A SKEW PRODUCT ENTROPY FOR NONSINGULAR TRANSFORMATIONS

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Abstract

We define a skew product entropy for conservative nonsingular transformations, show that it takes values in $\{0, \infty\}$ and use it to distinguish two classes on nonsingular transformations. Binary and ternary type III_λ odometers have zero skew product entropy, while nonsingular K-automorphisms have infinite skew product entropy.

1. *Introduction*

We introduce a skew product entropy for nonsingular endomorphisms. We show that the skew product entropy takes values in $\{0, \infty\}$ and use it to classify nonsingular transformations into two classes. For example, we show that type III_λ binary and ternary odometers have zero skew product entropy, while nonsingular K-automorphisms have infinite skew product entropy. We also obtain nonsingular automorphisms with the same ratio set but different skew product entropy.

In Section 2 we recall some properties of Krengel's entropy for infinite measure preserving transformations. We compute the Krengel entropy of $T \times S$, for T infinite measure preserving, in the case when S is a compact group rotation and when S has positive entropy. These questions are mentioned in [**14**, Section 5], and the difficulty lies in the fact that S is not a factor (in our and in Krengel's sense) of $T \times S$ (the factor algebra corresponding to S is not σ-finite for product measure).

In Section 3 we define the skew product entropy; this is defined by computing the Krengel entropy of the infinite measure preserving Maharam skew product of the transformation. We also compute the skew product entropy for binary and ternary odometers, and study basic properties of the skew product entropy.

In Section 4 we first show that the natural extension for nonsingular endomorphisms is unique and has the same type, completing our study in [**23, 24**]. We then define the notion of conservative nonsingular K-automorphisms. We show they are ergodic, and that any such T satisfies the following multiplier property: if S is a conservative ergodic nonsingular automorphism such that $T \times S$ is conservative, then $T \times S$ is ergodic; this property clearly implies weak mixing. This gives a partial answer to a question in [**3**], where they ask for a property of T that is equivalent to this ergodic multiplier property.

We assume all spaces X are standard Borel spaces; $\mathscr{B}(X)$ denotes the Borel σ-algebra of X, and all measures are σ-finite Borel measures. A *nonsingular endomorphism* (X, μ, T) is a map $T: X \to X$ defined on a space X such that $T^{-1}(\mathscr{B}(X)) \subseteq \mathscr{B}(X)$ and $\mu(N) = 0$ if and only if $\mu(T^{-1}(N)) = 0$ for every $N \in \mathscr{B}(X)$). A *nonsingular automorphism* is a nonsingular endomorphism such that $T^{-1}(\mathscr{B}(X)) =$

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$\mathscr{B}(X)(\operatorname{mod}\mu)$, (it is known that this condition is equivalent to the existence of a Borel set $Y \subseteq X$ of full measure such that $T\colon Y \to X$ is one-to-one); we denote by $\omega = d\mu \circ T/d\mu$ its Radon–Nikodym derivative. More generally (in the endomorphic case), a positive finite function ω satisfying

$$\int f \circ T\omega \, d\mu = \int f \, d\mu$$

for all integrable functions f is called a *markovian function* for (X, μ, T); we also say that (ω, μ) is a *markovian pair* for T and note that such ω may not be unique (this together with existence was shown in [**24**]). A nonsingular endomorphism is said to be *conservative* if, for every Borel set A of positive measure, there exists $n \geqslant 1$ such that $\mu(A \cap T^{-n}A) > 0$; a markovian pair is said to be *recurrent* if, for all positive integrable functions h, we have

$$\sum_{i \geqslant 0} h \circ T^i \omega_i = +\infty \text{ a.e.,} \quad \text{where } \omega_i = \prod_{j=0}^{i-1} \omega \circ T^j$$

(the existence of a recurrent markovian pair implies conservativity). Two markovian pairs (ω, μ) and (ω', μ') are said to be *cohomologous* if there is a *transfer function h* such that $\mu' = h\mu$ and $\omega' = \omega(h \circ T/h)$. A conservative ergodic nonsingular endomorphism (X, μ, T) is said to be *type* III if there is no σ-finite T-invariant measure equivalent to μ; it is *type* II if there exists σ-finite T-invariant measure equivalent to μ.

2. *Krengel entropy*

In this section we recall Krengel's definition of the entropy of a conservative infinite measure preserving endomorphism, and compute the entropy of the product of two such endomorphisms in two particular cases. The following definition is given for general nonsingular endomorphisms (X, μ, T, ω) and is used in Sections 3 and 4; however in this section, the endomorphisms are measure preserving and we assume that the corresponding ω is always 1.

DEFINITION 2.1. Let $(X_1, \mu_1, T_1, \omega_1)$ and $(X_2, \mu_2, T_2, \omega_2)$ be two nonsingular endomorphisms. We say that T_2 is a *factor* of T_1 (respectively, T_2 is *isomorphic* to T_1) if there exists a map $\phi\colon X_1 \to X_2$ called a *factor map* (respectively, *isomorphism*) which satisfies:

(i) $\phi^{-1}(\mathscr{B}(X_2)) \subseteq \mathscr{B}(X_1)(\operatorname{mod}\mu)$ (respectively, ϕ is bijective and bimeasurable),
(ii) $\mu_1 \circ \phi^{-1} = \mu_2$,
(iii) $\phi \circ T_1 = T_2 \circ \phi$ a.e.

We say that T_2 is a *markovian factor* of T_1 (ϕ a *markovian factor map*) if moreover:
(iv) $\omega_1 = \omega_2 \circ \phi$.

The factor map (respectively, isomorphism) is said to be *nonsingular* if (ii) is replaced by the requirement that $\mu_1 \circ \phi^{-1}$ is equivalent to μ_2.

For a nonsingular endomorphism (X, μ, T, ω) we extend these two definitions to sub-σ-algebras $\mathscr{F}$ such that μ restricted to $\mathscr{F}$ is σ-finite. Then $\mathscr{F}$ is called a *factor algebra* if $T^{-1}\mathscr{F} \subseteq \mathscr{F}(\operatorname{mod}\mu)$ and a *markovian factor algebra* if in additon ω is measurable with respect to $\mathscr{F}$. A factor algebra is called *invertible* if $T^{-1}\mathscr{F} = \mathscr{F}(\operatorname{mod}\mu)$, *noninvertible* if $T^{-1}\mathscr{F} \neq \mathscr{F}(\operatorname{mod}\mu)$.

We note that if $\phi: T_1 \to T_2$ is a factor map and T_2 is an automorphism, then ϕ is markovian if and only if ω_1 is measurable with respect to the factor algebra $\phi^{-1}\mathscr{B}(X_2)$. We shall see later the need to consider markovian factors when studying noninvertible transformations (for example, we obtain uniqueness of the natural extension only in the class of markovian factors, cf. 4.2).

If T is a finite measure preserving endomorphism of (X, μ) we let $h(T)$ denote the Kolmogorov–Sinai entropy of T with respect to the normalized measure $\mu/\mu(X)$.

We now recall the entropy defined by Krengel [**14**] for a conservative infinite measure preserving endomorphism. For any set A let $A^s = \bigcup_{i\geqslant 0} T^{-i}A$. By conservativity, $A \subseteq T^{-1}A^s$ and we can define the return time to A by $\tau_A(x) = \inf\{k \geqslant 1: T^k(x) \in A\}$ for all x in $T^{-1}A^s$ and $\tau_A = 1$ for all x in $X/T^{-1}A^s$. Then put $T_A(x) = T^{\tau_A}(x)$ for $x \in X$. The restriction of T_A to A is called the *induced map* on A. Let μ_A be the restriction of μ to $\mathscr{B}(A)$ (notice that the measure is not normalized nor necessarily finite), then (A, μ_A, T_A) is a conservative measure preserving endomorphism (see [**24**, 5.2(a) and 4.3]). We say that A is a *sweep out* set if $A^s = X (\mathrm{mod}\, \mu)$. (We note that if T is infinite measure preserving and admits a sweep out set of finite measure, then it is conservative [**16**].) If A has finite measure, $h(T_A)$ will denote the Kolmogorov–Sinai entropy of T_A restricted to A with respect to the normalized measure $\mu_A/\mu(A)$.

THEOREM 2.2 (Abramov's formula). *Let (X, μ, T) be a finite measure preserving endomorphism and B be a Borel set. Then*

(a) $\mu(X)\, h(T) \geqslant \mu(B)\, h(T_B)$,

(b) *if B sweeps out, then* $\mu(X)\, h(T) = \mu(B)\, h(T_B)$.

DEFINITION 2.3 [**14**]. Let T be a conservative finite or infinite measure preserving endomorphism. The *Krengel entropy* of T is defined by:

$$k_\mu(T) = \sup\{\mu(E)\, h(T_E): 0 < \mu(E) < +\infty\}.$$

We in fact have that $k_\mu(T) = \sup\{\mu(E)\, h(T_E): E \subseteq X_a \text{ and } 0 < \mu(E) < \infty\}$. We first notice that X can be decomposed into two disjoint invariant sets: $X = X_a \cup X_p$, where $X_a = \{x \in X: T^{n+k}(x) \neq T^k(x)\, \forall n \geqslant 1, k \geqslant 0\}$. By conservativity, X_a is actually equal to $\{x \in X: T^n(x) \neq x\, \forall n \geqslant 1\}$. Indeed, if $Y = \{x \in X: \exists n \geqslant 1 \text{ such that } T^n(x) = x\}$, then $TY \subseteq Y$, by conservativity $\mu(T^{-1}Y \setminus Y) = 0$ and $Y = \bigcap_{k\geqslant 0} T^{-k}Y (\mathrm{mod}\, \mu)$. If $E = E_a \cup E_p$ is a Borel set of finite measure, then $\mu(E)\, h(T_E) = \mu(E_a)\, h(T_{E_a}) + \mu(E_p)\, h(T_{E_p})$. Since for almost all $x \in E_p$ we have that x is a periodic point for T_{E_p}, it follows that $h(T_{E_p}) = 0$. Also T is called *aperiodic* if $X_a = X (\mathrm{mod}\, \mu)$. We now recall some results about Krengel's entropy. We first prove the following technical lemma.

LEMMA 2.4. *Let (X, μ, T) be a finite measure preserving endomorphism and $(E_n)_{n\geqslant 0}$ be an increasing sequence of Borel sets such that $\bigcup_{n\geqslant 0} E_n = X (\mathrm{mod}\, \mu)$. Then* $\mu(X)\, h(T) = \lim_{n\to+\infty} \mu(E_n)\, h(T_{E_n})$.

Proof. We first construct by induction a sequence of sets $(F_n)_{n\geqslant 0}$ such that $X = \bigcup_{n\geqslant 0} F_n^s$, the $\{F_n^s\}_{n\geqslant 0}$ are pairwise disjoint and $\bigcup_{k=0}^n F_k \subseteq E_n \subseteq \bigcup_{k=0}^n F_k^s$. We choose $F_0 = E_0$ and define: $F_n = E_n \backslash (\bigcup_{k=0}^{n-1} F_k^s)$ for all $n \geqslant 1$. We have only to check that the sets $\{F_n^s\}_{n \geqslant 0}$ are pairwise disjoint (mod μ): indeed, for $m < n$ we have $F_n \cap F_m^s = \varnothing$; moreover $F_m^s \cap F_n^s = (F_n \cap F_m^s)^s \cup (F_n^s \cap F_m)^s$ and by conservativity, $F_m \subseteq \bigcap_{k \geqslant 0} T^{-k} F_m^s$ (mod μ), in particular, implies that

$$F_m \cap F_n^2 = F_m \bigcap_{k \geqslant 0} T^{-k} F_m^s \cap F_n^s \subseteq (F_m^s \cap F_n)^s \ (\mathrm{mod}\ \mu).$$

We now conclude the proof using Abramov's formula. By disjointness of the invariant sets $\{F_n^s\}_{n \geqslant 0}$ we get $\mu(X)\,h(T) = \sum_{n \geqslant 0} \mu(F_n^s)\,h(T \mid F_n^s)$ and since E_n is a sweep out set of T restricted to $\bigcup_{k=0}^{n} F_k^s$ we obtain

$$\mu\left(\bigcup_{k=0}^{n} F_k^s\right) h\left(T \mid \bigcup_{k=0}^{n} F_k^s\right) = \sum_{k=0}^{n} \mu(F_k^s)\,h(T \mid F_k^s) = \mu(E_n)\,h(T_{E_n}).$$

THEOREM 2.5 [**14**]. *Let (X, μ, T) be a conservative infinite measure preserving endomorphism.*

(a) *If T is aperiodic, then there exists a sweep out set of finite measure.*
(b) *If E is a sweep out set of finite measure, then $k_\mu(T) = \mu(E)\,h(T_E)$.*
(c) *If F is a sweep out set of infinite measure, then $k_\mu(T) = k_{\mu_F}(T_F)$.*
(d) *If (Y, ν, S) is a factor, then $k_\nu(S) \leqslant k_\mu(T)$.*
(e) *If $k_\mu(T) = 0$, then T is an automorphism.*
(f) *For all $n \geqslant 0$, the map T^n is conservative and $k_\mu(T^n) = n k_\mu(T)$. If, moreover, T is an automorphism, then $k_\mu(T^{-1}) = k_\mu(T)$.*

Proof. Parts (a) and (b) are in [**14**, Theorems 2.1 and 3.1]. Parts (d), (e) and (f) are in [**14**, Proposition 5.1, Theorem 7.1, and Proposition 5.2].

Part (c). Since $(T_F)_E = T_E$ for every set $E \subseteq F$, we get $k_{\mu_F}(T_F) \leqslant k_\mu(T)$. Conversely, let E be a set of finite measure, $(F_k)_{k \geqslant 0}$ an increasing sequence of sets of finite measure which exhaust F and $E_k = E \cup F_k^s$. We notice that $(E_k)_{k \geqslant 0}$ is an increasing sequence, $\bigcup_{k \geqslant 0} E_k = E$ and that F_k is a sweep out set for $T_{E_k \cup F_k}$ restricted to $E_k \cup F_k$. We thus obtain

$$\mu(E_k)\,h(T_{E_k}) \leqslant \mu(E_k \cup F_k)\,h(T_{E_k \cup F_k}) = \mu(F_k)\,h(T_{F_k}) \leqslant k_{\mu_F}(T_F),$$

and finally use Lemma 2.4 to get $\mu(E)\,h(T_E) \leqslant k_{\mu_F}(T_F)$. Taking the supremum over E we have $k_\mu(T) \leqslant k_{\mu_F}(T_F)$.

We note that in the following Proposition 2.6(a), S is not a factor (according to Definition 2.1) of $T \times S$. This formula appears in Parry's book [**19**], but there he uses a different definition for the entropy. The following question remains open: does there exist a zero entropy finite measure preserving transformation S and a zero entropy infinite measure preserving transformation T such that the Krengel entropy of $T \times S$ is infinite?

PROPOSITION 2.6. *Let (X, μ, T) be a conservative infinite measure preserving endomorphism and (Y, ν, S) be a finite measure preserving endomorphism. Then*

(a) $k_{\mu \times \nu}(T \times S) \geqslant \mu(X)\,k_\nu(S) + \nu(Y)\,k_\mu(T)$ *(with the convention $\infty \cdot 0 = 0$);*
(b) *if $h(S) > 0$, then $k_{\mu \times \nu}(T \times S) = +\infty$;*
(c) *if S is a rotation of a compact abelian group Y with respect to Haar measure ν, then $k_{\mu \times \nu}(T \times S) = \nu(Y)\,k_\mu(T)$.*

Proof. Let $E \subseteq X$ be of finite positive measure, R be the induced map of $T \times S$ on $E \times Y$ and τ the return time to E. Then $R(x, y) = (T_E(x), S^{\tau(x)}(y))$.

(a) We first prove the following formula (cf. [5, (**)]):

$$h(R) = h(T_E) + \sup\left\{ \lim_{n \to +\infty} \int_E \frac{1}{n} H\left[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}\right] d\mu_E(x) \colon \mathcal{Q} \text{ is a finite partition of } Y \right\},$$

where $S_x^{-i} = (S_x^i)^{-1}$, $S_x^i(y) = S^{\tau_i(x)}(y)$ and $\tau_i(x) = \sum_{j=0}^{i-1} \tau \circ T_E^j(x)$.

Let $\mathcal{P}, \mathcal{Q}$ be finite partitions of E, Y, and $\mathcal{T}(Y)$ denote the trivial algebra of Y. Then

$$\begin{aligned} &H\left[\bigvee_{i=0}^{n-1} R^{-i}(\mathcal{P} \otimes \mathcal{P})\right] \\ &= H\left[\bigvee_{i=0}^{n-1} T_E^{-i}\mathcal{P}\right] + H\left[\bigvee_{i=0}^{n-1} R^{-i}(E \otimes \mathcal{Q}) \,\middle|\, \bigvee_{i=0}^{n-1} R^{-i}(\mathcal{P} \otimes Y)\right] \\ &\geqslant H\left[\bigvee_{i=0}^{n-1} T_E^{-i}\mathcal{P}\right] + H\left[\bigvee_{i=0}^{n-1} R^{-i}(E \otimes \mathcal{Q}) \,\middle|\, \mathcal{B}(E) \otimes \mathcal{T}(Y)\right]. \end{aligned}$$

The conditional measure of a Borel set $B \in \mathcal{B}(E) \otimes \mathcal{B}(Y)$ with respect to $\mathcal{B}(E) \otimes \mathcal{T}(Y)$ is given by the formula: $\mu_E \times \nu(B \mid \mathcal{B}(E) \otimes \mathcal{T}(Y))(x, y) = \nu(B_x)$, where B_x denotes the vertical slice of B. Since $(\bigvee_{i=0}^{n-1} R^{-i}(E \otimes \mathcal{Q}))_x = \bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}$ and the sequence $(H[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}])_{n \geqslant 0}$ is subadditive, by taking the limit on n and all partitions $\mathcal{P}$ we obtain:

$$h(R) \geqslant h(T_E) + \lim_{n \to \infty} \int \frac{1}{n} H\left[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}\right] d\mu_E(x).$$

Conversely, if $\mathcal{R}$ is a finite partition of $E \times Y$ we have

$$\begin{aligned} &H\left[\bigvee_{i=0}^{n-1} R^{-i}\mathcal{R}\right] \\ &\leqslant H\left[\bigvee_{i=0}^{n-1} R^{-i}(\mathcal{P} \otimes \mathcal{Q})\right] + H\left[\bigvee_{i=0}^{n-1} R^{-i}\mathcal{R} \,\middle|\, \bigvee_{i=0}^{n-1} R^{-i}(\mathcal{P} \otimes \mathcal{Q})\right] \\ &\leqslant H\left[\bigvee_{i=0}^{n-1} T_E^{-i}\mathcal{P}\right] + H\left[\bigvee_{i=0}^{n-1} R^{-i}(E \otimes \mathcal{Q}) \,\middle|\, \bigvee_{i=0}^{n-1} R^{-1}(\mathcal{P} \otimes Y)\right] + nH[\mathcal{R} \mid \mathcal{P} \otimes \mathcal{Q}]. \end{aligned}$$

Since the sequence $(H[\bigvee_{i=0}^{n-1} R^{-1}(E \otimes \mathcal{Q}) \mid \bigvee_{i=0}^{n-1} R^{-i}(P \otimes Y)])_{n \geqslant 0}$ is subadditive, by taking the limit on n, we obtain

$$h(\mathcal{R}, R) \leqslant h(\mathcal{P}, T_E) + H[\mathcal{R} \mid \mathcal{P} \otimes \mathcal{Q}] + (1/n) H\left[\bigvee_{i=0}^{n-1} R^{-i}(E \otimes \mathcal{Q}) \,\middle|\, \bigvee_{i=0}^{n-1} R^{-i}(\mathcal{P} \otimes Y)\right]$$

for all $n \geqslant 1$. We now choose an increasing sequence of partitions $(\mathcal{P}_k)_{k \geqslant 0}$ which generate $\mathcal{B}(E)$ and obtain:

$$h(\mathcal{R}, R) \leqslant h(T_E) + H[\mathcal{R} \mid \mathcal{B}(E) \otimes \mathcal{Q}] + \int (1/n) H\left[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}\right] d\mu_E(x).$$

We first take the infimum over n then the supremum over $\mathcal{Q}$ and over $\mathcal{R}$ and obtain the other inequality.

We next show that

$$\sup_{\mathcal{Q}}\left\{\lim_{n\to+\infty}\int_E (1/n)\,H\left[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}\right]d\mu_E(x)\right\} \geqslant \int_E \tau\, d\mu_E(x)\, h(S)$$

and we conclude the proof of assertion (a) by applying Kac's Lemma **[24]**: $\int_E \tau\, d\mu = \mu(E^s)$ and by choosing an increasing sequence of finite sets $(E_k)_{k\geqslant 0}$ which exhaust X.

Let $\mathcal{Q}$ be a finite partition of Y and $\mathcal{Q}_k = \bigvee_{i=0}^{k-1} S^{-i}\mathcal{Q}$. Then

$$\begin{aligned} H\left[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}_k\right] &= \sum_{i=0}^{n-1} H\left[\mathcal{Q}_k \,\middle|\, \bigvee_{j=1}^{n-i-1} S^{-j}_{T_E^i(x)}\mathcal{Q}_k\right] \\ &= \sum_{i=0}^{n-1} H\left[\mathcal{Q}_{k\wedge\tau\circ T_E^i(x)} \,\middle|\, \bigvee_{j=1}^{n-i-1} S^{-j}_{T_E^i(x)}\mathcal{Q}_k\right] \\ &\geqslant \sum_{i=0}^{n-1} H\left[\mathcal{Q}_{k\wedge\tau\circ T_E^i(x)} \,\middle|\, \bigvee_{j\geqslant k\wedge\tau\circ T_E^i(x)} S^{-j}\mathcal{Q}\right] \\ &= \sum_{i=0}^{n-1} k\wedge\tau\circ T_E^i(x)\, H\left[\mathcal{Q}\,\middle|\,\bigvee_{j\geqslant 1} S^{-j}\mathcal{Q}\right] \\ &= \sum_{i=0}^{n-1} k\wedge\tau\circ T_E^i(x)\, h(\mathcal{Q},S). \end{aligned}$$

We conclude by integrating over E and by taking the supremum over $\mathcal{Q}$ and then over k.

(b) It is an immediate corollary of 2.6(a).

(c) It is enough to show that $\lim_{n\to+\infty}(1/n)H[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}+ = 0$ for every finite partition $\mathcal{Q}$. We first assume that $Y = \mathbf{T}$, (here $\mathbf{T}$ is the one-dimensional torus), S is the rotation defined by $S(y) = y+\alpha$ and $\mathcal{Q}_p$ is the partition

$$\{[k/2^p,(k+1)/2^p)\colon 0\leqslant k < 2^p\}.$$

(The proof of this case appears in **[14]**.) We denote by increasing order $k_0 = 0 < k_1 < \ldots < k_q = 1$, where $q = q(x,p)$, the points of

$$\{k/2^p - \tau_i(x)\,\alpha(\text{mod}\ 1)\colon 0\leqslant k < 2^p, 0\leqslant i < n\}.$$

The partition $\mathcal{R} = \{[k_i, k_{i+1})\colon 0\leqslant i < q\}$ is finer than the partition $\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}_p$ and therefore $H[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}_p] \leqslant H[\mathcal{R}] \leqslant \ln(n2^p)$. In particular

$$\lim_{n\to+\infty}(1/n)\,H\left[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}_p\right] = 0.$$

If Y is a general compact abelian group, S the rotation $S(y) = y+\alpha$ and $\hat{Y}$ the dual group of characters of Y, then $\hat{Y}$ is countable, $\hat{Y} = \{\gamma_1,\gamma_2,\ldots\}$ and generates $\mathcal{B}(Y)$. Let $\mathcal{Q}_p^j = \{\gamma_j^{-1}([k/2^p,(k+1)/2^p))\colon 0\leqslant k < 2^p\}$ and $\mathcal{Q}_p = \mathcal{Q}_p^1 \vee \ldots \vee \mathcal{Q}_p^p$, then $(\mathcal{Q})_{p\geqslant 1}$ is an increasing sequence of partitions and $\bigvee_{p\geqslant 1}\mathcal{Q}_p = \mathcal{B}(Y)$. Each $\gamma_j\colon Y\to\mathbf{T}$ defines a factor of Y: $\gamma_j\circ S = R_j\circ\gamma_j$ with $R_j(y) = y+\gamma_j(a)$ and $H[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}_p] \leqslant \sum_{j=1}^{p} H[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}_p^j]$. Using the beginning of this part we obtain $\lim_{n\to+\infty}(1/n)H[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}_p] = 0$. If $\mathcal{Q}$ is another finite partition then $H[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}] \leqslant H[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}_p] + nH[\mathcal{Q}\,|\,\mathcal{Q}_p]$. Since $\lim_{p\to+\infty} H[\mathcal{Q}\,|\,\mathcal{Q}_p] = 0$ we finally obtain $\lim_{n\to+\infty}(1/n)H[\bigvee_{i=0}^{n-1} S_x^{-i}\mathcal{Q}] = 0$.

Hajian, Ito and Kakutani in [8] introduced and showed the importance of ergodic infinite measure preserving transformations which commute with a totally dissipative one. The following will be needed in Section 3.

LEMMA 2.7. *Let (X,μ,T) be a conservative infinite measure preserving endomorphism and $Q: X \to X$ a nonsingular automorphism such that $Q \circ T = T \circ Q$.*
(a) *If T is ergodic, then $\mu \circ Q^{-1} = \alpha\mu$ for some $\alpha > 0$.*
(b) *If $\mu \circ Q^{-1} = \alpha\mu$ for $\alpha \neq 1$, then $k_\mu(T) \in \{0, +\infty\}$.*

Proof. (a) Since $\nu = (d\mu \circ Q^{-1}/d\mu)\,\mu$ is T-invariant conservative ergodic, by the Hopf ergodic theorem, for any sets A and B of finite measure for μ and ν, we have that $\sum_{i=0}^{n-1} \chi_A \circ T^i / \sum_{i=0}^{n-1} \chi_B \circ T^i$ converges a.e. to $\mu(A)/\mu(B)$ and to $\nu(A)/\nu(B)$.
(b) Since $Q: (X,\mu,T) \to (X,\alpha\mu,T)$ is a (measure preserving) isomorphism, and $k_\mu(T) = k_{\alpha\mu}(T) = \alpha k_\mu(T)$ the result follows.

3. *Skew product entropy*

We now investigate a number that we call the *skew product entropy* of a nonsingular endomorphism. We are mainly interested in distinguishing between zero and positive entropy for nonsingular automorphisms. We note that while skew product entropy takes only two values, Krengel's entropy for an infinite measure transformation takes essentially three values since there is no canonical choice of the measure up to scalar multiple.

If (X,μ,T,ω) is a nonsingular endomorphism we denote by (X^*,μ^*,T_ω) the infinite measure preserving *Maharam skew product*: $X^* = X \times \mathbb{R}^+$, $\mu^* = \mu \times \text{Leb}$ and $T_\omega(x,t) = (T(x), t/\omega(x))$. We notice that T is not a factor of T_ω and recall that T_ω is conservative if and only if (ω,μ) is recurrent [23, Theorem 2].

DEFINITION 3.1. Let (X,μ,T) be the nonsingular endomorphism and (ω,μ) a recurrent pair for T. The *skew product entropy* of T with respect to (ω,μ) is defined by

$$s_{(\omega,\mu)}(T) = k_{\mu^*}(T_\omega).$$

Here also, to compute the skew product entropy, we may assume that T is aperiodic. Indeed $X_a^* = X_a \times \mathbb{R}^+$; since $X_p^n = \bigcup_{k\geq 0} T^{-k}\{x \in X: T^n(x) = x\}$ is T-invariant ($T^{-1}X_p^n = X_p^n$) and by conservativity $X_p^n = \{x \in X: T^n(x) = x\}$ (mod μ) we obtain for all positive integrable functions h that

$$\int_{X_p^n} h\omega_n \, d\mu = \int_{T^{-n}X_p^n} h \circ T^n \omega_n \, d\mu = \int_{X_p^n} h \, d\mu$$

and therefore $\omega_n(x) = 1$ for all x satisfying $T^n(x) = x$.

In the case when, for a given μ, there exists a unique recurrent (ω,μ) (for example, when T is an automorphism, or when T is measure preserving [24, Theorem 5.6]), we shall write $s_\mu(T)$ instead of $s_{(\omega,\mu)}(T)$. In the following proposition we collect some properties of the skew product entropy; the natural extension is defined in Section 4.

PROPOSITION 3.2. *Let T be a nonsingular endomorphism recurrent with respect to (ω, μ).*

(a) *The skew product entropy $s_{(\omega,\mu)}(T) \in \{0, +\infty\}$.*

(b) *If (ω', μ') is cohomologous to (ω, μ), then $s_{(\omega,\mu)}(T) = s_{(\omega',\mu')}(T)$.*

(c) *If T is measure preserving, then $s_\mu(T) = 0$ if and only if $k_\mu(T) = 0$.*

(d) *Let (Y, ν, S) be a finite measure preserving endomorphism. Then $T \times S$ is recurrent with respect to $(\omega \times 1, \mu \times \nu)$. If $h(S) > 0$, then $s_{(\omega\times 1, \mu\times\nu)}(T \times S) = +\infty$. If S is a rotation of a compact abelian group, then $s_{(\omega\times 1,\mu\times\nu)}(T \times S) = s_{(\omega,\mu)}(T)$.*

(e) *If $(\bar{X}, \bar{\mu}, \bar{T}, \bar{\omega})$ is a markovian factor of (X, μ, T, ω), then $s_{(\bar{\omega},\bar{\mu})}(\bar{T}) \leqslant s_{(\omega,\mu)}(T)$.*

(f) *If E is a sweep out set, then the induced map T_E on E is recurrent with respect to (ω_E, μ_E) (where $\omega_E(x) = \omega_{\tau_E(x)}(x)$) and $s_{(\omega_E,\mu_E)}(T_E) = s_{(\omega,\mu)}(T)$.*

(g) *If $s_{(\omega,\mu)}(T) = 0$, then T is an automorphism.*

(h) *For all $n > 0$, the pair (ω_n, μ) is recurrent for T^n and $s_{(\omega_n,\mu)}(T^n) = ns_{(\omega,\mu)}(T)$. If T is invertible, then T^{-1} is conservative and $s_\mu(T^{-1}) = s_\mu(T)$.*

(i) *If $(\tilde{X}, \tilde{\mu}, \tilde{T})$ is a natural extension of (X, μ, T, ω), then $s_{\tilde{\mu}}(\tilde{T}) = s_{(\omega,\mu)}(T)$.*

Proof. (a) The automorphism $Q(x, t) = (x, 2t)$ commutes with T_ω and satisfies $d\mu^* \circ Q^{-1}/d\mu^* = \frac{1}{2}$. Lemma 2.7 implies that $k_{\mu^*}(T_\omega) \in \{0, +\infty\}$.

(b) Let h be the transfer function between the cohomologous pairs, that is, $\mu' = h\mu$ and $\omega' = \omega(h \circ T/h)$. It follows that the map $\psi(x, t) = (x, t/h(x))$ is an isomorphism of $(X^*, \mu \times \text{Leb}, T_\omega)$ and $(X^*, \mu' \times \text{Leb}, T_{\omega'})$, and therefore their Krengel entropies agree.

(c) We may assume that T is aperiodic; by [**14**, Theorem 2.1] for each $n \geqslant 1$ there exists a sweep out set A_n such that $\mu(A_n) \leqslant 2^{-n}$. Let $B_n = A_n \times (n-1, n]$ and $B = \bigcup_{n\geqslant 1} B_n$. Then B is a sweep out set of finite measure for (X^*, μ^*, T_1), where $T_1(x, t) = (T(x), t)$. Let S denote the induced map on B: $S(x, t) = (T_{A_n}(x), t)$ on each $(x, t) \in B_n$. Since B_n is invariant for S, we have

$$s_\mu(T) = \mu^*(B)\,h(S) = \sum_{n\geqslant 1} \mu^*(B_n)\,h(S \mid B_n) = \sum_{n\geqslant 1} \mu(A_n)\,h(T_{A_n}) = \sum_{n\geqslant 1} k_\mu(T).$$

(d) The transformation $(T \times S)_{\omega\times 1}$ is isomorphic to $T_\omega \times S$ and it suffices to apply Proposition 2.6(b).

(e) Let $\phi: X \to \bar{X}$ be a markovian factor map. Then $\phi^*: X^* \to \bar{X}^*$ defined by $\phi^*(x, t) = (\phi(x), t)$ induces a factor between (X^*, μ^*, T_ω) and $(\bar{X}^*, \bar{\mu}^*, \bar{T}_{\bar{\omega}})$. By Theorem 2.5(d), $k_{\mu^*}(T_\omega) \geqslant k_{\bar{\mu}^*}(\bar{T}_{\bar{\omega}})$.

(f) Since the Maharam skew product of $(E, \mu_E, T_E, \omega_E)$ is isomorphic to (X^*, μ^*, T_ω) induced on $F = E \times \mathbb{R}^+$ and since F sweeps out,

$$s_{(\omega_E,\mu_E)}(T_E) = k_{\mu_F^*}((T_\omega)_F) = k_{\mu^*}(T_\omega) = s_{(\omega,\mu)}(T).$$

(g) If $k_{\mu^*}(T_\omega) = 0$ then, by Theorem 2.5(e), T_ω is injective on a set $Y^* \subseteq X^*$ of full measure. By Fubini's Theorem, we choose $Y \subseteq X$ of full measure such that $Y_x^* = \{t \in \mathbb{R}^+ : (x, t) \in Y^*\}$ has full Lebesgue measure for all $x \in Y$. For every $(x, y) \in Y$ with $x \neq y$, we can find $s \in Y_x^*$ and $t \in Y_y^*$ such that $s/\omega(x) = t/\omega(y)$. Since $T_\omega(x, s) \neq T_\omega(y, t)$, we obtain $T(x) \neq T(y)$. We have proven that T is injective on Y.

(h) If f is a positive integrable function, then by recurrence of (ω, μ) for T we have $\sum_{i\geqslant 0} f \circ T^i \omega_i = +\infty$ a.e. In particular $\sum_{k\geqslant 0}(\sum_{i=0}^{n-1} f \circ T^i\omega_i) \circ T^{kn}\omega_{kn} = +\infty$ a.e. and shows that (ω_n, μ) is recurrent for T^n. Furthermore $(T^n)_{\omega_n} = (T_\omega)^n$; if T is invertible, $\bar{\omega} = d\mu^* \circ T^{-1}/d\mu^* = 1/\omega \circ T^{-1}$ and $(T_\omega)^{-1} = (T^{-1})_{\bar{\omega}}$.

(i) We may assume that T is aperiodic. Let $\phi\colon \tilde{X} \to X$ be a markovian factor and $\tilde{\mathscr{F}} = \phi^{-1}\mathscr{B}(X)$ the associated factor algebra. Then $\phi^*\colon \tilde{X}^* \to X^*$ defined by $\phi^*(x,t) = (\phi(x),t)$ induces a factor map between $\tilde{\mathscr{X}}^* = (\tilde{X}^*, \tilde{\mu}^*, \tilde{T}_{\tilde{\omega}})$ and $\mathscr{X}^* = (X^*, \mu^*, T_\omega)$. Since $\tilde{\mathscr{F}}^* = \tilde{\mathscr{F}} \otimes \mathscr{B}(\mathbb{R}^+) = \phi^{*-1}\mathscr{B}(X^*)$ is an exhaustive factor algebra (see part 2 of the proof of Lemma 4.3), $\tilde{\mathscr{X}}^*$ is a natural extension of $\mathscr{X}^*$. Let $E \subseteq X^*$ be a sweep out set of finite measure; $\tilde{E} = \phi^{*-1}(E)$ is also a sweep out set of finite measure. We denote by ν the restriction of μ to E and by S the induced map on E (idem for $\tilde{\nu}$ and $\tilde{S}$). Then ϕ^* induces a factor map between $\tilde{\mathscr{Y}} = (\tilde{E}, \tilde{\nu}, \tilde{S})$ and $\mathscr{Y} = (E, \nu, S)$. For the same reasons (see part 2 of the proof of Lemma 4.3), $\tilde{\mathscr{Y}}$ is a natural extension of $\mathscr{Y}$ and so

$$k_{\mu^*}(T_\omega) = \mu^*(E)\,h(S) = \mu^*(\tilde{E})\,h(\tilde{S}) = k_{\tilde{\mu}^*}(\tilde{T}_{\tilde{\omega}}).$$

COROLLARY 3.3. *Let (X, μ, T) be a nonsingular automorphism. If $s_\mu(T) = 0$, then T has no noninvertible markovian factors.*

We now study the skew product entropy for type III_λ, with $0 < \lambda < 1$, conservative ergodic nonsingular automorphisms. We know that we can always take a cohomologous pair (ω', μ'), which will be referred to as a *special pair* for T, such that μ is a probability measure and the values of ω' are powers of λ (cf. [**13**, Proposition 2.2]). Thus from now on we assume that (ω, μ) is a special pair and set $\Lambda = \{\lambda^i\colon i \in \mathbf{Z}\}$. We give the name *special Maharam skew product* to the measure preserving automorphism $(\hat{X}, \hat{\mu}, \hat{T}_\omega)$ defined by $\hat{X} = X \times \Lambda$, $\hat{\mu} = \sum_{t\in\Lambda} \mu \times t\delta_t$, and $\hat{T}_\omega(x,t) = (T(x), t/\omega(x))$.

In [**8**], Hajian, Ito and Kakutani constructed what we call the special Maharam skew product in the case of the type III_λ binary odometers. They also constructed the map T^τ of Lemma 3.9(b) and showed it to be ergodic. (The map T^τ is also studied in [**2**].)

THEOREM 3.4. *The special Maharam skew product $(\hat{X}, \hat{\mu}, \hat{T}_\omega)$ is a conservative ergodic infinite measure preserving automorphism.*

Proof. Let $f\colon X \to \mathbb{R}$ and $g\colon \Lambda \to \mathbb{R}$ be bounded Borel functions. Using the change of variable $t \mapsto t\omega(x)$, we obtain:

$$\int f \otimes g \circ \hat{T}_\omega \, d\hat{\mu} = \sum_{t\in\Lambda} \int f(Tx)\, g(t/\omega(x))\, t\, d\mu(x)$$

$$= \int f(Tx)\, \omega(x)\, d\mu(x) \sum_{t\in\Lambda} t g(t) = \int f \otimes g\, d\hat{\mu}.$$

To prove ergodicity of $\hat{T}_\omega$, we first define nonsingular endomorphisms $S_t\colon X \to X$ with $t \in \Lambda$, in the following way: $S_t(x) = T^{\tau_t(x)}(x)$, where

$$\tau_t(x) = \min\{n \geqslant 1\colon \omega_n(x) = t\}.$$

Since t belongs to the ratio set of T, we have that τ_t is finite a.e. and for any Borel set A of positive measure, there exists $n \geqslant 1$ such that $\mu(A \cap S_t^{-1} \circ S_1^{-n}(A)) > 0$. It follows that $A \subseteq \bigcup_{n\geqslant 0} S_t^{-1} \circ S_t^{-1}(A) \pmod{\mu}$, for any $A \in \mathscr{B}(X)$. If $A^* \in \mathscr{B}(X^*)$ is $\hat{T}_\omega$-invariant, we denote by A_t the horizontal slice $A_t = \{x \in X\colon (x,t) \in A^*\}$. Since $\hat{T}_\omega^{\tau_t(x)}(x,s) = (S_t(x), s/t)$ for almost all $x \in X$ and all $s, t \in \Lambda$, we obtain $A_t = S_t^{-1}(A_1)$, $A_1 = S_1^{-1}(A_1)$ and therefore $A_t \subseteq \bigcup_{n\geqslant 0} S_t^{-1} \circ S_1^{-n}(A_1) = A_1 \pmod{\mu}$. Conversely, $A_1 = S_{1/t}^{-1}(A_t)$,

$A_t = S_1^{-1}(A_t)$ and $A_1 \subseteq \bigcup_{n\geqslant 0} S_{1/t}^{-1} \circ S_1^{-n}(A_t) = A_t$. We just have proved that $A_t = A_1 \,(\mathrm{mod}\,\mu)$ for all $t \in \Lambda$, $A^* = A_1 \times \Lambda$, and therefore A_1 is T-invariant. Since T is ergodic, A^* belongs to the trivial σ-algebra.

Conservativity follows from ergodicity since the measure μ is non atomic. Ergodicity can also be obtained from the more general [**22**, Corollary 5.4] since the essential range of the cocycle ω_n is all of Λ.

PROPOSITION 3.5. *Let T be a conservative ergodic nonsingular type* III_λ *automorphism. If (ω,μ) is a special pair, then (X^*,μ^*,T_ω) is isomorphic to* $(\hat{X}\times(\lambda,1],\hat{\mu}\times\mathrm{Leb},\hat{T}_\omega\times\mathrm{Id})$. *Furthermore* $s_\mu(T) = (1-\lambda)\,k_{\hat{\mu}}(\hat{T}_\omega)$.

Proof. Let $\phi\colon \hat{X}\times(\lambda,1]\to X^*$ be defined by $\phi(x,t,s) = (x,ts)$. Since for any $y\in\mathbb{R}^+$ there is a unique n such that $\lambda^{n+1} < y \leqslant \lambda^n$, it follows that ϕ is invertible. Also ϕ commutes with the actions and is measure preserving, so $s_\mu(T) = k_{\mu^*}(T_\omega) = (1-\lambda)\,k_{\hat{\mu}}(\hat{T}_\omega)$.

COROLLARY 3.6. *Let T be a conservative ergodic nonsingular type* III_λ *automorphism. Let τ be defined a.e. on X by $\tau(x) = \min\{n \geqslant 1\colon \omega_n(x) = 1\}$. Then (X,μ,T^τ) is a finite measure preserving ergodic automorphism, and $s_\mu(T) = (1-\lambda)\,h(T^\tau)$.*

Proof. The map $(x,1)\mapsto(T^\tau(x),1)$ is the induced map of $\hat{T}_\omega$ on $X\times\{1\}$, and $X\times\{1\}$ is a sweep out set for $\hat{T}_\omega$ by definition of the ratio set.

We now compute the skew product entropy for some examples.

DEFINITION 3.7. Let $0<\lambda<1$. We give the name *binary odometer* to (X,μ,T), the nonsingular automorphism defined by $X = \prod_0^{+\infty}\{0,1\}$, $\mu = \prod_0^{+\infty}\{1/(1+\lambda),\lambda/(1+\lambda)\}$ and T the addition by $100\ldots$ with carry. We denote a point of X by $x = x_0x_1\ldots$.

It is well known (cf. [**10**, **13**]) that the binary odometers are conservative ergodic type III_λ nonsingular automorphisms. (We should note that here we are considering only a special class of binary odometers; for the general class see [**10**, **13**].) Let ω be the Radon–Nikodym derivative of T with respect to μ and $\omega = d\mu\circ T/d\mu$. Let $1\overset{n}{\ldots}1$ and $0\overset{n}{\ldots}0$ denote strings of n consecutive numbers 1 and of n consecutive numbers 0, respectively.

LEMMA 3.8. *Let $\mathscr{P} = \{P_0,P_1,\ldots\}$ be the partition into cylinder sets $P_n = [1\overset{n}{\ldots}10] = \{x\colon x_0\ldots x_n = 1\overset{n}{\ldots}1, x_n = 0\}$. Then on P_n, it follows that ω equals λ^{-n+1}.*

Proof. If $A = [1\overset{n}{\ldots}10i_1\ldots i_k]$ then $TA = [0\overset{n}{\ldots}01i_1\ldots i_k]$ and $\mu(TA) = \lambda^{-n+1}\mu(A)$.

LEMMA 3.9. *Let τ be as in Corollary 3.6 and define a partition $\mathscr{Q} = \{Q_{pq}\colon p\geqslant 0, q\geqslant 0\}$ of X, by $Q_{pq} = [0\overset{p}{\ldots}01\overset{q}{\ldots}110]$. Then*

(a) *for all $x\in Q_{pq}$, we have $\tau(x) = 2^p+2^q-1$,*

(b) $T^\tau(0\overset{p}{\ldots}01\overset{q}{\ldots},110\ldots) = (1\overset{q}{\ldots}10\overset{p}{\ldots}001\ldots)$.

Proof. We first introduce some notation. For $p\geqslant 0$ define cylinder sets A_k^p by $A_0^p = [0\overset{p}{\ldots}0]$ and $A_k^p = T^kA_0^p$ for $0\leqslant k<2^p$. For each integer $k = \sum_{i=0}^\infty k_i2^i$ let

$\alpha(k) = \operatorname{card}\{i \geqslant 0 : k_i = 1\}$. Then $\mu(A_k^p) = (1/1+\lambda)^{p+1-\alpha(k)}(\lambda/1+\lambda)^{\alpha(k)}$. Since ω is constant on each set A_k^p for $0 \leqslant k < 2^p$, for any Borel set $B \subseteq A_k^p$ we have

$$\frac{\mu(TB)}{\mu(B)} = \frac{\mu(TA_k^p)}{\mu(A_k^p)} = \frac{\mu(A_{k+1}^p)}{\mu(A_k^p)} = \lambda^{\alpha(k+1)-\alpha(k)}.$$

We notice that $Q_{pq} = T^m A_0^{p+q+2}$ with $m = 2^p(2^{q+1}-1)$. For all $1 \leqslant k < 2^p$, we have $\alpha(m+k) = \alpha(m)+\alpha(k)$ and $\mu(T^k Q_{pq})/\mu(Q_{pq}) = \lambda^{\alpha(m+k)-\alpha(m)} = \lambda^{\alpha(k)} < 1$. Since $T^{2^p}Q_{pq}$ is a subset of A_0^{p+q+1}, for all $0 \leqslant r < 2^q$ we have

$$\frac{\mu(T^{2^p+r}Q_{pq})}{\mu(Q_{pq})} = \frac{\mu(A_r^{p+q})}{\mu(A_0^{p+q})}\frac{\mu(T^{2^p}Q_{pq})}{\mu(Q_{pq})} = \lambda^{\alpha(r)-q}.$$

We conclude the proof by noting that for all $0 \leqslant r < 2^q$, we have $\alpha(r) \leqslant q$ and $\alpha(r) = q$ if and only if $r = 2^q - 1$.

THEOREM 3.10. *Let T be the type* III_λ *binary odometer. Then $s_\mu(T) = 0$.*

Proof. The idea of the proof is to induce the measure preserving $S = T^\tau$ on the sets $A_1 = [1]$ and $A_0 = [0]$. If $x \in A_1$, then x can be written $x = (10\overset{p}{\ldots}01\overset{q}{\ldots}110\ldots)$; then

$$n_1(x) = \min\{k \geqslant 1 : S^k(x) \in A_1\} = p(x)+1,$$

$$S_1(x) = S^{n_1(x)}(x) = (11\overset{q}{\ldots}10\overset{p}{\ldots}001\ldots).$$

This implies that S_1 on A_1 is isomorphic to S; in particular $h(S_1) = h(S)$ (where $h(S_1)$ is computed with respect to the normalized measure on A_1). On the other hand, Abramov's formula tells us that

$$h(S) = \mu(A_1)\,h(S_1) = \frac{\lambda}{1+\lambda}h(S).$$

To conclude, it is enough to show that the entropy of S is finite. To do this we show that $(A_0, A_1\}$ is a generating partition for S. We note that, if $\mathscr{P}$ is a generating partition, by the Kolmogorov–Sinai theorem: $h(S) = h_\mu(S, \mathscr{P}) \leqslant \log 2$. We write $x \sim y$ when x and y are in the same element of the partition $\mathscr{P} = \{A_0, A_1\}$. In order to prove that $\mathscr{P}$ is a (two-sided) generating partition, we have to prove that for all $x, y \in X$, if $S^k(x) \sim S^k(y)$ for all k, then $x = y$. We notice that the induced map S_0 on A_0 is also isomorphic to S: for $x = (00\overset{p}{\ldots}01\overset{q}{\ldots}110)$ we have:

$$n_0(x) = \min\{k \geqslant 1 : S^k(x) \in A_0\} = q(x)+1,$$

$$S_0(x) = S^{n_0(x)}(x) = (01\overset{q}{\ldots}10\overset{p}{\ldots}001\ldots).$$

As usual we denote by σ the shift on X and notice that $\sigma \circ S_i = S \circ \sigma$ on A_i for $i = 0, 1$. Assume that x, y satisfy $S^k(x) \sim S^k(y)$ for all k. We first prove that $S^k \circ \sigma(x) \sim S^k \circ \sigma(y)$ for all k. Suppose that $x, y \in A_1$, since they have the same return times $n_1(S_1^k(x)) = n_1(S_1^k(y))$. In particular $p(S_1^k(x)) = p(S_1^k(y))$ and therefore $S^k \circ \sigma(x) = \sigma \circ S_1^k(x) \sim \sigma \circ S_1^k(y) = S^k \circ \sigma(y)$. Suppose that $x, y, \in A_0$, then $n_0(S_0^k(x)) = n_0(S_0^k(y))$ for all k. In particular $q(S_0^k(x)) = q(S_0^k(y))$ and therefore

$$S^{k+1} \circ \sigma(x) = \sigma \circ S_0^{k+1}(x) \sim \sigma \circ S_0^{k+1}(y) = S^{k+1} \circ \sigma(y) \quad \text{for all } k.$$

By induction we obtain $S^k \circ \sigma^l(x) \sim S^k \circ \sigma^l(y)$ for all k and all $l \geqslant 0$. In particular $\sigma^l(x) \sim \sigma^l(y)$ for all $l \geqslant 0$; thus $x = y$.

COROLLARY 3.11. *Let T be the type III_λ binary odometer, $0 < \lambda < 1$.*

(a) *If R is a rotation on a compact abelian group, then $s_{\mu\times\nu}(T\times R) = 0$. Furthermore, if R is an irrational rotation on the circle, then $T\times R$ is ergodic.*

(b) *If S is a finite measure preserving endomorphism with positive entropy, then $s_{\mu\times\nu}(T\times S) = +\infty$. Furthermore, if S is a lightly mixing automorphism, then $T\times S$ is ergodic and has the same ratio set as T.*

Proof. The skew product entropy equalities follow from Proposition 2.6 and Theorem 3.10. For part (a) let R be rotation by an irrational α. By a theorem of Furstenberg and Weiss, cf. [**1**, Section 2], if S is ergodic invertible finite measure preserving, and T is ergodic invertible nonsingular, then $T\times S$ is ergodic if and only if $\sigma_S(E(T)) = 0$, where σ_S is the maximal spectral type of S, and $E(T)$ is the L^∞ eigenvalue group of T. It is well known that the maximal spectral type of R is supported on $\{e^{in\alpha}: n \text{ an integer}\}$. It is also known that if T is the type III_λ binary odometer, then $E(T)$ consists of the dyadic rationals. Thus $T\times R$ is ergodic.

For part (b) we recall that a transformation S is *lightly mixing* if for all sets of positive measure A and B we have $\liminf_{n\to\infty}\mu(T^nA\cap B) > 0$, and S mildly mixing is equivalent to $\liminf_{n\to\infty}\mu(T^nA\cap A^c) > 0$ for all sets A with $\mu(A)\,\mu(A^c)\neq 0$. Therefore S must be mildly mixing and so $T\times S$ is ergodic [**3**]. Finally, the equality of the ratio sets of $T\times S$ and T follows from [**6**, Lemma 2; **7**, Lemma 2.2] since $\lambda\neq 0$ and S is lightly mixing.

We next consider another example of a nonsingular odometer with three symbols whose Radon–Nikodym derivative takes values in $\{\lambda, 1/\lambda\}$. This will allow us to compute the skew product entropy for the type III_λ Chacon nonsingular automorphisms.

DEFINITION 3.12. Let $X = \prod_0^{+\infty}\{0,1,2\}$, $T\colon X\to X$ the addition by $100\ldots$ with carry, $\mu = \prod_0^{+\infty}\{\lambda/(1+2\lambda), 1/(1+2\lambda), \lambda/(1+2\lambda)\}$. We denote by $\omega = (d\mu\circ T/d\mu)$ the Radon–Nikodym derivative and by $\tau(x)$ the smallest integer $n\geqslant 1$ such that $\omega_n(x) = 1$. The transformation T is a type III_λ for $0<\lambda<1$, conservative ergodic nonsingular automorphism, that we call the *ternary type* III_λ *odometer*.

THEOREM 3.13. *Let (X,μ,T) be the ternary type III_λ odometer. For all $n\geqslant 0$ we set $\omega(x) = 1/\lambda$ on $[2\overset{n}{\ldots}20]$ and $\omega(x) = \lambda$ on $[2\overset{n}{\ldots}21]$. Then τ is defined by:*

$$\tau(x) = 3^q + 1 \text{ and } T^\tau(x) = 0\overset{q}{\ldots}02\ldots \qquad \text{for all } x = 2\overset{q}{\ldots}20\ldots,$$

$$\tau(x) = 2.3^p \text{ and } T^\tau(x) = 1\overset{p}{\ldots}10\overset{q+1}{\ldots}01\ldots \quad \text{for all } x = 1\overset{p+1}{\ldots}12\overset{q}{\ldots}20\ldots,$$

$$\tau(x) = 2.3^p \text{ and } T^\tau(x) = 1\overset{p+1}{\ldots}10\overset{q}{\ldots}02\ldots \quad \text{for all } x = 1\overset{p}{\ldots}12\overset{q+1}{\ldots}21\ldots.$$

Furthermore, $s_\mu(T) = 0$.

Proof. The proof is similar to that given in Lemma 3.9. For concreteness we let $\lambda = \frac{1}{2}$. We define $A_0^p = [0\overset{p}{\ldots}0]$ and for all $k = \sum_{i=0}^{+\infty} k_i 3^i$ with $k_i\in\{0,1,2\}$, we define $\alpha(k) = \mathrm{card}\{i\geqslant 0: k_i = 1\}$. We notice that $\alpha(k+3^p l) = \alpha(k)+\alpha(l)$ for all $0\leqslant k < 3^p$ and $l\geqslant 0$; we notice also that $\alpha(k)\leqslant p$ for all $k<3^p$ and $\alpha(k) = p$ if and only if $k = \sum_{i=1}^{p-1} 3^i$.

Case (i): $B = [2 \overset{q}{\ldots} 20]$. Then $TB = [0 \overset{q}{\ldots} 01] \subseteq A_0^q$. For all $x \in B$ and $0 \leqslant k < 3^p$, we have

$$\omega_{k+1}(x) = \frac{\mu(T^{k+1}B)}{\mu(B)} = \frac{\mu(A_k^q)}{\mu(A_0^q)} \frac{\mu(TB)}{\mu(B)} = 2^{\alpha(k)+1} \geqslant 2$$

and $\omega_{3^q+1}(x) = 1$. We thus obtain $\tau(x) = 3^q + 1$ and $T^\tau(x) = 0 \overset{q}{\ldots} 02x_{q+1}x_{q+2} \ldots$.

Case (ii): $B = [1 \overset{p+1}{\ldots} 12 \overset{q}{\ldots} 20]$. Then $B = T^m A_0^{p+q+2}$, where

$$m = 3^{p+1}(3^q - 1) + \textstyle\sum_{i=0}^{p} 3^i.$$

For all $x \in B$, for all $m < k < 3^{p+1+1}$, we have

$$\omega_k(x) = \frac{\mu(T^k A_0^{p+q+2})}{\mu(T^m A_0^{p+q+2})} = 2^{\alpha(k)-\alpha(m)},$$

where $\alpha(k) - \alpha(m) = \alpha(k - 3^{p+1}(3^q - 1)) - \alpha(\sum_{i=0}^{p} 3^i) < 0$. Also

$$T^{3^{p+q+1}} A_0^{p+q+2} = [0 \overset{p+q+1}{\ldots} 01] \subseteq A_0^{p+q+1}, \quad \mu([0 \overset{p+q+1}{\ldots} 01]) = 2^{-p}\mu([1 \overset{p+1}{\ldots} 12 \overset{q}{\ldots} 20]).$$

For all $0 \leqslant l < 3^p$ and $x \in B$ we have

$$\omega_{3^{p+q+1}+l}(x) = \frac{\mu(T^l A_0^{p+q+1})}{\mu(A_0^{p+q+1})} \frac{\mu(T^{3^{p+q+1}} A_0^{p+q+2})}{\mu(T^m A_0^{p+q+2})} = 2^{\alpha(l)-p} \leqslant 1$$

and $\omega_{3^{p+q+1}+l}(x) = 1$ if and only if $l = \sum_{i=0}^{p-1} 3^i$. We thus obtain

$$\tau(x) = 3^{p+q+1} - m + \sum_{1=0}^{p-1} 3^i = 3^{p+1} - 3^p = 2.3^p \quad \text{and} \quad T^\tau(x) = 1 \overset{p}{\ldots} 10 \overset{q+1}{\ldots} 01x_{p+q+2} \ldots.$$

Case (iii): $B = [1 \overset{p}{\ldots} 12 \overset{q+1}{\ldots} 21]$. Then $B = T^m A_0^{p+q+2}$ with

$$m = 3^{p+q+1} + 3^p(3^{q+1} - 1) + \textstyle\sum_{i=0}^{p-1} 3^i.$$

For all $x \in B$ and $0 < k < 3^p - \sum_{i=0}^{p-1} 3^i$ we have

$$\omega_k(x) = \frac{\mu(T^{k+m} A_0^{p+q+2})}{\mu(T^m A_0^{p+q+2})} = 2^{\alpha(k+m)-\alpha(m)},$$

where $\alpha(k+m) - \alpha(m) = \alpha(k + \sum_{i=0}^{p-1} 3^i) - \alpha(\sum_{i=0}^{p-1} 3^i) < 0$. Also

$$T^{2.3^{p+q+1}} B = [0 \overset{p+q+1}{\ldots} 02] \subseteq A_0^{p+q+1} \quad \text{and} \quad \mu(T^{2.3^{p+q+1}} B) = 2^{-p-1}\mu(B).$$

For all $0 \leqslant k < 3^{p+1}$ and $x \in B$ we have

$$\omega_{2.3^{p+q+1}+k}(x) = \frac{\mu(T^k A_0^{p+q+1})}{\mu(A_0^{p+q+1})} = 2^{\alpha(k)-p-1} = 1$$

if and only if $k = \sum_{i=0}^{p} 3^i$. We thus obtain $\tau(x) = 2.3^{p+q+1} + \sum_{i=0}^{p} 3^i - m = 2.3^p$ and $T^\tau(x) = 1 \overset{p+1}{\ldots} 10 \overset{q}{\ldots} 02x_{p+q+2} \ldots$.

Let $T^\tau = S$ and $Y = \{x \in X\colon S^k(x) \notin [1] \text{ for all } k \in \mathbb{Z}\}$. By ergodicity of T^τ, we have $\mu(Y) = 0$ and $Y = \{x \in X\colon \sigma^k(x) \notin [1] \text{ for all } k \geqslant 0\}$. We next show that $\{[0], [1], [2]\}$ is a generating partition for S restricted to the invariant set $Z = X \setminus Y$.

Let $n_i\colon Z \to \mathbb{N}$ be the first return time to $[i] \cap Z$ and S_i the first return map. The computations show that S_i is conjugate to S: $\sigma \circ S_i = S \circ \sigma$ on $[i] \cap Z$ and that:

$$n_1(x) = 2^{q+1} + 1 \quad \text{for } x \in [12\overset{q}{\dots}20],$$

$$n_1(x) = 1 \quad \text{for } x \in [11\overset{p+1}{\dots}12\overset{q}{\dots}20] \cup [11\overset{p}{\dots}12\overset{q+1}{\dots}21]$$

and for $i = 0, 2$ we have:

$$n_i(x) = 2 \quad \text{for } x \in [i2\overset{q}{\dots}20],$$

$$n_i(x) = 2p + 3 \quad \text{for } x \in [i1\overset{p+1}{\dots}12\overset{q}{\dots}20] \cup [i1\overset{p}{\dots}12\overset{q+1}{\dots}21].$$

In order to prove that $h(S_{|Z}) = 0$, it is enough to show that $S^k(x) \sim S^k(y)$ for all k implies that $S^k \circ \sigma(x) \sim S^k \circ \sigma(y)$ for all k. Following the method in Theorem 3.10, it is enough to show that $x, y \in [i]$ and $n_i \circ S_i^k(x) = n_i \circ S_i^k(y)$ for all k implies that $\sigma(x) \sim \sigma(y)$.

Case 1: $i = 0$, $(x, y) \in [0] \cap Z$ *and* $n_0 \circ S_0^k(x) = n_0 \circ S_0^k(y)$ *for all* k (the case $i = 2$ is similar). Suppose that $n_0(x) = n_0(y) = 2p + 3$; if $p \geqslant 1$ then $\sigma(x), \sigma(y) \in [1]$; if $p = 0$ then the case $x = 012\overset{q}{\dots}20\dots$ and $y = 02\overset{r+1}{\dots}21\dots$ cannot happen since $n_0 \circ S_0(x)$ is even and $n_0 \circ S_0(y)$ is odd: in both cases $\sigma(x) \sim \sigma(y)$. Suppose that $n_0(x) = n_0(y) = = 2$, since $(x, y) \in Z$ there exist $a \geqslant 1$ and $b \geqslant 1$ such that

$$x = 0x_0 \dots x_{a-1} 1 \dots, \quad y = 0y_0 \dots y_{b-1} 1 \dots,$$

where $x_i, y_i \in \{0, 2\}$, $x_0 \dots x_{a-1} \neq 2\overset{a}{\dots}2$ and $y_0 \dots y_{b-1} \neq 2\overset{b}{\dots}2$. We claim that $a = b$ and $x_i = y_i$ for $0 \leqslant i < a$. We first notice that $n_0(0x_0 \dots x_{a-1} 1 \dots) = 2$ if $x_0 \dots x_{a-1} \neq 2\overset{a}{\dots}2$ and $n_0(02\overset{a}{\dots}21\dots) = 3$. Let $k = \sum_{i=0}^{a-1} \frac{1}{2} x_i 2^i$ and $l = \sum_{i=0}^{b-1} \frac{1}{2} y_i 2^i$ then $x = S_0^k(00\overset{a}{\dots}01\dots)$ and $y = S_0^l(00\overset{b}{\dots}01\dots)$. Since $S_0^{2^a - k}(x) = 02\overset{a}{\dots}21\dots$ and $S_0^{2^b - l}(y) = 02\overset{b}{\dots}21\dots$ then $2^a - k = 2^b - l$ and the claim is proven. In both cases $\sigma(x) \sim \sigma(y)$.

Case 2: $i = 1$, $(x, y) \in [1] \cap Z$ *and* $n_1 \circ S_1^k(x) = n_1 \circ S_1^k(y)$ *for all* k. Suppose that $n_1(x) = n_1(y) > 1$, then

$$2^{q(x)+1} = 2^{q(y)+1}, \quad q(x) = q(y) \quad \text{and} \quad \sigma(x) \sim \sigma(y).$$

Suppose that $n_1(x) = n_1(y) = 1$; we claim that the case $x = 11\overset{p+1}{\dots}12\overset{q}{\dots}20\dots$ and $y = 12\overset{r+1}{\dots}21\dots$ cannot happen. Either $p = 0$, $S_1(x) = 110\overset{q+1}{\dots}01\dots$ and $S_1(y) = 11\overset{r}{\dots}02\dots$, and since $n_1 \circ S_1(x) = 3$ and $n_1 \circ S_1(y) = 1$, we obtain a contradiction; or $p \geqslant 1$, $S_1^{-1}(x) = 11\dots$ and $S_1^{-1}(y) = 10\dots$, and we obtain a contradiction since $n_1 \circ S_1^{-1}(x) = 1$ and $n_1 \circ S_1^1(y) = 3$.

We now consider a family of maps that we call nonsingular type III_λ Chacon automorphisms for $0 < \lambda < 1$. This family was first suggested in **[21]**, where it was claimed that they have rational (nonsingular) minimal self-joinings, and hence have no nontrivial proper invertible factors. However, they were not defined and were replaced by a more general family of maps (that includes type III_λ for $0 \leqslant \lambda \leqslant 1$ and type II_∞). Recently, A. del Junco and the first named author have used this family (in a slightly more general form than in Definition 3.15) to construct various examples. However, the methods used are those of joinings, and while they serve well to control invertible factors (see e.g. **[21]**), as far as we know, they cannot be used to control noninvertible factors (see Example 3.14).

EXAMPLE 3.14. Let (X, μ, T) be the two-sided finite measure preserving Bernoulli shift with two symbols and measure $(\frac{1}{2}, \frac{1}{2})$. Then $\mathscr{F} = \bigvee_{i=0}^{+\infty} T^{-i}\{[0],[1]\}$ is a noninvertible factor algebra. The relatively independent joining ν over the factor algebra $\mathscr{F}$ is defined by

$$\nu(A \times B) = \int \mathbf{E}(1_A \mid \mathscr{F})\, \mathbf{E}(1_B \mid \mathscr{F})\, d\mu$$

for any Borel sets A and B. Then we have that $\nu([0]_1 \times [1]_1) = 0$ but $\nu(T[0]_1 \times T[1]_1) = \frac{1}{4}$, so that the joining is not nonsingular for $T \times T$.

DEFINITION 3.15. Let (X, μ, T) be the ternary type III_λ odometer. We define T' to be the exduced transformation (cf. **[12]**) of T on the symbols $2^k 1$. That is, let $X' = X \cup [2^*1']$, where $[2^*1']$ is a disjoint copy of the sequences starting with $2^k 1$ for $k \geqslant 0$. Extend the measure and the Borel sets in a natural way to X'. For $x \in X'$: if $x \in X \backslash [2^*1]$ let $T'(x) = T(x)$, if $x = (2^k 1 x_1 x_2 \ldots)$ let $T'(x) = (2^k 1' x_1 x_2 \ldots)$, and if $x = (2^k 1' x_1 x_2 \ldots)$ let $T'(x) = T(2^k 1 x_1 x_2 \ldots)$. We call (X', μ', T') the *nonsingular type* III_λ *Chacon* automorphism.

It follows that T' is a type III_λ for $0 < \lambda < 1$, conservative ergodic nonsingular automorphism. We would like to thank Andrés del Junco for observing the following consequence of Theorem 3.13.

COROLLARY 3.16. *Let T' be the type* III_λ *Chacon automorphism. Then* $s_{\mu'}(T') = 0$.

Proof. It follows from Definition 3.15 that the transformation T' induced on X is the transformation of Theorem 3.13, which has zero skew product entropy. Proposition 3.2(f) completes the proof.

4. *Nonsingular* K-*automorphisms*

We develop in this section the notions of natural extension and K-automorphism for nonsingular maps. We first recall two definitions. A factor algebra $\mathscr{F}$ is called *exact* if

$$\bigcap_{k \geqslant 0} T^{-k}\mathscr{F} = \{\emptyset, X\} \pmod{\mu}$$

and *exhaustive* (when T is an automorphism) if

$$\bigvee_{0}^{\infty} T^{n}\mathscr{F} = \mathscr{B}(X) \pmod{\mu}.$$

DEFINITION 4.1. Let (X, μ, T, ω) be a nonsingular endomorphism. A nonsingular automorphism $(\tilde{X}, \tilde{\mu}, \tilde{T})$ is called a *natural extension* of (X, μ, T, ω) if there is a markovian factor map $\phi: \tilde{X} \to X$ such that $\phi^{-1}(\mathscr{B}(X))$ is an exhaustive factor algebra of $\mathscr{B}(\tilde{X})$.

Since in the measure preserving case all factor maps are markovian, Definition 4.1 coincides with Rohklin's definition of the natural extension [**20**]. In the nonsingular case, it is necessary to require the factor map to be markovian; at the end of this section we mention an example of a type III automorphism (Hamachi's example, cf. 4.9) that has an exhaustive factor that is finite measure preserving – this factor is not markovian, and in general, as we shall see below, it could never be markovian. Further, it is necessary to consider the natural extension of T with (or with respect to) ω. The natural extensions of T with respect to different markovian functions may have different properties cf. [**24**, Example 5.4]. However, if (ω', μ') is cohomologous to (ω, μ), then the natural extensions of (X, μ, T, ω) and (X, μ', T, ω') are in a canonical way *nonsingular* isomorphic but not necessarily (measure preserving) isomorphic. We refer to Maharam [**17**] for another construction of an invertible extension.

THEOREM 4.2. *Let (X, μ, T, ω) be a nonsingular endomorphism. Then there exists a unique natural extension up to isomorphism.*

Proof. Existence was proved in [**23**, **24**]. Let $(\bar{X}, \mu, \bar{T}, \bar{\omega})$ be the invertible extension of (X, μ, T, ω) defined in [**24**, Theorem 5.9] using inverse limits and let $(\tilde{X}, \tilde{\mu}, \tilde{T}, \tilde{\omega})$ be any natural extension. Let $\bar{\phi}: \bar{X} \to X$ and $\tilde{\phi}: \tilde{X} \to X$ be the factor maps. We define $\psi: \tilde{X} \to \bar{X}$ by $\psi(\tilde{x}) = (\tilde{\phi} \circ \tilde{T}^{-n}(\tilde{x}))_{n \geqslant 0}$. It is clear that $\bar{\phi} \circ \psi = \tilde{\phi}$. If f is a positive $\mathscr{B}(X)$-measurable function, then

$$\int f \circ \bar{\phi} \circ \bar{T}^{-n} \circ \psi \, d\tilde{\mu} = \int f \circ \bar{\phi} \circ \psi \circ \tilde{T}^{-n} \, d\tilde{\mu} = \int f \circ \bar{\phi} \circ \psi \tilde{\omega}_n \, d\tilde{\mu} = \int f \circ \tilde{\phi} \tilde{\omega}_n \, d\tilde{\mu}$$

$$= \int f \omega_n \, d\mu = \int f \circ \bar{\phi} \bar{\omega}_n \, d\bar{\mu} = \int f \circ \bar{\phi} \circ \bar{T}^{-n} \, d\bar{\mu}.$$

Since $\bigcup_{n \geqslant 0} \bar{T}^n \circ \bar{\phi}^{-1}(\mathscr{B}(X))$ generates $\mathscr{B}(\bar{X})$, the above calculations show that ψ is a measure preserving map. Since $\bigcup_{n \geqslant 0} \tilde{T}^n \circ \tilde{\phi}^{-1}(\mathscr{B}(X))$ generates $\mathscr{B}(\tilde{X})$, we have that $\psi^{-1}: \mathscr{B}(\bar{X}) \to \mathscr{B}(\tilde{X})$ defines a one-to-one and onto map between the Borel measure algebras of standard spaces and therefore shows that ψ is invertible.

The following lemma is the main lemma which transposes the properties of a nonsingular endomorphism to its natural extension.

LEMMA 4.3. *Let (X, μ, T) be a conservative nonsingular automorphism, $\mathscr{F} \subseteq \mathscr{B}(X)$ be an exhaustive Markovian factor algebra and $h: X \to \mathbb{R}^+$ a positive $\mathscr{B}(X)$-measurable function. If $(h/h \circ T)$ is $\mathscr{F}$-measurable, then h is $\mathscr{F}$-measurable also.*

Proof. The proof is divided into three parts.

Part 1. We assume that μ is finite T-invariant ($\omega = 1$) and that h is T-invariant ($h = h \circ T$). Since $\bigcup_{n \geqslant 1} T^n \mathscr{F}$ generates $\mathscr{B}(X)$, then for every $\alpha < \beta$, $B = \{x: \alpha < h < \beta\}$ and $\varepsilon > 0$, there exist $n \geqslant 1$ and F, where F is $\mathscr{F}$-measurable, such that $\mu(B\Delta T^n F) < \varepsilon$. By invariance of μ and B, we have $\mu(B\Delta F) < \varepsilon$. In particular B and therefore h is $\mathscr{F}$-measurable (mod μ).

Part 2. We assume that μ is σ-finite T-invariant and that h is T-invariant. Since $\mathscr{F}$ is σ-finite, we can find a partition of X, say $\{X_k\}_{k\geq 0}$, of $\mathscr{F}$-measurable sets which have finite measure. We call $T_k\colon X_k \to X_k$ the induced map, $\tau_k\colon X_k \to \mathbb{N}$ the return time and $\mathscr{F}_k, \mu_k, h_k$ the restriction of $\mathscr{F}, \mu, h$ to X_k. By conservativity of T, we know that μ_k is T_k-invariant. Since h_k is T_k-invariant, it is enough to show that $\mathscr{F}_k$ is an exhaustive factor algebra for T_k. If $F \subseteq X_k$ is $\mathscr{F}$-measurable, then

$$T_k^{-1}(F) = \bigcup_{n\geq 1} (\tau_k = n) \cap T^{-n}F \in \mathscr{F}.$$

If $B \subseteq X_k$ is $\mathscr{B}(X)$-measurable, for every $\varepsilon > 0$, there exists F, where F is $\mathscr{F}$-measurable, such that $\mu(B\Delta T^n F) < \varepsilon$. Since $X_k \cap T^n F = \bigcup_{1\leq p, 0\leq q} T_k^p G_{p,q}$, where

$$G_{p,q} = (X\backslash F) \cap \cdots \cap (X\backslash T^{-q+1}F) \cap T^{-q}F \cap (\tau_k \geq q+1) \cap \left(\sum_{i=0}^{p-1} \tau_k \circ T_k^i = n+q\right),$$

is $\mathscr{F}$-measurable, $\mu(B\Delta G) < \varepsilon$ for some $G \in \bigvee_{n\geq 0} T_k^n \mathscr{F}_k$.

Part 3 (general case). We define the following extension $(\bar{X}, \bar{\mu}, \bar{T})$ where $\bar{X} = X \times \mathbb{R}^+ \times \mathbb{R}^+$, $\bar{\mu} = h\mu \otimes \mathrm{Leb} \otimes \mathrm{Leb}$ and $\bar{T}(x,s,t) = (T(x), s/\omega(x), th(x)/h\circ T(x))$ and also $\bar{\mathscr{F}} = \mathscr{F} \otimes \mathscr{B} \otimes \mathscr{B}$, $\bar{h}(x,s,t) = th(x)$. We notice that $\bar{\mu}, \bar{h}$ are $\bar{T}$-invariant and show that $\bar{\mathscr{F}}$ is a σ-finite exhaustive factor for $\bar{T}$. Let $a\colon X \to \mathbb{R}$, $b\colon \mathbb{R}^+ \to \mathbb{R}$ and $c\colon \mathbb{R}^+ \to \mathbb{R}$ be integrable functions, then, for every $\varepsilon > 0$, there exist $a_1\colon X \to \mathbb{R}$, where a_1 is $\mathscr{F}$-measurable, and $n \geq 1$ such that

$$\int |a - a_1 \circ T^{-n}|\, d\mu \leq \varepsilon.$$

Define $b_1(x,s,t) = b(s/\omega_n(x))$, $c_1(x,s,t) = c(th(x)/h\circ T^n(x))$, *then*

$$\int |abc - (a_1 b_1 c_1)\circ \bar{T}^{-n}|\, d\bar{\mu} < \varepsilon \int |b(s)\, c(t)|\, ds\, dt$$

and $a_1 b_1 c_1$ is $\bar{\mathscr{F}}$-measurable. By Part 2, it is enough to show that $\bar{T}$ is conservative: indeed we notice that $\bar{T}$ is the Maharam skew product of $(X^*, h\mu^*, T_\omega, h\circ T/h)$, where (X^*, μ', T_ω) is the Maharam skew product of (X, μ, T).

Parts (a) and (b) of the following theorem were proved in [**23**]; we outline a proof here for completeness.

THEOREM 4.4. *Let (X, μ, T, ω) be a nonsingular endomorphism and $(\tilde{X}, \mu, \tilde{T})$ be its natural extension. Then*

(a) *ω is recurrent if and only if $\tilde{T}$ is conservative;*

(b) *if ω is recurrent, then T is ergodic if and only if $\tilde{T}$ is ergodic;*

(c) *if ω is recurrent and T is ergodic, then T is type* III *if and only if $\tilde{T}$ is type* III.

Proof. Let $\phi\colon \tilde{X} \to X$ be a markovian factor map such that $\mathscr{F} = \phi^{-1}\mathscr{B}(X)$ is an exhaustive factor algebra, and $\tilde{\omega} = \omega \circ \phi = d\tilde{\mu}\circ \tilde{T}/d\tilde{\mu}$.

(a) We have that ω is recurrent if and only if $\tilde{\omega}$ is recurrent if and only if $\tilde{T}$ is conservative (cf. [**24**, Corollary 5.4]).

(b) If ω is recurrent, as $\tilde{T}$ is conservative, by Lemma 4.3, $\tilde{B} \in \mathscr{B}(\tilde{X})$ is $\tilde{T}$-invariant if and only if there exists $B \in \mathscr{B}(X)$, where B is T-invariant, such that $\tilde{B} = \phi^{-1}(B)$. It follows that $\tilde{\mathscr{I}} = \phi^{-1}(\mathscr{I}) \pmod{\tilde{\mu}}$ where $\tilde{\mathscr{I}}$ (respectively, $\mathscr{I}$) denotes the σ-algebra of $\tilde{T}$-invariant (respectively T-invariant) sets.

(c) Suppose that T is ergodic and ω recurrent. By [**24**, Theorem 5.6], if T is type II, then ω is a coboundary $\omega = h/h \circ T$, $\tilde{\omega} = \tilde{h}/\tilde{h} \circ \tilde{T}$ with $\tilde{h} = h \circ \phi$, and $\tilde{T}$ preserves the measure $\tilde{h}\tilde{\mu}$: hence $\tilde{T}$ is type II. Conversely, if $\tilde{T}$ is type II, then $\tilde{\omega}$ is an $\mathscr{F}$-measurable coboundary $\tilde{\omega} = \tilde{h}/\tilde{h} \circ \tilde{T}$; by conservativity and Lemma 4.3, $\tilde{h}$ is $\mathscr{F}$-measurable, $\tilde{h} = h \circ \phi$, $\omega = h/h \circ T$, T preserves the measure $h\mu$: therefore T is type II.

DEFINITION 4.5. Let (X, μ, T) be a nonsingular automorphism such that $\mathscr{B}(X) \neq \{\varnothing, X\}$. Then T is said to be a K-*automorphism* if T is conservative and admits a factor algebra $\mathscr{F}$ that is exhaustive and exact and such that $d\mu \circ T/d\mu$ is $\mathscr{F}$-measurable.

REMARK 4.6. (a) The definition of a K-automorphism is with respect to a fixed measure μ. For a given factor $\mathscr{F}$, the chosen measure μ is the only measure in its class for which $d\mu \circ T/d\mu$ is $\mathscr{F}$-measurable. This follows from Lemma 4.3.

(b) We could have defined K-automorphisms without requiring them to be conservative, and then a K-automorphism would have been either conservative or totally dissipative: the dissipative part $\{x: \sum_{k \geqslant 0} h \circ T^k \omega_k < +\infty\}$ (for some positive integrable $\mathscr{F}$-measurable function h) is indeed $\mathscr{F}$-measurable and invariant. Parry studied in [**18**] K-automorphisms in the context of infinite invariant measure, and allowed them to be dissipative. He also proved in his context (infinite measure preserving K-automorphisms) a statement analogous to Lemma 4.3.

(c) Proposition 4.8(a) for the case of infinite measure preserving K-automorphisms was proved in [**18**], and for a special case of nonsingular K-automorphisms in [**15**].

(d) It follows from the definitions and Theorem 4.4, that if (X, μ, T, ω) is an exact nonsingular endomorphism and ω is recurrent then its natural extension is a nonsingular K-automorphism. If T is a nonsingular K-automorphism then it is the natural extension of an exact nonsingular endomorphism with respect to a recurrent markovian pair.

The notion of weakly mixing was extended to nonsingular automorphisms in [**3**], where the authors also give other characterizations equivalent to the definition below. They also asked for a property that is equivalent to the ergodic multiplier property mentioned in Proposition 4.8(b), a question we thank Aaronson for pointing out to us.

DEFINITION 4.7. Let (X, μ, T) be a nonsingular automorphism. Then T is said to be *weakly mixing* if for every finite measure preserving ergodic automorphism (Y, ν, S), we have $(X \times Y, \mu \times \nu, T \times S)$ is ergodic.

PROPOSITION 4.8. *Let (X, μ, T) be a nonsingular* K-*automorphism. Then*

(a) *T is ergodic,*

(b) *for every ergodic nonsingular automorphism (Y, ν, S), if $T \times S$ is conservative then $T \times S$ is ergodic,*

(c) *T is weakly mixing,*

(d) $s_\mu(T) = +\infty$.

Proof. (a) If A is an invariant set, by Lemma 4.3, A belongs $\mathscr{F}$ and so has to be trivial.

(b) Let $\mathscr{F}^* = \mathscr{F} \otimes \mathscr{B}(Y)$, then $\mathscr{F}^*$ is exhaustive markovian for $T \times S$ and its tail σ-algebra is $\{\emptyset, X\} \otimes \mathscr{B}(Y) (\text{mod}\, \mu \times \nu)$. Suppose that E is $T \times S$-invariant; since $T \times S$ is conservative, then by Lemma 4.3, $E \in \mathscr{F}^*$ and therefore $E \in \{\emptyset, X\} \otimes \mathscr{B}(Y)$. By ergodicity of S, it follows that E has to be trivial.

(c) This follows from (b).

(d) The mapping T is the natural extension of a nonsingular endomorphism S that corresponds to the exhaustive factor $\mathscr{F}$. Since S is not invertible a.e., the result follows from Proposition 3.2 Parts (g) and (i).

We consider now Hamachi's example **[9]** of constructing an exact nonsingular type III endomorphism whose natural extension (with respect to a recurrent markovian function) is the original Hamachi example.

EXAMPLE 4.9. Let (X, μ, T) be Hamachi's example **[9]** defined by $X = \prod_{-\infty}^{+\infty} \{0, 1\}$, T is the left shift, and $\mu = \prod_{-\infty}^{+\infty} \mu_k$ is a product, where $\mu_k = \{\frac{1}{2}, \frac{1}{2}\}$ for all $k \geqslant 0$, and μ_k for $k < 0$ is chosen carefully so that the shift is nonsingular conservative ergodic type III for the resulting measure. Hamachi shows that

$$\omega(x) = d\mu \circ T / d\mu = \prod_{-\infty}^{0} \mu_{k-1}(x_k) / \mu_k(x_k).$$

In particular ω is $\mathscr{B}^- = \bigvee_{-\infty}^{0} T^{-i}\mathscr{P}$-measurable, where $\mathscr{P}$ is the time zero partition $\mathscr{P} = \{[0], [1]\}$. Define $\mathscr{B}^+ = \bigvee_{0}^{+\infty} T^{-i}\mathscr{P}$. Let (Y, ν, S, ϖ) be the nonsingular endomorphism defined by: $Y = \prod_{-\infty}^{-} \{0, 1\}$, S is the right shift, $\phi \colon X \to Y$ is the factor map $\phi(x) = (\ldots, x_{-1}, x_0)$, $\nu = \mu \circ \phi^{-1}$, and $\varpi(y) = 1/\omega \circ T^{-1}(x)$ for any $x \in \phi^{-1}(y)$. (A similar construction can be done from Krengel's example **[15]**.)

PROPOSITION 4.10. *With the above notation (Y, ν, S, ϖ) is a recurrent exact type* III *endomorphism whose natural extension is (X, μ, T^{-1}).*

Proof. We note that (X, μ, T^{-1}) is a nonsingular K-automorphism with exhaustive and exact factor $\mathscr{B}^- = \phi^{-1}(\mathscr{B}(Y))$. Moreover, ω is $\mathscr{B}^-$-measurable and ϕ is a markovian factor; ϖ is recurrent and type III by Theorem 4.4. By Kolmogorov's 0–1 law, S is exact.

The need to consider markovian extensions is illustrated by the following example. The automorphism (X, μ, T) has an exhaustive and exact factor algebra $\mathscr{B}^+$ which is the $\{\frac{1}{2}, \frac{1}{2}\}$ one-sided Bernoulli shift, but it is not a markovian factor.

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