ldots, x_n)$ must be monotone.

Part 2. Assume by contradiction that $(x_0, \ldots, x_n)$ is not strictly monotone. Then, $(x_0, \ldots, x_n)$ contains a subsequence of the form $(x_{i-1}, x_i, \ldots, x_{i+r}, x_{i+r+1})$ with $r \geq 1$ and $x_{i-1} \neq x_i = \ldots = x_{i+r} \neq x_{i+r+1}$. To simplify the proof, we assume $x_{i-1} < x_{i+r+1}$. We want to built a configuration $(x'_{i-1}, x'_i, \ldots, x'_{i+r}, x'_{i+r+1})$ so that $x'_{i-1} = x_{i-1}$, $x'_{i+r+1} = x_{i+r+1}$ and

$$E_\omega(x_{i-1}, x_i, \ldots, x_{i+r}, x_{i+r+1}) > E_\omega(x'_{i-1}, x'_i, \ldots, x'_{i+r}, x'_{i+r+1}).$$

Indeed, since $(x_{i-1}, \ldots, x_{i+r+1})$ is minimizing, we have

$$E_\omega(x_{i-1}, \ldots, x_{i+r+1}) = E_\omega(x_{i-1}, x_i + \epsilon, x_{i+1} - \epsilon, \ldots, x_{i+r} - \epsilon, x_{i+r+1}) + o(\epsilon^2).$$

Let

$$\alpha = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} \frac{\partial^2 E_\omega}{\partial x \partial y}(x, x_i)\, dx < 0,$$
$$\beta = \frac{1}{x_{i+r+1} - x_{i+r}} \int_{x_{i+r}}^{x_{i+r+1}} \frac{\partial^2 E_\omega}{\partial x \partial y}(x_{i+r}, y)\, dy < 0.$$

By Aubry crossing lemma,

$$\begin{aligned} &E_\omega(x_{i-1}, x_i + \epsilon) + E_\omega(x_i + \epsilon, x_{i+1} - \epsilon) \\ &\quad = E_\omega(x_{i-1}, x_{i+1} - \epsilon) + E_\omega(x_i + \epsilon, x_i + \epsilon) - 2\epsilon(x_i - x_{i-1})\alpha + o(\epsilon). \end{aligned}$$

Since $x_i = x_{i+r}$, obviously $E_\omega(x_i + \epsilon, x_i + \epsilon) = E_\omega(x_{i+r} + \epsilon, x_{i+r} + \epsilon)$. Again by Aubry crossing lemma,

$$\begin{aligned} &E_\omega(x_{i+r} + \epsilon, x_{i+r} + \epsilon) + E_\omega(x_{i+r} - \epsilon, x_{i+r+1}) \\ &\quad = E_\omega(x_{i+r} - \epsilon, x_{i+r} + \epsilon) + E_\omega(x_{i+r} + \epsilon, x_{i+r+1}) \\ &\qquad - 2\epsilon(x_{i+r+1} - x_{i+r})\beta + o(\epsilon). \end{aligned}$$

Then, for ϵ small enough, we have

$$E_\omega(x_{i-1}, \ldots, x_{i+r+1}) > E_\omega(x_{i-1}, x_i - \epsilon, \ldots, x_{i-r-1} - \epsilon, x_{i+r} + \epsilon, x_{i+r+1}),$$

which contradicts that $(x_{i-1}, \ldots, x_{i+r+1})$ is minimizing. We have thus proved that $(x_0, \ldots, x_n)$ is strictly monotone. $\square$

4. Locally Constant Lagrangians and Quasicrystals

We present in the first subsection a general framework that includes Example 4 and naturally appears in the context of quasicrystals and strongly pattern equivariant functions. In the second subsection, we recall the construction of Kakutani–Rohlin towers, transverse measures, and homology matrices for uniquely ergodic $\mathbb{R}$-actions, which will be useful to prove Theorem 12.

4.1. One-Dimensional Quasicrystals

Our purpose in this section is to provide a rich variety of examples of almost crystalline interaction models (Definition 11). The two main concepts are: *the* *hull of a quasicrystal* and *a strongly equivariant function* (see [4,20,21] for a deeper understanding of these notions).

We first recall the definition of a quasicrystal (see [14]). Let $\omega \subset \mathbb{R}$ be a discrete subset of $\mathbb{R}$. A *ρ-patch*, or a *pattern* for short, is a finite set $\mathtt{P}$ of the form $\omega \cap \overline{B_\rho(x)}$ for some $x \in \omega$ and some constant $\rho > 0$, where $B_\rho(x)$ denotes the open ball of radius ρ centered in x. We say that $y \in \omega$ is an *occurrence* of $\mathtt{P}$ if $\omega \cap \overline{B_\rho(y)}$ is equal to $\mathtt{P}$ up to a translation. A *quasicrystal* is a discrete set $\omega \subset \mathbb{R}$ satisfying

- *finite local complexity*: for any $\rho > 0$, ω has just a finite number of ρ-patches up to translations;
- *repetitivity*: for all $\rho > 0$, there exists $M(\rho) > 0$ such that any closed ball of radius $M(\rho)$ contains at least one occurrence of every ρ-patch of ω;
- *uniform pattern distribution*: for any pattern $\mathtt{P}$ of ω, uniformly in $x \in \mathbb{R}$, the following positive limit exists

$$\lim_{r \to +\infty} \frac{\#\left(\{y \in \mathbb{R} : y \text{ is an occurrence of } \mathtt{P}\} \cap B_r(x)\right)}{\mathrm{Leb}(B_r(x))} = \nu(\mathtt{P}) > 0.$$

We notice that the finite local complexity is equivalent to the fact that the intersection of the difference set $\omega - \omega$ with any bounded set is finite. The set of quasicrystals can be equipped with an $\mathbb{R}$-action: $\tau_t(\omega) := \omega - t$, for every $t \in \mathbb{R}$, by translating every point in ω by t. A quasicrystal is said to be *aperiodic* if $\tau_t(\omega) = \omega$ implies $t = 0$, and *periodic* otherwise. The lattice $\mathbb{Z}$ or the Beatty sequence $\omega(\alpha) = \{k \in \mathbb{Z} : \lfloor k\alpha \rfloor - \lfloor (k-1)\alpha \rfloor = 1\}$, $\alpha \in (0,1)$, is basic examples of one-dimensional quasicrystals. When α is irrational (as in Example 4), $\omega(\alpha)$ is an aperiodic quasicrystal for which the repetitivity and the uniform pattern distribution are obtained thanks to the minimality and the unique ergodicity of an irrational rotation on the circle. For details, we refer to [21].

The first nontrivial concept we need is given by the hull of a quasicrystal. Given a quasicrystal $\omega_* \subset \mathbb{R}$, we equip the set $\tilde{\Omega}(\omega_*) := \{\tau_t(\omega_*) : t \in \mathbb{R}\}$ of all the translations of ω_* with the Gromov–Hausdorff topology. Roughly speaking, two quasicrystals in this set are close if and only if they have the same pattern, up to a small translation, in a large neighborhood of the origin. More precisely, we define a metric as follows (for details, see [4,19]): the distance between two translations $\omega, \underline{\omega} \in \tilde{\Omega}(\omega_*)$ is the real number

$$\mathrm{dist}(\omega, \underline{\omega}) := \inf \left\{ \frac{1}{r+1} : \exists\, |t|, |\underline{t}| < \frac{1}{r} \text{ s.t. } (\omega + t) \cap B_r(0) = (\underline{\omega} + \underline{t}) \cap B_r(0) \right\}.$$

The Gromov–Hausdorff topology is equivalent to the topology given by this distance. We call *hull* $\Omega(\omega_*)$ of the quasicrystal ω_* the completion of $\tilde{\Omega}(\omega_*)$. The finite local complexity hypothesis implies that $\Omega(\omega_*)$ is a compact metric space. Each element $\omega \in \Omega(\omega_*)$ is a quasicrystal with the same patterns as ω_* up to translations. Each map $\tau_t : \Omega(\omega_*) \to \Omega(\omega_*)$ is a homeomorphism. The orbit of ω_* is by definition dense in $\Omega(\omega_*)$. The repetitivity hypothesis is actually equivalent to the minimality of the $\mathbb{R}$-action τ_t. The uniform pattern distribution is equivalent to the unique ergodicity of τ_t (the $\mathbb{R}$-action has a unique invariant probability measure). We refer to [4,20] for a more detailed analysis. We summarize these facts in the following proposition.

Proposition 26 ([4,20]). *Let ω_* be a quasicrystal of $\mathbb{R}$. Then, the dynamical system $(\Omega(\omega_*), \{\tau_t\}_{t\in\mathbb{R}})$ is minimal and uniquely ergodic.*

We call *canonical transversal* $\Xi_0(\omega_*)$ of the hull $\Omega(\omega_*)$ the set of quasicrystals ω in $\Omega(\omega_*)$ such that the origin 0 belongs to ω. A basis of the topology on $\Xi_0(\omega_*)$ is given by cylinder sets $\Xi_{\omega,\rho}$ with $\omega \in \Xi_0(\omega_*)$ and $\rho > 0$. In general, that is, for every $\omega \in \Omega(\omega_*)$ and $\rho > 0$ such that $\omega \cap B_\rho(0) \neq \emptyset$, a *transverse cylinder set* $\Xi_{\omega,\rho}$ is defined by

$$\Xi_{\omega,\rho} := \{\underline{\omega} \in \Omega(\omega_*) : \omega \cap \overline{B_\rho(0)} = \underline{\omega} \cap \overline{B_\rho(0)}\}.$$

If $\omega \in \Xi_0(\omega_*)$, then $\Xi_{\omega,\rho} \subset \Xi_0(\omega_*)$.

The designation of transversal comes from the obvious fact that the set $\Xi_0(\omega_*)$ is transverse to the action: for any real t small enough, we have $\tau_t(\omega) \notin \Xi_0(\omega_*)$ for any $\omega \in \Xi_0(\omega_*)$. This gives a Poincaré section.

Proposition 27 ([20]). *The canonical transversal $\Xi_0(\omega_*)$ and the transverse cylinder sets $\Xi_{\omega,\rho}$ associated with an aperiodic quasicrystal ω_* are Cantor sets. If ω_* is a periodic quasicrystal, these sets are finite.*

This allows us to give a more dynamical description of the hull in one dimension by considering the *return time* function $\Theta : \Xi_0(\omega_*) \to \mathbb{R}^+$ defined by

$$\Theta(\omega) := \inf\{t > 0 : \tau_t(\omega) \in \Xi_0(\omega_*)\}, \quad \forall\, \omega \in \Xi_0(\omega_*).$$

The finite local complexity implies that this function is locally constant. The *first return map* $T\colon \Xi_0(\omega_*) \to \Xi_0(\omega_*)$ is then given by

$$T(\omega) := \tau_{\Theta(\omega)}(\omega), \quad \forall\, \omega \in \Xi_0(\omega_*).$$

Remark that the unique invariant probability measure on $\Omega(\omega_*)$ induces a finite measure on $\Xi_0(\omega_*)$ that is T-invariant (see [14]).

It is straightforward to check that the dynamical system $(\Omega(\omega_*), \{\tau_t\}_{t\in\mathbb{R}})$ is conjugate to the suspension of the map T on the set $\Xi_0(\omega_*)$ with the time map given by the function Θ. Thus, when ω_* is periodic, the hull $\Omega(\omega_*)$ is homeomorphic to a circle. Otherwise, $\Omega(\omega_*)$ has a laminated structure: it is locally the Cartesian product of a Cantor set by an interval.

Transverse cylinder sets are base construction pieces of the notion of flow boxes introduced in Definition 9. In the aperiodic case, if $\omega \in \Omega(\omega_*)$, $r > 0$, and ρ is large enough, the set

$$U_{\omega,\rho,r} := \{\underline{\omega} - t : t \in B_r(0),\ \underline{\omega} \in \Xi_{\omega,\rho}\}$$

is open and homeomorphic to $B_r(0) \times \Xi_{\omega,\rho}$ by the map $(t, \underline{\omega}) \to \tau_t(\underline{\omega}) = \underline{\omega} - t$. Their collection forms a basis of the topology of $\Omega(\omega_*)$. The set $U_{\omega,\rho,r}$ is called *a flow box of basis* $\Xi_{\omega,\rho}$. The following lemma shows that these flow boxes are admissible and therefore form a flow box decomposition (Definition 9).

Lemma 28 ([4]). *Let ω_* be an aperiodic quasicrystal. Let $U_i := U_{\omega_i,\rho_i,r_i}$, $i = 1,2$, be two flow boxes such that $U_1 \cap U_2 \neq \emptyset$. Then, there exists a real number $a \in \mathbb{R}$ such that, for every $\underline{\omega}_i \in \Xi_{\omega_i,\rho_i}$, for every $|t_i| < r_i$, $i = 1,2$,*

$$\underline{\omega}_1 - t_1 = \underline{\omega}_2 - t_2 \quad \Longrightarrow \quad t_2 = t_1 - a.$$

The second nontrivial concept we need is the notion of strongly equivariant function as introduced in [19]. Let ω_* be a quasicrystal. We recall that a potential $V_{\omega_*} : \mathbb{R} \to \mathbb{R}$ is said to be *strongly ω_*-equivariant* if there exists a constant $R > 0$ (called the interaction range) such that

$$V_{\omega_*}(x) = V_{\omega_*}(y), \quad \forall\, x, y \in \mathbb{R} \text{ with } (B_R(x) \cap \omega_*) - x = (B_R(y) \cap \omega_*) - y.$$

Of course any periodic potential is strongly equivariant with respect to a discrete lattice of periods. In Example 4, the function $V_{\omega(\alpha)}$ is strongly $\omega(\alpha)$-equivariant with range $R = \lfloor \frac{1}{\alpha} \rfloor + 1$. Let us mention another example from [19], which holds for any quasicrystal ω_*. Let $\delta := \sum_{x\in\omega_*} \delta_x$ be the Dirac comb supported on the points of a quasicrystal ω_* and let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support. Then, one may check that the convolution product $\delta * g$ is a smooth strongly ω_*-equivariant function. Actually, any strongly ω-equivariant function can be defined by a similar procedure [19].

We recall in the following lemma that a strongly ω_*-equivariant function always arises from a global function defined on the space $\Omega(\omega_*)$.

Lemma 29 ([14,19]). *Let ω_* be a quasicrystal and $V_{\omega_*} : \mathbb{R} \to \mathbb{R}$ be a continuous strongly ω_*-equivariant function with range R. Then, there exists a unique continuous function $V : \Omega(\omega_*) \to \mathbb{R}$ such that*

$$V_{\omega_*}(x) = V \circ \tau_x(\omega_*), \quad \forall\, x \in \mathbb{R}.$$

Besides, V is constant on transverse cylinder sets $\Xi_{\omega,R+S}$, with $\omega \in \Omega(\omega_*)$ and $S \geq 0$. If V_{ω_*} is C^2, then V is C^2 along the flow: $x \in \mathbb{R} \mapsto V(\tau_x(\omega))$ is C^2, $\forall\, \omega$.

The global function given by Lemma 29 satisfies the *locally transversally constant* property that is at the origin of Definition 10. We indeed observe on each flow box $U_{\omega,R+S,S}$

$$V\left(\tau_x(\underline{\omega})\right) = V\left(\tau_x(\underline{\omega}')\right), \quad \forall\, |x| < S,\ \forall\, \underline{\omega}, \underline{\omega}' \in \Xi_{\omega,R+S},$$

thanks to the fact that $\tau_x(\underline{\omega}') \in \Xi_{\tau_x(\underline{\omega}),R}$ whenever $\underline{\omega}, \underline{\omega}' \in \Xi_{\omega,R+S}$ and $|x| < S$. More generally, we introduce the following definition.

Definition 30. Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be an almost periodic interaction model. A function $V : \Omega \to \mathbb{R}$ is said to be locally transversally constant with respect to a flow box decomposition $\{U_i\}_{i\in I}$, where $U_i = \tau(B_{R_i} \times \Xi_i)$, if

$$\forall\, i \in I,\ \forall\, \omega, \omega' \in \Xi_i,\ \forall\, |x| < R_i, \quad V(\tau_x(\omega)) = V(\tau_x(\omega')).$$

The Examples 3 and 4 are of the form

$$L(\omega, t) = W(t) + V_1(\omega) + V_2(\tau_t(\omega)) \tag{18}$$

with locally transversally constant functions V_1 and V_2. The next lemma shows that such a Lagrangian L is locally transversally constant as in Definition 10.

Lemma 31. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be an almost periodic interaction model admitting a flow box decomposition. Let $V_1, V_2 : \Omega \to \mathbb{R}$ be two locally transversally constant functions on the same flow box decomposition, and $W = \mathbb{R} \to \mathbb{R}$ be any function. Define $L(\omega, t) = W(t) + V_1(\omega) + V_2(\tau_t(\omega))$. Then, L is locally transversally constant.*

Proof. Assume V_1 and V_2 are locally transversally constant on a flow box decomposition $\{U_i\}_{i\in I}$. Let $\tau(B_R \times \Xi)$ be a flow box which is compatible with respect to $\{U_i\}_{i\in I}$. If $|x|, |y| < R$ and $\omega, \omega' \in \Xi$, then

$$E_\omega(x, y) = W(y - x) + V_{1,\omega}(x) + V_{2,\omega}(y).$$

There exist $i \in I$, $|t_i| < R_i$ and $\tilde{\Xi}_i$ a clopen subset of Ξ_i such that $\tau_x(\Xi) = \tau_{t_i}(\tilde{\Xi}_i)$. Then, $\tau_x(\omega) = \tau_{t_i}(\omega_i)$ and $\tau_x(\omega') = \tau_{t_i}(\omega_i')$ for some $\omega_i, \omega_i' \in \tilde{\Xi}_i$. We have

$$V_{1,\omega}(x) = V_{1,\omega_i}(t_i) = V_{1,\omega_i'}(t_i) = V_{1,\omega'}(x).$$

Similarly $V_{2,\omega}(y) = V_{2,\omega'}(y)$. We have thus proved $E_{\omega'}(x, y) = E_\omega(x, y)$. □

We conclude this section by describing a family of quasicrystalline interaction models $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ for which the conclusions of Theorem 12 hold. We say that a C^2 function $W : \mathbb{R} \to \mathbb{R}$ is *superlinear* and *weakly convex* if

$$W'' > 0 \text{ a.e.} \quad \text{and} \quad \lim_{|t|\to+\infty} |W'(t)| = +\infty. \tag{19}$$

Corollary 32. *Let ω_* be a quasicrystal, $V_{1*}, V_{2*} : \mathbb{R} \to \mathbb{R}$ be two C^2 strongly ω_*-equivariant functions, and $W : \mathbb{R} \to \mathbb{R}$ be a C^2 superlinear, weakly convex function. Let $\Omega(\omega_*)$ be the hull of ω_* and $\{\tau_t\}_{t\in\mathbb{R}}$ be the canonical $\mathbb{R}$-action on $\Omega(\omega_*)$. Let $V_1, V_2 : \Omega(\omega_*) \to \mathbb{R}$ be the extension of V_{1*}, V_{2*} as explained in Lemma 29. Define*

$$L(\omega, t) = W(t) + V_1(\omega) + V_2(\tau_t(\omega)).$$

Then, $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ is an almost crystalline interaction model.

4.2. Kakutani–Rohlin Tower Description

Flow boxes are open sets obtained by taking the union of every orbit of size R starting from any point belonging to a closed transverse Poincaré section. The restricted topology on a transverse section must be special: it must admit a basis of clopen sets. We recall in Lemma 35 how to construct a suspension with locally constant return maps called Kakutani–Rohlin tower. When the flow is uniquely ergodic, we describe in the Lemmas 36 and 37 how this Kakutani–Rohlin tower enables to characterize the unique transverse measure associated with each transverse section.

We gather in the following lemma basic results about flow boxes that are particular cases of *tilable laminations* (see [4]). We leave the proof of the lemma to the reader (or see proofs in [4]).

Lemma 33. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be a minimal $\mathbb{R}$-action. Assume that the action is not periodic ($t \in \mathbb{R} \mapsto \tau_t(\omega) \in \Omega$ is injective for every $\omega \in \Omega$). Then,*

1. *If $\tau(B_R \times \Xi)$ is a flow box, then there exists $R' \geq R$ such that $\Omega = \tau(B_{R'} \times \Xi)$.*
2. *If $\tau(B_R \times \Xi)$ is a flow box, then $\tau : \mathbb{R} \times \Xi \to \Omega$ is open and $\tau(B_R \times \Xi')$ is again a flow box for every clopen subset $\Xi' \subset \Xi$.*
3. *If $\tau(B_R \times \Xi)$ is a flow box, then, for every $R' > 0$ and $\omega \in \Xi$, there exists a clopen set $\Xi' \subset \Xi$ containing ω such that $\tau(B_{R'} \times \Xi')$ is again a flow box.*
4. *If $\tau(B_{2R+2R'} \times \Xi)$ and $\tau(B_{2R+2R'} \times \Xi')$ are flow boxes, and $U = \tau(B_R \times \Xi)$ and $U' = \tau(B_{R'} \times \Xi')$ are admissible flow boxes, then*
$$U \cap U' = \tau(\tilde{B} \times \tilde{\Xi}) = \tau(\tilde{B}' \times \tilde{\Xi}')$$
for some clopen sets $\tilde{\Xi}$, $\tilde{\Xi}'$ and some open convex subsets $\tilde{B} \subset B_R$, $\tilde{B}' \subset B_{R'}$.
5. *If $\{U_i\}_{i \in I}$ is a flow box decomposition, then, for every $\omega \in \Omega$ and $R > 0$, there exits a flow box $\tau(B_R \times \Xi)$, with a transverse section Ξ containing ω, that is compatible with respect to $\{U_i\}_{i \in I}$.*

The existence of a flow box decomposition enables us to build a global transverse section of the flow with locally constant return times.

Definition 34. Let $(\Omega, \{\tau_t\}_{t \in \mathbb{R}})$ be a one-dimensional minimal $\mathbb{R}$-action possessing a flow box decomposition $\{U_i\}_{i \in I}$. We call Kakutani–Rohlin tower a partition $\{F_\alpha\}_{\alpha \in A}$ of Ω of the form
$$F_\alpha = \tau\big([0, H_\alpha) \times \Sigma_\alpha\big) = \cup_{0 \leq t < H_\alpha} \tau_t(\Sigma_\alpha)$$
for some height $H_\alpha > 0$ and some transverse section Σ_α (closed set admitting a basis of clopen subsets), where $\tau\big((0, H_\alpha) \times \Sigma_\alpha\big)$ is a flow box (open and homeomorphic to $(0, H_\alpha) \times \Sigma_\alpha$), and $\cup_{\alpha \in A} \tau(\{H_\alpha\} \times \Sigma_\alpha) = \cup_{\alpha \in A} \tau(\{0\} \times \Sigma_\alpha) = \cup_{\alpha \in A} \Sigma_\alpha$. Moreover, we say that a Kakutani–Rohlin tower is compatible with respect to $\{U_i\}_{i \in I}$ if, for every $\alpha \in A$, there exist $i \in I$, $t_i \in \mathbb{R}$ and a clopen subset $\tilde{\Xi}_i \subset \Xi_i$ such that $\Sigma_\alpha = \tau_{t_i}(\tilde{\Xi}_i)$ and $[t_i, t_i + H_\alpha) \subset [-R_i, R_i)$.

The proof of the existence of a Kakutani–Rohlin tower for one-dimensional minimal $\mathbb{R}$-actions is similar to the construction given in [14] for quasicrystals.

Lemma 35. *Let $(\Omega, \{\tau_t\}_{t \in \mathbb{R}})$ be a one-dimensional minimal $\mathbb{R}$-action possessing a flow box decomposition $\{U_i\}_{i \in I}$. Then, there exists a Kakutani–Rohlin tower $\{F_\alpha\}_{\alpha \in A}$ which is compatible with respect to $\{U_i\}_{i \in I}$.*

The existence of a Kakutani–Rohlin tower enables us to build a global transverse section $\cup_{\alpha \in A} \Sigma_\alpha$ with a return time constant on each Σ_α and equal to H_α. The induction of the $\mathbb{R}$-action on a particular section Σ_{α_0} gives a second Kakutani–Rohlin tower with larger heights. We explain in the next paragraph the notations that will be used for these successive towers.

If $\{F_\alpha^0\}_{\alpha \in A^0}$ is a Kakutani–Rohlin tower of order 0, denote $F_\alpha^0 := \tau\big([0, H_\alpha^0) \times \Sigma_\alpha^0\big)$. We say that $\Sigma^0 := \cup_\alpha \Sigma_\alpha^0$ is the basis of the tower. Let ω_* be a reference point of the base Σ^0. Consider α_0 such that $\omega_* \in \Sigma_{\alpha_0}^0$. The construction of the tower of order 1 is done by inducing the flow on $\Sigma^1 := \Sigma_{\alpha_0}^0$.

We obtain a partition of Σ^1 given by $\{\Sigma^1_\beta\}_{\beta\in A^1}$, where $\beta=(\alpha_0,\ldots,\alpha_p)$, $p\geq 1$, $\alpha_p=\alpha_0$, $\alpha_i\neq\alpha_0$ for $i=1,\ldots,p-1$,

$$\Sigma^1_\beta=\Sigma^0_{\alpha_0}\cap\tau^{-1}_{H^0_{\alpha_0}}(\Sigma^0_{\alpha_1})\cap\ldots\cap\tau^{-1}_{H^0_{\alpha_0}+\ldots+H^0_{\alpha_{p-1}}}(\Sigma^0_{\alpha_p}).$$

By minimality, there is a finite collection of such nonempty sets Σ^1_β. Define then

$$\begin{gathered} H^1_\beta:=H^0_{\alpha_0}+\ldots+H^0_{\alpha_{p-1}}, \\ F^1_\beta:=\tau\big([0,H^1_\beta)\times\Sigma^1_\beta\big)=\bigcup_{i=0}^{p-1}\tau\big([t_i,t_i+H^0_{\alpha_i})\times\Sigma^0_{\alpha_i}\big),\ \text{with}\ t_i=\sum_{j=0}^{i-1}H^0_{\alpha_j}. \end{gathered} \tag{20}$$

We have just obtained a new Kakutani–Rohlin tower $\{F^1_\beta\}_{\beta\in A^1}$ of basis $\Sigma^0_{\alpha_0}$. We induce again on the section $\Sigma^1_{\beta_0}$ that contains ω_* and build the tower of order 2. We shall write $\{F^l_\alpha\}_{\alpha\in A^l}$ for the successive towers that are built using this procedure and F^l_* for the tower of height H^l_* whose basis Σ^l_* contains ω_*. The preceding construction gives $\min_{\alpha\in A^{l+1}}H^{l+1}_\alpha\geq H^l_*$ and in particular $H^{l+1}_*\geq H^l_*$. It may happen that $H^l_*=H^{l+1}_*=H^{l+2}_*=\ldots$. In that case, the flow is a suspension over Σ^l_* of constant return time H^l_* (and Ω is isomorphic to $\Sigma^l_*\times S^1$). In order to exclude this situation, we split the basis $\Sigma^l_{\alpha_0}$ which contains ω_* into two disjoint clopen sets $\Sigma^l_{\alpha_0}=\Sigma^l_{\alpha_0'}\cup\Sigma^l_{\alpha_0''}$. We obtain again a Kakutani–Rohlin tower, and we induce as before on the subset which contains ω_*. If $(\Omega,\{\tau_t\}_{t\in\mathbb{R}})$ is not periodic, we may choose the splitting so that $H^{l+1}_*>H^l_*$ at each step of the construction.

We assume that the flow $(\Omega,\{\tau_t\}_{t\in\mathbb{R}})$ is uniquely ergodic. Let λ be the unique ergodic invariant probability measure. The average frequency of return times to a transverse section of a flow box measures the thickness of the section. The next lemma gives a precise definition of a family of transverse measures $\{\nu_\Xi\}_\Xi$ parameterized by every transverse section Ξ. The proof is standard, and we leave it to the reader.

Lemma 36. *Let $(\Omega,\{\tau_t\}_{t\in\mathbb{R}})$ be a minimal and uniquely ergodic $\mathbb{R}$-action admitting a flow bx decomposition. For every transverse section Ξ, the set of return times to Ξ is given by*

$$\mathcal{R}_\Xi(\omega):=\{t\in\mathbb{R}:\tau_t(\omega)\in\Xi\},\quad\forall\,\omega\in\Omega.$$

Then, for every nonempty clopen set $\Xi'\subset\Xi$, the following limit exists uniformly with respect to $\omega\in\Omega$ and is positive:

$$\nu_\Xi(\Xi'):=\lim_{T\to+\infty}\frac{\#(\mathcal{R}_{\Xi'}(\omega)\cap B_T)}{Leb(B_T)}>0.$$

Moreover, ν_Ξ extends to a σ-finite measure on Ξ of finite mass, called transverse measure to Ξ, and, for every flow box $U=\tau(B_R\times\Xi)$,

$$\lambda(\tau(B'\times\Xi'))=Leb(B')\nu_\Xi(\Xi'),\quad \textit{for all Borel sets } B'\subset B_R,\ \Xi'\subset\Xi.$$

Let $\{F^l_\alpha\}_{\alpha\in A^l}$ be a tower of order l and $\{F^{l+1}_\beta\}_{\beta\in A^{l+1}}$ be the subsequent tower as introduced in (20). The homology matrix explained in lemma 2.7 of

[14] may be here similarly defined. Indeed, for every $\alpha \in A^l$ and $\beta \in A^{l+1}$, $\beta = (\alpha_0, \ldots, \alpha_p)$, $\alpha_0 = \alpha_p$, $\alpha_i \neq \alpha_0$ for $i = 1, \ldots, p-1$, we denote

$$M^l_{\alpha,\beta} := \#\{0 \leq k \leq p-1 : \alpha_k = \alpha\}.$$

A flow box of order $l+1$, $\tau\big([0, H^{l+1}_\beta) \times \Sigma^{l+1}_\beta\big)$, is obtained as a disjoint union of flow boxes of order l of the type $\tau\big([t_i, t_i + H^l_{\alpha_i}) \times \Sigma^l_{\alpha_i}\big)$. The integer $M^l_{\alpha,\beta}$ counts the number of times a flow box of order $l+1$ indexed by β cuts a flow box of order l indexed by α. The main result that we shall need is given by the following lemma.

Lemma 37. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be a minimal and uniquely ergodic $\mathbb{R}$-action. Let $\{F^l_\alpha\}_{\alpha\in A^l}$ be a sequence of Kakutani–Rohlin towers built as in (20). Let ν^l be the transverse measure associated with the transverse section $\cup_{\alpha\in A^l}\Sigma^l_\alpha$. If $\nu^l_\alpha := \nu^l(\Sigma^l_\alpha)$, then*

$$\nu^l_\alpha = \sum_{\beta\in A^{l+1}} M^l_{\alpha,\beta}\nu^{l+1}_\beta.$$

Proof. Let $\Xi = \cup_{\beta\in A^{l+1}}\Sigma^{l+1}_\beta$. For $\omega \in \Xi$, let $0 = t_0, t_1, t_2, \ldots$ be its successive return times to Ξ. We introduce as in Lemma 36 the set of return times to the transverse section Σ^l_α, say, $\mathcal{R}^l_\alpha(\omega) := \{t \in \mathbb{R} : \tau_t(\omega) \in \Sigma^l_\alpha\}$. The set $\mathcal{R}^{l+1}_\beta(\omega)$ is defined similarly. Since

$$\#\big(\mathcal{R}^l_\alpha(\omega) \cap [0, t_n)\big) = \sum_{\beta\in A^{l+1}} M^l_{\alpha,\beta}\ \#\big(\mathcal{R}^{l+1}_\beta(\omega) \cap [0, t_n)\big),$$

we divide by t_n and apply Lemma 36 to conclude. □

5. Almost Crystalline Interaction Models

This section is devoted to the proof of the second main result of this paper, Theorem 12. By recalling Definition 11, we consider a one-dimensional almost crystalline interaction model $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$. By hypothesis, L is transversally constant with respect to a flow box decomposition $\{U_i = \tau(B_{R_i} \times \Xi_i)\}_{i\in I}$.

If for some $\omega \in \Omega$ and $x \in \mathbb{R}$, $E_\omega(x, x) = \bar{E}$, then $\delta_{(\tau_x(\omega),0)} \in \mathbb{M}_{min}(L)$, $\tau_x(\omega)$ belongs to the projected Mather set, and the configuration $x_{k,\omega} = x$ fulfills the two items of Theorem 12. We thus assume from now on

$$\forall\, \omega \in \Omega,\ \forall x \in \mathbb{R}, \quad E_\omega(x, x) > \bar{E}.$$

We first prove in Proposition 39 that a finite configuration $(x^n_0, \ldots, x^n_n)$ which realizes the minimum of the energy among all configurations of the same length must be strictly monotone and must have bounded jumps, $|x^n_k - x^n_{k-1}| \leq R$, uniformly in n. We next prove in Proposition 42 that $\liminf_{n\to+\infty} \frac{1}{n}|x^n_n - x^n_0| > 0$. We finally conclude this section with the proof of Theorem 12.

Lemma 38. *There exists $R > 0$ such that, if $\omega \in \Omega$, if $(x_0, \ldots, x_n) \in \mathbb{R}$ is minimizing for E_ω and $|x_n - x_0| \geq R$, then $(x_0, \ldots, x_n)$ is strictly monotone.*

Proof. Since $\{U_i\}_{i\in I}$ is a finite cover, we may choose R large enough so that every orbit of size R meets every box entirely: for every ω, for every $|y-x| \ge R$, for every $i \in I$, there exists $t_i \in \mathbb{R}$ such that $(t_i - R_i, t_i + R_i) \subset [x,y]$ and $\tau_{t_i}(\omega) \in \Xi_i$.

We first show that there cannot exist $r \ge 0$ and $0 < k < n-r$ such that

$$x_k < x_{k-1}, \quad x_k = \ldots = x_{k+r} \quad \text{and} \quad x_k < x_{k+r+1}.$$

Otherwise, Aubry crossing lemma implies that

$$E_\omega(x_{k-1}, x_k) + E_\omega(x_k, x_{k+r+1}) > E_\omega(x_{k-1}, x_{k+r+1}) + E_\omega(x_k, x_k).$$

We rewrite the configuration $(x_0, \ldots, x_{k-1}, x_{k+r+1}, \ldots, x_n)$ as $(y_0, \ldots, y_{n-r-1})$. Let U_i be a flow box containing $\tau_{x_k}(\omega)$. There exists $|s| < R_i$ and $\omega' \in \Xi_i$ such that $\tau_{x_k}(\omega) = \tau_s(\omega')$. By the choice of R, there exists t such that $(t - R_i, t + R_i) \subset [x_0, x_n]$ and $\tau_t(\omega) \in \Xi_i$. Let $z_0 = \ldots = z_r := t + s$ and $1 \le l \le n-r-1$ be such that $y_{l-1} < z_0 \le y_l$. Using the fact that L is transversally constant on U_i, we have

$$E_\omega(x_k, x_k) = E_{\omega'}(s, s) = E_{\tau_t(\omega)}(s, s) = E_\omega(z_0, z_0).$$

By applying again Aubry crossing lemma, we obtain

$$E_\omega(y_{l-1}, y_l) + E_\omega(z_0, z_0) \ge E_\omega(y_{l-1}, z_0) + E_\omega(z_0, y_l),$$

(possibly with a strict inequality if $z_0 < y_l$). We have just obtained a new configuration $(y_0, \ldots, y_{l-1}, z_0, \ldots, z_r, y_l, \ldots, y_{n-r-1})$ of n points with a strictly lower energy, which contradicts the fact that $(x_0, \ldots, x_n)$ is minimizing.

Similarly, there cannot exist $r \ge 0$ and $0 < k < n-r$ such that

$$x_k > x_{k-1}, \quad x_k = \ldots = x_{k+r} \quad \text{and} \quad x_k > x_{k+r+1}.$$

There cannot exist either a sub-configuration $(x_{k-1}, x_k, \ldots, x_{k+r}, x_{k+r+1})$, $r \ge 1$, of the form $x_{k-1} \ne x_{k+r+1}$ and $x_k = \ldots = x_{k+r}$ strictly between x_{k-1} and x_{k+r+1} thanks to Proposition 25. We are thus left to a configuration of the form

$$x_0 = \ldots = x_r < \ldots < x_{n-r'} = \ldots = x_n$$
or
$$x_0 = \ldots = x_r > \ldots > x_{n-r'} = \ldots = x_n$$

for some $r, r' \ge 0$.

Assume by contradiction that $x_0 = x_1$ (the case $x_{n-1} = x_n$ is done similarly). Exactly as before, there exist U_i containing $\tau_{x_0}(\omega)$, $|s| < R_i$ and $\omega' \in \Xi_i$ such that $\tau_{x_0}(\omega) = \tau_s(\omega')$, and there exists $t \in \mathbb{R}$ such that $(t - R_i, t + R_i) \subset [\min\{x_0, x_n\}, \max\{x_0, x_n\}]$ and $\tau_t(\omega) \in \Xi_i$. One can show in an analogous way that, whenever $z := t + s$ belongs to $(\min\{x_{l-1}, x_l\}, \max\{x_{l-1}, x_l\}]$ for $2 \le l \le n$, $E_\omega(x_0, x_1, \ldots, x_n) \ge E_\omega(x_1, \ldots, x_{l-1}, z, x_l, \ldots, x_n)$, with strict inequality if $z < \max\{x_{l-1}, x_l\}$. Since $(x_0, x_1, \ldots, x_n)$ is a minimizing configuration, this implies that $z = \max\{x_{l-1}, x_l\} \notin \{x_0, x_n\}$, and $(x_1, \ldots, x_{l-1}, z, x_l, \ldots, x_n)$ is a minimizing configuration. The first part of this proof shows that this cannot happen.

The proof that $(x_0, \ldots, x_n)$ is strictly monotone is complete. $\square$

Proposition 39. *There exists $R > 0$ such that, for every $\omega \in \Omega$, $n \geq 2$, and $(x_0, \ldots, x_n) \in \mathbb{R}$, if*

$$E_\omega(x_0, \ldots, x_n) = \min_{(y_0,\ldots,y_n)} E_\omega(y_0, \ldots, y_n) \quad \textit{and} \quad \max_{0\leq k<l\leq n} |x_k - x_l| \geq R,$$

then $(x_0, \ldots, x_n)$ is strictly monotone and $\sup_{1\leq k\leq n} |x_k - x_{k-1}| \leq R$.

Proof. Consider $\omega \in \Omega$, $n \geq 2$, and $(x_0, \ldots, x_n)$ realizing the minimum of the energy among all configurations of length n in the environment ω.

Part 1. We show there exists $R' > 0$ (independent of ω and n) such that $|x_1 - x_0| \leq R'$ and $|x_2 - x_1| \leq R'$. Indeed, we have

$$E_\omega(x_0, x_1) \leq E_\omega(x_1, x_1) \quad \text{and} \quad E_\omega(x_0, x_1, x_2) \leq E_\omega(x_2, x_2, x_2),$$

which implies

$$E_\omega(x_0, x_1) \leq \sup_{x\in\mathbb{R}} E_\omega(x, x)$$

and

$$E_\omega(x_1, x_2) \leq 2 \sup_{x\in\mathbb{R}} E_\omega(x, x) - \inf_{x,y\in\mathbb{R}} E_\omega(x, y).$$

The existence of R' follows then from the coercivity of L, which is uniform with respect to ω. Similarly, we have $|x_{n-1} - x_{n-2}| \leq R'$ and $|x_n - x_{n-1}| \leq R'$.

Part 2. We show there exists $R'' > 0$ such that, if $(x_0, \ldots, x_m)$ is strictly monotone, then $|x_i - x_{i-1}| \leq R''$ for every $1 \leq i \leq m$. We can find a collection of transverse sections $\{\Xi'_i\}_{i\in I'}$ such that $\{U'_i = \tau(B_{2R'} \times \Xi'_i)\}_{i\in I'}$ is a flow box decomposition, $\{\tau(B_{R'} \times \Xi'_i)\}_{i\in I'}$ is a covering of Ω, and L is transversally constant with respect to $\{U'_i\}_{i\in I'}$. We choose $R'' > 0$ large enough so that every orbit of length R'' meets entirely each U'_i.

Let $\tau(B_{R'} \times \Xi'_i)$ be a flow box containing $\tau_{x_1}(\omega)$: there exist $|s_1| < R'$ and $\omega' \in \Xi'_i$ such that $\tau_{x_1}(\omega) = \tau_{s_1}(\omega')$. From part 1, we deduce that U'_i contains $\{\tau_{x_0}(\omega), \tau_{x_1}(\omega), \tau_{x_2}(\omega)\}$. Denote $s_0 := s_1 + x_0 - x_1$ and $s_2 := s_1 + x_2 - x_1$, so that $|s_0|, |s_2| < 2R'$, $\tau_{x_0}(\omega) = \tau_{s_0}(\omega')$ and $\tau_{x_2}(\omega) = \tau_{s_2}(\omega')$. Assume by contradiction $|x_i - x_{i-1}| > R''$. Then, there exists $t \in \mathbb{R}$ such that $(t-2R', t+2R') \subset [\min\{x_{i-1}, x_i\}, \max\{x_{i-1}, x_i\}]$ and $\tau_t(\omega) \in \Xi'_i$. Let $z_0 = t + s_0$, $z_1 = t + s_1$ and $z_2 = t + s_2$. Notice that (x_{i-1}, x_i) and (z_0, z_1, z_2) are ordered in the same way. As L is transversally constant on U'_i, we obtain

$$E_\omega(x_0, x_1, x_2) = E_{\omega'}(s_0, s_1, s_2) = E_{\tau_t(\omega)}(s_0, s_1, s_2) = E_\omega(z_0, z_1, z_2).$$

Aubry crossing lemma applied twice gives

$$\begin{aligned} E_\omega(x_{i-1}, x_i) + E_\omega(z_0, z_1, z_2) &> E_\omega(x_{i-1}, z_1) + E_\omega(z_0, x_i) + E_\omega(z_1, z_2), \\ &> E_\omega(x_{i-1}, z_1, x_i) + E_\omega(z_0, z_2). \end{aligned}$$

As L is transversally constant, $E_\omega(z_0, z_2) = E_\omega(x_0, x_2)$ and we obtain

$$E_\omega(x_{i-1}, x_i) + E_\omega(x_0, x_1, x_2) > E_\omega(x_{i-1}, z_1, x_i) + E_\omega(x_0, x_2).$$

The configuration $(x_0, x_2, \ldots, x_{i-1}, z_1, x_i, \ldots, x_m)$ has a strictly lower energy, which contradicts the fact that $(x_0, \ldots, x_m)$ is minimizing. We obtain similarly that, if $(x_m, \dots, x_n)$ is strictly monotone, then $|x_{i-1} - x_i| \le R''$ for every $m+1 \le i \le n$.

Part 3. Let R''' be the constant given by Lemma 38. Take $R > 2R'' + 4R'''$. If $|x_n - x_0| > R'''$, then $(x_0, \dots, x_n)$ is strictly monotone by Lemma 38 and the jumps $|x_i - x_{i-1}|$ are uniformly bounded by R''. The proof is finished.

Assume by contradiction that $|x_n - x_0| \le R'''$. Let $a = \min_{0 \le k \le n} x_k$ and $b = \max_{0 \le k \le n} x_k$. Since $\operatorname{diam}(\{x_k : 0 \le k \le n\}) \ge R$, one of the two inequalities $|a - x_0| > R/2$ or $|b - x_0| > R/2$ must be satisfied. Assume to simplify $|b - x_0| > R/2$ (the case $|a - x_0| > R/2$ is done similarly). Hence, $b = x_m$ for some $0 < m < n$. Since $(x_0, \dots, x_m)$ and $(x_m, \dots, x_n)$ are minimizing and satisfy $|x_m - x_0| > R'''$ and $|x_m - x_n| > R'''$, these two configurations are strictly monotone. Then, part 2 tells us that the jumps $|x_i - x_{i-1}|$ are uniformly bounded by R''. In particular, $|x_{m+1} - x_m| \le R''$. The configuration $(x_0, \dots, x_{m+1})$ is minimizing and, since $|x_m - x_0| > R'' + 2R'''$, it satisfies $|x_{m+1} - x_0| > R'''$. By Lemma 38, it must be strictly monotone, which is in contradiction with the maximum x_m.

Thus, $|x_n - x_0| > R'''$, $(x_0, \dots, x_n)$ is strictly monotone and $|x_i - x_{i-1}| \le R''$. □

The proof of the fact that $|x_k - x_{k-1}|$ is uniformly bounded uses the same ideas as in Lemma 3.1 of [14]. The fact that L is transversally constant enables us to translate sub-configurations without modifying the total energy. For a minimizing and strictly monotone configuration, by minimality of the energy, two consecutive points cannot enclose a translated sub-configuration of three points. More precisely, we have the following lemma that extends Lemma 3.2 of [14].

Lemma 40. *For $R > 0$, let $\tau(B_R \times \Xi)$ be a flow box compatible with respect to $\{U_i\}_{i \in I}$. Let $(x_0, \dots, x_n)$ be a strictly monotone minimizing configuration for some environment $\omega \in \Omega$. Let $(a - R, a + R)$ and $(b - R, b + R)$ be two disjoint intervals such that $\tau_a(\omega) \in \Xi$ and $\tau_b(\omega) \in \Xi$. Assume that $(a - R, a + R)$ is a subset of $[x_0, x_n]$. Let A be the number of sites $0 \le k \le n$ such that x_k belongs to $(a - R, a + R)$ and let B be defined similarly. Then, $B \le A + 2$. In particular, if $(b - R, b + R) \subset [x_0, x_n]$, then $|A - B| \le 2$.*

Proof. To simplify we assume that $(x_0, \dots, x_n)$ is strictly increasing. The proof is done by contradiction by assuming $B \ge A + 3$. Denote

$$\begin{aligned} \{y_1, \dots, y_A\} &:= \{x_0, \dots, x_n\} \cap (a - R, a + R) \quad \text{and} \\ \{y'_1, \dots, y'_B\} &:= \{x_0, \dots, x_n\} \cap (b - R, b + R). \end{aligned}$$

Let y_0 be the greatest $x_k \le a - R$ and y_{A+1} be the smallest $x_k \ge a + R$. We write $s_k := y'_k - b$ and $z_k := a + s_k$ for $k = 1, \dots, B$. The partition into $A + 1$ disjoint intervals $\cup_{k=1}^{A+1} (y_{k-1}, y_k]$ must contain $A + 3$ distinct points $\{z_1, \dots, z_{A+3}\}$. We have therefore to consider two cases.

Case 1. Either of some interval $(y_{k-1}, y_k]$, $2 \le k \le A$, contains three points (z_{i-1}, z_i, z_{i+1}). By Aubry crossing lemma,

$$\begin{aligned} E_\omega(y_{k-1}, y_k) + E_\omega(z_{i-1}, z_i) &> E_\omega(y_{k-1}, z_i) + E_\omega(z_{i-1}, y_k), \\ E_\omega(z_{i-1}, y_k) + E_\omega(z_i, z_{i+1}) &\ge E_\omega(z_{i-1}, z_{i+1}) + E_\omega(z_i, y_k). \end{aligned}$$

Since L is transversally constant on $\tau(B_R \times \Xi)$, we obtain

$$\begin{aligned} E_\omega(y'_{i-1}, y'_i, y'_{i+1}) + E_\omega(y_{k-1}, y_k) &= E_\omega(z_{i-1}, z_i, z_{i+1}) + E_\omega(y_{k-1}, y_k) \\ &> E_\omega(z_{i-1}, z_{i+1}) + E_\omega(y_{k-1}, z_i, y_k) \\ &= E_\omega(y'_{i-1}, y'_{i+1}) + E_\omega(y_{k-1}, z_i, y_k). \end{aligned}$$

We have obtained a configuration (if, for instance, $b < a$) of the form

$$(x_0, \ldots, y'_{i-1}, y'_{i+1}, \ldots, y'_B, \ldots, y_1, \ldots, y_{k-1}, z_i, y_k, \ldots, x_n)$$

with strictly lower energy, which contradicts the fact that $(x_0, \ldots, x_n)$ is minimizing.

Case 2. Or there exist two distinct intervals $(y_{k-1}, y_k]$ and $(y_{l-1}, y_l]$, with $2 \le k < l \le A$, that contain each two points (z_{i-1}, z_i) and (z_{j-1}, z_j), respectively. Notice that we may have $y_k = y_{l-1}$, but we must have $z_i < z_{j-1}$, $z_{i+1} \in (a - R, a + R)$, and possibly $z_{i+1} = z_{j-1}$. We want to obtain a contradiction by showing that one can decrease the sum of energies $E_\omega(y'_{i-1}, \ldots, y'_j) + E_\omega(y_{k-1}, \ldots, y_l)$ while fixing the four boundary points.

In the case $z_i = y_k$, we perturb the point z_i slightly by a small quantity ϵ and allow an increase in the energy of order ϵ^2. Since (z_{i-1}, z_i, z_{i+1}) is minimizing, we have

$$E_\omega(z_{i-1}, z_i, z_{i+1}) = E_\omega(z_{i-1}, z_i - \epsilon, z_{i+1}) + o(\epsilon^2).$$

By Aubry crossing lemma, either $z_i < y_k$ or the reminder in Lemma 22 takes the form

$$\begin{aligned} &\text{reminder} := (z_{i-1} - y_{k-1})(z_i - y_k)\alpha > 0, \\ &\text{where} \quad \alpha = \frac{1}{(z_{i-1} - y_{k-1})(z_i - y_k)} \int_{y_{k-1}}^{z_{i-1}} \int_{y_k}^{z_i} \frac{\partial^2 E_\omega}{\partial x \partial y}(x, y)\, dy dx < 0, \end{aligned}$$

(in that case, we define $\epsilon := 0$), or $z_i = y_k$, and the reminder becomes

$$\begin{aligned} &\text{reminder} := -\epsilon(z_{i-1} - y_{k-1})\alpha + o(\epsilon) > o(\epsilon^2), \\ &\text{where} \quad \alpha = \frac{1}{z_{i-1} - y_{k-1}} \int_{y_{k-1}}^{z_{i-1}} \frac{\partial^2 E_\omega}{\partial x \partial y}(x, y_k)\, dx < 0. \end{aligned}$$

In both cases,

$$\begin{aligned} &E_\omega(y_{k-1}, y_k) + E_\omega(z_{i-1}, z_i - \epsilon) \\ &\quad = E_\omega(y_{k-1}, z_i - \epsilon) + E_\omega(z_{i-1}, y_k) + \text{reminder}, \\ &\quad E_\omega(y_{k-1}, y_k) + E_\omega(z_{i-1}, z_i, z_{i+1}) > E_\omega(y_{k-1}, z_i - \epsilon, z_{i+1}) + E_\omega(z_{i-1}, y_k). \end{aligned}$$

Again by Aubry crossing lemma,

$$E_\omega(y_{l-1}, y_l) + E_\omega(z_{j-1}, z_j) \ge E_\omega(y_{l-1}, z_j) + E_\omega(z_{j-1}, y_l),$$

with possibly equality if $z_j = y_l$. Since L is transversally constant, we obtain

$$\begin{aligned} &E_\omega(y'_{i-1}, \ldots, y'_j) \\ &\qquad + E_\omega(y_{k-1}, \ldots, y_l) \\ &\quad = E_\omega(z_{i-1}, \ldots, z_j) + E_\omega(y_{k-1}, \ldots, y_l) \\ &\quad > E_\omega(z_{i-1}, y_k, \ldots, y_{l-1}, z_j) + E_\omega(y_{k-1}, z_i - \epsilon, z_{i+1}, \ldots, z_{j-1}, y_l) \\ &\quad = E_\omega(y'_{i-1}, w_k, \ldots, w_{l-1}, y'_j) + E_\omega(y_{k-1}, z_i - \epsilon, z_{i+1}, \ldots, z_{j-1}, y_l), \end{aligned}$$

with $t_k := y_k - a$, $w_k := b + t_k, \ldots, t_{l-1} := y_{l-1} - a$, $w_{l-1} := b + t_{l-1}$. Hence, we have a configuration $(\ldots, y'_{i-1}, w_k, \ldots, w_{l-1}, y'_j, \ldots, y_{k-1}, z_i - \epsilon, z_{i+1}, \ldots, z_{j-1}, y_l, \ldots)$ with strictly lower energy, which contradicts the fact that $(x_0, \ldots, x_n)$ is minimizing. $\square$

We recall that we have assumed $\inf_{\omega\in\Omega,\ x\in\mathbb{R}} E_\omega(x, x) > \bar{E}$.

Lemma 41. *Let $\omega \in \Omega$. For $n \geq 1$, let $(x_0^n, \ldots, x_n^n)$ be a configuration realizing the minimum of $E_\omega(x_0, \ldots, x_n)$ over all $(x_0, \ldots, x_n)$. Then, $\lim_{n\to+\infty} |x_n^n - x_0^n| = +\infty$.*

Proof. The proof is done by contradiction. Let $\omega \in \Omega$ and $R > 0$. Assume there exist infinitely many n's for which every configuration $(x_0^n, \ldots, x_n^n)$ realizing the minimum of $E_\omega(x_0, \ldots, x_n)$ satisfies $|x_n^n - x_0^n| \leq R$. If $(x_0^n, \ldots, x_n^n)$ is not monotone, thanks to Lemma 23, we can find distinct indices $\{i_0, \ldots, i_r\}$ of $\{0, \ldots, n\}$ such that $i_0 = 0$, $i_r = n$, $(x_{i_0}^n, \ldots, x_{i_r}^n)$ is monotone (possibly not strictly monotone) and

$$E_\omega(x_0^n, \ldots, x_n^n) \geq E_\omega(x_{i_0}^n, \ldots, x_{i_r}^n) + \sum_{i\notin\{i_0,\ldots,i_r\}} E_\omega(x_i^n, x_i^n).$$

Let $\epsilon > 0$ be chosen so that $E_\omega(x, y) \geq \bar{E} + \epsilon$ for every $|y - x| \leq \epsilon$. Thus, if θ_n denotes the number of indices $1 \leq k \leq r$ such that $|x_{i_k}^n - x_{i_{k-1}}^n| > \epsilon$, it is clear that $\theta_n \leq R/\epsilon$. Since

$$n\bar{E} \geq E_\omega(x_0^n, \ldots, x_n^n) \geq (n - \theta_n)(\bar{E} + \epsilon) + \theta_n \inf_{x,y\in\mathbb{R}} E_\omega(x, y),$$

we obtain a contradiction by letting $n \to +\infty$. $\square$

We show in the following proposition that a configuration $(x_0^n, \ldots, x_n^n)$ realizing the minimum of the energy $E_\omega(x_0, \ldots, x_n)$ among all configurations of length n admits a rotation number from below in the sense that

$$\liminf_{n\to+\infty} \frac{|x_n^n - x_0^n|}{n} > 0. \tag{21}$$

This means that, for such a finite minimizing configuration, the average distance between consecutive atoms is bounded from below. The existence of a rotation number for an infinite minimizing configuration $(x_k)_{k\in\mathbb{Z}}$ has been established in [14]. The following proposition extends partially this result in two directions: Firstly, the interaction model is more general, and secondly, whereas in [14] the rotation number is obtained for an infinite configuration, we get the rotation number from below for a sequence of finite configurations.

Proposition 42. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be an almost crystalline interaction model satisfying $\inf_{\omega\in\Omega,\ x\in\mathbb{R}} E_\omega(x,x) > \bar{E}$. Given $\omega \in \Omega$, for $n \geq 1$, suppose $(x_0^n, \ldots, x_n^n)$ is a configuration realizing the minimum of $E_\omega(x_0, \ldots, x_n)$ over all $(x_0, \ldots, x_n)$. Then,*

1. $\bar{E} = \lim_{n\to+\infty} \frac{1}{n} E_\omega(x_0^n, \ldots, x_n^n) = \sup_{n\geq 1} \frac{1}{n} E_\omega(x_0^n, \ldots, x_n^n)$,
2. *for n sufficiently large, $(x_0^n, \ldots, x_n^n)$ is strictly monotone,*
3. *there is $R > 0$ (independent of ω) such that $\sup_{n\geq 1} \sup_{1\leq k\leq n} |x_k^n - x_{k-1}^n| \leq R$,*
4. $\liminf_{n\to+\infty} \frac{1}{n} |x_n^n - x_0^n| > 0$.

Proof. To avoid trivialities, we assume that the flow $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ is not periodic.

Step 1. The first item has been proved in Proposition 13; the limit exists as a supremum by superadditivity. Moreover, from Lemma 41, $|x_n^n - x_0^n| \to +\infty$. From Proposition 39, the configuration $(x_0^n, \ldots, x_n^n)$ must be strictly monotone and have uniformly bounded jumps R. We are left to prove the last item of the proposition.

Step 2. By definition of an almost crystalline interaction model, L is transversally constant with respect to some flow box decomposition $\{U_i\}_{i\in I}$ (Definitions 9 and 10). Let $\{F_\alpha\}_{\alpha\in A}$ be a Kakutani–Rohlin tower that is compatible with respect to $\{U_i\}_{i\in I}$ (Definition 34) and let $\Sigma = \cup_{\alpha\in A}\Sigma_\alpha$ be its basis. We may assume that $\min_{\alpha\in A} H_\alpha$ is as large as we want and, in particular, larger than R (see the construction (20)). We also assume that n is sufficiently large so that every tower F_α of basis Σ_α is completely cut by the trajectory $\tau_t(\omega)$ for $t \in (\min\{x_0^n, x_n^n\}, \max\{x_0^n, x_n^n\})$. We consider ν the transverse measure to Σ (as defined in Lemma 36) and we denote $\nu_\alpha := \nu(\Sigma_\alpha)$.

Step 3. Let $S^n < T^n$ be the two return times to Σ (namely, $\tau_{S^n}(\omega) \in \Sigma$ and $\tau_{T^n}(\omega) \in \Sigma$) that are chosen so that $[S^n, T^n)$ is the smallest interval containing the sequence $(x_k^n)_{k=0}^n$. From the definition of a Kakutani–Rohlin tower, $[S^n, T^n)$ can be written as a disjoint union of intervals of type $I_{\alpha,i} := [t_{\alpha,i}, t_{\alpha,i} + H_\alpha)$, where the list $\{t_{\alpha,i}\}_i$, $i = 1, \ldots, C_\alpha^n$, denotes the successive return times to Σ_α between S^n and T^n. We distinguish two exceptional intervals among this list: the two intervals which contain x_0^n and x_n^n. If $x_0^n < x_n^n$, then $N_{\alpha,i}^n$ denotes the number of points $(x_k^n)_{k=1}^n$ belonging to $I_{\alpha,i}$ and N_α^n denotes the maximum of $N_{\alpha,i}^n$. If $x_n^n < x_0^n$, then $N_{\alpha,i}^n$ and N_α^n are defined similarly by considering in this case $(x_k^n)_{k=0}^{n-1}$. From Lemma 40, we obtain $N_\alpha^n - 2 \leq N_{\alpha,i}^n \leq N_\alpha^n$ for every nonexceptional interval $I_{\alpha,i}$. We show that $\sup_{n\geq 1} N_\alpha^n < +\infty$ for every $\alpha \in A$. The proof is done by contradiction.

Let $E_{\alpha,i}^n$ be the energy of the configuration localized in $I_{\alpha,i}$. More precisely, assume first $x_0^n < x_n^n$; index the part of $(x_k^n)_{k=1}^n$ in $I_{\alpha,i}$ by $(x_{k,\alpha,i}^n)_{k=1}^N$ with $N = N_{\alpha,i}^n$; denote by $x_{0,\alpha,i}^n$ the nearest point strictly smaller than $x_{1,\alpha,i}^n$ and define the partial energy $E_{\alpha,i}^n := E_\omega(x_{0,\alpha,i}^n, \ldots, x_{N,\alpha,i}^n)$. If $x_n^n < x_0^n$, the part of $(x_k^n)_{k=0}^{n-1}$ in $I_{\alpha,i}$ is indexed by $(x_{k,\alpha,i}^n)_{k=0}^{N-1}$ with $N = N_{\alpha,i}^n$; denote by $x_{N,\alpha,i}^n$ the nearest point strictly larger than $x_{N-1,\alpha,i}^n$ and define $E_{\alpha,i}^n$ similarly.

Thanks to the hypothesis $\inf_{x\in\mathbb{R}} E_\omega(x,x) > \bar{E}$, one can choose $\epsilon > 0$ such that $E_\omega(x,y) \geq \bar{E} + \epsilon$ as soon as $|y-x| \leq \epsilon$. Let $\bar{H} := \max_{\alpha\in A} H_\alpha$. Then, if $\theta^n_{\alpha,i}$ denotes the number of consecutive points $x^n_{k,\alpha,i}$ in $I_{\alpha,i}$ satisfying $|x^n_{k,\alpha,i} - x^n_{k-1,\alpha,i}| > \epsilon$, obviously $\theta^n_{\alpha,i} \leq \bar{H}/\epsilon$. Thus, since $n = \sum_{\alpha\in A} \sum_{1\leq i\leq C^n_\alpha} N^n_{\alpha,i}$, we have that

$$\begin{aligned} n\bar{E} &\geq E_\omega(x^n_0,\ldots,x^n_n) = \sum_{\alpha\in A}\ \sum_{1\leq i\leq C^n_\alpha} E^n_{\alpha,i} \\ &\geq \sum_{\alpha\in A}\ \sum_{1\leq i\leq C^n_\alpha} \Big[\theta^n_{\alpha,i} \inf_{x,y\in\mathbb{R}} E_\omega(x,y) + \big(N^n_{\alpha,i} - \theta^n_{\alpha,i}\big)(\bar{E}+\epsilon)\Big] \\ &= n(\bar{E}+\epsilon) + \sum_{\alpha\in A} \sum_{1\leq i\leq C^n_\alpha} \theta^n_{\alpha,i}\underline{E} \geq n(\bar{E}+\epsilon) + \sum_{\alpha\in A} C^n_\alpha \frac{\bar{H}}{\epsilon}\underline{E}, \end{aligned} \tag{22}$$

where $\underline{E} := (\inf_{x,y\in\mathbb{R}} E_\omega(x,y) - \bar{E} - \epsilon) < 0$. For α fixed, among the intervals $(I_{\alpha,i})_i$, $i = 1,\ldots,C^n_\alpha$, at most two of them are exceptional and the other intervals satisfy $N^n_{\alpha,i} \geq N^n_\alpha - 2$. We thus get $n \geq \sum_{\alpha\in A}(C^n_\alpha - 2)(N^n_\alpha - 2)$. For n sufficiently large, we have

$$\frac{C^n_\alpha}{T^n - S^n} \leq (1+\epsilon)\nu_\alpha, \quad \frac{C^n_\alpha - 2}{T^n - S^n} \geq (1-\epsilon)\nu_\alpha \quad \text{and}$$

$$\frac{1}{n}\sum_{\alpha\in A} C^n_\alpha \leq \frac{(1+\epsilon)\sum_{\alpha\in A}\nu_\alpha}{(1-\epsilon)\sum_{\alpha\in A}\nu_\alpha(N^n_\alpha - 2)}.$$

If $N^n_\alpha \to +\infty$ for some α and a subsequence $n \to +\infty$, then $\frac{1}{n}\sum_{\alpha\in A} C^n_\alpha \to 0$ and we obtain a contradiction with the previous inequality (22).

Step 4. For every α, $I_{\alpha,i} \subset [x^n_0, x^n_n]$ except maybe for at most two of them. Then,

$$\frac{|x^n_n - x^n_0|}{n} \geq \frac{\sum_{\alpha\in A}(C^n_\alpha - 2)H_\alpha}{\sum_{\alpha\in A} C^n_\alpha N^n_\alpha}.$$

Denote $\bar{N}_\alpha := \limsup_{n\to+\infty} N^n_\alpha$. From step 3, we know that $\bar{N}_\alpha < +\infty$. By dividing by $(T^n - S^n)$ and by letting $n \to +\infty$, we obtain

$$\liminf_{n\to+\infty} \frac{|x^n_n - x^n_0|}{n} \geq \frac{\sum_{\alpha\in A}\nu_\alpha H_\alpha}{\sum_{\alpha\in A}\nu_\alpha \bar{N}_\alpha} = \frac{1}{\sum_{\alpha\in A}\nu_\alpha\bar{N}_\alpha} > 0.$$

□

Now we are able to prove Theorem 12. Thanks to Theorem 8 and the above results, we only have to show that the intersection of each $\{\tau_t\}_t$-orbit with the projected Mather set is a nonempty relatively dense subset of the orbit.

Proof of Theorem 12. Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be an almost crystalline interaction model. We discuss two cases.

Case 1. Either $\inf_{\omega\in\Omega}\inf_{x\in\mathbb{R}} E_\omega(x,x) = \bar{E}$. Then, $E_{\omega_*}(x_*,x_*) = \bar{E}$ for some ω_* and x_*. By hypothesis, L is transversally constant with respect to a flow box decomposition $\{U_i = \tau(B_{R_i} \times \Xi_i)\}_{i\in I}$. Let $i \in I$ be such that $\tau_{x_*}(\omega_*) \in U_i$. Let $|t_i| < R_i$ and $\omega_i \in \Xi_i$ be such that $\tau_{x_*}(\omega_*) = \tau_{t_i}(\omega_i)$. Then,

$$\bar{E} = E_{\omega_*}(x_*, x_*) = E_{\omega_i}(t_i, t_i) = E_{\omega}(t_i, t_i), \quad \forall\, \omega \in \Xi_i.$$

We have just proved that $\delta_{(\tau_{t_i}(\omega),0)}$ is a minimizing measure for every $\omega \in \Xi_i$. The projected Mather set contains $\tau_{t_i}(\Xi_i)$. By minimality of the flow, we have $\Omega = \tau(B_R \times \Xi_i)$, for some $R > 0$, thanks to item 1 of Lemma 33. The projected Mather set thus meets every sufficiently long orbit of the flow.

Case 2. Or $\inf_{\omega\in\Omega} \inf_{x\in\mathbb{R}} E_\omega(x,x) > \bar{E}$. Proposition 42 shows that, if $\omega_* \in \Omega$ has been fixed, if for every $n \geq 1$ a sequence $(x_k^n)_{0\leq k<n}$ of points of $\mathbb{R}$ realizing the minimum $E_{\omega_*}(x_0^n, \ldots, x_n^n) = \min_{x_0,\ldots,x_n} E_{\omega_*}(x_0, \ldots, x_n)$ has been fixed, then

- $\bar{E} = \lim_{n\to+\infty} \frac{1}{n} E_{\omega_*}(x_0^n, \ldots, x_n^n)$,
- $(x_k^n)_{0\leq k<n}$ is strictly monotone for n large enough,
- there is $R > 0$ (independent of ω_*) such that $\sup_{n\geq 1}\ \sup_{1\leq k\leq n} |x_k^n - x_{k-1}^n| < 2R$,
- $\rho := \liminf_{n\to+\infty} \frac{1}{n}|x_n^n - x_0^n| > 0$.

Let μ_{n,ω_*} be the probability measure on $\Omega \times \mathbb{R}$ defined by

$$\mu_{n,\omega_*} := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{(\tau_{x_k^n}(\omega_*),\, x_{k+1}^n - x_k^n)}.$$

Notice that $\int L\, d\mu_{n,\omega_*} = \frac{1}{n} E_{\omega_*}(x_0^n, \ldots, x_n^n)$. Since the consecutive jumps of x_k^n are uniformly bounded, the sequence of measures $(\mu_{n,\omega_*})_{n\geq 1}$ is tight. By taking a subsequence, we may assume that $\mu_{n,\omega_*} \to \mu_\infty$ with respect to the weak topology. Moreover, μ_∞ is holonomic and minimizing. Let $\Xi \subset \Omega$ be a transverse section of a flow box $\tau(B_R \times \Xi)$. Let $\mathcal{R}_\Xi(\omega_*)$ be the set of return times to Ξ as defined in Lemma 36. Let $pr^1 : \Omega \times \mathbb{R} \to \Omega$ be the first projection. Then,

$$\begin{aligned} pr^1_*(\mu_{n,\omega_*})(\tau(B_R \times \Xi)) &= \frac{1}{n} \#\big\{k : x_k^n \in \cup_{t\in\mathcal{R}_\Xi(\omega_*)} B_R(t)\big\} \\ &\geq \frac{1}{n} \#(B_{T_n}(c_n) \cap \mathcal{R}_\Xi(\omega_*)), \end{aligned}$$

with $T_n := \frac{1}{2}|x_n^n - x_0^n|$ and $c_n := \frac{1}{2}(x_0^n + x_n^n)$. The previous inequality comes from the fact that the intervals $B_R(t)$ are disjoint and contain at least one x_k^n. Then,

$$pr^1_*(\mu_{n,\omega_*})(\tau(B_R \times \Xi)) \geq \frac{2T_n}{n}\ \frac{\#(B_{T_n}(0) \cap \mathcal{R}_\Xi(\tau_{c_n}(\omega_*))}{\mathrm{Leb}(B_{T_n}(0))}.$$

By taking the limit as $n \to +\infty$, one obtains $pr^1_*(\mu_\infty)(\overline{\tau(B_R \times \Xi)}) \geq \rho\nu_\Xi(\Xi) > 0$. Therefore, since Ξ is arbitrary, every orbit of the flow of length $2R$ meets the projected Mather set. $\square$

Appendix A. The Ergodic and Sup-Inf Formulas

We give a second proof of the equality $\bar{K} = \bar{L}$ in Proposition 13. We will use basic properties of the Kantorovich–Rubinstein topology on the set of probabilities measures on a Polish space (Z, d) and a version of the Topological Minimax Theorem which is a generalization of Sion's classical result [25]. For a recent review on the last topic, see [26]. We state a particular case of theorem 5.7 there.

Theorem A.1 (Topological Minimax Theorem [26]). *Let X and Y be Hausdorff topological spaces. Let $F(x, y) : X \times Y \to \mathbb{R}$ be a real-valued function. Define $\eta := \sup_{y\in Y} \inf_{x\in X} F(x, y)$ and assume there exists a real number $\alpha^* > \eta$ such that*

1. *$\forall\, \alpha \in (\eta, \alpha^*)$, for every finite set $\emptyset \neq H \subset Y$, $\cap_{y\in H}\{x \in X : F(x, y) \leq \alpha\}$ is either empty or connected;*
2. *$\forall\, \alpha \in (\eta, \alpha^*)$, for every set $K \subset X$, $\cap_{x\in K}\{y \in Y : F(x, y) > \alpha\}$ is either empty or connected;*
3. *for any $y \in Y$ and $x \in X$, $F(x, y)$ is lower semi-continuous in x and upper semi-continuous in y;*
4. *there exists a finite set $M \subset Y$ such that $\cap_{y\in M}\{x \in X : F(x, y) \leq \alpha^*\}$ is compact and nonempty.*

Then,

$$\inf_{x\in X} \sup_{y\in Y} F(x, y) = \sup_{y\in Y} \inf_{x\in X} F(x, y).$$

We recall basic facts on the Kantorovich–Rubinstein topology (see [28] or [1]). Given a Polish space Z and a point $z_0 \in Z$, let us consider the set of probability measures on the Borel sets of Z that admit a finite first moment, i.e.,

$$\mathcal{P}^1(Z) = \left\{\mu : \int_Z d(z_0, z)\, d\mu(z) < +\infty\right\}.$$

Notice that this set does not depend on the choice of the point z_0. The Wasserstein distance or Kantorovitch–Rubinstein distance on $\mathcal{P}^1(Z)$ is a distance between two probabilities $\mu, \nu \in \mathcal{P}^1(Z)$ defined by

$$W_1(\mu, \nu) := \inf\left\{\int_{Z\times Z} d(x, y)\, d\gamma(x, y) :\ \gamma \in \Gamma(\mu, \nu)\right\},$$

where $\Gamma(\mu, \nu)$ denotes the set of all the probability measures γ on $Z \times Z$ with marginals μ and ν on the first and second factors, respectively.

Recall that a continuous function $L \colon Z \to \mathbb{R}$ is said to be superlinear on a Polish space Z if the map defined by $z \in Z \mapsto L(z)/\big(1 + d(z, z_0)\big) \in \mathbb{R}$ is proper. Notice that this definition is also independent of the choice of z_0 and, by considering the distance $\hat{d} := \min(d, 1)$ on Z, any proper function is superlinear for $\hat{d}$. The following lemma is easy to prove and gives us a sufficient condition for relative compactness in $\mathcal{P}^1(Z)$ (see theorem 6.9 in [28] or [1] for a more detailed discussion).

Lemma A.2. *Let Z be a Polish space, $L : Z \to \mathbb{R}$ be a continuous function, and $X := \{\mu \in \mathcal{P}^1(Z) : \int L\, d\mu < +\infty\}$ be equipped with the Kantorovich–Rubinstein distance. Then*

1. *the map $\mu \in X \mapsto \int L\, d\mu$ is lower semi-continuous;*
2. *if L is a superlinear, then, for every $\alpha \in \mathbb{R}$, the set $\{\mu \in X : \int L\, d\mu \leq \alpha\}$ is compact (the map $\mu \in X \mapsto \int L\, d\mu$ is proper).*

Second Proof of $\bar{K} = \bar{L}$ in Proposition 13. Lemma A.2 applied to the C^0 superlinear Lagrangian $L : \Omega \times \mathbb{R}^d \to \mathbb{R}$ guarantees the existence of a minimizing probability for L. This minimizing measure is holonomic since the set of holonomic measures is a closed subset of $\mathcal{P}^1(\Omega \times \mathbb{R}^d)$ for the Kantorovich–Rubinstein distance. Notice that, for every $u \in C^0(\Omega)$,

$$\begin{aligned}\inf_{\omega\in\Omega,\ t\in\mathbb{R}^d}(L+u-u\circ\tau)(\omega,t) &= \inf_{\omega\in\Omega,\ t\in\mathbb{R}^d}\int (L+u-u\circ\tau)\, d\delta_{(\omega,t)} \\ &\geq \inf_{\mu\in\mathcal{P}^1(\Omega\times\mathbb{R}^d)}\int (L+u-u\circ\tau)\, d\mu \\ &\geq \inf_{\omega\in\Omega,\ t\in\mathbb{R}^d}(L+u-u\circ\tau)(\omega,t).\end{aligned}$$

Let $X := \{\mu \in \mathcal{P}^1(\Omega \times \mathbb{R}^d) : \int L\, d\mu < +\infty\}$ and $Y := C^0(\Omega)$. Then

$$\bar{K} = \sup_{u\in Y}\inf_{\mu\in X}\int (L+u-u\circ\tau)\, d\mu \leq \min_{\omega\in\Omega} L(\omega,0).$$

Define $\alpha^* := \min_{\omega\in\Omega} L(\omega,0)+1 > \bar{K}$ and

$$F : (\mu,u) \in X\times Y \mapsto \int (L+u-u\circ\tau)\, d\mu.$$

Since F is affine in both variables, it satisfies items 1 and 2 of Theorem A.1. 3 is also satisfied since $F(\mu,u)$ is lower semi-continuous in μ and continuous in u. By taking $M = \{0\}$, the singleton set reduced to the null function in Y, the set $\cap_{u\in M}\{\mu \in X : F(\mu,u) \leq \alpha^*\}$ is compact and nonempty, so that item 4 is satisfied. The Topological Minimax Theorem therefore implies

$$\bar{K} = \inf_{\mu\in X}\sup_{u\in Y}\int (L+u-u\circ\tau)\, d\mu. \tag{A.1}$$

We show that every $\mu \in X$ such that $\sup_{u\in Y}\int (L+u-u\circ\tau)\, d\mu < +\infty$ is holonomic. If not, there would exist a function $u \in C^0(\Omega)$ such that $\int (u-u\circ\tau)\, d\mu > 0$. Multiplying $(u-u\circ\tau)$ by a positive scalar λ and letting $\lambda \to +\infty$ would lead to a contradiction. Thus, the infimum in (“Appendix A.1”) may be taken over holonomic measures with respect to which L is integrable. We finally conclude that

$$\bar{K} = \inf_{\mu\in X}\sup_{u\in Y}\int (L+u-u\circ\tau)\, d\mu = \inf_{\mu\in\mathbb{M}_{hol}}\int L\, d\mu = \bar{L}.$$

□

References

[1] Ambrosio, L., Gigli, N., Savaré, G.: Gradient flows in metric spaces and in the space of probability measures. Lectures in mathematics ETH Zürich. Birkhäuser, Basel (2005)

[2] Aubry, S., Le Daeron, P.Y.: The discrete Frenkel–Kontorova model and its extensions: I. Exact results for the ground states. Phys. D **8**, 381–422 (1983)

[3] Auslander, L., Hahn, F.: Real functions coming from flows on compact spaces and concepts of almost periodicity. Trans. Am. Math. Soc. **106**, 415–426 (1963)

[4] Bellissard, J., Benedetti, R., Gambaudo, J.M.: Spaces of tilings, finite telescopic approximations and gap-labeling. Commun. Math. Phys. **261**, 1–41 (2006)

[5] Davini, A., Siconolfi, A.: Exact and approximate correctors for stochastic Hamiltonians: the 1-dimensional case. Math. Ann. **345**, 749–782 (2009)

[6] Davini, A., Siconolfi, A.: Metric techniques for convex stationary ergodic Hamiltonians. Calc. Var. Partial Differ. Equ. **40**, 391–421 (2011)

[7] Davini, A., Siconolfi, A.: Weak KAM theory topics in the stationary ergodic setting. Calc. Var. Partial Differ. Equ. **44**, 319–350 (2012)

[8] de la Llave, R., Su, X.: KAM theory for quasi-periodic equilibria in one-dimensional quasi-periodic media. SIAM J. Math. Anal. **44**, 3901–3927 (2012)

[9] Fathi, A.: Solutions KAM faibles conjuguées et barrières de Peierls. Comptes Rendus des Séances de l'Académie des Sciences, Série I, Mathématique **325**, 649–652 (1997)

[10] Fathi, A.: The weak KAM theorem in Lagrangian dynamics. Cambridge University Press **(in press)**

[11] Frenkel, Ya I.: On the theory of plastic deformation and twinning I. Zh. Eksp. Teor. Fiz. **8**, 89–95 (1938)

[12] Frenkel, Ya I., Kontorova, T.A.: On the theory of plastic deformation and twinning II. Zh. Eksp. Teor. Fiz. **8**, 1340–1349 (1938)

[13] Frenkel, Ya I., Kontorova, T.A.: On the theory of plastic deformation and twinning III. Zh. Eksp. Teor. Fiz. **8**, 1349–1359 (1938)

[14] Gambaudo, J.M., Guiraud, P., Petite, S.: Minimal configurations for the Frenkel–Kontorova model on a quasicrystal. Commun. Math. Phys. **265**, 165–188 (2006)

[15] Garibaldi, E., Thieullen, Ph: Minimizing orbits in the discrete Aubry–Mather model. Nonlinearity **24**, 563–611 (2011)

[16] Gomes, D.A.: Viscosity solution methods and the discrete Aunbry–Mather problem. Discrete Contin. Dyn. Syst. Ser. A **13**, 103–116 (2005)

[17] Gomes, D.A.: Generalized Mather problem and selection principles for viscosity solutions and Mather measures. Adv. Calc. Var. **1**, 291–307 (2008)

[18] Gomes, D.A., Oliveira, E.R.: Mather problem and viscosity solutions in the stationary setting. São Paulo J. Math. Sci. **6**, 301–334 (2012)

[19] Kellendonk, J.: Pattern-equivariant functions and cohomology. J. Phys. A Math. Gen. **36**, 5765–5772 (2003)

[20] Kellendonk, J., Putnam, I.F.: Tilings, C^*-algebras and K-theory. In: Baake, M., Moody, R.V. (eds.) Directions in Mathematical Quasicrystals, CRM Monograph Series 13, pp. 177–206. AMS, Providence (2000)

[21] Lagarias, J.C., Pleasants, P.A.B.: Repetitive Delone sets and quasicrystals. Ergod. Theory Dyn. Syst. **23**, 831–867 (2003)

[22] Lions, P.L., Souganidis, P.E.: Correctors for the homogenization of Hamilton–Jacobi equations in the stationary ergodic setting. Commun. Pure Appl. Math. **56**, 1501–1524 (2003)

[23] Mañé, R.: Generic properties and problems of minimizing measures of Lagrangian systems. Nonlinearity **9**, 273–310 (1996)

[24] Mather, J.N.: Existence of quasiperiodic orbits for twist homeomorphisms of the annulus. Topology **21**, 457–467 (1982)

[25] Sion, M.: On general minimax theorems. Pac. J. Math. **8**, 171–176 (1958)

[26] Tuy, H.: Topological minimax theorems: old and new. Vietnam J. Math. **40**, 391–405 (2012)

[27] van Erp, T.S.: Frenkel–Kontorova model on quasi-periodic substrate potential. PhD thesis, Katholieke Universiteit Nijmegen (1999)

[28] Villani, C.: Optimal transport: old and new. Grundlehren der mathematischen Wissenschaften **338**. Springer, Berlin (2008)

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