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Eigenfunctions of the Laplacian and associated Ruelle operator

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Abstract
Let Γ be a co-compact Fuchsian group of isometries on the Poincaré disk $\mathbb{D}$ and Δ the corresponding hyperbolic Laplace operator. Any smooth eigenfunction f of Δ, equivariant by Γ with real eigenvalue $\lambda = -s(1-s)$, where $s = \frac{1}{2}+\mathrm{i}t$, admits an integral representation by a distribution $\mathcal{D}_{f,s}$ (the Helgason distribution) which is equivariant by Γ and supported at infinity $\partial\mathbb{D} = \mathbb{S}^1$. The geodesic flow on the compact surface $\mathbb{D}/\Gamma$ is conjugate to a suspension over a natural extension of a piecewise analytic map $T : \mathbb{S}^1 \to \mathbb{S}^1$, the so-called Bowen–Series transformation. Let $\mathcal{L}_s$ be the complex Ruelle transfer operator associated with the Jacobian $-s \ln |T'|$. Pollicott showed that $\mathcal{D}_{f,s}$ is an eigenfunction of the dual operator $\mathcal{L}_s^*$ for the eigenvalue 1. Here we show the existence of a (nonzero) piecewise real analytic eigenfunction $\psi_{f,s}$ of $\mathcal{L}_s$ for the eigenvalue 1, given by an integral formula

$$\psi_{f,s}(\xi) = \int \frac{J(\xi,\eta)}{|\xi-\eta|^{2s}}\, \mathcal{D}_{f,s}\,(\mathrm{d}\eta),$$

where $J(\xi,\eta)$ is a $\{0,1\}$-valued piecewise constant function whose definition depends upon the geometry of the Dirichlet fundamental domain representing the surface $\mathbb{D}/\Gamma$.

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1. Introduction

Consider the Laplace operator Δ defined by

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

on the Lobatchevskii upper half-plane $\mathbb{H} = \{w = x + \mathrm{i}y \in \mathbb{C};\ y > 0\}$, equipped with the hyperbolic metric $\mathrm{d}s_{\mathbb{H}} = \frac{|\mathrm{d}w|}{y}$, and the eigenvalue problem

$$\Delta f = -s(1-s)f,$$

where s is of the form $s = \frac{1}{2} + \mathrm{i}t$, with t real. We shall also consider the same corresponding Laplace operator

$$\Delta = \frac{1}{4}(1-|z|^2)^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

and eigenvalue problem

$$\Delta f = -s(1-s)f,$$

defined on the Poincaré disk $\mathbb{D} = \{z = x + y\mathrm{i} \in \mathbb{C};\ |z| < 1\}$, equipped with the metric $\mathrm{d}s_{\mathbb{D}} = 2\frac{|\mathrm{d}z|}{1-|z|^2}$.

Helgason showed in [11] and [12] that any eigenfunction f associated with this eigenvalue problem can be obtained by means of a generalized Poisson representation

$$\begin{cases} f(w) = \displaystyle\int_{-\infty}^{\infty}\left(\frac{(1+t^2)y}{(x-t)^2+y^2}\right)^s \mathcal{D}^{\mathbb{H}}_{f,s}(t), & \text{for } w \in \mathbb{H}, \\ \text{or} & \\ f(z) = \displaystyle\int_{\partial\mathbb{D}}\left(\frac{1-|z|^2}{|z-\xi|^2}\right)^s \mathcal{D}^{\mathbb{D}}_{f,s}(\xi), & \text{for } z \in \mathbb{D}, \end{cases}$$

where $\mathcal{D}^{\mathbb{D}}_{f,s}$ or $\mathcal{D}^{\mathbb{H}}_{f,s}$ are analytic distributions called from now on *Helgason's distributions*. We have used the canonical isometry between $z \in \mathbb{D}$ and $w \in \mathbb{H}$, namely $w = \mathrm{i}\frac{1-z}{1+z}$ or $z = \frac{\mathrm{i}-w}{\mathrm{i}+w}$. The hyperbolic metric is given in $\mathbb{H}$ and in $\mathbb{D}$ by

$$\mathrm{d}s^2_{\mathbb{H}} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}, \qquad \mathrm{d}s^2_{\mathbb{D}} = \frac{4(\mathrm{d}x^2 + \mathrm{d}y^2)}{(1-|z|^2)^2}.$$

We shall be interested in a more restricted problem, where the eigenfunction f is also automorphic with respect to a co-compact Fuchsian group Γ, i.e. a discrete subgroup of the group of Möbius transformations (see [5, 20, 25]) with compact fundamental domain. It is known that the eigenvalues $\lambda = s(1-s) = \frac{1}{4} + t^2$ form a discrete set of positive real numbers with finite multiplicity and accumulating at $+\infty$ (see [13]).

Pollicott showed [21] that Helgason's distribution can be seen as a generalized eigenmeasure of the dual complex Ruelle transfer operator associated with a subshift of the finite type defined at infinity. Let T_{L} be the left Bowen–Series transformation that acts on the boundary $\mathbb{S}^1 = \partial\mathbb{D}$ and is associated with a particular set of generators of Γ. The precise definition of T_{L} has been given in [8, 22–24], and more geometrical descriptions have then been given in [1, 18]. Specific examples of the Bowen–Series transformation have been studied in [4, 17] for the modular surface and in [3] for a symmetric compact fundamental domain of genus two. The map T_{L} is known to be piecewise Γ-Möbius constant, Markovian with respect to a partition $\{I^{\mathrm{L}}_k\}$ of intervals of $\mathbb{S}^1$, on which the restriction of T_{L} is constant and equal to an element γ_k of Γ, transitive and orbit equivalent to Γ. Let $\mathcal{L}^{\mathrm{L}}_s$ be the *complex Ruelle transfer operator* associated with the map T_{L} and the potential $A_{\mathrm{L}} = -s\ln|T'_{\mathrm{L}}|$, namely

$$(\mathcal{L}^{\mathrm{L}}_s\psi)(\xi') = \sum_{T_{\mathrm{L}}(\xi)=\xi'} \mathrm{e}^{A_{\mathrm{L}}(\xi)}\psi(\xi) = \sum_{T_{\mathrm{L}}(\xi)=\xi'} \frac{\psi(\xi)}{|T'_{\mathrm{L}}(\xi)|^s},$$

where the summation is taken over all preimages ξ of ξ' under T_{L}. Here T'_{L} denotes the Jacobian of T_{L} with respect to the canonical Lebesgue measure on $\mathbb{S}^1$. In the case of an automorphic eigenfunction f of Δ, Pollicott showed that the corresponding Helgason distribution $\mathcal{D}_{f,s}$ satisfies the dual functional equation

$$(\mathcal{L}_s^{\mathrm{L}})^*(\mathcal{D}_{f,s}) = \mathcal{D}_{f,s}$$

or, according to Pollicott's terminology, the parameter s is a *(dual) Perron–Frobenius value*, that is, 1 is an eigenvalue for the dual Ruelle transfer operator.

Although suggested in [21], it is not clear whether s could be a *Perron–Frobenius value*, that is, whether 1 could also be an eigenvalue for $\mathcal{L}_s^{\mathrm{L}}$, not only for $(\mathcal{L}_s^{\mathrm{L}})^*$. Our goal in this paper is to show that this is actually the case.

The three main ingredients we use are the following:

- Otal's proof of Helgason's distribution in [19], giving more precise information on $\mathcal{D}_{f,s}$ and enabling us to integrate piecewise $\mathcal{C}^1$ test functions, instead of real analytic globally defined test functions;
- a more careful reading of [1, 8, 18, 24], or a careful study of a particular example in [16], which enables us to construct a piecewise Γ-Möbius baker transformation ('arithmetically' conjugate to the geodesic billiard);
- the existence of a kernel that we introduced in [3], which enables us to permute past and future coordinates and transfer a dual eigendistribution to a piecewise real analytic eigenfunction. Haydn (in [10]) has introduced a similar kernel in a more abstract setting, without geometric considerations.

More precisely, we prove the following theorem:

Theorem 1. *Let Γ be a co-compact Fuchsian group of the hyperbolic disk $\mathbb{D}$ and Δ the corresponding hyperbolic Laplace operator. Let $\lambda = s(1-s)$, with $s = \frac{1}{2} + \mathrm{i}t$, and let f be an eigenfunction of $-\Delta$, automorphic with respect to Γ, that is, $\Delta f = -\lambda f$ and $f \circ \gamma = f$, for every $\gamma \in \Gamma$. Then there exists a (nonzero) piecewise real analytic eigenfunction $\psi_{f,s}$ on $\mathbb{S}^1$ that is a solution of the functional equation*

$$\mathcal{L}_s^{\mathrm{L}}(\psi_{f,s}) = \psi_{f,s},$$

where $\mathcal{L}_s^{\mathrm{L}}$ is the complex Ruelle transfer operator associated with the left Bowen–Series transformation $T_{\mathrm{L}} : \mathbb{S}^1 \to \mathbb{S}^1$ and the potential $A_{\mathrm{L}} = -s \ln |T_{\mathrm{L}}'|$.

Moreover, $\psi_{f,s}$ admits an integral representation via Helgason's distribution $\mathcal{D}_{f,s}^{\mathbb{D}}$, representing f at infinity, and a geometric positive kernel $k(\xi, \eta)$ defined on a finite set of disjoint rectangles $\cup_k I_k^{\mathrm{L}} \times Q_k^{\mathrm{R}} \subset \mathbb{S}^1 \times \mathbb{S}^1$, namely,

$$\psi_{f,s}(\xi) = \int_{Q_k^{\mathrm{R}}} k^s(\xi,\eta)\, \mathcal{D}_{f,s}^{\mathbb{D}}(\eta) = \int_{Q_k^{\mathrm{R}}} \frac{1}{|\xi - \eta|^{2s}}\, \mathcal{D}_{f,s}^{\mathbb{D}}(\eta),$$

for every $\xi \in I_k^{\mathrm{L}}$, where I_k^{L} and Q_k^{R} are intervals of $\mathbb{S}^1$ with disjoint closure, and $\{I_k^{\mathrm{L}}\}_k$ is a partition of $\mathbb{S}^1$ where T_{L} is injective, Markovian and piecewise Γ-Möbius constant.

Lewis [14] and, later, Lewis and Zagier [15], started a different approach to understand Maass wave forms. They were able to identify in a bijective way Maass wave forms of $PSL(2, \mathbb{Z})$ and solutions of a functional equation with three terms closely related to Mayer's transfer operator. Their setting is strongly dependent on the modular group. Our theorem 1 may be viewed as part of their programme for co-compact Fuchsian groups. The Helgason distribution has been used by Zelditch in [26] to generalize microlocal analysis on hyperbolic surfaces, by Flaminio and Forni in [9], to study invariant distributions by the horocycle flow, and by Anantharaman and Zelditch in [2], to understand the 'quantum unique ergodicity conjecture'.

2. Preliminary results

Let Γ be a co-compact Fuchsian group of the Poincaré disk $\mathbb{D}$. We denote by $\mathrm{d}(w, z)$ the hyperbolic distance between two points of $\mathbb{D}$, given by the Riemannian metric $\mathrm{d}s^2 = 4(\mathrm{d}x^2 + \mathrm{d}y^2)/(1 - |z|^2)^2$. Let $M = \mathbb{D}/\Gamma$ be the associated compact Riemann surface, $N = T^1M$ the unit tangent bundle and Δ the Laplace operator on M. Let $f : M \to \mathbb{R}$ be an eigenfunction of $-\Delta$ or, in other words, a Γ-automorphic function $f : \mathbb{D} \to \mathbb{R}$ satisfying $\Delta f = -s(1 - s)f$ for the eigenvalue $\lambda = s(1 - s) > \frac{1}{4}$ and such that $f \circ \gamma = f$, for every $\gamma \in \Gamma$. We know that f is $\mathcal{C}^\infty$ and uniformly bounded on $\mathbb{D}$. Thanks to Helgason's representation theorem, f can be represented as a superposition of horocycle waves, given by the *Poisson kernel*

$$P(z, \xi) := \mathrm{e}^{b_\xi(\mathcal{O},z)} = \frac{1 - |z|^2}{|z - \xi|^2},$$

where $b_\xi(w, z)$ is the *Busemann cocycle* between two points w and z inside the Poincaré disk, observed from a point at infinity $\xi \in \mathbb{S}^1$, defined by

$$b_\xi(w, z) := \text{‘}\mathrm{d}(w, \xi) - \mathrm{d}(z, \xi)\text{’} = \lim_{t \to \xi} \mathrm{d}(w, t) - \mathrm{d}(z, t),$$

where the limit is uniform in $t \to \xi$ in any hyperbolic cone at ξ. Helgason's theorem states that

$$f(z) = \int_{\mathbb{D}} P^s(z, \xi)\, \mathcal{D}_{f,s}(\xi) = \langle \mathcal{D}_{f,s}, P^s(z, .)\rangle$$

for some analytic distribution $\mathcal{D}_{f,s}$ acting on real analytic functions on $\mathbb{S}^1$. Unfortunately, Helgason's work is too general and is valid for any eigenfunction not necessarily equivariant by a group. For bounded $\mathcal{C}^2$ functions f, Otal [19] has shown that the distribution $\mathcal{D}_{f,s}$ has stronger properties and can be defined in a simpler manner.

We first recall some standard notation in hyperbolic geometry. We call $\mathrm{d}(z, z_0)$ the hyperbolic distance between two points: for instance, the distance from the origin is given by $\mathrm{d}(\mathcal{O}, \tanh(\frac{r}{2})\mathrm{e}^{\mathrm{i}\theta}) = r$. Let $\mathcal{C}(\mathcal{O}, r)$ denote the set of points in $\mathbb{D}$ at hyperbolic distance r from the origin,

$$\mathcal{C}(\mathcal{O}, r) = \{z \in \mathbb{D};\ |z| = \tanh(\tfrac{r}{2})\}$$

and, more generally, given any interval I at infinity and any point $z_0 \in \mathbb{D}$, let $\mathcal{C}(z_0, r, I)$ denote the angular arc at the hyperbolic distance r from z_0 delimited at infinity by I, that is,

$$\mathcal{C}(z_0, r, I) = \{z \in \mathbb{D};\ z \in [[z_0, \xi]] \text{ for some } \xi \in I \text{ and } \mathrm{d}(z, z_0) = r\},$$

where $[[z_0, \xi]]$ denotes the geodesic ray from z_0 to the point ξ at infinity. Let $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$ denote the exterior normal derivative to $\mathcal{C}(\mathcal{O}, r)$ and $|\mathrm{d}z|_{\mathbb{D}} = \sinh(r)\, \mathrm{d}\theta$ the hyperbolic arc length on $\mathcal{C}(\mathcal{O}, r)$.

Theorem 2 ([19]). *Let f be a bounded $\mathcal{C}^2$ eigenfunction satisfying $\Delta f = -s(1-s)f$. Then:*

1. *There exists a continuous linear functional $\mathcal{D}_{f,s}$ acting on $\mathcal{C}^1$ functions of $\mathbb{S}^1$, defined by*
$$\int \psi(\xi)\, \mathcal{D}_{f,s}(\xi) := \lim_{r\to+\infty} \frac{1}{c(s)} \int_{\mathcal{C}(\mathcal{O},r)} \psi(z)\mathrm{e}^{-sr} \left(\frac{\partial f}{\partial n} + sf\right) |\mathrm{d}z|_{\mathbb{D}},$$
where $c(s)$ is a nonzero normalizing constant such that $\langle \mathcal{D}_{f,s}, \mathbf{1}\rangle = f(0)$, and $\psi(z)$ is any $\mathcal{C}^1$ extension of $\psi(\xi)$ to a neighbourhood of $\mathbb{S}^1$.
2. *$\mathcal{D}_{f,s}$ represents f in the following sense:*
$$f(z) = \int \left[P(z, \xi)\right]^s \mathcal{D}_{f,s}(\xi), \qquad \forall\, z \in \mathbb{D}.$$
$\mathcal{D}_{f,s}$ is unique and is called the Helgason distribution of f.

3. *For all* $0 \leqslant \alpha \leqslant 2\pi$*, the following limit exists:*

$$\tilde{\mathcal{D}}_{f,s}(\alpha) := \lim_{r\to+\infty} \frac{1}{c(s)} \int_0^\alpha \mathrm{e}^{-sr} \left(\frac{\partial f}{\partial n} + sf\right) \left(\tanh\left(\frac{r}{2}\right) \mathrm{e}^{\mathrm{i}\theta}\right) \sinh(r) \mathrm{d}\theta.$$

The convergence is uniform in $\alpha \in [0, 2\pi]$ *and* $\tilde{\mathcal{D}}_{f,s}(0) = 0$.

4. $\tilde{\mathcal{D}}_{f,s}$ *can be extended to* $\mathbb{R}$ *as a* $\frac{1}{2}$*-Hölder continuous function satisfying:*
   (a) $\tilde{\mathcal{D}}_{f,s}(\theta + 2\pi) = \tilde{\mathcal{D}}_{f,s}(\theta) + f(0)$*, for every* $\theta \in \mathbb{R}$,
   (b) *for any* $\mathcal{C}^1$ *function* $\psi : \mathbb{S}^1 \to \mathbb{C}$*, denoting* $\tilde{\psi}(\theta) = \psi(\exp \mathrm{i}\theta)$*,*

$$\int \psi(\xi)\, \mathcal{D}_{f,s}(\xi) = \tilde{\psi}(0) f(0) - \int_0^{2\pi} \frac{\partial \tilde{\psi}}{\partial \theta} \tilde{\mathcal{D}}_{f,s}(\theta)\, \mathrm{d}\theta.$$

Using similar technical tools as Otal, one can prove the following extension of $\mathcal{D}_{f,s}$ on piecewise $\mathcal{C}^1$ functions, that is, on functions not necessarily continuous but which admit a $\mathcal{C}^1$ extension on each interval $[\xi_k, \xi_{k+1}]$ of some finite and ordered subdivision $\{\xi_0, \xi_1, \ldots, \xi_{r-1}\}$ of $\mathbb{S}^1$.

Proposition 3. *Let* f *and* $\mathcal{D}_{f,s}$ *be as in theorem 2.*

1. *For any interval* $I \subset \mathbb{S}^1$ *and any function* $\psi : I \to \mathbb{C}$*, which is* $\mathcal{C}^1$ *on the closure of* I *and null outside* I*, the following limit exists:*

$$\int \psi(\xi)\, \mathcal{D}_{f,s}(\xi) := \frac{1}{c(s)} \lim_{r\to+\infty} \int_{\mathcal{C}(\mathcal{O},r,I)} \psi(z) \mathrm{e}^{-sr} \left(\frac{\partial f}{\partial n} + sf\right) |\mathrm{d}z|_{\mathbb{D}},$$

where again $\psi(z)$ *is any* $\mathcal{C}^1$ *extension of* $\psi(\xi)$ *to a neighbourhood of* $\mathbb{S}^1$.

2. *For any* $0 \leqslant \alpha < \beta \leqslant 2\pi$ *and any* $\mathcal{C}^1$ *function* ψ *on the interval* $I = [\exp(\mathrm{i}\alpha), \exp(\mathrm{i}\beta)]$*,*

$$\int \psi(\xi)\, \mathcal{D}_{f,s}(\xi) = \tilde{\psi}(\beta)\tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\psi}(\alpha)\tilde{\mathcal{D}}_{f,s}(\alpha) - \int_\alpha^\beta \frac{\partial \tilde{\psi}}{\partial \theta} \tilde{\mathcal{D}}_{f,s}(\theta)\, \mathrm{d}\theta,$$

where $\tilde{\mathcal{D}}_{f,s}$ *and* $\tilde{\psi}(\theta)$ *have been defined in theorem 2.*

Proof. Given $\alpha \in [0, 2\pi]$, let $I = \{\mathrm{e}^{\mathrm{i}\theta} \mid 0 \leqslant \theta \leqslant \alpha\}$ be an interval in S^1, and ψ a $\mathcal{C}^1$ function defined on a neighbourhood of $\mathbb{S}^1$. Denote $\tilde{\psi}(r, \theta) = \psi(\tanh(\frac{r}{2})\mathrm{e}^{\mathrm{i}\theta})$ and $K(r, \theta) = \mathrm{e}^{-sr}\left(\frac{\partial f}{\partial n} + sf\right)\left(\tanh(\frac{r}{2}\mathrm{e}^{\mathrm{i}\theta}\right) \sinh(r)$. Then

$$\begin{aligned}
\frac{1}{c(s)} \int_{\mathcal{C}(\mathcal{O},r,I)} & \psi(z) \mathrm{e}^{-sr} \left(\frac{\partial f}{\partial n} + sf\right) |\mathrm{d}z|_{\mathbb{D}} \\
&= \int_0^\alpha \tilde{\psi}(r, \beta) K(r, \beta)\, \mathrm{d}\beta \\
&= \int_0^\alpha \left[\tilde{\psi}(r, \alpha) + \int_\beta^\alpha -\frac{\partial \tilde{\psi}}{\partial \theta}(r, \theta)\, \mathrm{d}\theta\right] K(r, \beta)\, \mathrm{d}\beta \\
&= \tilde{\psi}(r, \alpha) \int_0^\alpha K(r, \beta)\, \mathrm{d}\beta - \int_0^\alpha \frac{\partial \tilde{\psi}}{\partial \theta}(r, \theta) \left[\int_0^\theta K(r, \beta)\, \mathrm{d}\beta\right] \mathrm{d}\theta.
\end{aligned}$$

Since $\int_0^\alpha K(r, \beta)\, \mathrm{d}\beta \to \tilde{\mathcal{D}}_{f,s}(\alpha)$ uniformly in $\alpha \in [0, 2\pi]$, the left-hand side of the previous equality converges to

$$\int \psi(\xi) \mathbf{1}_{\{\xi\in I\}} \mathcal{D}_{f,s}(\xi) = \tilde{\psi}(\alpha)\tilde{\mathcal{D}}_{f,s}(\alpha) - \int_0^\alpha \frac{\partial \tilde{\psi}}{\partial \theta}(\theta) \tilde{\mathcal{D}}_{f,s}(\theta)\, \mathrm{d}\theta.$$

The second part of the proposition follows subtracting such an expression from another one, such as:

$$\int \psi(\xi)\mathbf{1}_{\{\xi=\mathrm{e}^{\mathrm{i}\theta};\,0\leqslant\theta\leqslant\beta\}}\tilde{\mathcal{D}}_{f,s}(\xi) - \int \psi(\xi)\mathbf{1}_{\{\xi=\mathrm{e}^{\mathrm{i}\theta};\,0\leqslant\theta\leqslant\alpha\}}\tilde{\mathcal{D}}_{f,s}(\xi). \qquad \blacksquare$$

If, in addition, we assume that f is equivariant with respect to a co-compact Fuchsian group Γ, Pollicott observed in [21] that $\mathcal{D}_{f,s}$, acting on real analytic functions, is equivariant by Γ, that is, satisfies $\gamma^*(\mathcal{D}_{f,s})(\xi) = |\gamma'(\xi)|^s\mathcal{D}_{f,s}(\xi)$, for all $\gamma \in \Gamma$. Because Otal's construction is more precise and implies that Helgason's distribution also acts on piecewise $\mathcal{C}^1$ functions, the above equivariance property can be improved in the following way.

Proposition 4. *Let $f : \mathbb{D} \to \mathbb{R}$ be a $\mathcal{C}^2$ function, $I \subset \mathbb{S}^1$ an interval and $\psi : I \to \mathbb{C}$ a $\mathcal{C}^1$ function on the closure of I. If f satisfies $f \circ \gamma = f$, for some $\gamma \in \Gamma$ (f is not necessarily automorphic), then*

$$\langle \mathcal{D}_{f,s}, \frac{\psi\circ\gamma^{-1}}{|\gamma'\circ\gamma^{-1}|^s}\mathbf{1}_{\gamma(I)}\rangle = \langle \mathcal{D}_{f,s}, \psi\mathbf{1}_I\rangle.$$

The main difficulty here is to transfer the equivariance property $f\circ\gamma = f$ to an equivalent property for the extension of $\mathcal{D}_{f,s}$ to piecewise $\mathcal{C}^1$ functions. If $I = \mathbb{S}^1$ and ψ is real analytic, then, by uniqueness of the representation, proposition 4 is easily proved. It seems that just knowing the fact that $\mathcal{D}_{f,s}$ is the derivative of some Hölder function is not enough to reach a conclusion. The following proof uses Otal's approach and, essentially, the extension of $\mathcal{D}_{f,s}$ described in part 1 of proposition 3.

Proof of proposition 4. First we prove the proposition for $\psi = 1$. Let $g(z) = \exp(-s\mathrm{d}(\mathcal{O}, z))$. By the definition of $\mathcal{D}_{f,s}$, we obtain

$$\begin{aligned}\int \mathbf{1}_I(\xi)\,\mathcal{D}_{f,s}(\xi) &= \lim_{r\to+\infty}\frac{1}{c(s)}\int_{\mathcal{C}(\mathcal{O},r',I)}\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)|\mathrm{d}z|_{\mathbb{D}} \\ &= \lim_{r\to+\infty}\frac{1}{c(s)}\int_{\mathcal{C}(\mathcal{O}',r',\gamma(I))}\left(g'\frac{\partial f}{\partial n} - f\frac{\partial g'}{\partial n}\right)|\mathrm{d}z|_{\mathbb{D}},\end{aligned}$$

where $r' = r + \mathrm{d}(\mathcal{O}, \mathcal{O}')$, $\mathcal{O}' = \gamma(\mathcal{O})$ and $g' = g\circ\gamma^{-1}$. Notice that the domain bounded by the circle $\mathcal{C}(\mathcal{O}', r')$ contains the circle $\mathcal{C}(\mathcal{O}, r)$. Let $\overline{PQ}$ be the positively oriented arc $\mathcal{C}(\mathcal{O}, r, \gamma(I))$ and $\overline{P'Q'}$ be the arc $\mathcal{C}(\mathcal{O}', r', \gamma(I))$. Then the two geodesic segments $[[P, P']]$ and $[[Q, Q']]$ belong to the annulus $r \leqslant \mathrm{d}(z, \mathcal{O}) \leqslant r + 2\mathrm{d}(\mathcal{O}, \mathcal{O}')$ and their length is uniformly bounded.

We now use Green's formula to compute the right-hand side of the above expression. Let Ω denote the domain delimited by P, P', Q', Q using the corresponding arcs and geodesic segments, and let $\mathrm{d}v = \sinh(r)\,\mathrm{d}r\,\mathrm{d}\theta$ be the hyperbolic volume element. We obtain

$$\begin{aligned}\int_{\overline{P'Q'}}\left(g'\frac{\partial f}{\partial n} - f\frac{\partial g'}{\partial n}\right)|\mathrm{d}z|_{\mathbb{D}} &= \int_{\overline{PQ}}\left(g'\frac{\partial f}{\partial n} - f\frac{\partial g'}{\partial n}\right)|\mathrm{d}z|_{\mathbb{D}} \\ &\quad - \int_{[[P,P']]}\cdots|\mathrm{d}z|_{\mathbb{D}} - \int_{[[Q',Q]]}\cdots|\mathrm{d}z|_{\mathbb{D}} + \iint_{\Omega}\left(g'\Delta f - f\Delta g'\right)\mathrm{d}v.\end{aligned}$$

When r tends to infinity, the last three terms at the right-hand side tend to 0, since along the geodesic segments $[P, P']$ and $[Q, Q']$, the gradient $\nabla g'$ is uniformly bounded by $\exp(-\frac{1}{2}r)$ and

$$g'\Delta f - f\Delta g' = sg'f\sinh(\mathrm{d}(z, \mathcal{O}'))^{-2} \qquad \text{and} \qquad \frac{\partial}{\partial n}g' + sg'$$

are uniformly bounded by a constant times $\exp(-\frac{5}{2}r)$ in the domain Ω, for the first expression, and by a constant times $\exp(-\frac{3}{2}r)$ on $\mathcal{C}(\mathcal{O}, r)$, for the second expression. It follows that

$$\begin{aligned}\int \mathbf{1}_I(\xi)\,\mathcal{D}_{f,s}(\xi) &= \lim_{r\to+\infty}\frac{1}{c(s)}\int_{\mathcal{C}(\mathcal{O},r,\gamma(I))} g'\left(\frac{\partial f}{\partial n}+sf\right)|\mathrm{d}z|_{\mathbb{D}}\\ &= \lim_{r\to+\infty}\frac{1}{c(s)}\int_{\mathcal{C}(\mathcal{O},r,\gamma(I))}\left[\psi(z)\right]^s \mathrm{e}^{-sr}\left(\frac{\partial f}{\partial n}+sf\right)|\mathrm{d}z|_{\mathbb{D}},\end{aligned}$$

where $\psi(z) = \exp\left(\mathrm{d}(\mathcal{O}, z) - \mathrm{d}(\mathcal{O}, \gamma^{-1}(z))\right)$. Now we observe that

$$\begin{cases}\psi(z) = \exp s\,(\mathrm{d}(\mathcal{O}, z) - \mathrm{d}(\gamma(\mathcal{O}), z))\,, & \text{for } z\in\mathbb{D},\\ \psi(\xi) = \exp b_\xi(\mathcal{O}, \gamma(\mathcal{O})) = |\gamma'\circ\gamma^{-1}(\xi)|^{-1}, & \text{for } \xi\in\partial\mathbb{D},\end{cases}$$

actually coincides with a real analytic function $\Psi(z)$ defined in a neighbourhood of $\mathbb{S}^1$, given explicitly by

$$\Psi(z) = \left(\frac{(1+|z|)^2}{(1+|\gamma^{-1}(z)|)^2|\gamma'\circ\gamma^{-1}(z)|}\right)^s.$$

Thus we have proved that

$$\int \mathbf{1}_I(\xi)\,\mathcal{D}_{f,s}(\xi) = \int \frac{\mathbf{1}_{\gamma(I)}(\xi)}{|\gamma'\circ\gamma^{-1}(\xi)|^s}\,\mathcal{D}_{f,s}(\xi).$$

Now we prove the general case. We use the same notation for the lifting $\gamma : \mathbb{R} \mapsto \mathbb{R}$ of a Möbius transformation $\gamma : \mathbb{S}^1 \mapsto \mathbb{S}^1$. The lifting satisfies $\gamma(\alpha + 2\pi) = \gamma(\alpha) + 2\pi$, $\exp(\mathrm{i}\gamma(\alpha)) = \gamma(\exp(\mathrm{i}\alpha))$ and $\gamma'(\alpha) = |\gamma'(\alpha)|$, for all $\alpha\in\mathbb{R}$. Using proposition 3, we obtain

$$\begin{aligned}&\tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\mathcal{D}}_{f,s}(\alpha)\\ &\quad= \frac{\tilde{\mathcal{D}}_{f,s}\circ\gamma(\beta)}{\gamma'(\beta)^s} - \frac{\tilde{\mathcal{D}}_{f,s}\circ\gamma(\alpha)}{\gamma'(\alpha)^s} - \int_{\gamma(\alpha)}^{\gamma(\beta)}\frac{\partial}{\partial\theta}\left(\frac{1}{(\gamma'\circ\gamma^{-1}(\theta))^s}\right)\tilde{\mathcal{D}}_{f,s}(\theta)\,\mathrm{d}\theta.\end{aligned}$$

For any $\mathcal{C}^1$ function $\psi(\xi)$ defined on I, we denote $\tilde{\psi}(\theta) = \psi(\exp\mathrm{i}\theta)$, and obtain

$$\begin{aligned}LHS &:= \int \psi(\xi)\mathbf{1}_I(\xi)\,\mathcal{D}_{f,s}(\xi)\\ &= \tilde{\psi}(\beta)\tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\psi}(\alpha)\tilde{\mathcal{D}}_{f,s}(\alpha) - \int_\alpha^\beta \frac{\partial\tilde{\psi}}{\partial\theta}\tilde{\mathcal{D}}_{f,s}(\theta)\,\mathrm{d}\theta\\ &= \tilde{\psi}(\beta)\tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\psi}(\alpha)\tilde{\mathcal{D}}_{f,s}(\alpha) - \int_{\gamma(\alpha)}^{\gamma(\beta)}\frac{\partial}{\partial\theta}\left(\tilde{\psi}\circ\gamma^{-1}(\theta)\right)\tilde{\mathcal{D}}_{f,s}(\gamma^{-1}\theta)\,\mathrm{d}\theta\\ &= \tilde{\psi}(\beta)\left(\tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\mathcal{D}}_{f,s}(\alpha)\right) - \int_{\gamma(\alpha)}^{\gamma(\beta)}\frac{\partial\tilde{\psi}(\gamma^{-1}\theta)}{\partial\theta}\left(\tilde{\mathcal{D}}_{f,s}(\gamma^{-1}\theta) - \tilde{\mathcal{D}}_{f,s}(\alpha)\right)\mathrm{d}\theta.\end{aligned}$$

We now use the above equivariance and replace both $\tilde{\mathcal{D}}_{f,s}(\beta) - \tilde{\mathcal{D}}_{f,s}(\alpha)$ and $\tilde{\mathcal{D}}_{f,s}(\gamma^{-1}\theta) - \tilde{\mathcal{D}}_{f,s}(\alpha)$ by the corresponding formula involving $\tilde{\mathcal{D}}_{f,s}\circ\gamma(\beta)$, $\tilde{\mathcal{D}}_{f,s}\circ\gamma(\alpha)$, $\tilde{\mathcal{D}}_{f,s}(\theta)$. Thus

$$\begin{aligned}LHS &= \frac{\tilde{\psi}(\beta)\tilde{\mathcal{D}}_{f,s}\circ\gamma(\beta)}{\gamma'(\beta)^s} - \frac{\tilde{\psi}(\alpha)\tilde{\mathcal{D}}_{f,s}\circ\gamma(\alpha)}{\gamma'(\alpha)^s} - \int_{\gamma(\alpha)}^{\gamma(\beta)}\frac{\partial}{\partial\theta}\left(\frac{\tilde{\psi}(\gamma^{-1}\theta)}{\gamma'(\gamma^{-1}\theta)^s}\right)\tilde{\mathcal{D}}_{f,s}(\theta)\,\mathrm{d}\theta\\ &= \int\frac{\psi\circ\gamma^{-1}(\xi)}{|\gamma'\circ\gamma^{-1}(\xi)|^s}\mathbf{1}_{\gamma(I)}\,\mathcal{D}_{f,s}(\xi).\end{aligned}$$

■

Following [1,8,18,22–24] for the general case and [16] for a specific example we recall the definition of the left T_{L} and right T_{R} Bowen–Series transformations. The hyperbolic surface we are interested in is given by the quotient of the hyperbolic disk $\mathbb{D}$ by a co-compact Fuchsian group Γ. Given a point $\mathcal{O} \in \mathbb{D}$, let

$$D_{\Gamma,\mathcal{O}} = \{z \in \mathbb{D}; \mathrm{d}(z, \mathcal{O}) < \mathrm{d}(z, \gamma(\mathcal{O})), \quad \forall\, \gamma \in \Gamma\}$$

denote the corresponding Dirichlet domain, a convex fundamental domain with compact closure in $\mathbb{D}$, admitting an even number of geodesic sides and an even number of vertices, some of which may be elliptic. More precisely, the boundary of $D_{\Gamma,\mathcal{O}}$ is a disjoint union of semi-closed geodesic segments $S^{\mathrm{L}}_{-r}, \cdots, S^{\mathrm{L}}_{-1}, S^{\mathrm{L}}_{1}, \cdots, S^{\mathrm{L}}_{r}$, closed to the left and open to the right, or, equivalently, to a union of semi-closed geodesic segments $S^{\mathrm{R}}_{-r}, \cdots, S^{\mathrm{R}}_{-1}, S^{\mathrm{R}}_{1}, \cdots, S^{\mathrm{R}}_{r}$, closed to the right and open to the left; for each k, the intervals S^{L}_{k} and S^{R}_{k} have the same endpoints and S^{L}_{k} is associated with S^{R}_{-k} by an element $a_k \in \Gamma$ satisfying $a_k(S^{\mathrm{L}}_{k}) = S^{\mathrm{R}}_{-k}$. The elements a_k generate Γ and satisfy $a_{-k} = a_k^{-1}$, for $k = \pm 1, \cdots, \pm r$.

To define the two Bowen–Series transformations T_{L} and T_{R} geometrically, we need to impose a geometric condition on Γ: following [8, 22, 24], we say that Γ satisfies the *even corner* property if, for each $1 \leqslant |k| \leqslant r$, the complete geodesic line through S^{L}_{k} is equal to a disjoint union of Γ-translates of the sides S^{L}_{l}, with $1 \leqslant |l| \leqslant r$. Some Γ do not satisfy this geometric property. Nevertheless, any two co-compact Fuchsian groups Γ and Γ', with identical signature, are geometrically isomorphic, that is, there exists a group isomorphism $h_* : \Gamma \to \Gamma'$ and a quasi-conformal orientation preserving homeomorphism $h : \mathbb{D} \to \mathbb{D}$ admitting an extension to a conjugating homeomorphism $h : \partial\mathbb{D} \to \partial\mathbb{D}$, that is,

$$h(\gamma(z)) = h_*(\gamma)(h(z)), \qquad \forall\, \gamma \in \Gamma.$$

An important observation in [8, 22, 24] is that any co-compact Fuchsian group is geometrically isomorphic to a Fuchsian group with identical signature and satisfying the *even corner* property. We are going to recall the Bowen and Series construction in the case that Γ possesses the *even corner* property and will show that their main conclusions remain valid under geometric isomorphisms.

The complete geodesic line associated with a side S^{L}_{k} cuts the boundary at infinity $\mathbb{S}^1$ at two points s^{L}_{k} and s^{R}_{k}, positively oriented with respect to s^{L}_{k}, the oriented geodesic line $]]s^{\mathrm{L}}_{k}, s^{\mathrm{R}}_{k}[[$ seeing the origin $\mathcal{O}$ to the left. Both end points s^{L}_{k} and s^{R}_{k} are neutrally stable with respect to the associated generator a_k, that is, $|a_k'(s^{\mathrm{L}}_{k})| = |a_k'(s^{\mathrm{R}}_{k})| = 1$. The family of open intervals $]s^{\mathrm{L}}_{k}, s^{\mathrm{R}}_{k}[$ covers $\mathbb{S}^1$; since these intervals $]s^{\mathrm{L}}_{k}, s^{\mathrm{R}}_{k}[$ overlap each other, there is no canonical partition adapted to this covering. Nevertheless, we may associate two well-defined partitions, the left partition $\mathcal{A}_{\mathrm{L}}$ and the right partition $\mathcal{A}_{\mathrm{R}}$. The former consists of disjoint half-closed intervals,

$$\mathcal{A}_{\mathrm{L}} = \{A^{\mathrm{L}}_{-r}, \cdots, A^{\mathrm{L}}_{-1}, A^{\mathrm{L}}_{1}, \cdots, A^{\mathrm{L}}_{r}\},$$

given by $A^{\mathrm{L}}_{k} = [s^{\mathrm{L}}_{k}, s^{\mathrm{L}}_{l(k)}[$ where $s^{\mathrm{L}}_{l(k)}$ denotes the nearest point s^{L}_{l} after s^{L}_{k}, according to a positive orientation. Each A^{L}_{k} belongs to the unstable domain of the hyperbolic element a_k, that is, $|a_k'(\xi)| \geqslant 1$, for each $\xi \in A^{\mathrm{L}}_{k}$. By definition, the left Bowen–Series transformation $T_{\mathrm{L}} : \mathbb{S}^1 \mapsto \mathbb{S}^1$ is given by

$$T_{\mathrm{L}}(\xi) = a_k(\xi), \qquad \text{if } \xi \in A^{\mathrm{L}}_{k}.$$

Analogously, $\mathbb{S}^1$ can be partitioned into half-closed intervals

$$\mathcal{A}_{\mathrm{R}} = \{A^{\mathrm{R}}_{-r}, \cdots, A^{\mathrm{R}}_{-1}, A^{\mathrm{R}}_{1}, \cdots, A^{\mathrm{R}}_{r}\},$$

where $A^{\mathrm{R}}_k =]s^{\mathrm{R}}_{j(k)}, s^{\mathrm{R}}_k]$, and $s^{\mathrm{R}}_{j(k)}$ denotes the nearest s^{R}_j before s^{R}_k, according to a positive orientation. The right Bowen–Series transformation is given by

$$T_{\mathrm{R}}(\eta) = a_k(\eta), \qquad \text{if } \eta \in A^{\mathrm{R}}_k.$$

The two partitions A^{L} and A^{R} generate two ways of coding a trajectory. Let $\gamma_{\mathrm{L}} : \mathbb{S}^1 \mapsto \Gamma$ and $\gamma_{\mathrm{R}} : \mathbb{S}^1 \mapsto \Gamma$ be the left and right symbolic coding defined by

$$\gamma_L[\xi] = a_k, \qquad \text{if } \xi \in A^{\mathrm{L}}_k, \qquad \text{and} \qquad \gamma_{\mathrm{R}}[\eta] = a_k, \qquad \text{if } \eta \in A^{\mathrm{R}}_k.$$

In particular, $T_{\mathrm{R}}(\eta) = \gamma_{\mathrm{R}}[\eta](\eta)$ and $T_{\mathrm{L}}(\xi) = \gamma_{\mathrm{L}}[\xi](\xi)$, for each $\xi \in \mathbb{S}^1$. Also, it is known that T^2_{R} and T^2_{L} are expanding. Series, in [22–24], and later, Adler and Flatto in [1], proved that T_{L} (respectively, T_{R}) is Markov with respect to a partition of $\mathcal{I}^{\mathrm{L}} = \{I^{\mathrm{L}}_k\}^q_{k=1}$ (respectively, $\mathcal{I}^{\mathrm{R}} = \{I^{\mathrm{R}}_l\}^q_{l=1}$) that is finer than $\mathcal{A}_{\mathrm{L}}$ (respectively, $\mathcal{A}_{\mathrm{R}}$). The semi-closed intervals I^{L}_k and I^{R}_l are of the same kind as A^{L}_k and A^{R}_l, and have the same closure.

Definition 5. *A dynamical system* $(\mathbb{S}^1, T, \{I_k\})$ *is said to be a piecewise* Γ*-Möbius Markov transformation if* $T : \mathbb{S}^1 \to \mathbb{S}^1$ *is a surjective map, and* $\{I_k\}$ *is a finite partition of* $\mathbb{S}^1$ *into intervals such that:*

1. *for each* k, $T(I_k)$ *is a union of adjacent intervals* I_l*;*
2. *for each* k, *the restriction of* T *to* I_k *coincides with an element* $\gamma_k \in \Gamma$*;*
3. *some finite iterate of* T *is uniformly expanding.*

Theorem 6 ([8,24]). *For any co-compact Fuchsian group* Γ*, there exists a piecewise* Γ*-Möbius Markov transformation* $(\mathbb{S}^1, T, \{I_k\})$ *which is transitive and orbit equivalent to* Γ.

The Ruelle transfer operator can be defined for any piecewise $\mathcal{C}^2$ Markov transformation $(\mathbb{S}^1, T, \{I_k\})$ and any potential function A. Actually, we need a particular complex transfer operator given by the potential

$$A = -s \ln |T'|.$$

For any function $\psi : \mathbb{S}^1 \to \mathbb{C}$, define

$$(\mathcal{L}_s(\psi))(\xi') = \sum_{T(\xi)=\xi'} \mathrm{e}^{A(\xi)} \psi(\xi) = \sum_{T(\xi)=\xi'} \frac{\psi(\xi)}{|T'(\xi)|^s},$$

where the summation is taken over all preimages ξ of ξ' under T. We modify $\mathcal{L}_s$ slightly, so that it acts on the space of piecewise $\mathcal{C}^1$ functions. Let $\{I_k\}^q_{k=1}$ be a partition of S^1. Given a piecewise C^1 function and $\oplus^q_{k=1}\psi_k \in \oplus^q_{k=1}\mathcal{C}^1(\bar{I}_k)$ set

$$\mathcal{L}^{\mathrm{L}}_s \psi = \oplus^q_{l=1}\phi_l, \qquad \text{where } \phi_l = \sum_{I_l \subset T(I_k)} \frac{\psi_k \circ T^{-1}_{k,l}}{|T' \circ T^{-1}_{k,l}|^s},$$

and $T^{-1}_{k,l}$ denotes the restriction to I_l of the inverse of $T : I_k \to T(I_k) \supset I_l$.

Proposition 7. *Let* Γ *be a co-compact Fuchsian group. Let* $s = \frac{1}{2} + \mathrm{i}t$ *and* f *be an automorphic eigenfunction of* $-\Delta$*, that is,* $\Delta f = -s(1-s) f$*. Let* $(\mathbb{S}^1, T, \{I_k\})$ *be a piecewise* Γ*-Möbius Markov transformation and* $\mathcal{L}_s$ *be the Ruelle transfer operator corresponding to the observable* $A = -s \ln |T'|$*. Then the Helgason distribution* $\mathcal{D}_{f,s}$ *satisfies*

$$(\mathcal{L}_s)^* \mathcal{D}_{f,s} = \mathcal{D}_{f,s}.$$

Proof. Let $\oplus_{k=1}^{q}\psi_k$ be a piecewise $\mathcal{C}^1$ function in $\oplus_{k=1}^{q}\mathcal{C}^1(\bar{I}_k)$. Using proposition 4,

$$\begin{aligned}\int (\mathcal{L}_s\psi)(\xi)\,\mathcal{D}_{f,s}(\xi) &= \sum_{l=1}^{q}\int_{I_l}(\mathcal{L}_s\psi)_l(\xi)\,\mathcal{D}_{f,s}(\xi)\\ &= \sum_{T(I_k)\supset I_l}\int_{I_l}\frac{\psi_k\circ T_{k,l}^{-1}}{|T'\circ T_{k,l}^{-1}|^s}(\xi)\,\mathcal{D}_{f,s}(\xi)\\ &= \sum_{T(I_k)\supset I_l}\int_{T^{-1}(I_l)\cap I_k}\psi_k(\xi)\,\mathcal{D}_{f,s}(\xi)\\ &= \sum_{k=1}^{q}\int_{I_k}\psi_k(\xi)\,\mathcal{D}_{f,s}(\xi) = \int \psi(\xi)\,\mathcal{D}_{f,s}(\xi). \qquad\blacksquare\end{aligned}$$

Series in [24], Adler and Flatto in [1] and Morita in [18] noticed that T_{L} admits a natural extension $\hat{T}:\hat{\Sigma}\mapsto\hat{\Sigma}$ strongly related to T_{R}. We also showed the existence of such a $\hat{T}$ in [16], and it was an important step in the proof of theorem 3 of [16]. The following definition explains how the two maps T_{L} and T_{R} are glued together in an abstract way.

Definition 8. *Let Γ be a co-compact Fuchsian group. A dynamical system $(\hat{\Sigma},\hat{T},\{I_k^{\mathrm{L}}\},\{I_l^{\mathrm{R}}\},J)$ is said to be a piecewise Γ-Möbius baker transformation if it admits a description as follows.*

1. *$\{I_k^{\mathrm{L}}\}$ and $\{I_l^{\mathrm{R}}\}$ are finite partitions of $\mathbb{S}^1$ into disjoint intervals; $J(k,l)$ is a $\{0,1\}$-valued function, and $\hat{\Sigma}$ is the subset of $\mathbb{S}^1\times\mathbb{S}^1$ defined by*
$$\hat{\Sigma} = \coprod_{J(k,l)=1} I_k^{\mathrm{L}}\times I_l^{\mathrm{R}}.$$
2. *For each k, $Q_k^{\mathrm{R}} = \coprod\{I_l^{\mathrm{R}};\,J(k,l)=1\}$ is an interval whose closure is disjoint from $\bar{I}_k^{\mathrm{L}}$. For each l, $Q_l^{\mathrm{L}} = \coprod\{I_k^{\mathrm{L}};\,J(k,l)=1\}$ is an interval whose closure is disjoint from $\bar{I}_l^{\mathrm{R}}$. Let $I^{\mathrm{L}}(\xi)=I_k^{\mathrm{L}}$ and $Q^{\mathrm{R}}(\xi)=Q_k^{\mathrm{R}}$, for $\xi\in I_k^{\mathrm{L}}$. Let $I^{\mathrm{R}}(\eta)=I_l^{\mathrm{R}}$ and $Q^{\mathrm{L}}(\eta)=Q_l^{\mathrm{L}}$, for $\eta\in I_l^{\mathrm{R}}$.*
3. *$\hat{T}:\hat{\Sigma}\to\hat{\Sigma}$ is bijective and is given by*
$$\begin{cases}\hat{T}(\xi,\eta) = (T_{\mathrm{L}}(\xi), S_{\mathrm{R}}(\xi,\eta)),\\ \hat{T}^{-1}(\xi',\eta') = (S_{\mathrm{L}}(\xi',\eta'), T_{\mathrm{R}}(\eta')),\end{cases}$$
for certain maps $T_{\mathrm{L}},T_{\mathrm{R}}:\mathbb{S}^1\to\mathbb{S}^1$ and $S_{\mathrm{L}},S_{\mathrm{R}}:\hat{\Sigma}\to\mathbb{S}^1$.
4. *$(\mathbb{S}^1,T_{\mathrm{L}},\{I_k^{\mathrm{L}}\})$ and $(\mathbb{S}^1,T_{\mathrm{R}},\{I_l^{\mathrm{R}}\})$ are piecewise Γ-Möbius Markov transformations. There exist two functions $\gamma_{\mathrm{L}}:\mathbb{S}^1\to\Gamma$, respectively $\gamma_{\mathrm{R}}:\mathbb{S}^1\to\Gamma$, that are piecewise constant on each I_k^{L}, respectively $\{I_l^{\mathrm{R}}\}$, and satisfying*
$$\begin{cases}\hat{T}(\xi,\eta) = (\gamma_{\mathrm{L}}[\xi](\xi), \gamma_{\mathrm{L}}[\xi](\eta)),\\ \hat{T}^{-1}(\xi',\eta') = (\gamma_{\mathrm{R}}[\eta'](\xi'), \gamma_{\mathrm{R}}[\eta'](\eta')).\end{cases}$$

The maps T_{L} and T_{R} are called the left and right Bowen–Series transformations, whereas γ_{L} and γ_{R} are the left and right Bowen–Series codings. Finally, we say that J is the incidence matrix, which we extend as a function on $\mathbb{S}^1\times\mathbb{S}^1$ defining

$$\begin{cases}J(\xi,\eta)=1, & \text{if } (\xi,\eta)\in\hat{\Sigma},\\ J(\xi,\eta)=0, & \text{if } (\xi,\eta)\notin\hat{\Sigma}.\end{cases}$$

Notice that this definition is equivariant by geometric isomorphisms. For co-compact Fuchsian groups satisfying the *even corner* property, Adler and Flatto in [1], Series in [24] (and, for a particular example, in [16]) obtained geometrically the existence of a piecewise Γ-Möbius baker transformation with left T_{L} and right T_{R} maps orbit equivalent to Γ. By geometric isomorphism considerations, we obtain more generally the following.

Proposition 9 ([1,16,24]). *For any co-compact Fuchsian group Γ, there exists a piecewise Γ-Möbius baker transformation with left and right Bowen–Series transformations that are transitive and orbit equivalent to Γ.*

The two maps T_{L} and T_{R} are related to the action of the group Γ on the boundary $\mathbb{S}^1$. The baker transformation $(\hat{\Sigma}, \hat{T})$ encodes this action into a unique dynamical system. For later reference, we state two further properties of this encoding.

Remark 10.

1. The two codings γ_{L} and γ_{R} are reciprocal, in the following sense:
$$\gamma_{\mathrm{R}}[\eta'] = \gamma_{\mathrm{L}}^{-1}[\xi], \qquad \text{whenever } (\xi', \eta') = \hat{T}(\xi, \eta).$$
2. For any ξ' and η in $\mathbb{S}^1$, there is a bijection between the two finite sets
$$\{\xi;\, (\xi, \eta) \in \hat{\Sigma} \text{ and } T_{\mathrm{L}}(\xi) = \xi'\}, \qquad \{\eta';\, (\xi', \eta') \in \hat{\Sigma} \text{ and } T_{\mathrm{R}}(\eta') = \eta\}.$$

In order to better understand this baker transformation, we briefly explain how $(\hat{\Sigma}, \hat{T})$ is conjugate to a specific Poincaré section of the geodesic flow on the surface $N = T^1 M$. We assume for the rest of this section that Γ satisfies the *even corner* property.

Since $D_{\Gamma,\mathcal{O}}$ is a convex fundamental domain, every geodesic (modulo Γ) cuts $\partial D_{\Gamma,\mathcal{O}}$ at two distinct points p and q, unless the geodesic is tangent to one of the sides of $D_{\Gamma,\mathcal{O}}$. These tangent geodesics correspond to a finite union of closed geodesics. We could have parametrized the set of oriented geodesics by all pairs $(p, q) \in \partial D_{\Gamma,\mathcal{O}} \times \partial D_{\Gamma,\mathcal{O}}$, with p and q not belonging to the same side of $D_{\Gamma,\mathcal{O}}$, but we prefer to introduce the space X of all $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1$ oriented geodesics $[[y, x]]$, either cutting the interior of $D_{\Gamma,\mathcal{O}}$ or passing through one of the corners of $D_{\Gamma,\mathcal{O}}$ and seeing $\mathcal{O}$ to the left. Using this notation, we define the two intersection points $p = p(x, y) \in \partial D_{\Gamma,\mathcal{O}}$ and $q = q(x, y) \in \partial D_{\Gamma,\mathcal{O}}$ for every oriented geodesic $[[y, x]]$, $(x, y) \in X$, such that $[[q, p]] = [[y, x]] \cap \bar{D}_{\Gamma,\mathcal{O}}$ has the same orientation as $[[y, x]]$.

For a geodesic passing through a corner, $p = q$, unless the geodesic is tangent to a side of $D_{\Gamma,\mathcal{O}}$. We are now in a position to define a geometric Poincaré section $B : X \to X$. If $(x, y) \in X$, the geodesic $[[y, x]]$ leaves $D_{\Gamma,\mathcal{O}}$ at $p = p(x, y) \in S_i$, for some side S_i^{L}. Since S_i^{L} and S_{-i}^{R} are permuted by the generator a_i, the new geodesic $a_i([[y, x]]) = [[y', x']]$ enters again the fundamental domain at a new point $q' = q(x', y')$ with $q' = a_i(p) \in S_{-i}^{\mathrm{R}}$. By definition, $B(x, y) = (x', y')$ and the map $B : X \to X$ is called a geodesic billiard, such as the codings for T_{L} and T_{R}, we introduce two geometric codings $\gamma_B : X \to \Gamma$ and $\bar{\gamma}_B : X \to \Gamma$ given by

$$\begin{cases} \gamma_B[x, y] = a_i & \text{if } p(x, y) \in S_i^{\mathrm{L}}, \\ \bar{\gamma}_B[x, y] = a_i & \text{if } q(x, y) \in S_i^{\mathrm{R}}. \end{cases}$$

Now the geodesic billiard can be defined by

$$\begin{cases} B(x, y) = (\gamma_B[x, y](x), \gamma_B[x, y](y)), \\ B^{-1}(x', y') = (\bar{\gamma}_B[x', y'](x'), \bar{\gamma}_B[x', y'](y')). \end{cases}$$

Notice that $\bar{\gamma}_B \circ B = \gamma_B^{-1}$. The map B is very close to being a baker transformation: B and B^{-1} have the same structure as $\hat{T}$ and $\hat{T}^{-1}$, and γ_B (respectively, $\bar{\gamma}_B$) plays the role of γ_{L} (respectively, γ_{R}). The main difference is that $\gamma_B[x, y]$ depends on both x and y, but $\gamma_{\mathrm{L}}[\xi]$ depends only on ξ. Nevertheless, we have the following crucial result.

Theorem 11 ([1,16,24]). *There exists a* Γ*-Möbius baker transformation* $(\hat{\Sigma}, \hat{T})$ *conjugate to* (X, B)*. More precisely, there exists a map* $\rho : X \to \Gamma$ *such that* $\pi(x, y) = (\rho[x, y](x), \rho[x, y](y))$*, defines a conjugating map* $\pi : X \to \hat{\Sigma}$ *between* $\hat{T}$ *and* B*, such that* $\hat{T} \circ \pi = \pi \circ B$*. Equivalently,* $\gamma_{\mathrm{L}} \circ \pi$ *and* γ_B *are cohomologous over* (X, B)*, that is,* $\gamma_{\mathrm{L}} \circ \pi\rho = \rho \circ B\gamma_B$*, and* $\gamma_{\mathrm{R}} \circ \pi$ *and* $\bar{\gamma}_B$ *are cohomologous over* (X, B)*, that is,* $\gamma_{\mathrm{R}} \circ \pi\rho = \rho \circ B^{-1}\bar{\gamma}_B$*.*

3. Proof of theorem 1

We want to associate with any eigenfunction f of the Laplace operator a nonzero piecewise real analytic function $\psi_{f,s}$ that is a solution of the functional equation

$$\mathcal{L}_s^{\mathrm{L}}(\psi_{f,s}) = \psi_{f,s}, \qquad \text{where } \mathcal{L}_s^{\mathrm{L}}(\psi)(\xi') = \sum_{T_{\mathrm{L}}(\xi)=\xi'} \frac{\psi(\xi)}{|T_{\mathrm{L}}'(\xi)|^s}.$$

The main idea is to use a kernel $k(\xi, \eta)$ introduced in theorem 7 of [3], as well by Haydn in [10], and by Bogomolny and Carioli in [6,7], in the context of double-sided subshifts of the finite type. We begin by extending this definition to include baker transformations.

Definition 12. *Let* $(\hat{\Sigma}, \hat{T})$ *be a piecewise* Γ*-Möbius baker transformation, with* T_{L} *and* T_{R} *the left and right Bowen–Series transformations. Let* $A_{\mathrm{L}} : \mathbb{S}^1 \to \mathbb{C}$ *and* $A_{\mathrm{R}} : \mathbb{S}^1 \to \mathbb{C}$ *be two potential functions. We say that* A_{L} *and* A_{R} *are in involution if there exists a nonzero kernel* $k : \hat{\Sigma} \to \mathbb{C}^*$*, called an involution kernel, such that*

$$k(\xi, \eta)\mathrm{e}^{A_{\mathrm{L}}(\xi)} = k(\xi', \eta')\mathrm{e}^{A_{\mathrm{R}}(\eta')}, \qquad \text{whenever } (\xi', \eta') = \hat{T}(\xi, \eta) \in \hat{\Sigma}.$$

The kernel k *is extended to* $\mathbb{S}^1 \times \mathbb{S}^1$ *by* $k(\xi, \eta) = 0$*, for* $(\xi, \eta) \notin \hat{\Sigma}$*.*

Remark 13.

1. Let $W(\xi, \eta) = \ln k(\xi, \eta)$, for $(\xi, \eta) \in \hat{\Sigma}$. Then A_{L} and A_{R} are cohomologous, that is $A_{\mathrm{L}} - A_{\mathrm{R}} \circ \hat{T} = W \circ \hat{T} - W$.
2. If $A_{\mathrm{L}}(\xi)$ is Hölder, then there exists a Hölder function $A_{\mathrm{R}}(\eta)$ (depending only on η) in involution with A_{L} with a Hölder involution kernel.
3. If $\mathcal{L}_{\mathrm{L}}$ and $\mathcal{L}_{\mathrm{R}}$ are the two Ruelle transfer operators associated with A_{L} and A_{R}, if A_{L} and A_{R} are in involution with respect to a kernel k, and if ν is an eigenmeasure of $\mathcal{L}_{\mathrm{R}}$, that is, $\mathcal{L}_{\mathrm{R}}^*(\nu) = \lambda\nu$, then $\psi(\xi) = \int k(\xi, \eta)\,\mathrm{d}\nu(\eta)$ is an eigenfunction of $\mathcal{L}_{\mathrm{L}}$, that is, $\mathcal{L}_{\mathrm{L}}(\psi) = \lambda\psi$.

These remarks appeared first in [10] and were later rediscovered in [3], in the context of a subshift of the finite type. The proofs in this general context can be easily reproduced. The third remark suggests a strategy to obtain the eigenfunction $\psi_{f,s}$, by taking $A_{\mathrm{L}} = -s \ln |T_{\mathrm{L}}'|$, $A_{\mathrm{R}} = -s \ln |T_{\mathrm{R}}'|$ and replacing ν by the distribution $\mathcal{D}_{f,s}$. All there is left to prove is that $-\ln |T_{\mathrm{L}}'|$ and $-\ln |T_{\mathrm{R}}'|$ are in involution with respect to a piecewise $\mathcal{C}^1$ involution kernel. It so happens that this involution kernel exists and is given by the Gromov distance.

Definition 14. *The Gromov distance* $\mathrm{d}(\xi, \eta)$ *between two points* ξ *and* η *at infinity is given by*

$$\mathrm{d}^2(\xi, \eta) = \exp\Big(-b_\xi(\mathcal{O}, z) - b_\eta(\mathcal{O}, z)\Big),$$

for any point z *on the geodesic line* $[[\xi, \eta]]$*. Notice that this definition depends on the choice of the origin* $\mathcal{O}$ *(but not on* $z \in [[\xi, \eta]]$*).*

In the Poincaré disk model, $(\xi, \eta) \in \mathbb{S}^1 \times \mathbb{S}^1$, or in the upper half-plane, $(s, t) \in \mathbb{R} \times \mathbb{R}$, the Gromov distance takes the simple form

$$\mathrm{d}^2(\xi, \eta) = \frac{1}{4}|\xi - \eta|^2, \qquad \text{or} \qquad \mathrm{d}^2(s, t) = \frac{|s-t|^2}{(1+s^2)(1+t^2)}.$$

Lemma 15. *Let $T_{\mathrm{L}} : \mathbb{S}^1 \to \mathbb{S}^1$ and $T_{\mathrm{R}} : \mathbb{S}^1 \to \mathbb{S}^1$ be the two left and right Bowen–Series transformations of a Γ-Möbius Markov baker transformation $(\hat{\Sigma}, \hat{T})$. Then the two potential functions $A_{\mathrm{L}}(\xi) = -\ln |T_{\mathrm{L}}'(\xi)|$ and $A_{\mathrm{R}}(\eta) = -\ln |T_{\mathrm{R}}'(\eta)|$ are in involution and*

$$A_{\mathrm{L}}(\xi) - A_{\mathrm{R}}(\eta') = W(\xi', \eta') - W(\xi, \eta), \qquad \text{for } (\xi', \eta') = \hat{T}(\xi, \eta) \in \hat{\Sigma},$$

where $W(\xi, \eta) = b_\xi(\mathcal{O}, z) + b_\eta(\mathcal{O}, z)$ and z is any point of the geodesic line $[[\xi, \eta]]$. In particular, $k(\xi, \eta) = \exp(W(\xi, \eta)) = 4/\mathrm{d}^2(\xi, \eta)$ is an involution kernel.

Proof of lemma 15. To simplify the notation, we call $(\xi', \eta') = \hat{T}(\xi, \eta)$, $\gamma_{\mathrm{L}} = \gamma_{\mathrm{L}}[\xi]$, and $\gamma_{\mathrm{R}} = \gamma_{\mathrm{R}}[\eta']$. We also recall the relation $\gamma_{\mathrm{R}} = \gamma_{\mathrm{L}}^{-1}$. Then, choosing any point $z \in [[\xi, \eta]]$, we get

$$\begin{aligned} A_{\mathrm{L}}(\xi) - A_{\mathrm{R}}(\eta') &= -b_\xi(\mathcal{O}, \gamma_{\mathrm{L}}^{-1}\mathcal{O}) + b_{\eta'}(\mathcal{O}, \gamma_{\mathrm{R}}^{-1}\mathcal{O}) \\ &= -b_\xi(\mathcal{O}, z) - b_\xi(z, \gamma_{\mathrm{L}}^{-1}\mathcal{O}) + b_{\eta'}(\mathcal{O}, \gamma_{\mathrm{L}}(z)) + b_{\eta'}(\gamma_{\mathrm{L}}(z), \gamma_{\mathrm{R}}^{-1}\mathcal{O}) \\ &= W(\xi', \eta') - W(\xi, \eta), \end{aligned}$$

where $W(\xi', \eta') = b_{\eta'}(\mathcal{O}, \gamma_{\mathrm{L}}(z)) - b_\xi(z, \gamma_{\mathrm{L}}^{-1}\mathcal{O})$ and $W(\xi, \eta) = b_\xi(\mathcal{O}, z) - b_{\eta'}(\gamma_{\mathrm{L}}(z), \gamma_{\mathrm{R}}^{-1}\mathcal{O})$. ■

Notice that if $A(\xi)$ and $\bar{A}(\eta)$ are in involution by a positive kernel $k(\xi, \eta)$, then $sA(\xi)$ and $s\bar{A}(\eta)$ are in involution by $k(\xi, \eta)^s$.

Lemma 16. *Let $T_{\mathrm{L}} : \mathbb{S}^1 \to \mathbb{S}^1$ and $T_{\mathrm{R}} : \mathbb{S}^1 \to \mathbb{S}^1$ be the two left and right Bowen–Series transformations of a Γ-Möbius Markov baker transformation $(\hat{\Sigma}, \hat{T})$. Let $A_{\mathrm{L}} : \mathbb{S}^1 \to \mathbb{R}$ and $A_{\mathrm{R}} : \mathbb{S}^1 \to \mathbb{R}$ be two potential functions in involution with respect to a kernel $k(\xi, \eta)$. Let $\mathcal{L}_{\mathrm{L}}$ and $\mathcal{L}_{\mathrm{R}}$ be the two Ruelle transfer operators associated with A_{L} and A_{R}. Then, for any $\xi' \in \mathbb{S}^1$ and $\eta \in \mathbb{S}^1$,*

$$\mathcal{L}_{\mathrm{R}}(k(\xi', \cdot))(\eta) = \mathcal{L}_{\mathrm{L}}(k(\cdot, \eta))(\xi').$$

Proof. Given $\xi' \in \mathbb{S}^1$ and $\eta \in \mathbb{S}^1$, the two finite sets

$$\{\eta' \in \mathbb{S}^1;\ T_{\mathrm{R}}(\eta') = \eta,\ \ J(\xi', \eta') = 1\}, \qquad \{\xi \in \mathbb{S}^1;\ T_{\mathrm{L}}(\xi) = \xi',\ \ J(\xi, \eta) = 1\}$$

are in bijection. Thus, we obtain

$$\begin{aligned} \mathcal{L}_{\mathrm{R}}(k(\xi', \cdot))(\eta) &= \sum_{T_{\mathrm{R}}(\eta') = \eta} k(\xi', \eta') \mathrm{e}^{A_{\mathrm{R}}(\eta')} \\ &= \sum_{T_{\mathrm{L}}(\xi) = \xi'} k(\xi, \eta) \mathrm{e}^{A_{\mathrm{L}}(\xi)} = \mathcal{L}_{\mathrm{L}}(k(\cdot, \eta))(\xi'). \end{aligned}$$ ■

Theorem 1 now follows immediately from lemmas 15 and 16.

Proof of theorem 1. We first prove that $\psi_{f,s}(\xi) = \int k(\xi, \eta)^s \mathcal{D}_{f,s}(\eta)$, with $k(\xi, \eta) = J(\xi, \eta)/\mathrm{d}^2(\xi, \eta)$, is a solution of the equation $\mathcal{L}_s^{\mathrm{L}} \psi_f = \psi_f$. In fact, we have

$$\begin{aligned} \psi_{f,s}(\xi') &= \int k^s(\xi', \eta')\, \mathcal{D}_{f,s}(\eta') = \int \mathcal{L}_s^{\mathrm{R}}(k^s(\xi', \cdot))(\eta)\, \mathcal{D}_{f,s}(\eta) \\ &= \int \mathcal{L}_s^{\mathrm{L}}(k^s(\cdot, \eta)(\xi')\, \mathcal{D}_{f,s}(\eta) = (\mathcal{L}_s^{\mathrm{L}} \psi_{f,s})(\xi'). \end{aligned}$$

We next prove that $\psi_{f,s} \neq 0$. Suppose on the contrary that $\psi_{f,s}(\xi') = 0$ for each $\xi' \in \mathbb{S}^1$. Following Haydn [10], we introduce step functions of the form

$$\bar{\chi}(\xi', \eta') = \chi \circ pr_1 \circ \hat{T}^{-1}(\xi', \eta'),$$

where $\chi = \chi(\xi)$ depends only on ξ. For instance, for some fixed ξ', let χ be the characteristic function of the interval $I^{\mathrm{L}}(n, \xi) = \cap_{k=0}^{n} T_{\mathrm{L}}^{-k}(I^{\mathrm{L}} \circ T_{\mathrm{L}}^{k}(\xi))$, for some ξ such that $T_{\mathrm{L}}^{n}(\xi) = \xi'$. Let $Q^{\mathrm{R}}(\xi) = \{\eta \in \mathbb{S}^1;\ J(\xi, \eta) = 1\}$ and write

$$\gamma_{\mathrm{L}}[n, \xi] = \gamma_{\mathrm{L}}[T_{\mathrm{L}}^{n-1}(\xi)] \cdots \gamma_{\mathrm{L}}[T_{\mathrm{L}}(\xi)]\gamma_{\mathrm{L}}[\xi], \qquad Q^{\mathrm{R}}(n, \xi) = \gamma_{\mathrm{L}}[n, \xi] Q^{\mathrm{R}}(\xi).$$

Then $\bar{\chi}$ equals the characteristic function of the rectangle $I^{\mathrm{L}}(\xi') \times Q^{\mathrm{R}}(n, \xi)$ and $Q^{\mathrm{R}}(\xi')$ is equal to the disjoint union of the intervals $Q^{\mathrm{R}}(n, \xi)$, for all ξ such that $T_{\mathrm{L}}^{n}(\xi) = \xi'$. We also denote by $\Delta(\xi')$ the set of endpoints of $Q^{\mathrm{R}}(n, \xi)$, for all $T_{\mathrm{L}}^{n}(\xi) = \xi'$, and observe that $\Delta(\xi')$ is a dense subset of $Q^{\mathrm{R}}(\xi')$. Using the same ideas as in lemma 16, we obtain

$$\int \bar{\chi}(\xi', \eta') k^{s}(\xi', \eta') \mathcal{D}_{f,s}(\eta') = (\mathcal{L}_{s}^{\mathrm{L}})^{n}(\chi \psi_{f,s})(\xi') = 0, \qquad \forall\, \xi' \in \mathbb{S}^1.$$

In particular, if $\tilde{\alpha}(\xi') < \tilde{\beta}(\xi') < \tilde{\alpha}(\xi') + 2\pi$ are chosen such that $\exp \mathrm{i}\tilde{\alpha}(\xi')$ and $\exp \mathrm{i}\tilde{\beta}(\xi')$ are the two endpoints of the interval $Q^{\mathrm{R}}(\xi')$, if $\tilde{k}(\theta) = k(\xi', \exp \mathrm{i}\theta)$, then

$$\tilde{k}(\beta)\tilde{\mathcal{D}}_{f,s}(\beta) = \tilde{k}(\tilde{\alpha}(\xi'))\tilde{\mathcal{D}}_{f,s}(\tilde{\alpha}(\xi')) + \int_{\tilde{\alpha}(\xi')}^{\beta} \frac{\partial \tilde{k}}{\partial \theta} \tilde{\mathcal{D}}_{f,s}(\theta)\, \mathrm{d}\theta$$

for every $\beta \in [\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')] \cap \Delta(\xi')$. Since $\tilde{k}(\theta) \neq 0$, for each $\theta \in [\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')]$, we conclude that the above equality applies to all $\beta \in [\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')]$, the two functions $\tilde{k}(\beta)\tilde{\mathcal{D}}_{f,s}(\beta)$ and $\tilde{\mathcal{D}}_{f,s}(\beta)$ are $\mathcal{C}^1$, and

$$\int_{\tilde{\alpha}(\xi'))}^{\beta} k(\theta) \frac{\partial \tilde{\mathcal{D}}_{f,s}}{\partial \theta}\, \mathrm{d}\theta = 0, \qquad \forall\, \beta \in [\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')].$$

Therefore, $\tilde{\mathcal{D}}_{f,s}(\theta)$ is a constant function on each $[\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')]$, thus everywhere on $\mathbb{S}^1$. It follows that the distribution $\mathcal{D}_{f,s}$ would have to be equal to zero, which is impossible, because it represents a nonzero eigenfunction f. ■

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References

[1] Adler R and Flatto L 1991 Geodesic flows, interval maps and symbolic dynamics *Bull. Am. Math. Soc.* **25** 229–334

[2] Anantharaman N and Zelditch S 2007 Patterson–Sullivan distributions and quantum ergodicity *Ann. H Poincaré* **8** 361–426

[3] Baraviera A, Lopes A O and Thieullen Ph 2006 A large deviation principle for equilibrium states of Hölder potentials: the zero temperature case *Stochastics Dyn.* **6** 77–96

[4] Bedford T, Keane M and Series C 1991 *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces* (Oxford: Oxford University Press)

[5] Bekka M and Mayer M 2000 *Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces* (Oxford: Oxford University Press)

[6] Bogomolny E B and Carioli M 1992 Quantum maps of geodesic flows on surfaces of constant negative curvature *4th Int. Conf. 'Path Integrals from meV to MeV' (Tutzing, May 1992)* pp 18–21

[7] Bogomolny E B and Carioli M 1993 Quantum maps from transfer operator *Physica* D **67** 88–112

[8] Bowen R and Series C 1979 Markov maps associated to Fuchsian groups *Publ. Math. IHES* **50** 153–70

[9] Flaminio L and Forni G 2003 Invariant distributions and the time averages for horocycle flows *Duke Math. J.* **119** 465–526

[10] Haydn N 1990 Gibbs' functionals on subshifts *Commun. Math. Phys.* **134** 217–36

[11] Helgason S 1972 *Analysis on Lie Groups and Homogeneous Spaces* (Providence, RI: American Mathematical Society)

[12] Helgason S 1981 *Topics in Harmonic Analysis on Homogeneous Spaces* (Basel: Birkhauser)

[13] Iwaniec H 2002 *Spectral Methods of Automorphic Forms* (Providence, RI: American Mathematical Society)

[14] Lewis J B 1997 Spaces of holomorphic functions equivalent to the even Maass cusp forms *Invent. Math.* **127** 271–306

[15] Lewis J and Zagier D 2001 Period functions for Maass wave forms *Ann. Math.* **153** 191–258

[16] Lopes A O and Thieullen P 2006 Mather theory and the Bowen–Series transformation *Ann. Inst. H Poincaré, Anal. Non-Linéaire* **23** 663–82

[17] Mayer D H 1991 Thermodynamic formalism approach to Selberg's zeta function for PSL $(2, \mathbb{Z})$ *Bull. Am. Math. Soc.* **25** 55–60

[18] Morita T 1997 Markov systems and transfer operators associated with cofinite Fuchsian groups *Ergod. Theory Dyn. Syst.* **17** 1147–81

[19] Otal J-P 1998 Sur les fonctions propres du laplacien du disque hyperbolique *C. R. Acad. Sci. Paris Sér. I Math.* **327** 161–6

[20] Patterson S J 1976 The limit set of a Fuchsian group *Acta Math.* **136** 241–73

[21] Pollicott M 1991 Some applications of thermodynamic formalism to manifolds with constant negative curvature *Adv. Math.* **85** 161–92

[22] Series C 1981 Symbolic dynamics for geodesic flows *Acta Math.* **146** 103–28

[23] Series C 1985 The modular surface and continuous fractions *J. Lond. Math. Soc.* **31** 69–80

[24] Series C 1986 Geometrical Markov coding on surfaces of constant negative curvature *Ergod. Theory Dyn. Syst.* **6** 601–25

[25] Sullivan D 1979 The density at infinity of a discrete group of hyperbolic motions *Publ. Math. IHES* **50** 171–202

[26] Zelditch S 1987 Uniform distribution of eigenfunctions on compact hyperbolic surfaces *Duke Math. J.* **55** 919–41