Nonlinearity

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To cite this article: Xifeng Su and Philippe Thieullen 2018 *Nonlinearity* **31** 2126

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **31** (2018) 2126–2155 https://doi.org/10.1088/1361-6544/aaacbb

Convergence of discrete Aubry–Mather model in the continuous limit

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Received 18 July 2017, revised 8 January 2018
Accepted for publication 2 February 2018
Published 6 April 2018

CrossMark

Recommended by Professor Dmitry V Treschev

Abstract

We develop two approximation schemes for solving the cell equation and the discounted cell equation using Aubry–Mather–Fathi theory. The Hamiltonian is supposed to be Tonelli, time-independent and periodic in space. By Legendre transform it is equivalent to find a fixed point of some nonlinear operator, called Lax-Oleinik operator, which may be discounted or not. By discretizing in time, we are led to solve an additive eigenvalue problem involving a discrete Lax–Oleinik operator. We show how to approximate the effective Hamiltonian and some weak KAM solutions by letting the time step in the discrete model tend to zero. We also obtain a selected discrete weak KAM solution as in Davini *et al* (2016 *Invent. Math.* **206** 29–55), and show that it converges to a particular solution of the cell equation. In order to unify the two settings, continuous and discrete, we develop a more general formalism of the short-range interactions.

Keywords: discrete weak KAM theory, Frenkel–Kontorova models, Aubry–Mather theory, discounted Lax–Oleinik operator, ergodic cell equation, short-range interactions, additive eigenvalue problem

Mathematics Subject Classification numbers: 37J, 49L, 52C

1. Introduction

In this article, we consider a Hamiltonian $H(x,p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ which is C^2, periodic in x, time-independent and satisfies the following assumptions:

(L1) **Positive definiteness:** $H(x,p)$ is strictly convex with respect to p, i.e. the second partial derivative $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite as a quadratic form uniformly in $x \in \mathbb{T}^d$ and $\|p\| \leqslant R$, for every $R > 0$;
(L2) **Superlinear growth:** $H(x,p)$ is superlinear with respect to p, uniformly in x, that is,

$$\lim_{\|p\| \to +\infty} \inf_{x \in \mathbb{T}^d} \frac{H(x,p)}{\|p\|} = +\infty.$$

We will say that $H(x,p)$ is a *Tonelli Hamiltonian*. We denote by $L(x,v)$ the Legendre–Fenchel transform of $H(x,p)$. We call $L(x,v)$ the *Lagrangian* of the system; $L(x,v)$ is again C^2, strictly convex with respect to v, and superlinear. A more general framework could be chosen where $\mathbb{T}^d \times \mathbb{R}^d$ is replaced by the cotangent space T^*M of some compact manifold M, but this approach would increase the complexity of the notations. To illustrate the two approximation schemes we are going to present, we choose the following basic Hamiltonian:

$$H(x,p) = \frac{1}{2}\|p+P\|^2 - K(1-\cos(2\pi N \cdot x)),$$

where $P \in \mathbb{R}^d, N \in \mathbb{Z}^d$ and $K \in \mathbb{R}$ are three parameters. The Lagrangian becomes

$$L(x,v) = \frac{1}{2}\|v\|^2 - P \cdot v + K(1-\cos(2\pi N \cdot x)).$$

We consider the following two equations: the *PDE cell equation* and the *discounted PDE cell equation*,

$$H(x, \mathrm{d}u(x)) = \bar{H}, \tag{1}$$

$$\delta u_\delta(x) + H(x, d_x u_\delta(x)) = 0, \tag{2}$$

where $u(x)$ and $u_\delta(x)$ solve (1) and (2) in the viscosity sense. Our main objective is to describe an ergodic approximation scheme for each equation.

Equation (1) is a degenerate PDE equation of first order with two unknowns $(\bar{H}, u)$. The constant $\bar{H}$ is unique and is called *the effective Hamiltonian*. The function $u(x)$ is C^0 periodic but may not be unique. Equation (2) is more regular and admits a unique C^0 periodic solution $u_\delta(x)$. Equation (1) was first studied by Lions, Papanicolaou and Varadhan [LPV87]. A comprehensive treatment may be found in Crandall *et al* [CIL92], Bardi *et al* [BCD97] or Barles [Bar94]. Some recent overviews may be found in the articles [Ish13, Bar13].

A new approach has been initiated by Mather and Fathi [Mat91, Mat93, Fat97a, Fat97b, Fat08] to solve equation (1). Fathi showed that (1) is equivalent to an additive eigenvalue problem for a semi-group of nonlinear operators,

$$u(x) - t\bar{H} = T^t[u](x), \quad \forall t > 0, \ \forall x \in \mathbb{R}^d, \tag{3}$$

$$T^t[u](x) := \inf_{\substack{\gamma \in C^{ac}([-t,0],\mathbb{R}^d) \\ \gamma(0)=x}} \Big[u(\gamma(-t)) + \int_{-t}^0 L(\gamma, \dot{\gamma}) \, \mathrm{d}s \Big], \tag{4}$$

(where the infimum is taken over absolutely continuous paths over $[-t,0]$ with the terminal point $x \in \mathbb{R}^d$). For the Tonelli Hamiltonian, the infimum is actually attained by a C^2 curve thanks to the Tonelli–Weierstrass theorem.

Equation (3) is called the *ergodic cell equation*, and T^t is called the *(backward) Lax–Oleinik semi-group*. Fathi calls the unknown $u(x)$ the *weak KAM solution*, and $\bar{H}$ is as before the effective Hamiltonian. Mañé [Mn96] first recognized the importance of this constant $\bar{H}$. After Contreras and Iturriaga [CI99], $\bar{H}$ is called the *Mañé critical value*: $\bar{H}$ has the explicit value

$$-\bar{H} := \lim_{t\to+\infty} \inf_{\gamma\in C^{ac}([-t,0],\mathbb{R}^d)} \Big[\frac{1}{t}\int_{-t}^{0} L(\gamma,\dot{\gamma})\,\mathrm{d}s\Big]. \tag{5}$$

Equation (2) has been studied by [LPV87, CIL92, Bar94, BCD97]. The solution is unique and given explicitly by the integral formula

$$u_\delta(x) = \inf_{\substack{\gamma\in C^2((-\infty,0],\mathbb{R}^d)\\ \gamma(0)=x}} \int_{-\infty}^{0} \mathrm{e}^{s\delta} L(\gamma(s),\dot{\gamma}(s))\,\mathrm{d}s, \tag{6}$$

where the infimum is taken over C^2 paths ending at x with a uniformly bounded first and second derivative. The two equations (1) and (2) are related; but very recently, the authors of [DFIZ16b] showed that $u_\delta(x)$, correctly normalized, converges to a selected solution $u^*(x)$ of (3),

$$\lim_{\delta\to 0}\Big(u_\delta(x)+\frac{\bar{H}}{\delta}\Big) = u^*(x) \quad (\text{exists in the } C^0 \text{ topology}). \tag{7}$$

We will call this selected solution u^*, the *balanced weak KAM solution*.

Our main objective is to develop approximation schemes that solve (1) and (2). In the first scheme, we compute an approximated effective Hamiltonian of (5) and an approximated weak KAM solution of (3). In the second scheme, we compute an approximated discounted weak KAM solution of (6) and show a similar selection principle. In both cases we discretize in time—either the semi-group (4) or the integral formula (6)—and rewrite the two problems in the framework of the Frenkel–Kontorova model.

The Frenkel–Kontorova model has been studied in solid state physics in 1D by [FK38] and then more rigorously by Aubry and Le Daeron [ALD83], Chou and Griffiths [CG86], and in a higher dimension by Gomes [Gom05], Garibaldi and Thieullen [GT11]. Similar problems under the name of Aubry–Mather theory have been studied using transport theory by Bernard and Buffoni [BB07] and Zavidovique [Zav12]. The Frenkel–Kontorova model describes the space of the configurations of an infinite chain of atoms $(x_n)_{n\in\mathbb{Z}}$ at the ground-level energy. In this model x_n denotes the position of the nth atom of the chain in $\mathbb{R}^d$, and $E(x_n,x_{n+1})$ denotes a *short-range interaction* between two successive atoms. The interaction $E(x,y)$ models both the internal interaction between the nearest atoms and the external interaction with the substrate. The *original Frenkel–Kontorova model* [FK38] is given by

$$E(x,y) = \frac{1}{2}\|y-x\|^2 - P\cdot(y-x) + K(1-\cos(2\pi N\cdot x)).$$

In solid state physics, it is more appropriate to write the elastic interaction as $\frac{1}{2}\|y-x-P\|^2$ instead of $\frac{1}{2}\|y-x\|^2 - P\cdot(y-x)$, where P denotes the mean distance at rest between two successive atoms of the chain. In Mather theory, P represents a cohomological term.

The main problem in the Frenkel–Kontorova model is to understand the set of configurations that minimizes the total interaction $\sum_{n\in\mathbb{Z}} E(x_n,x_{n+1})$ in a precise sense. Chou and Griffiths [CG86] were the first to highlight the importance of the following two quantities: $\bar{E}$, the effective interaction of the system (or the ground-state energy in Gibbs theory), and $u(x)$, the effective potential, which is a continuous periodic function that calibrates the interaction energy. They showed that $(\bar{E},u)$ can be seen as two unknowns of a discrete additive eigenvalue equation, now called the *discrete (backward) Lax–Oleinik equation*,

$$u(y) + \bar{E} = \inf_{x \in \mathbb{R}^d} \big\{ u(x) + E(x,y) \big\}, \quad \forall y \in \mathbb{R}^d. \tag{8}$$

The goal of the first scheme is to show that one can solve (3) by solving (8) with the following interaction $E(x,y) = \mathcal{L}_\tau(x,y)$ and by letting $\tau \to 0$. We call it *discrete action*,

$$\mathcal{L}_\tau(x,y) := \tau L\Big(x, \frac{y-x}{\tau}\Big), \quad \forall \tau > 0. \tag{9}$$

If $(\bar{\mathcal{L}}_\tau, u_\tau)$ is a solution of (8), one obtains in particular

$$\lim_{\tau \to 0} \frac{\bar{\mathcal{L}}_\tau}{\tau} = -\bar{H}, \quad \lim_{\tau_i \to 0} u_{\tau_i} = u \text{ (for some subsequence } \tau_i \searrow 0).$$

The discrete action associated with the basic example is given, for instance, by

$$E_\tau(x,y) = \frac{1}{2\tau}\|y-x\|^2 - P\cdot(y-x) + \tau K(1-\cos(2\pi\ \cdot x)).$$

We recognize the original Frenkel–Kontorova model by taking $\tau = 1$. Notice that (3) can trivially be written as a discrete Lax–Oleinik equation with the following short-range interaction $E(x,y) = \mathcal{E}_\tau(x,y)$. We call it *minimal action*

$$\mathcal{E}_\tau(x,y) := \inf_{\substack{\gamma \in C^{ac}([0,\tau],\mathbb{R}^d) \\ \gamma(0)=x,\ \gamma(\tau)=y}} \int_0^\tau L(\gamma(t),\dot{\gamma}(t))\,\mathrm{d}t, \quad \forall \tau > 0,\ \forall x,y \in \mathbb{R}^d. \tag{10}$$

The infimun can be realized by some C^2 curve thanks to the Tonelli–Weierstrass theorem. We will use $\mathcal{L}_\tau(x,y)$ as a numerical tool to solve (3). Several algorithms can be used to solve (8), like Ishikawa's iterative method. We will use $\mathcal{E}_\tau(x,y)$ as a theoretical tool to prove the convergence of the scheme.

The goal of the second scheme is to extend, in the discrete case, the main result of Davini *et al* in their first paper [DFIZ16b]. We became aware of a second paper [DFIZ16a] related to ours after this paper had been completed. However, in the latter paper, the authors do not consider the convergence issues of the approximation scheme. We will show in particular that the solution $u_{\tau,\delta}$ of the *discounted discrete Lax–Oleinik equation*

$$u_{\tau,\delta}(y) = \inf_{x \in \mathbb{R}^d} \big\{(1-\tau\delta)u_{\tau,\delta}(x) + \mathcal{L}_\tau(x,y\big\}, \forall \tau > 0,\ \forall y \in \mathbb{R}^d \tag{11}$$

satisfies for every $\tau > 0$, $\lim_{\delta \to 0}\Big(u_{\tau,\delta} - \frac{\bar{\mathcal{L}}_\tau}{\tau\delta}\Big) = u^*_\tau$ and $\lim_{\substack{\tau,\delta \to 0 \\ \tau/\delta \to 0}}\Big(u_{\tau,\delta} - \frac{\bar{\mathcal{L}}_\tau}{\tau\delta}\Big) = u^*$.

We would like to thank the referees for their careful reading and for the two references [Mat88, Mos86] they suggested to include.

2. Main results

The two previous short-range interactions $\mathcal{L}_\tau(x,y)$ and $\mathcal{E}_\tau(x,y)$ belong to a class of parametrized interactions that we are going to discuss. In the following definition we focus on the fact that $\|y-x\|$, (the sup norm) and τ should have the same order of magnitude as $\tau \to 0$: we call this property *short-range*.

Definition 1. We call a one-parameter family of functions $E_\tau(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ indexed by $\tau > 0$ a *short-range interaction* satisfying:

(H1) $E_\tau(x,y)$ is continuous in (x,y) for every $\tau > 0$;
(H2) $E_\tau(x,y)$ is translational periodic for every $\tau > 0$:

$$E_\tau(x+k, y+k) = E_\tau(x,y), \quad \forall k \in \mathbb{Z}^d \quad \text{and} \quad \forall x, y \in \mathbb{R}^d;$$

(H3) $E_\tau(x,y)$ is coercive for every $\tau > 0$:

$$\lim_{R\to+\infty} \inf_{\|x-y\|\geqslant R} E_\tau(x,y) = +\infty;$$

(H4) $E_\tau(x,y)$ is uniformly bounded: for every $R > 0$

$$\inf_{\tau\in(0,1]} \inf_{x,y\in\mathbb{R}^d} \tfrac{1}{\tau}E_\tau(x,y) > -\infty, \qquad \sup_{\tau\in(0,1]} \sup_{\|y-x\|\leqslant\tau R} \tfrac{1}{\tau}E_\tau(x,y) < +\infty;$$

(H5) $E_\tau(x,y)$ is uniformly superlinear:

$$\lim_{R\to+\infty} \inf_{\tau\in(0,1]} \inf_{\|x-y\|\geqslant\tau R} \frac{E_\tau(x,y)}{\|x-y\|} = +\infty;$$

(H6) $E_\tau(x,y)$ is uniformly Lipschitz: for every $R > 0$, there exists a constant $C(R) > 0$ such that for every $\tau \in (0,1]$ and for every $x, y, z \in \mathbb{R}^d$,

– *if* $\|y-x\| \leqslant \tau R$ and $\|z-x\| \leqslant \tau R$ then

$$|E_\tau(x,z) - E_\tau(x,y)| \leqslant C(R)\|z-y\|,$$

– *if* $\|z-x\| \leqslant \tau R$ and $\|z-y\| \leqslant \tau R$ then

$$|E_\tau(x,z) - E_\tau(y,z)| \leqslant C(R)\|y-x\|.$$

We call the *periodic interaction* associated with $E_\tau(x,y)$, the doubly periodic function

$$E^*_\tau(x,y) := \inf_{k\in\mathbb{Z}^d} E_\tau(x, y+k).$$

The following proposition says that the two short-range interactions $\mathcal{L}_\tau(x,y)$ and $\mathcal{E}_\tau(x,y)$ are comparable in the sense that $|\mathcal{L}_\tau(x,y) - \mathcal{E}_\tau(x,y)| = O(\tau^2)$ uniformly on $\|y-x\| = O(\tau)$. In dimension $d = 1$, Moser [Mos86] proved that every monotone smooth twist mapping can be obtained as the time-one map of a periodic Hamiltonian.

Proposition 2 (Comparison estimate). *Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Tonelli Hamiltonian and L be the associated Lagrangian.*

i. The two short-range interactions $(\mathcal{L}_\tau(x,y))_{\tau>0}$ and $(\mathcal{E}_\tau(x,y))_{\tau>0}$, defined in (9) and (10), respectively, satisfy the hypotheses (H1)–(H6).
ii. For every $R > 0$, there exists a constant $C(R) > 0$ such that if $\tau \in (0,1]$, $x, y \in \mathbb{R}^d$ satisfy $\|y-x\| \leqslant \tau R$, then

$$|\mathcal{E}_\tau(x,y) - \mathcal{L}_\tau(x,y)| \leqslant \tau^2 C(R).$$

We recall two important definitions associated with an interaction: the discrete Lax–Oleinik operator, and the discrete weak KAM solution. The vocabulary is chosen so that it coincides with the new terminology used by Fathi in the case of a continuous time Lax–Oleinik operator.

Definition 3. Let $(E_\tau(x,y))_{\tau>0}$ be a short-range interaction satisfying (H1)–(H3).

- We call the *discrete (backward) Lax–Oleinik operator*,

$$T_\tau[u](y) := \min_{x\in\mathbb{R}^d} \big\{u(x) + E_\tau(x,y)\big\}, \qquad \forall y \in \mathbb{R}^d,$$

acting on continuous periodic functions $u \in C^0(\mathbb{R}^d)$.

- We call any periodic continuous function u_τ solution of the additive eigenvalue problem the *discrete (backward) weak KAM solution for* $E_\tau(x,y)$,

$$T_\tau[u_\tau] = u_\tau + \bar{E}_\tau, \tag{12}$$

for some $\bar{E}_\tau \in \mathbb{R}$.

Note that T_τ has the same definition if $E_\tau(x,y)$ is replaced by $E^*_\tau(x,y)$.

We have defined two Lax–Oleinik operators: the first one in the continuous case T^t in (4), using a superscript t; the second one in the discrete case T_τ in (3) using a subscript τ. For the minimal action $\mathcal{E}_\tau(x,y)$ we obviously have $T^\tau = T_\tau$.

We recall a classical result on the existence of discrete weak KAM solutions for the Lax–Oleinik operator. Different proofs may be found, for instance in [Nus91, Gom05] or [GT11].

Proposition 4 (Lax–Oleinik equation for short-range interactions). *We consider a short-range interaction $(E_\tau(x,y))_{\tau>0}$ satisfying the hypotheses (H1)–(H3).*

i. For every $\tau > 0$, there exists a unique scalar $\bar{E}_\tau$ such that equation $T_\tau[u_\tau] = u_\tau + \bar{E}_\tau$ admits a continuous periodic solution u_τ.

ii. $\bar{E}_\tau$ is called effective interaction and can be computed in many ways

$$\begin{aligned}\bar{E}_\tau &= \sup_{u\in C^0(\mathbb{T}^d)} \inf_{x,y\,\in\,\mathbb{R}^d} \big\{E_\tau(x,y) - [u(y)-u(x)]\big\}, \\ &= \sup_{v\in \mathcal{B}(\mathbb{R}^d)} \inf_{x,y\,\in\,\mathbb{R}^d} \big\{E_\tau(x,y) - [v(y)-v(x)]\big\}, \\ &= \lim_{k\to+\infty} \inf_{z_0,\dots,z_k\,\in\,\mathbb{R}^d} \tfrac{1}{k}\sum_{i=0}^{k-1} E_\tau(z_i,z_{i+1}).\end{aligned} \tag{13}$$

$\mathcal{B}(\mathbb{R}^d)$ denotes the space of bounded functions that are not necessarily periodic. Note that we could have used $E^*_\tau(x,y)$ instead of $E_\tau(x,y)$ in one of these formulas.

The first two formulas are called the *sup-inf formula*, and are analogue to the sup-inf formula introduced by [CIPP98] for continuous-time Tonelli Hamiltonian systems. The third formula is called the *mean interaction per site formula*. Another characterization will be given in the lemma 14.

The conclusions of proposition 4 hold for both the discrete and the minimal action. There is no reason *a priori* for the two effective interactions $\bar{\mathcal{L}}_\tau$ and $\bar{\mathcal{E}}_\tau$ to be comparable. The mean interaction per site formula suggests the consideration of minimizing paths $(z_0,\cdots,z_k)$. The following proposition shows that the jumps $\|z_k - z_{k-1}\|$ of such minimizing paths are uniformly comparable to τ. We will be able to apply the proposition 2 and obtain $|\bar{\mathcal{L}}_\tau - \bar{\mathcal{E}}_\tau| = O(\tau^2)$.

Proposition 5 (*A priori* compactness for short-range interactions). *We consider a short-range interaction $(E_\tau(x,y))_{\tau>0}$ satisfying the hypotheses (H1)–(H6).*

i. There exist constants $C, R > 0$ such that if $\tau \in (0,1]$ and u_τ is a discrete weak KAM solution of $E_\tau(x,y)$, then

(a) u_τ is Lipschitz and $\mathrm{Lip}(u_\tau) \leqslant C$,

(b) $\forall y \in \mathbb{R}^d$, $x \in \arg\min_{x\in\mathbb{R}^d}\big\{u_\tau(x) + E_\tau(x,y)\big\} \quad\Rightarrow\quad \|y-x\| \leqslant \tau R$.

ii. *For every Lipschitz periodic function u,* $\lim_{\tau\to 0} T_\tau[u] = u$ *uniformly. More precisely, for every constant* $\kappa > 0$*, there exist constants* $R_\kappa, C_\kappa > 0$*, such that if u is any Lipschitz function satisfying* $\mathrm{Lip}(u) \leqslant \kappa$*, and* $\tau \in (0, 1]$*, then*
(a) $\forall y \in \mathbb{R}^d$*,* $x \in \arg\min_{x\in\mathbb{R}^d}\{u(x) + E_\tau(x, y)\} \quad\Rightarrow\quad \|y - x\| \leqslant \tau R_\kappa$*,*
(b) $\| T_\tau[u] - u \|_\infty \leqslant \tau C_\kappa$*.*

Notice that the effective Hamiltonian (5) can be written in the terminology of short-range interactions using the minimal action,

$$-\bar{H} = \lim_{\tau\to+\infty} \frac{1}{\tau} \min_{x,y\in\mathbb{R}^d} \mathcal{E}_\tau(x, y).$$

We show more generally how to solve equation (3) and how to obtain formula (5) for any short-range interaction which is a min-plus convolution semi-group.

Definition 6.

- We call the *min-plus convolution* of two interactions $\mathcal{E}_1$ and $\mathcal{E}_2$, the interaction

$$\mathcal{E}_1 \otimes \mathcal{E}_2(x, y) := \inf_{z\in\mathbb{R}^d} [\mathcal{E}_1(x, z) + \mathcal{E}_2(z, y)], \quad \forall x, y \in \mathbb{R}^d.$$

- A short-range interaction $(E_\tau(x, y))_{\tau>0}$ is said to be a *min-plus convolution semi-group* if

$$E_{\tau+\sigma} = E_\tau \otimes E_\sigma, \quad \forall \tau, \sigma > 0.$$

Mather, [Mat88] p 206, calls a *conjunction* what we call a min-plus convolution. The following observation is trivial and will not be proved.

Lemma 7. *Let H be a Tonelli Hamiltonian. Then the minimal action* $(\mathcal{E}_\tau(x, y))_{\tau>0}$ *is a min-plus convolution semi-group.*

The following proposition extends (3) and (5) for any short-range interaction which is a min-plus convolution semi-group. The proposition states that there exists a common additive eigenfunction associated with a unique linear eigenvalue.

Proposition 8. *Let* $(E_\tau(x, y))_{\tau>0}$ *be a short-range interaction satisfying (H1)–(H6). Assume the interaction is a min-plus convolution semi-group. Consider the equation*

$$T_\tau[u] = u + \tau\bar{E}_1, \quad \forall \tau > 0, \tag{14}$$

where u is a C^0 *periodic function (independent of* τ*) and* $\bar{E}_1 \in \mathbb{R}$*.*

i. There exists a Lipschitz periodic function u solution of (14). Moreover

$$\bar{E}_\tau = \tau\bar{E}_1, \quad \forall \tau > 0.$$

ii. Let u_τ *be any discrete weak KAM solution of* $E_\tau(x, y)$*. Assume* $u_{\tau_i} \to u$ *uniformly along a subsequence* $\tau_i \to 0$*. Then u is a Lipschitz solution of (14).*

iii. $\lim_{\tau\to+\infty} \frac{1}{\tau} \min_{x,y\in\mathbb{R}^d} E_\tau(x, y) = \bar{E}_1$.

We summarize in the following theorem the previous results we have obtained for any short-range interactions to the particular case of discrete and minimal actions. We show how the solutions of the PDE cell equation (1) can be approximated by discrete weak KAM solutions u_τ of (14). The speed of convergence to the effective Hamiltonian $\bar{H}$ is of the order $O(\tau)$. The convergence to the viscosity solution u is obtained by taking a subsequence as $\tau \to 0$.

Theorem 9 (First approximation scheme). *Let $H(x,p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Tonelli Hamiltonian and $L(x,v)$ be the associated Lagrangian. We consider the two equations*

$$u_\tau(y) + \bar{\mathcal{L}}_\tau = \min_{x\in\mathbb{R}^d} \big\{ u_\tau(x) + \mathcal{L}_\tau(x,y) \big\}, \quad \forall y \in \mathbb{R}^d,\ \forall \tau > 0, \tag{E1}$$

$$u(y) - \tau\bar{H} = \min_{x\in\mathbb{R}^d} \big\{ u(x) + \mathcal{E}_\tau(x,y) \big\}, \quad \forall y \in \mathbb{R}^d,\ \forall \tau > 0, \tag{E2}$$

where u_τ, u are C^0 periodic functions.

i. There is a unique $\bar{\mathcal{L}}_\tau$, such that (E1) admits a solution u_τ. Moreover

$$\bar{\mathcal{L}}_\tau = \lim_{k\to+\infty} \inf_{z_0,\ldots,z_k \in \mathbb{R}^d} \frac{1}{k} \sum_{i=0}^{k-1} \mathcal{L}_\tau(z_i, z_{i+1}).$$

ii. There is a unique $\bar{H}$ such that (E2) admits a solution u. Moreover

$$-\bar{H} = \lim_{\tau\to+\infty} \frac{1}{\tau} \min_{x,y\in\mathbb{R}^d} \mathcal{E}_\tau(x,y).$$

iii. There exists a constant $C > 0$ such that

$$\left| \frac{\bar{\mathcal{L}}_\tau}{\tau} + \bar{H} \right| \leqslant C\tau, \quad \forall \tau \in (0,1].$$

iv. There exist constants $C, R > 0$, such that for every $\tau \in (0,1]$ and for every solution $v = u_\tau$ of (E1), or $v = u$ of (E2),

(a) $\mathrm{Lip}(v) \leqslant C$, in particular $\|v\|_\infty \leqslant C$ if $\min(v) = 0$,
(b) $\forall y \in \mathbb{R}^d$, if $x \in \arg\min_{x\in\mathbb{R}^d} \big\{ v(x) + E_\tau(x,y) \big\}$ then $\|y - x\| \leqslant \tau R$.

v. There exists a subsequence $\tau_i \to 0$ and a subsequence u_{τ_i} solution of (E1), such that $u_{\tau_i} \to u$ uniformly. Moreover every such u is a solution of (E2).

Theorem 9 is proved in section 3. The convergence of the discrete solution to the solution of the ergodic cell equation has been addressed by Gomes [Gom05] and Camilli *et al* [CCDG08], but their proofs require a particular form of the Lagrangian that we do not assume. Several other numerical schemes have been studied for computing the effective Hamiltonian, see [GO04, Ror06, FR10], but the properties (i)–(v) are not stated explicitly, see also [BFZ16] for a mechanical Lagrangian of the form $L(t,x,v) = W(v) + V(t,x)$.

Note that the discrete (backward) Lax–Oleinik equation (12) possesses a second form: the *discrete forward Lax–Oleinik equation*,

$$u_\tau(x) - \bar{E}_\tau = \max_{y\in\mathbb{R}^d} \big\{ u_\tau(y) - E_\tau(x,y) \big\}, \quad \forall x \in \mathbb{R}^d.$$

Theorem 9 is also valid for the forward Lax–Oleinik equation with the same effective interaction $\bar{E}_\tau$ and possibly a different solution u_τ that is called the *discrete forward weak KAM*. From now on we only study the backward problem.

Our second objective is to show, by introducing a discounted factor δ in the discrete Lax–Oleinik equation (12), that we do not need to take a subsequence in time to obtain a solution of the PDE cell equation. A discrete version of [DFIZ16b] is also proved in [DFIZ16a], but they do not study the convergence issues as $\tau \to 0$. Some related results can be found in [AAOIM14, MT14] with a different setting.

Our approach is actually more general and applies to any short-range interaction. We first extend the definition of the Lax–Oleinik operator.

Definition 10. Let $(E_\tau(x,y))_{\tau>0}$ be a short-range interaction satisfying (H1)–(H3). We call the *discounted discrete Lax–Oleinik operator*, the nonlinear operator

$$T_{\tau,\delta}[u](y) := \inf_{x\in\mathbb{R}^d} \big\{(1-\tau\delta)u(x) + E_\tau(x,y)\big\}, \quad \forall y \in \mathbb{R}^d,$$

defined for every C^0 periodic function u, for every $\tau > 0$ and $\delta \in (0,1]$. By coerciveness the infimum is actually attained. As before, we do not change $T_{\tau,\delta}$ by using the periodic interaction $E^*_\tau(x,y)$ instead of $E_\tau(x,y)$.

It is easy to show that $T_{\tau,\delta}$ admits a unique fixed point $u_{\tau,\delta}$ that we call the *discounted discrete weak KAM solution*. On the other hand, it is not so easy to show that it possesses uniform estimates, as in proposition 5,

Proposition 11 (*A priori* compactness in the discounted case). *Let $(E_\tau(x,y))_{\tau>0}$ be a short-range interaction satisfying (H1)–(H6). Then there exist constants $R > 1$ and $C > 0$ such that for every $\tau, \delta \in (0,1]$,*

i. $T_{\tau,\delta}$ admits a unique fixed point $u_{\tau,\delta}$ which is C^0 periodic,

$$u_{\tau,\delta}(x) := \inf_{(x_{-k})_{k=0}^{+\infty}\in(\mathbb{R}^d)^{\mathbb{N}},\, x_0=x} \sum_{k=0}^{\infty} (1-\tau\delta)^k E_\tau(x_{-(k+1)}, x_{-k}), \quad \forall x \in \mathbb{R}^d.$$

ii. $\inf_{x,y\in\mathbb{R}^d} \frac{E_\tau(x,y)}{\tau\delta} \leqslant u_{\tau,\delta} \leqslant \sup_{x\in\mathbb{R}^d} \frac{E_\tau(x,x)}{\tau\delta}$,
iii. $u_{\tau,\delta}$ is uniformly Lipschitz with $\mathrm{Lip}(u_{\tau,\delta}) \leqslant C$,
iv. $\forall y \in \mathbb{R}^d, \quad x \in \arg\min_{x\in\mathbb{R}^d}\big\{(1-\tau\delta)u_{\tau,\delta}(x) + E_\tau(x,y)\big\} \quad \Rightarrow \quad \|y-x\| \leqslant \tau R.$

A configuration $(x_{-k})_{k=0}^{\infty}$ realizing the infimum in (i) is called the discounted backward calibrated configuration. Such a configuration is also calibrated for the periodic interaction $E^*_\tau(x,y)$ instead of $E_\tau(x,y)$.

As in [DFIZ16b], we characterize the limit of the unique fixed point of $T_{\tau,\delta}$ in terms of the minimizing plan, the Mañé potential. We recall these two definitions, see [GT11] for more details. We usually introduce the notions of minimizing measures, the Mather set or the Aubry set, in the space $\mathbb{T}^d \times \mathbb{R}^d$. This space is the correct space if want to understand the cohomology of these notions. We instead consider here the projection on $\mathbb{T}^d \times \mathbb{T}^d$ of these objects that we recall.

Definition 12. A probability measure π defined on $\mathbb{T}^d \times \mathbb{T}^d$ is said to be a *stationary plan* if $pr^1_*(\pi) = pr^2_*(\pi)$. (We denote by $pr^1, pr^2 : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{T}^d$, the two canonical projections.)

Definition 13. We call the *periodic Mañé potential*, a doubly periodic function

$$\Phi^*_\tau(x,y) := \inf_{n\geqslant 1} \inf_{\substack{(x_0,\dots,x_n)\in(\mathbb{R}^d)^{n+1} \\ x_0=x,\ x_n=y}} \sum_{k=0}^{n-1} \big[E^*_\tau(x_k,x_{k+1}) - \bar{E}_\tau\big], \quad \forall\, x,y \in \mathbb{R}^d.$$

We recall how the effective Hamiltonian can be computed using the stationary plan. See [BB07, GT11] for a proof.

Lemma 14. *Let $(E_\tau(x,y))_{\tau>0}$ be a short-range interaction satisfying (H1)–(H3). Let $E^*_\tau(x,y)$ be the associated periodic interaction. Then*

$$\bar{E}_\tau = \inf\left\{ \int\int_{\mathbb{T}^d\times\mathbb{T}^d} E^*_\tau(x,y)\,\pi(\mathrm{d}x,\mathrm{d}y) \ :\ \pi \text{ is a stationary plan} \right\}.$$

Note that the infimum in lemma 14 can be realized by compactness. We recall several classical notions. See [BB07, GT11] for two distinct approaches.

Definition 15. Let π be a stationary plan on $\mathbb{T}^d\times\mathbb{T}^d$.

- π is said to be *minimizing* if it realizes the infimum in lemma 14. Define
$$\mathcal{M}^*(E_\tau) := \{\pi :\ \pi \text{ is a minimizing plan}\}.$$
- π is said to be *extremal* if it is minimizing and cannot be written as the strict barycentre $\pi = \alpha\pi_1 + (1-\alpha)\pi_2$ of a minimizing plan, π_1 and π_2, with $\alpha\in(0,1)$, $\pi_1\neq\pi_2$.
- We call the *Mather set*, the compact set in $\mathbb{T}^d\times\mathbb{T}^d$
$$\mathrm{Mather}^*(E_\tau) := \overline{\cup\{\mathrm{supp}(\pi) : \pi\in\mathcal{M}^*(E_\tau)\}}.$$
We call the *projected Mather set*, the set $pr^1(\mathrm{Mather}^*(E_\tau))$.
- We call the *Aubry set*, the compact set in $\mathbb{T}^d\times\mathbb{T}^d$
$$\mathrm{Aubry}^*(E_\tau) := \left\{(x,y)\in\mathbb{T}^d\times\mathbb{T}^d : E^*_\tau(x,y) - \bar{E}_\tau + \Phi^*_\tau(y,x) = 0\right\}.$$
We call the *projected Aubry set*, the set $pr^1(\mathrm{Aubry}^*(E_\tau))$.
- We call the *Aubry class*, the class of an equivalence relation on $pr^1(\mathrm{Aubry}^*(E_\tau))$,
$$x\sim y \iff \Phi^*_\tau(x,y) + \Phi^*_\tau(y,x) = 0.$$

We can show (see [GT11] in the discrete setting).

Lemma 16. *Let $(E_\tau(x,y))_{\tau>0}$ be a short-range interaction satisfying (H1)–(H3). Then*

i. $\Phi^*_\tau(x,y)$ *is continuous with respect to* (x,y)*,*
ii. $pr^1(\mathrm{Aubry}^*(E_\tau)) = \{x\in\mathbb{T}^d : \Phi^*_\tau(x,x) = 0\}$*,*
iii. For any Aubry class A*,* $\forall x,y,z\in A$*,* $\Phi^*_\tau(x,y)+\Phi^*_\tau(y,z) = \Phi^*_\tau(x,z)$*,*
iv. $\mathrm{Mather}^*(E_\tau)\subset\mathrm{Aubry}^*(E_\tau)$*,*
v. $\forall x\in pr^1(\mathrm{Aubry}^*(E_\tau))$*,* $y\mapsto\Phi^*_\tau(x,y)$ *is a discrete weak KAM solution,*
vi. (representation formula) if u_τ *is any discrete weak KAM solution, then*

$$\begin{aligned} u_\tau(y) &= \min_{x\in\mathbb{R}^d}\{u(x)+\Phi^*_\tau(x,y)\},\\ &= \min_{x\in pr^1(\mathrm{Mather}^*(E_\tau))}\{u(x)+\Phi^*_\tau(x,y)\}, \quad \forall y\in\mathbb{R}^d. \end{aligned}$$

The following lemma gives a new type of discrete weak KAM solution. Though it is simple to prove, the lemma is new and justifies *a priori* the notion of a balanced weak KAM solution.

Lemma 17. *Let $(E_\tau(x,y))_{\tau>0}$ be a short-range interaction satisfying (H1)–(H3). Let π be an extremal plan. Let $\mu = pr^1_*(\pi)$.*

i. $\mathrm{supp}(\mu)$ *belongs to an Aubry class.*
ii. $y\mapsto\int\Phi^*_\tau(z,y)\,\mu(\mathrm{d}z)$ *is a discrete weak KAM solution.*
iii. $\iint\Phi^*_\tau(x,y)\,\mu(\mathrm{d}x)\mu(\mathrm{d}y) = 0.$

By taking the supremum or infimum of discrete weak KAM solutions, we again obtain a discrete weak KAM solution. The balanced weak KAM solution (7) is of this type.

Proposition 18. *Define* $u^*_\tau(x) := \inf\left\{ \int_{\mathbb{T}^d} \Phi^*_\tau(z,x)\, pr^1_*(\pi)(\mathrm{d}z) : \pi \in \mathcal{M}^*(E_\tau)\right\}$*. Then*

i. u^*_τ *is a discrete weak KAM solution,*

ii. $u^*_\tau(y) = \sup\left\{ w(y) : w + \bar{E}_\tau = T_\tau[w],\ \int_{\mathbb{T}^d} w(x)\, pr^1_*(\pi)(\mathrm{d}x) \leqslant 0,\ \forall \pi \in \mathcal{M}^*(E_\tau)\right\}$,

iii. $\sup\{\int u^*_\tau(y)\ pr^1_*(\pi)(\mathrm{d}y)\ :\ \pi \text{is an extremal plan}\} = 0.$

u^*_τ *is called a balanced discrete weak KAM solution.*

The following proposition extends to short-range interactions, with the main result obtained by [DFIZ16b] in the continuous case and by [DFIZ16a] in the discrete case.

Proposition 19. *Let* $(E_\tau(x,y))_{\tau>0}$ *be a short-range interaction satisfying (H1)–(H3). Let* u^*_τ *be the balanced discrete weak KAM solution defined in proposition 18. Then,*

$$\forall \tau \in (0,1], \quad \lim_{\delta\to 0}\left(u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta}\right) = u^*_\tau, \quad \text{in the } C^0 \text{ topology.}$$

In the following theorem, we summarize the approximation scheme we have obtained in the case of the discrete action $\mathcal{L}_\tau(x,y)$.

Theorem 20 (Second approximation scheme). *Let* $H(x,p)$ *be a Tonelli Hamiltonian, and* $L(x,v)$ *be the associated Lagrangian. Let* $u_{\tau,\delta}$ *and* u_δ *be the unique* C^0 *periodic solutions of*

$$u_{\tau,\delta}(y) = \min_{x\in\mathbb{R}^d}\left\{(1-\tau\delta)u_{\tau,\delta}(x) + \mathcal{L}_\tau(x,y)\right\}, \quad \forall y \in \mathbb{R}^d,\ \forall \tau,\delta \in (0,1], \tag{E1}$$

$$u_\delta(y) = \inf_{\substack{\gamma\in C^2((-t,0],\mathbb{R}^d)\\ \gamma(0)=y}}\left\{\mathrm{e}^{-t\delta}u_\delta(\gamma(t)) + \int_{-t}^0 \mathrm{e}^{s\delta}L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s\right\},\ \forall y \in \mathbb{R}^d,\ t>0. \tag{E2}$$

Consider the equations with the C^0 *periodic unknowns* u_τ *and* u*,*

$$u_\tau(y) + \bar{\mathcal{L}}_\tau = \min_{x\in\mathbb{R}^d}\left\{u_\tau(x) + \mathcal{L}_\tau(x,y)\right\}, \quad \forall y\in\mathbb{R}^d,\ \forall\tau\in(0,1], \tag{E3}$$

$$u(y) - t\bar{H} = \min_{x\in\mathbb{R}^d}\left\{u(x) + \mathcal{E}_t(x,y)\right\}, \quad \forall y \in \mathbb{R}^d,\ \forall t>0. \tag{E4}$$

i. Let $\delta \in (0,1]$*,* $x\in\mathbb{R}^d$*. Let* $(x^{\tau,\delta}_{-n})_{n\geqslant 0}$ *be a backward calibrated configuration for the equation (E3) starting at* $x^{\tau,\delta}_0 = x$*. Let* $\gamma_{\tau,\delta}(t)$ *be the piecewise linear approximation satisfying* $\gamma_{\tau,\delta}(-n\tau) = x^{\tau,\delta}_{-n}$*. Then there exists a sequence* $\tau_i \to 0$ *such that*

(a) $\gamma_{\tau_i,\delta}(t) \to \gamma_\delta(t)$ uniformly on every compact subset of $(-\infty,0]$,
(b) $\gamma_\delta \in C^2((-\infty,0],\mathbb{R}^d)$, $\|\dot\gamma_\delta\|_\infty \leqslant C$, $\|\ddot\gamma_\delta\|_\infty \leqslant C$
(c) $u_\delta(x) = \mathrm{e}^{-t\delta}u_\delta(\gamma_\delta(-t)) + \int_{-t}^0 \mathrm{e}^{s\delta}L(\gamma_\delta(s),\dot\gamma_\delta(s))\,\mathrm{d}s, \quad \forall t\geqslant 0.$

ii. There exist constants $C>0, R>1$ *such that for every* $\tau,\delta\in(0,1]$*,* *(a)* $u_{\tau,\delta}$ *is uniformly Lipschitz with* $\mathrm{Lip}(u_{\tau,\delta}) \leqslant C$,

(b) $\forall y \in \mathbb{R}^d, \quad x \in \arg\min_{x\in\mathbb{R}^d}\big\{(1-\tau\delta)u_{\tau,\delta}(x) + \mathcal{L}_\tau(x,y)\big\} \quad \Rightarrow \quad \|y-x\| \leqslant \tau R$,

(c) $\|u_{\tau,\delta} - u_\delta\|_\infty \leqslant C\frac{\tau}{\delta}$ and $\Big\| \Big(u_{\tau,\delta} - \frac{\bar{\mathcal{L}}_\tau}{\tau\delta}\Big) - \Big(u_\delta + \frac{\bar H}{\delta}\Big) \Big\|_\infty \leqslant C\frac{\tau}{\delta}$.

(iii) Let $\tau \in (0,1]$ *and* u^*_τ *be defined in proposition 18. Then*

$$\lim_{\delta\to 0}\Big(u_{\tau,\delta} - \frac{\bar{\mathcal{L}}_\tau}{\tau\delta}\Big) = u^*_\tau, \quad \text{in the } C^0 \text{ topology.}$$

(iv) Let u^* *be the solution of (E4) defined by (7). Then*

$$\lim_{\substack{\tau\to 0,\ \delta\to 0\\ \tau/\delta\to 0}}\Big(u_{\tau,\delta} - \frac{\bar{\mathcal{L}}_\tau}{\tau\delta}\Big) = u^*, \quad \text{in the } C^0 \text{ topology.}$$

Theorem 20 is proved in section 4. Item (i) shows how to obtain a C^2 minimizer in the continuous discounted case from a discrete calibrated configuration, item (ii) improves similar estimates in [Ror06, FR10, BFZ16]. Item (iii) generalizes [DFIZ16a] and is a particular case of proposition 19, item (iv) is a corollary of (iic) and [DFIZ16b].

3. First approximation scheme

This section is devoted to the proof of theorem 9 and the necessary tools presented before. The *a priori* estimates in proposition 2 are easy to prove for the Tonelli Hamiltonian. We recall the following result; see [Fat08, Mat91] in the autonomous case and [BFZ16] in the nonautonomous case for more details.

Lemma 21 (*A priori* compactness for minimizers). *Let* $H(x,p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ *be a Tonelli Hamiltonian. For every* $R > 0$, *there exists a constant* $C(R) > 0$ *such that for every* $\tau > 0$, $x, y \in \mathbb{R}^d$ *satisfying* $\|y-x\| \leqslant \tau R$, *and for every minimizer* $\gamma : [0,\tau] \to \mathbb{R}^d$ *satisfying*

$$\gamma(0) = x, \ \gamma(\tau) = y, \ \int_0^\tau L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s = \mathcal{E}_\tau(x,y),$$

we have $\|\dot\gamma\| \leqslant C(R)$ *and* $\|\ddot\gamma\| \leqslant C(R)$.

Proof of proposition 2. Properties (H1)–(H6) are trivially satisfied for the discrete action $\mathcal{L}_\tau(x,y)$. Properties (H1)–(H3) and (H5) are also easy to prove for the minimal action $\mathcal{E}_\tau(x,y)$ using the superlinearity of $L(x,v)$.

Part 1: proof of property (H4). Let $\tau > 0$, $x, y \in \mathbb{R}^d$, $\|y-x\| \leqslant \tau R$. Since $\gamma(s) := x + s\frac{y-x}{\tau}$ is a particular path joining x to y, we obtain

$$\sup_{\tau>0,\ \|y-x\|\leqslant\tau R} \tfrac{1}{\tau}\mathcal{E}_\tau(x,y) \leqslant \sup_{x\in\mathbb{R}^d,\ \|v\|\leqslant R} L(x,v).$$

Let $\tau > 0$ and $x, y \in \mathbb{R}^d$. By superlinearity, $L(x,v) \geqslant \|v\| - C$ for some constant $C > 0$. Then $\int_0^\tau L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s \geqslant \|y-x\| - \tau C$ for every absolutely continuous path $\gamma : [0,\tau] \to \mathbb{R}^d$ satisfying $\gamma(0) = x$ and $\gamma(\tau) = y$. One obtains

$$\inf_{\tau>0,\ x,y\in\mathbb{R}^d} \frac{1}{\tau}\mathcal{E}_\tau(x,y) \geqslant -C.$$

Part 2: proof of property (H6). Let $\tau \in (0,1]$, $x,y,z \in \mathbb{R}^d$ such that $\|y-x\| \leqslant \tau R$ and $\|z-x\| \leqslant \tau R$. By Tonelli–Weierstrass, there exists a C^2 minimizer $\gamma : [0,\tau] \to \mathbb{R}^d$ starting at x, ending at y, and satisfying $\int_0^\tau L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s = \mathcal{E}_\tau(x,y)$. Define the path $\xi : [0,\tau] \to \mathbb{R}^d$ by $\xi(s) = \gamma(s) + s\frac{z-y}{\tau}$. By lemma 21, there exists a constant $C(R) > 0$ such that $\|\dot\gamma\| \leqslant C(R)$. Then

$$\mathcal{E}_\tau(x,z) - \mathcal{E}_\tau(x,y) \leqslant \int_0^\tau \big[L(\xi(s),\dot\xi(s)) - L(\gamma(s),\dot\gamma(s))\big]\,\mathrm{d}s \leqslant \tilde C(R)\|z-y\|,$$

where $\tilde C(R) = \sup_{x\in\mathbb{R}^d,\ \|v\|\leqslant C(R)+R} \|DL(x,v)\|$.

Part 3: proof of item (ii). Let $R > 0$ and $C(R)$ be the constants given by lemma 21. Let $\tau \in (0,1]$ and $\|y-x\| \leqslant \tau R$. We know that $\mathcal{E}_\tau(x,y)$ admits a C^2 minimizer $\gamma : [0,\tau] \to \mathbb{R}^d$ satisfying $\gamma(0)=x$, $\gamma(\tau)=y$, $\mathcal{E}_\tau(x,y) = \int_0^\tau L(\gamma,\dot\gamma)\,\mathrm{d}s$, $\|\dot\gamma\| \leqslant C(R)$ and $\|\ddot\gamma\| \leqslant C(R)$. Let $V_0 = \dot\gamma(0)$. Then

$$\|\gamma(s)-x\| = \|\gamma(s)-\gamma(0)\| \leqslant sC(R) \leqslant \tau C(R),$$
$$\|\dot\gamma(s)-V_0\| \leqslant sC(R),\ \ \Big\|\frac{y-x}{\tau} - V_0\Big\| \leqslant \tau C(R) \ \text{ and } \ \Big\|\dot\gamma(s) - \frac{y-x}{\tau}\Big\| \leqslant 2\tau C(R).$$

We are now in a position to compare the two actions

$$|\mathcal{E}_\tau(x,y) - \mathcal{L}_\tau(x,y)| \leqslant \int_0^\tau \Big|L(\gamma(s),\dot\gamma(s)) - L\Big(x,\frac{y-x}{\tau}\Big)\Big|\,\mathrm{d}s \leqslant \tau^2\tilde C(R),$$

with $\tilde C(R) := 2\sup_{x\in\mathbb{R}^d,\ \|v\|\leqslant R+C(R)} \|DL\|\ C(R)$.

The *a priori* estimates of proposition 5 are the main technical results.

Proof of proposition 5. We begin by fixing the constants C and R: let

$$\begin{aligned}
C_1 &:= 2\sup_{\tau\in(0,1],\ \|y-x\|\leqslant\tau} \frac{E_\tau(x,y)-\bar E_\tau}{\tau},\\
R &:= \inf\Big\{R>1 : \inf_{\tau\in(0,1],\ \|y-x\|>\tau R} \frac{E_\tau(x,y)-\bar E_\tau}{\|y-x\|} > C_1\Big\},\\
C &:= \max\Big(C_1, \sup_{\|y-x\|,\ \|z-x\|\leqslant\tau(R+1)} \frac{E_\tau(x,y)-E_\tau(x,z)}{\|z-y\|}\Big).
\end{aligned} \tag{15}$$

Notice that C_1 is finite thanks to (H4), R is finite thanks to (H5) and C is finite thanks to (H6).

Part 1. We show a partial proof of item (ia), namely

$$\|y-x\| > \tau \ \Rightarrow\ u_\tau(y) - u_\tau(x) \leqslant C_1\|y-x\|.$$

Indeed, by choosing $n \geqslant 2$ such that $(n-1)\tau < \|y-x\| \leqslant n\tau$ and by choosing $x_i = x + \frac{i}{n}(y-x)$, we obtain $n\tau \leqslant 2\|y-x\|$,

$$u_\tau(x_{i+1}) - u_\tau(x_i) \leqslant E_\tau(x_i,x_{i+1}) - \bar E_\tau, \quad \text{and}$$
$$u_\tau(y) - u_\tau(x) \leqslant n\tau \sup_{\|y-x\|\leqslant\tau} \frac{E_\tau(x,y)-\bar E_\tau}{\tau} \leqslant C_1\|y-x\|.$$

Part 2. We prove item (ib). Let $y \in \mathbb{R}^d$. Let x be a calibrated point for $u_\tau(y)$, that is, x satisfies

$$u_\tau(y) - u_\tau(x) = E_\tau(x, y) - \bar{E}_\tau.$$

Choose some $R > 1$ as in (15) and assume by contradiction that $\|y - x\| > \tau R$. Then the first part of the proof may be used and we obtain the absurd inequality

$$C_1 \|y - x\| \geqslant u_\tau(y) - u_\tau(x) > C_1 \|y - x\|.$$

Part 3. We end the proof of item (ia). Let $y, z \in \mathbb{R}^d$; either $\|z - y\| > \tau$ and we are done by step 1, or $\|z - y\| \leqslant \tau$. Let x be a calibrated point for $u_\tau(y)$. Then $\|y - x\| \leqslant \tau R$, $\|z - x\| \leqslant \tau(R + 1)$,

$$u_\tau(y) - u_\tau(x) = E_\tau(x, y) - \bar{E}_\tau, \quad u_\tau(z) - u_\tau(x) \leqslant E_\tau(x, z) - \bar{E}_\tau,$$
$$u_\tau(z) - u_\tau(y) \leqslant E_\tau(x, z) - E_\tau(x, y) \leqslant C\|z - y\|.$$

By permuting z and y, we just have proved that $\mathrm{Lip}(u_\tau) \leqslant C$.
Part 4. We prove item (ic). Let $\kappa > 0$. We define $R_\kappa > 0$ as before

$$R_\kappa := \inf \Big\{ R' > 1 : \inf_{\tau \in (0,1],\ \|y - x\| > \tau R'} \frac{E_\tau(x, y) - E_\tau(y, y)}{\|y - x\|} > \kappa \Big\}.$$

Let u be a periodic function satisfying $\mathrm{Lip}(u) \leqslant \kappa$ and y be any point in $\mathbb{R}^d$. Let x be a point realizing the minimum of $\min_x \big\{ u(x) + E_\tau(x, y) \big\}$. Assume by contradiction that $\|y - x\| > \tau R_\kappa$, then on the one hand

$$E_\tau(x, y) - E_\tau(y, y) > \kappa \|y - x\|,$$

and on the other hand $u(x) + E_\tau(x, y) \leqslant u(y) + E_\tau(y, y)$ and

$$\kappa \|y - x\| \geqslant u(y) - u(x) \geqslant E_\tau(x, y) - E_\tau(y, y),$$

which is impossible. We then estimate $\| T_\tau[u] - u \|_\infty$. On the one hand

$$T_\tau[u](y) - u(y) \leqslant E_\tau(y, y).$$

On the other hand, if x realizes the minimum of $\min_{x \in \mathbb{R}^d} [u(x) + E_\tau(x, y)]$

$$\begin{aligned} T_\tau[u](y) - u(y) &= u(x) - u(y) + E_\tau(x, y) \\ &\geqslant -\kappa \|y - x\| + \inf_{x, y \in \mathbb{R}^d} E_\tau(x, y), \end{aligned}$$
$$\frac{1}{\tau} \big[T_\tau[u](y) - u(y) \big] \geqslant -\kappa R_\kappa + \inf_{\tau \in (0,1]} \inf_{x, y \in \mathbb{R}^d} \frac{1}{\tau} E_\tau(x, y).$$

We conclude by taking

$$C_\kappa := \kappa R_\kappa + \sup_{\tau \in (0,1]} \sup_{y \in \mathbb{R}^d} \tfrac{1}{\tau} E_\tau(y, y) - \inf_{\tau \in (0,1]} \inf_{x, y \in \mathbb{R}^d} \tfrac{1}{\tau} E_\tau(x, y).$$

□

Proposition 8 is new for short-range interactions. The proof we present gives another proof of the existence of Fathi's weak KAM solutions in the particular case of the minimal action.

Proof of proposition 8.

Part 1. We prove property (i) for $\tau \in \mathbb{Q}$. Let

$$\bar{E}_\tau(M) := \min\Big\{ \sum_{j=1}^{M} E_\tau(x_{j-1}, x_j) \;:\; x_j \in \mathbb{R}^d \Big\} \qquad \forall\, M \in \mathbb{Z}_+.$$

It is enough to prove $\bar{E}_{N\tau} = N\bar{E}_\tau$ for every positive integer N and $\tau > 0$ that is not necessarily rational. We choose an integer $M > 0$

$$(z_0, \ldots, z_M) \in \arg\min\Big\{ \sum_{i=1}^{M} E_{N\tau}(z_{i-1}, z_i) \;:\; z_i \in \mathbb{R}^d \Big\},$$

and by the min-plus convolution of $E_{N\tau}$, we choose $(x_{i,0}, \ldots, x_{i,N})$ so that

$$E_{N\tau}(z_{i-1}, z_i) = \sum_{j=1}^{N} E_\tau(x_{i,j-1}, x_{i,j}), \;\; x_{i,0} = z_{i-1} \;\text{ and }\; x_{i,N} = z_i.$$

Then $\bar{E}_{N\tau}(M) = \sum_{i=1}^{M}\sum_{j=1}^{N} E_\tau(x_{i,j-1}, x_{i,j}) \geqslant \bar{E}_\tau(MN)$. By dividing by MN and by taking $M \to +\infty$, one obtains $\bar{E}_{N\tau} \geqslant N\bar{E}_\tau$. Conversely, we choose

$$(x_0, \ldots, x_{M-1}) \in \arg\min\Big\{ \sum_{i=1}^{M-1} E_\tau(x_{i-1}, x_i) \;:\; x_i \in \mathbb{R}^d \Big\},$$

and the N integer translates $k_j \in \mathbb{Z}^d$, $j = 1 \ldots N$, such that $k_0 = 0$ and

$$\|(x_0 + k_j) - (x_{M-1} + k_{j-1})\| \leqslant 1.$$

We define a new chain $(z_0, \ldots, z_{MN})$ by concatenating the previous translations

$$z_{i-1+(j-1)M} := x_{i-1} + k_{j-1}M, \quad i = 1, \ldots, M, \;\; j = 1, \ldots, N.$$

Then, using the fact $\|z_{jM} - z_{M-1+(j-1)M}\| \leqslant 1$

$$\begin{aligned} N\bar{E}_\tau(M-1) &= \sum_{j=1}^{N}\sum_{i=1}^{M-1} E_\tau(z_{i-1+(j-1)M}, z_{i+(j-1)M}) \\ &\geqslant \sum_{j=1}^{N}\sum_{i=1}^{M} E_\tau(z_{i-1+(j-1)M}, z_{i+(j-1)M}) - N \sup_{\|y-x\|\leqslant 1} |E_\tau(x,y)|, \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^{N}\sum_{i=1}^{M} E_\tau(z_{i-1+(j-1)M}, z_{i+(j-1)M}) &= \sum_{i=1}^{M}\sum_{j=1}^{N} E_\tau(z_{j-1+(i-1)N}, z_{j+(i-1)N}) \\ &\geqslant \sum_{i=1}^{M} E_{N\tau}(z_{i-1}, z_i) \geqslant \bar{E}_{N\tau}(M). \end{aligned}$$

By dividing by M and by taking $M \to +\infty$, one obtains $N\bar{E}_\tau \geqslant \bar{E}_{N\tau}$.
Part 2. We prove an intermediate estimate, namely

$$\sup_{\tau>0} \|T_\tau[0] - \bar{E}_\tau\| \leqslant C,$$

where C is the constant given by the item (ia) of proposition 5. Let $\tau > 0$ and N be a positive integer such that $\tau/N \leqslant 1$. Let $u_{\tau/N}$ be a weak KAM solution of $T_{\tau/N}$ that we normalize by $\min u_{\tau/N} = 0$. Then

$$T_{\tau/N}[u_{\tau/N}] = u_{\tau/N} + \bar{E}_{\tau/N},$$
$$T_\tau[u_{\tau/N}] = (T_{\tau/N})^N[u_{\tau/N}] = u_{\tau/N} + N\bar{E}_{\tau/N} = u_{\tau/N} + \bar{E}_\tau.$$

Since $\|u_{\tau/N}\| \leqslant C$, we obtain

$$T_\tau[0] \leqslant T_\tau[u_{\tau/N}] \leqslant C + \bar{E}\tau,$$
$$T_\tau[0] \geqslant T_\tau[u_{\tau/N} - C] = u_{\tau/N} - C + \bar{E}_\tau \geqslant -C + \bar{E}_\tau,$$

and finally $\|T_\tau[0] - \bar{E}_\tau\|_\infty \leqslant C$, for every $\tau > 0$.

Part 3. We resume the proof of property (i) for $\tau \notin \mathbb{Q}$. We choose $p_i, q_i \in \mathbb{N}$, $q_i \to +\infty$, such that $p_i < q_i\tau < p_i + 1$. Denote by $\sigma_i = p_i + 1 - q_i\tau$. Then $T_{p_i+1} = T_{\sigma_i} \circ T_{q_i\tau}$. Since $\|T_{q_i\tau}[0] - q_i\bar{E}_\tau\|_\infty \leqslant C$, by applying T_{σ_i}, one obtains on the one hand

$$\|T_{p_i+1}[0] - q_i\bar{E}_\tau\|_\infty \leqslant C + \| T_{\sigma_i}[0] \|_\infty.$$

On the other hand $\|T_{p_i+1}[0] - (p_i + 1)\bar{E}_1\|_\infty \leqslant C$, which implies

$$\|(p_i + 1)\bar{E}_1 - q_i\bar{E}_\tau\|_\infty \leqslant 2C + \sup_{\sigma\in(0,1]} \| T_\sigma[0] \|_\infty.$$

Notice that item (ic) of proposition 5 implies that $\| T_\sigma[0] \|_\infty$ is uniformly bounded for $\sigma \in (0, 1]$. We conclude by dividing by q_i and letting q_i go to infinity.

Part 4. We prove item (ii). From the *a priori* compactness property of proposition 5, one can find a constant $C > 0$ such that every discrete weak KAM solution u_τ satisfies $\mathrm{Lip}(u_\tau) \leqslant C$. Since u_τ is defined up to a constant, we may assume that $\min(u_\tau) = 0$. By choosing a subsequence $\tau_i \to 0$, we may assume that $u_{\tau_i} \to u$ uniformly. Moreover, the second part of proposition 5 implies that $\| T_\sigma[v] - v \|_\infty \leqslant \sigma C$, for every $\sigma \in (0, 1]$ and every Lipshitz function v satisfying $\mathrm{Lip}(v) \leqslant C$. Let $t > 0$. There exist integers N_i such that $N_i\tau_i \leqslant t < (N_i + 1)\tau_i$. Let $\sigma_i = t - N_i\tau_i$. Then

$$T_{\tau_i}[u_{\tau_i}] = u_{\tau_i} + \tau_i\bar{E}_1, \quad T_{N_i\tau_i}[u_{\tau_i}] = u_{\tau_i} + N_i\tau_i\bar{E}_1,$$
$$T_t[u_{\tau_i}] = T_{t-N_i\tau_i}[u_{\tau_i}] + N_i\tau_i\bar{E}_1,$$
$$\| T_t[u_{\tau_i}] - u_{\tau_i} - t\bar{E}_1 \|_\infty \leqslant \| T_{\sigma_i}[u_{\tau_i}] - u_{\tau_i} \|_\infty + \sigma_i|\bar{E}_1|.$$

As $\sigma_i \to 0$, $u_{\tau_i} \to u$, $T_{\sigma_i}[u] \to u$, and $\|T_{\sigma_i}[u_{\tau_i}] - T_{\sigma_i}[u]\|_\infty \leqslant \|u_{\tau_i} - u\|_\infty$, we obtain $T_t[u] = u + t\bar{E}_1$.

Part 5. We prove item (iii). We first notice

$$\min_{x,y\in\mathbb{R}^d} E_t(x, y) = \min_{y\in\mathbb{R}^d} T_t[0](y).$$

On the one hand,

$$T_t[0] \leqslant T_t[u - \min(u)] = u + t\bar{E}_1 - \min(u) \leqslant \max(u) - \min(u) + t\bar{E}_1.$$

On the other hand,

$$T_t[0] \geqslant T_t[u - \max(u)] = u + t\bar{E}_1 - \max(u) \geqslant \min(u) - \max(u) + t\bar{E}_1.$$

In particular, $\|T_t[0] - t\bar{E}_1\|_\infty \leqslant \mathrm{osc}(u)$ and $\lim_{t\to+\infty} \min_{x,y\in\mathbb{R}^d} \frac{1}{t}E_t(x,y) = \bar{E}_1$.

We conclude this section with the proof of theorem 9.

Proof of theorem 9.

Part 1: proof of items (i) and (ii). The discrete action $\mathcal{L}_\tau(x,y)$ and the minimal action $\mathcal{E}_\tau(x,y)$ are particular cases of short-range interactions. Item (i) is proved in proposition 4. Item (ii) is proved in proposition 8.

Part 2: proof of item (iii). Let us show that there exists a constant $C > 0$ such that

$$|\bar{\mathcal{E}}_\tau - \bar{\mathcal{L}}_\tau| \leqslant \tau^2 C, \quad \forall \tau \in (0,1].$$

Let u_τ be a discrete weak KAM solution of $\mathcal{E}_\tau(x,y)$ and $(x_{-k})_{k=0}^{+\infty}$ be a calibrated configuration for u_τ. Thanks to propositions 5 and 2, there exist constants $R > 0$ and $C > 0$ independent of τ such that,

$$\begin{aligned}
&\|x_{-k} - x_{-k-1}\| \leqslant \tau R, \quad \forall k \geqslant 0,\\
&|\mathcal{E}_\tau(x,y) - \mathcal{L}_\tau(x,y)| \leqslant \tau^2 C, \quad \forall x,y \text{ satisfying } \|y-x\| \leqslant \tau R,\\
&\mathcal{E}_\tau(x_{-k-1}, x_{-k}) = u_\tau(x_{-k}) - u_\tau(x_{-k-1}) + \bar{\mathcal{E}}_\tau,\\
&\mathcal{L}_\tau(x_{-k-1}, x_{-k}) \leqslant \mathcal{E}_\tau(x_{-k-1}, x_{-k}) + \tau^2 C,\\
&\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}_\tau(x_{-k-1}, x_{-k}) \leqslant \bar{\mathcal{E}}_\tau + \tau^2 C(R) + \frac{2}{n}\|u_\tau\|_\infty.
\end{aligned}$$

By taking the limit $n \to +\infty$, and by using the mean action per site formula, we obtain $\bar{\mathcal{L}}_\tau \leqslant \bar{\mathcal{E}}_\tau + \tau^2 C$. By permuting the roles of $\mathcal{E}_\tau$ and $\mathcal{L}_\tau$ we conclude the proof of item (iii).

Part 3: Proof of item (iv). This follows directly from the *a priori* compactness property of proposition 5.

Part 4: Proof of item (v). We will use two Lax–Oleinik operators: T_τ, the discrete Lax–Oleinik operator associated with $\mathcal{L}_\tau$, and T^τ, the Lax–Oleinik semi-group associated with $\mathcal{E}_\tau$. We claim there exists a constant $C > 0$ such that for every small $\tau > 0$, for every discrete weak KAM solution u for $\mathcal{L}_\tau$,

$$\| T^\tau[u] - T_\tau[u] \|_\infty \leqslant \tau^2 C.$$

Indeed, we know from propositions 5 and 2, that there exist positive constants R and C such that for every $\tau \in (0,1]$ and for every discrete weak KAM solution u for $\mathcal{L}_\tau$,

- $\mathrm{Lip}(u) \leqslant C, \|u\|_\infty \leqslant C$,
- $\forall y \in \mathbb{R}^d, \quad x \in \arg\min_{x\in\mathbb{R}^d}\{u(x) + \mathcal{L}_\tau(x,y)\} \Rightarrow \|y-x\| \leqslant \tau R$,
- $\forall y \in \mathbb{R}^d, \quad x \in \arg\min_{x\in\mathbb{R}^d}\{u(x) + \mathcal{E}_\tau(x,y)\} \Rightarrow \|y-x\| \leqslant \tau R$,
- $\| T^\tau[u] - u \|_\infty \leqslant \tau C$,
- for every $x, y, \quad \|y-x\| \leqslant \tau R \Rightarrow \|\mathcal{E}_\tau(x,y) - \mathcal{L}_\tau(x,y)| \leqslant \tau^2 C$.

On the one hand, for every y and $x \in \arg\min_{x\in\mathbb{R}^d}\{u(x) + \mathcal{L}_\tau(x,y)\}$,

$$\begin{aligned}
T^\tau[u](y) &\leqslant u(x) + \mathcal{E}_\tau(x,y) \leqslant u(x) + \mathcal{L}_\tau(x,y) + \tau^2 C,\\
T^\tau[u](y) &\leqslant T_\tau[u](y) + \tau^2 C.
\end{aligned}$$

On the other hand, if $x \in \arg\min_{x\in\mathbb{R}^d} \big[u(x) + \mathcal{E}_\tau(x,y)\big]$,

$$T^\tau[u](y) = u(x) + \mathcal{E}_\tau(x,y) \geqslant u(x) + \mathcal{L}_\tau(x,y) - \tau^2 C,$$
$$T^\tau[u](y) \geqslant T_\tau[u](y) - \tau^2 C.$$

The claim is proved. Since $\mathrm{Lip}(u)$ is uniformly bounded independently of τ for any discrete weak KAM solution u for $\mathcal{L}_\tau$, we may choose a sequence of times $\tau_i \to 0$ and discrete weak KAM solutions u_i for $\mathcal{L}_{\tau_i}$, such that $u_i \to u$ uniformly for some periodic Lipschitz function u. Let $t > 0$ be fixed, and N_i be integers such that $N_i\tau_i \leqslant t < (N_i+1)\tau$. The non-expansiveness property of the Lax–Oleinik operator implies

$$\| T^t[u] - T^{N_i\tau_i}[u_i] \|_\infty \leqslant \| T^{t-N_i\tau_i}[u] - u \|_\infty + \|u - u_i\|_\infty \to 0.$$

The previous claim $\| T^{\tau_i}[u_i] - T_{\tau_i}[u_i] \|_\infty \leqslant \tau_i^2 C$ and the estimate $|\bar{\mathcal{E}}_{\tau_i} - \bar{\mathcal{L}}_{\tau_i}| \leqslant \tau_i^2 C$, proved in item (iii) of theorem 9, imply

$$\| T^{\tau_i}[u_i] - u_i - \tau_i \bar{\mathcal{E}}_1 \|_\infty \leqslant \tau_i^2 2C.$$

By iterating this inequality, one obtains

$$\| T^{N_i\tau_i}[u_i] - u_i - N_i\tau_i\bar{\mathcal{E}}_1 \|_\infty \leqslant N_i\tau_i^2 2C \leqslant t\tau_i 2C.$$

Since $u_i + N_i\tau_i\bar{\mathcal{E}}_1 \to u + t\bar{\mathcal{E}}_1$, one gets

$$T^t[u] = u + t\bar{\mathcal{E}}_1, \quad \forall t > 0.$$

4. Second approximation scheme

This section is devoted to the proof of theorem 20. Our approach follows article [DFIZ16b] to identify the selected discrete weak KAM solution, but with a slightly more precise description using Aubry classes and extremal plans.

We first improve the *a priori* estimates of proposition 5 to the discounted case.

Proof of proposition 11.

Part 1. The operator $T_{\tau,\delta}$ is contracting in the C^0 norm, i.e.

$$\| T_{\tau,\delta}[u] - T_{\tau,\delta}[v] \|_\infty \leqslant (1-\tau\delta)\| u - v \|_\infty, \quad \forall\, u, v \in C^0(\mathbb{T}^d).$$

Moreover, $T_{\tau,\delta}$ preserves the ball $\|u\|_\infty \leqslant \frac{C_0}{\delta}$ where

$$C_0 := \sup_{\tau\in(0,1]} \Big(\sup_{x\in\mathbb{R}^d} \frac{E_\tau(x,x)}{\tau}, - \inf_{x,y\in\mathbb{R}^d} \frac{E_\tau(x,y)}{\tau} \Big).$$

Indeed, we have

$$T_{\tau,\delta}[u](y) \leqslant (1-\tau\delta)\max(u) + \max_{x\in\mathbb{R}^d} E_\tau(x,x),$$
$$T_{\tau,\delta}[u](y) \geqslant (1-\tau\delta)\min(u) + \min_{x,y\in\mathbb{R}^d} E_\tau(x,y),$$
$$\|u\|_\infty \leqslant \frac{C_0}{\delta} \Rightarrow \| T_{\tau,\delta}[u] \|_\infty \leqslant (1-\tau\delta)\|u\|_\infty + \tau C_0 \leqslant \frac{C_0}{\delta}.$$

In particular, $T_{\tau,\delta}$ admits a unique fixed point $u_{\tau,\delta}$ which is inside $B(0, \frac{C_0}{\delta})$. We have proved item (i). The fixed point satisfies

$$u_{\tau,\delta}(y) = \min_{x \in \mathbb{R}^d} \big\{(1-\tau\delta)u_{\tau,\delta}(x) + E_\tau(x,y)\big\}, \quad \forall y \in \mathbb{R}^d.$$

By iterating backward, one obtains the explicit formula for $u_{\tau,\delta}$.
Part 2. We prove item (iii). We use the same reasoning as in the proof of proposition 5. We claim that for every point x, y satisfying $\|y-x\| \geqslant \tau$, we have

$$|u_{\tau,\delta}(y) - u_{\tau,\delta}(x)| \leqslant C_1\|y-x\|, \quad \text{with} \quad C_1 := \sup_{\tau\in(0,1]} \sup_{\|y-x\|\leqslant 2\tau} \Big(\frac{E_\tau(x,y)}{\tau} + C_0\Big).$$

Indeed, choose $n \geqslant 1$ so that $n\tau < \|y-x\| \leqslant (n+1)\tau$ and define $x_i = x + \frac{i}{n}(y-x)$. By applying n times the inequality

$$u_{\tau,\delta}(x_{i+1}) - u_{\tau,\delta}(x_i) \leqslant E_\tau(x_i, x_{i+1}) + \tau\delta\|u_{\tau,\delta}\|_\infty \leqslant \tau C_1$$

we obtain $u_{\tau,\delta}(y) - u_{\tau,\delta}(x) \leqslant C_1\|y-x\|$.
Define R using the uniform superlinearity (H5) by

$$R := \inf\Big\{R > 1 \ : \ \inf_{\tau\in(0,1]} \inf_{\|y-x\|\geqslant\tau R} \frac{E_\tau(x,y) - C_0\tau}{\|y-x\|} > C_1\Big\}.$$

We prove by contradiction that every $x \in \arg\min_x\big\{(1-\tau\delta)u_{\tau,\delta}(x) + E_\tau(x,y)\big\}$ satisfies $\|y-x\| \leqslant \tau R$. If not $\|y-x\| > \tau R > \tau$, $u_{\tau,\delta}(y) - u_{\tau,\delta}(x) \leqslant C_1\|y-x\|$ and by definition of R, we have

$$u_{\tau,\delta}(y) - u_{\tau,\delta}(x) \geqslant E_\tau(x,y) - \tau\delta\|u_{\tau,\delta}\|_\infty \geqslant E_\tau(x,y) - \tau C_0 > C_1\|y-x\|.$$

We obtain a contradiction, therefore $\|y-x\| \leqslant \tau R$, and the proof of item (iii) is complete.
Part 3. We prove item (iv). If $\|z-y\| \leqslant \tau$ and x is a point realizing the minimum in the definition of $u_{\tau,\delta}(y)$,

$$u_{\tau,\delta}(z) - u_{\tau,\delta}(y) \leqslant E_\tau(x,z) - E_\tau(x,y) \leqslant C\|z-y\|,$$

where

$$C := \max\Big(C_1, \sup_{\tau\in(0,1]} \sup_{\|y-x\|,\|z-x\|\leqslant\tau(R+1)} \frac{E_\tau(x,z) - E_\tau(x,y)}{\|y-x\|}\Big).$$

□

Proof of lemma 17. Let π be an extremal plan, and $\mu = pr^1_*(\pi)$.

Part 1. Let $\hat\Omega := (\mathbb{T}^d)^{\mathbb{N}}$, $\hat\sigma : \hat\Omega \to \hat\Omega$ be the left shift, and $pr^{1,2} : \hat\Omega \to \mathbb{T}^d \times \mathbb{T}^d$ be the projection onto the first two coordinates. We claim there exists an ergodic $\hat\sigma$-invariant probability measure $\hat\pi$ defined on $\hat\Omega$, which projects onto π by $pr^{1,2}$ and minimizes $\hat E_\tau(x) := E^*_\tau(x_0, x_1)$, $\forall x = (x_0, x_1, \ldots) \in \hat\Omega$.

Let $\pi(\mathrm{d}x, \mathrm{d}y) = \mu(\mathrm{d}x)\mathbb{P}(\mathrm{d}y|x)$ be a regular family of disintegrated measures of π along the projection pr^1. Define the Markov measure on $\hat\Omega$ by

$$\hat{\mathbb{P}}(\mathrm{d}x) = \mu(\mathrm{d}x_0)\mathbb{P}(\mathrm{d}x_1|x_0)\mathbb{P}(\mathrm{d}x_2|x_1)\cdots.$$

Then $\hat{\mathbb{P}}$ is a $\hat{\sigma}$-invariant probability measure which projects onto π and minimizes $\hat{E}_\tau$. Let $\hat{\mathbb{P}}(dx) = \int_{\hat{\Omega}} \hat{\mathbb{P}}_\omega(dx)\hat{\mathbb{P}}(d\omega)$ be an ergodic decomposition of $\hat{\mathbb{P}}$ (see [Mn87, theorem 6.1]). We claim that $\omega \mapsto pr_*^{1,2}(\hat{\mathbb{P}}_\omega)$ is a.e. constant. By contradiction there would exist $\varphi \in C^0(\mathbb{T}^d \times \mathbb{T}^d)$ and a constant $a \in \mathbb{R}$ such that

$$\hat{B} := \Big\{\omega \in \hat{\Omega} : \int \varphi(x,y)\, pr_*^{1,2}(\mathbb{P}_\omega)(dx,dy) < a\Big\}.$$

Both $\hat{B}$ and $\hat{B}^c$ have a positive measure. Since $\hat{\mathbb{P}}_\omega$ is $\hat{\sigma}$-invariant and minimizing, $pr_*^{1,2}(\hat{\mathbb{P}}_\omega)$ is a minimizing plan. Define

$$\pi_1(dx,dy) := \frac{1}{\hat{\mathbb{P}}(\hat{B})}\int_{\hat{B}} pr_*^{1,2}(\hat{\mathbb{P}}_\omega)(dx,dy)\,\hat{\mathbb{P}}(d\omega),$$
$$\pi_2(dx,dy) := \frac{1}{\hat{\mathbb{P}}(\hat{B}^c)}\int_{\hat{B}_c} pr_*^{1,2}(\hat{\mathbb{P}}_\omega)(dx,dy)\,\hat{\mathbb{P}}(d\omega).$$

Then π_1 and π_2 are distinct minimizing plans and

$$\pi = \hat{\mathbb{P}}(\hat{B})\pi_1 + \hat{\mathbb{P}}(\hat{B}^c)\pi_2, \text{ with } \hat{\mathbb{P}}(\hat{B}) \in (0,1)\text{nontrivial},$$

which contradicts the fact that π is extremal. For almost every ω we have obtained $pr^{1,2}(\hat{\mathbb{P}}_\omega) = \pi$ and $\hat{\mathbb{P}}_\omega$ is ergodic.

Part 2: proof of item (i). We have shown from part 1 that there exists an ergodic $\hat{\sigma}$-invariant measure $\hat{\pi}$ on $\hat{\Omega}$ such that $pr_*^1(\hat{\pi}) = \mu$, where $pr^1 : \hat{\Omega} \to \mathbb{T}^d$ is the first projection. Let $\epsilon > 0, x, y \in \text{supp}(\mu)$. Define

$$\hat{B}_x = \{(x_0, x_1, \cdots) : x_0 \in B(x,\epsilon)\}, \quad \hat{B}_y = \{(x_0, x_1, \cdots) : x_0 \in B(y,\epsilon)\}.$$

Then $\hat{B}_x, \hat{B}_y$ are open sets and have positive measures for $\hat{\pi}$. Choose a discrete weak KAM solution u_τ and define

$$\hat{\varphi}(z) := E_\tau^*(z_0,z_1) - [u_\tau(z_1) - u_\tau(z_0)] - \bar{E}_\tau, \quad \forall z = (z_0, z_1, \ldots) \in \hat{\Omega}.$$

By Atkinson's theorem [Atk76], since $\int \hat{\varphi}\, d\hat{\pi} = 0$, for a.e. $z \in \hat{B}_x$,

$$\exists 0 < m < n, \ \text{ s.t. } \hat{\sigma}^m(z) \in \hat{B}_y, \ \hat{\sigma}^n(z) \in \hat{B}_x, \ \text{ and } \ 0 \leqslant \sum_{k=0}^{n-1} \hat{\varphi} \circ \hat{\sigma}^k(z) < \epsilon.$$

We have obtained in particular, $z_0 \in B(x,\epsilon)$, $z_m \in B(y,\epsilon)$, $z_n \in B(x,\epsilon)$, and

$$\Phi_\tau^*(z_0,z_m) + \Phi_\tau^*(z_m,z_n) \leqslant \sum_{k=0}^{n-1} \hat{\varphi} \circ \hat{\sigma}^k(\omega) + [u_\tau(z_n) - u_\tau(z_0)] = O(\epsilon).$$

Letting $\epsilon \to 0$, we obtain $\Phi_\tau^*(x,y) + \Phi_\tau^*(y,x) = 0$ or $x \sim y$.

Part 3: proof of item (ii). Let A be the Aubry class containing $\text{supp}(\mu)$ and $\bar{z} \in A$ arbitrarily fixed. Then, as a function of y, using item (iii) of lemma 16,

$$\int \Phi_\tau^*(z,y)\,\mu(dz) = \int \Phi_\tau^*(z,\bar{z})\,\mu(dz) + \Phi_\tau^*(\bar{z},y), \forall y \in \mathbb{R}^d$$

is equal to the sum of $\Phi_\tau^*(\bar{z},y)$ and a constant, which is a discrete weak KAM solution thanks to item (v) of lemma 16.

Part 4: proof of item (iii). For every $x, y \in A$, $\Phi^*_\tau(x,y) + \Phi^*_\tau(y,x) = 0$. We conclude by integrating with respect to $\mu(\mathrm{d}x)\mu(\mathrm{d}y)$. □

Proof of proposition 18.

Part 1. We use the notations of part 1 in the proof of lemma 17. We claim that the infimum in the definition of u^*_τ can be realized at an extremal plan. Let π be a minimizing plan realizing the infimum. Let $\hat{\mathbb{P}}$ be a $\hat\sigma$-invariant measure on $\hat\Omega$ such that $pr^{1,2}_*(\hat{\mathbb{P}}) = \pi$. Then $\hat{\mathbb{P}}$ is minimizing. Let $\hat{\mathbb{P}}(\mathrm{d}x) = \int \hat{\mathbb{P}}_\omega(\mathrm{d}x)\,\hat{\mathbb{P}}(\mathrm{d}\omega)$ be an ergodic decomposition. Define $\pi_\omega := pr^{1,2}_*(\hat{\mathbb{P}}_\omega)$. Since $\hat{\mathbb{P}}_\omega$ is ergodic, π_ω is an extremal plan. Moreover, for x fixed,

$$\begin{aligned}
\pi(\mathrm{d}x,\mathrm{d}y) &= \int_{\hat\Omega} \pi_\omega(\mathrm{d}x,\mathrm{d}y)\,\hat{\mathbb{P}}(\mathrm{d}\omega),\\
u^*_\tau(x) &= \int_{\hat\Omega}\Big[\int_{\mathbb{T}^d} \Phi^*_\tau(z,x)\,pr^1_*(\pi_\omega)(\mathrm{d}z)\Big]\,\hat{\mathbb{P}}(\mathrm{d}\omega),\\
u^*_\tau(x) &= \int_{\mathbb{T}^d} \Phi^*_\tau(z,x)\,pr^1_*(\pi_\omega)(\mathrm{d}z), \quad \hat{\mathbb{P}}(\mathrm{d}\omega)\ \text{a.e.},\\
u^*_\tau(x) &= \inf\Big\{\int_{\mathbb{T}^d} \Phi^*_\tau(z,x)\,pr^1_*(\pi)(\mathrm{d}z) : \pi \in \mathcal{M}^*(E_\tau)\ \text{is extremal}\Big\}.
\end{aligned}$$

Part 2: proof of item (i). This follows from the fact that u^*_τ is obtained as an infimum of discrete weak KAM solutions thanks to part 1 and item (ii) of lemma 17.
Part 3: proof of item (ii). Let

$$w^*(x) := \sup\Big\{w(y) : w + \bar{E}_\tau = T_\tau[w],\ \int_{\mathbb{T}^d} w(x)\,pr^1_*(\pi)(\mathrm{d}x) \leqslant 0,\ \forall\pi \in \mathcal{M}^*(E_\tau)\Big\}.$$

We already know that u^*_τ is a discrete weak KAM solution. Using item (iii) of lemma 17, we have for every extremal plan $\pi \in \mathcal{M}^*(E_\tau)$

$$\int_{\mathbb{T}^d} u^*_\tau(x)\,pr^1_*(\pi)(\mathrm{d}x) \leqslant \iint \Phi^*_\tau(z,x)\,pr^1_*(\pi)(\mathrm{d}z)\,pr^1_*(\pi)(\mathrm{d}x) = 0.$$

Thus by taking convex combinations of the extremal plans, we get

$$\int_{\mathbb{T}^d} u_\tau(x)\,pr^1_*(\pi)(\mathrm{d}x) \leqslant 0, \quad \forall\pi \in \mathcal{M}^*(E_\tau).$$

We have proved that $u^*_\tau \leqslant w^*$. Conversely, if w is a discrete weak KAM solution satisfying $\int w(x)\,pr^1_*(\pi)(\mathrm{d}x) \leqslant 0,\ \forall\pi \in \mathcal{M}^*(E_\tau)$, then,

$$\begin{aligned}
w(x) &\leqslant w(z) + \Phi^*_\tau(z,x), \quad \forall x, z \in \mathbb{R}^d,\\
w(x) &\leqslant \int_{\mathbb{T}^d} \Phi^*_\tau(z,x)\,pr^1(\pi)(\mathrm{d}z), \quad \forall\pi\ \text{extremal plan}.
\end{aligned}$$

By taking the supremum over such w and the infimum over all extremal plans π, one obtains $w^* \leqslant u^*_\tau$ and therefore $w^* = u^*_\tau$.
Part 4: poof of item (iii). Assume by contradiction that for some $\epsilon > 0$, $\int u^*_\tau(x)\,pr^1_*(\pi)(\mathrm{d}x) \leqslant -\epsilon$ for every extremal plan π. Then

$$u_\tau^*(x) \leqslant u_\tau^*(z) + \Phi_\tau^*(z,x), \quad \forall x, z \in \mathbb{R}^d,$$
$$u_\tau^*(x) \leqslant -\epsilon + \int_{\mathbb{T}^d} \Phi_\tau^*(z,x)\, pr^1(\pi)(\mathrm{d}z), \quad \forall \pi \text{ extremal plan.}$$

We thus obtain a contradiction by taking the infimum over all extremal plans. □

Proof of proposition 19.

Part 1. Let C be the constant given by proposition 5. We claim that for every $\tau, \delta \in (0, 1]$,

$$\Big\| u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta} \Big\|_\infty \leqslant C.$$

Let u_τ be some discrete weak KAM solution. Let

$$y \in \arg\max_{y\in\mathbb{R}^d} \Big\{ u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y) \Big\}.$$

As a fixed point of $T_{\tau,\delta}$, the discounted discrete solution satisfies for every x,

$$\begin{aligned} u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y) \leqslant (1-\tau\delta)\Big[u_{\tau,\delta}(x) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(x) \Big] \\ + \big[E_\tau(x,y) - u_\tau(y) + u_\tau(x) - \bar{E}_\tau \big] - \tau\delta u_\tau(x). \end{aligned}$$

Let x be a backward calibrated point for y with respect to u_τ. Then, by definition of y, we have

$$\begin{aligned} u_{\tau,\delta}(x) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(x) \leqslant u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y), \\ u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y) \leqslant -u_\tau(x), \\ u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} \leqslant \operatorname{osc}(u_\tau) \leqslant C. \end{aligned}$$

On the other hand, let y be a point realizing the minimum of $u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y)$ and x be a discounted backward calibrated point for y, that is

$$u_{\tau,\delta}(y) = (1-\tau\delta) u_{\tau,\delta}(x) + E_\tau(x,y).$$

Then similar to what we have done in part 1, we obtain

$$\begin{aligned} u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y) = (1-\tau\delta)\Big[u_{\tau,\delta}(x) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(x) \Big] \\ + \big[E_\tau(x,y) - u_\tau(y) + u_\tau(x) - \bar{E}_\tau \big] - \tau\delta u_\tau(x). \end{aligned}$$

As $E_\tau(x,y) - u_\tau(y) + u_\tau(x) - \bar{E}_\tau \geqslant 0$, we obtain $u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y) \geqslant -u_\tau(x)$ or $u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} \geqslant -\operatorname{osc}(u_\tau) \geqslant -C$.

Part 2. We claim that for every $\tau, \delta \in (0, 1]$, $\pi \in \mathcal{M}^*(E_\tau)$, $\mu = pr^1_*(\pi)$,

$$\int_{\mathbb{T}^d} \Big[u_{\tau,\delta}(x) - \frac{\bar{E}_\tau}{\tau\delta} \Big] \mathrm{d}\mu(x) \leqslant 0.$$

By definition of the discounted discrete solution $u_{\tau,\delta}$, we have

$$u_{\tau,\delta}(y) \leqslant (1-\tau\delta)u_{\tau,\delta}(x) + E^*_\tau(x,y), \quad \forall x,y \in \mathbb{R}^d.$$

By integrating the previous inequality, we obtain

$$\int_{\mathbb{T}^d} u_{\tau,\delta}(y)\,\mu(\mathrm{d}y) \leqslant (1-\tau\delta)\int_{\mathbb{T}^d} u_{\tau,\delta}(x)\,\mu(\mathrm{d}x) + \iint_{\mathbb{T}^d\times\mathbb{T}^d} E^*_\tau(x,y)\,\pi(\mathrm{d}x,\mathrm{d}y).$$

The last integral is equal to $\bar{E}_\tau$ and $\tau\delta\int_{\mathbb{T}^d} u_{\tau,\delta}(x)\,\mu(\mathrm{d}x) \leqslant \bar{E}_\tau$.

Part 3. Let $\tau > 0$ be fixed. Let $\delta_i \to 0$ be a sequence converging to 0. For every δ_i, let $(x^i_{-k})_{k=0}^{+\infty}$ be a discounted backward calibrated configuration,

$$u_{\tau,\delta_i}(x^i_{-k}) = (1-\tau\delta_i)u_{\tau,\delta_i}(x^i_{-k-1}) + E_\tau(x^i_{-k-1}, x^i_{-k}).$$

Let π_i be the probability measure on $\mathbb{T}^d\times\mathbb{T}^d$ defined by

$$\pi_i := \sum_{k\geqslant 0}\tau\delta(1-\tau\delta)^k\delta_{(x^i_{-k-1},x^i_{-k})}.$$

We claim that every weak* accumulation measure π of $\{\pi_i\}_{i=1}^\infty$ is a minimizing plan. Assume that $\pi_i \to \pi$ as $i\to\infty$ to simplify the notations.

We first prove that π is a stationary plan. Let $\varphi : \mathbb{T}^d \to \mathbb{R}$ be a continuous function, then

$$\begin{aligned}\iint_{\mathbb{T}^d\times\mathbb{T}^d}\varphi(y)\,\pi_i(\mathrm{d}x,\mathrm{d}y) &= \sum_{k\geqslant 0}\tau\delta_i(1-\tau\delta_i)^k\varphi(x^i_{-k})\\ &= \tau\delta_i\varphi(x^i_0) + (1-\tau\delta_i)\sum_{k\geqslant 0}\tau\delta_i(1-\tau\delta_i)^k\varphi(x^i_{-k-1})\\ &= \tau\delta_i\varphi(x^i_0) + (1-\tau\delta_i)\iint_{\mathbb{T}^d\times\mathbb{T}^d}\varphi(x)\,\pi_{i-1}(\mathrm{d}x,\mathrm{d}y).\end{aligned}$$

We complete the proof by letting $\delta_i \to 0$. We next prove that π is minimizing:

$$\begin{aligned}\iint_{\mathbb{T}^d\times\mathbb{T}^d} E^*_\tau(x,y)\,\pi_i(\mathrm{d}x,\mathrm{d}y) &= \sum_{k\geqslant 0}\tau\delta_i(1-\tau\delta_i)^k E^*_\tau(x^i_{-k-1},x^i_{-k})\\ &= \sum_{k\geqslant 0}\tau\delta_i(1-\tau\delta_i)^k\big[u_{\tau,\delta_i}(x^i_{-k}) - (1-\tau\delta_i)u_{\tau,\delta_i}(x^i_{-k-1})\big] = \tau\delta_i u_{\tau,\delta_i}(x^i_0).\end{aligned}$$

We conclude the proof thanks to part 1 which implies $\tau\delta_i u_{\tau\delta_i} \to \bar{E}_\tau$ uniformly.

Part 4. Since $\mathrm{Lip}(u_{\tau,\delta})$ and $\big\|u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta}\big\|_\infty$ are uniformly bounded with respect to δ, there exists a subsequence $\delta_i \to 0$ and a C^0 periodic function u_τ such that

$$u_{\tau,\delta_i} - \frac{\bar{E}_\tau}{\tau\delta_i} \to u_\tau, \quad \text{in the } C^0\text{-topology}.$$

We first prove that u_τ is a discrete weak KAM solution. On the one hand, by letting $\delta_i \to 0$ in

$$u_{\tau,\delta_i}(y) - \frac{\bar{E}_\tau}{\tau\delta_i} \leqslant (1-\tau\delta_i)\big[u_{\tau,\delta_i}(x) - \frac{\bar{E}_\tau}{\tau\delta_i}\big] + E_\tau(x,y) - \bar{E}_\tau,$$

one obtains $u_\tau(y) - u_\tau(x) \leqslant E_\tau(x,y) - \bar{E}_\tau$, for every $x, y \in \mathbb{R}^d$. On the other hand, for every y there exists $x_i \in \mathbb{R}^d$ such that

$$u_{\tau,\delta_i}(y) - \frac{\bar{E}_\tau}{\tau\delta_i} = (1-\tau\delta_i)\Big[u_{\tau,\delta_i}(x_i) - \frac{\bar{E}_\tau}{\tau\delta_i}\Big] + E_\tau(x_i,y) - \bar{E}_\tau.$$

Proposition 11 implies there exists a constant $R > 0$, independent of δ, such that $\|y - x_i\| \leqslant \tau R$. By taking possibly a subsequence, one may assume $x_i \to x$ for some $x \in \mathbb{R}^d$. One then obtains $u_\tau(y) - u_\tau(x) = E_\tau(x,y) - \bar{E}_\tau$. The proof is finished.

We next prove that $u_\tau = u_\tau^*$ given by proposition 18. Let $\pi \in \mathcal{M}^*(E_\tau)$ and $\mu = pr_*^1(\pi)$. By letting $\delta_i \to 0$ in part 2, one obtains $\int_{\mathbb{T}^d} u_\tau(x)\,\mathrm{d}\mu(x) \leqslant 0$ and

$$u_\tau(y) \leqslant \sup\Big\{w(y) \,:\, T_\tau[w] = w + \bar{E}_\tau,\ \int_{\mathbb{T}^d} w(x)\,pr_*^1(\pi)(\mathrm{d}x) \leqslant 0,\ \forall \pi \in \mathcal{M}^*(E_\tau)\Big\}.$$

Conversely, let w be a discrete weak KAM solution satisfying $\int_{\mathbb{T}^d} w\,\mathrm{d}pr_*^1(\pi) \leqslant 0$ for every $\pi \in \mathcal{M}^*(E_\tau)$. Let $y \in \mathbb{R}^d$ and for every δ_i, $(x_{-k}^i)_{k\geqslant 0}$ be a discounted backward calibrated configuration starting at $y = x_0^i$. Then

$$\begin{aligned} u_{\tau,\delta_i}(x_{-k}^i) - \frac{\bar{E}_\tau}{\tau\delta_i} - w(x_{-k}^i) &= (1-\tau\delta_i)\Big[u_{\tau,\delta_i}(x_{-k-1}^i) - w(x_{-k-1}^i) - \frac{\bar{E}_\tau}{\tau\delta_i}\Big] \\ &\quad + \big[E_\tau(x_{-k-1}^i, x_{-k}^i) - w(x_{-k}^i) + w(x_{-k-1}^i) - \bar{E}_\tau\big] - \tau\delta_i w(x_{-k-1}^i). \end{aligned}$$

As $E_\tau(x_{-k-1}^i, x_{-k}^i) - w(x_{-k}^i) + w(x_{-k-1}^i) - \bar{E}_\tau \geqslant 0$, by iterating these inequalities, one obtains

$$u_{\tau,\delta_i}(y) - \frac{\bar{E}_\tau}{\tau\delta_i} - w(y) \geqslant \sum_{k\geqslant 0} -\tau\delta_i(1-\tau\delta_i)^k w(x_{-k-1}^i) = -\iint_{\mathbb{T}^d\times\mathbb{T}^d} w(x)\,\pi_i(\mathrm{d}x,\mathrm{d}y),$$

where π_i is the probability measure defined in part 3. As π_i converges to a minimizing plan π, one obtains $u_\tau(y) - w(y) \geqslant -\int_{\mathbb{T}^d} w\,\mathrm{d}pr_*^1(\pi) \geqslant 0$ and therefore $u_\tau \geqslant u_\tau^*$. Since u_τ^* is the only accumulation point of $u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta}$, the proof of proposition 19 is complete. □

The only results in theorem 20 to be proved are items (i) and (iic). Items (iia) and (iib) are particular cases of proposition 11. Item (iii) is a particular case of proposition 19. Item (iv) is a consequence of item (iic) and the existence of the balanced weak KAM solution (7).

Proof of item (i) of theorem 20.

Part 1. Let $\tau > 0$ and $\{x_n^{\tau,\delta}\}_{n\leqslant 0}$ be a discounted backward calibrated configuration for the discrete action $\mathcal{L}_\tau$ ending at x. We note

$$v_n^{\tau,\delta} := \frac{1}{\tau}\Big(x_{n+1}^{\tau,\delta} - x_n^{\tau,\delta}\Big), \quad \forall n \leqslant -1.$$

We show in this part that there exists a constant $C > 0$, independent of n, δ and x, such that $\|v_n^{\tau,\delta} - v_{n-1}^{\tau,\delta}\| \leqslant C\tau$ for all $n \leqslant -1$. Let $x_n := x_n^{\tau,\delta}$ and $v_n := v_n^{\tau,\delta}$. By the definition of the calibration we have

$$\begin{aligned} u_{\tau,\delta}(x_{n+1}) &= (1-\tau\delta)u_{\tau,\delta}(x_n) + \mathcal{L}_\tau(x_n, x_{n+1}) \\ &= (1-\tau\delta)^2 u_{\tau,\delta}(x_{n-1}) + (1-\tau\delta)\mathcal{L}_\tau(x_{n-1}, x_n) + \mathcal{L}_\tau(x_n, x_{n+1}) \\ &\leqslant (1-\tau\delta)u_{\tau,\delta}(x) + \mathcal{L}_\tau(x, x_{n+1}), \quad \forall x \in \mathbb{R}^d \\ &\leqslant (1-\tau\delta)^2 u_{\tau,\delta}(x_{n-1}) + (1-\tau\delta)\mathcal{L}_\tau(x_{n-1}, x) + \mathcal{L}_\tau(x, x_{n+1}), \quad \forall x \in \mathbb{R}^d. \end{aligned}$$

In other words $\{x_n^{\tau,\delta}\}_{n\leqslant 0}$ is minimizing in the following sense:

$$(1-\tau\delta)\mathcal{L}_\tau(x_{n-1},x_n)+\mathcal{L}_\tau(x_n,x_{n+1})\leqslant(1-\tau\delta)\mathcal{L}_\tau(x_{n-1},x)+\mathcal{L}_\tau(x,x_{n+1}),\quad \forall x\in\mathbb{R}^d,$$

and satisfies the *discounted discrete Euler–Lagrange equation*

$$\begin{aligned}
&(1-\tau\delta)\frac{\partial\mathcal{L}_\tau}{\partial y}(x_{n-1},x_n)+\frac{\partial\mathcal{L}_\tau}{\partial x}(x_n,x_{n+1})=0\\
\iff\quad &(1-\tau\delta)\frac{\partial L}{\partial v}(x_{n-1},v_{n-1})-\frac{\partial L}{\partial v}(x_n,v_n)+\tau\frac{\partial L}{\partial x}(x_n,v_n)=0\\
\iff\quad &\frac{1}{\tau}\Big[\frac{\partial L}{\partial v}(x_n,v_n)-\frac{\partial L}{\partial v}(x_{n-1},v_{n-1})\Big]=\frac{\partial L}{\partial x}(x_n,v_n)-\delta\frac{\partial L}{\partial v}(x_{n-1},v_{n-1}).
\end{aligned}\tag{16}$$

Proposition 11 shows there exists $R>0$ such that $\|v_n^{\tau,\delta}\|\leqslant R$, $\forall n\leqslant -1$. The property of positive definiteness (L1) implies the existence of a constant $\alpha(R)>0$ such that for every $x\in\mathbb{R}^d$, $v\in\mathbb{R}^d$ satisfying $\|v\|\leqslant R$

$$\frac{\partial^2 L}{\partial v\partial v}(x,v).(h,h)\geqslant\alpha(R)\|h\|^2,\quad \forall h\in\mathbb{R}^d.$$

By integrating over $t\in[0,1]$, the term $\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial v}\big(x_{n-1}+t(x_n-x_{n-1}),v_{n-1}+t(v_n-v_{n-1})\big)\Big)$ and by taking the scalar product with (v_n-v_{n-1}), one obtains

$$\alpha(R)\|v_n-v_{n-1}\|\leqslant\Big\|\frac{\partial L}{\partial x\partial v}\Big\|\,\|x_n-x_{n-1}\|+\tau\Big(\,\Big\|\frac{\partial L}{\partial x}\Big\|+\delta\Big\|\frac{\partial L}{\partial v}\Big\|\Big)$$

where all norms $\|\cdot\|$ are taken over $\mathbb{T}^d\times\big\{v\in\mathbb{R}^d\ :\ \ \|v\|\leqslant R\|\big\}$. As $\|x_n-x_{n-1}\|\leqslant\tau R$, thanks to item (iib) of proposition 11, one obtains $\|v_n-v_{n-1}\|\leqslant\tau C$, for some constant $C>0$, uniformly in n,δ and x.

Part 2. Let $\gamma^x_{\tau,\delta}:(-\infty,0]\to\mathbb{R}^d$ be the piecewise affine path interpolating the points x_n at time $n\tau$. We show that $\gamma^x_{\tau,\delta}$ is Lipschitz uniformly in n,δ and $\{x_n^{\tau,\delta}\}_{n\leqslant 0}$. To simplify, we write $\gamma=\gamma^x_{\tau,\delta}$. Let $s<t<0$. Either s,t belong to the same interval $((n-1)\tau,n\tau]$; as γ is affine with speed bounded by R, we obtain $\|\gamma(t)-\gamma(s)\|\leqslant|t-s|R$. Or s,t belong to different intervals; by introducing the points x_n corresponding to the intermediate times $s\leqslant n\tau\leqslant t$, one again obtains the same estimate.

Part 3. We choose a subsequence $\tau_i\to 0$ and a discounted backward calibrated configuration $\{x_n^i\}_{n\leqslant 0}$ such that $\gamma_i:=\gamma^x_{\tau_i,\delta}\to\gamma^x_\delta$ uniformly on any compact interval of $(-\infty,0]$ for some Lipschitz function γ^x_δ. We claim there exists a uniformly Lipschitz function $V:(-\infty,0]\to\mathbb{R}^d$ such that

$$\int_t^0 V(s)\,\mathrm{d}s=x-\gamma^x_\delta(t),\quad \forall t\leqslant 0.$$

Let $T\subset(-\infty,0)$ be a countable dense subset. Let $V_i:(-\infty,0)\to\mathbb{R}^d$ such that

$$V_i(t):=\frac{1}{\tau_i}\Big(x_n^i-x_{n-1}^i\Big),\quad \forall t\in[(n-1)\tau_i,n\tau_i),\ \forall n\leqslant 0.$$

By compactness of the ball $\{v\ :\ \|v\|\leqslant R\}$, by taking a subsequence if needed, we may assume $V_i(t)\to V(t)$ exists for every $t\in T$. Let $s<t<0$ and $m\leqslant n$ be nonpositive integers such that $(m-1)\tau_i\leqslant s<m\tau_i$ and $(n-1)\tau_i\leqslant t<n\tau_i$. Part 1 implies

$$\|V_i(t) - V_i(s)\| = \|v^i_{n-1} - v^i_{m-1}\| \leqslant (n-m)\tau_i C \leqslant |t-s|C + \tau_i C.$$

By letting $\tau_i \to 0$, one obtains $\|V(t) - V(s)\| \leqslant |t-s|C$ for every $s, t \in T$. Let $V : (-\infty, 0) \to \mathbb{R}^d$ be the unique Lipschitz extension of V. Then $V_i(t) \to V(t)$ for every $t \in (-\infty, 0)$. Since

$$\int_t^0 V_i(s)\,\mathrm{d}s = x - \gamma_i(t), \quad \forall t < 0,$$

the claim is proved and γ^x_δ is a $C^{1,1}$ path.

Part 4. Item (iia) of proposition 11 shows there exists a constant $C > 0$ such that $\mathrm{Lip}(u_{\tau_i,\delta}) \leqslant C$. By taking a subsequence if necessary, we may assume that $u_i := u_{\tau_i,\delta} \to u$ uniformly for some Lipschitz function u. We claim that

$$u(x) - \mathrm{e}^{t\delta} u(\gamma^x_\delta(t)) = \int_t^0 \mathrm{e}^{s\delta} L(\gamma^x_\delta(s), \dot\gamma^x_\delta(s))\,\mathrm{d}s, \quad \forall x \in \mathbb{R}^d,\ \forall t \leqslant 0.$$

Indeed using the notations in part 3, we have for every $n \leqslant -1$,

$$u_i(x) = (1-\tau_i\delta)^{-n} u_i \circ \gamma_i(n\tau_i) + \sum_{k=n}^{-1} (1-\tau_i\delta)^{-k-1} \tau_i L\big(\gamma_i(k\tau_i), V_i(k\tau_i)\big).$$

Let $t < 0$ be fixed, $n \leqslant 0$ be such that $(n-1)\tau_i \leqslant t < n\tau_i$. Then

$$I := \Big| \sum_{k=n}^{-1} (1-\tau_i\delta)^{-k-1} \tau_i L\big(\gamma_i(k\tau_i), V_i(k\tau_i)\big) - \int_{n\tau_i}^0 \mathrm{e}^{s\delta} L(\gamma_i(s), V_i(s))\,\mathrm{d}s \Big|$$

can be bounded from above by the following three terms I_1, I_2, I_3

$$\begin{aligned}
I_1 &= \sum_{k=n}^{-1} (1-\tau_i\delta)^{-k-1} \int_{k\tau_i}^{(k+1)\tau_i} \big| L\big(\gamma_i(k\tau_i), V_i(k\tau_i)\big) - L(\gamma_i(s), V_i(s)) \big|\,\mathrm{d}s \\
&\leqslant R \Big\| \frac{\partial L}{\partial x} \Big\| \frac{\tau_i}{\delta}, \\
I_2 &= \sum_{k=n}^{-1} \Big[(1-\tau_i\delta)^{-k-1} - (1-\tau_i\delta)^{-k} \Big] \int_{k\tau_i}^{(k+1)\tau_i} \big| L(\gamma_i(s), V_i(s)) \big|\,\mathrm{d}s \\
&\leqslant \tau_i \|L\| \Big(1 - (1-\tau_i\delta)^{-n} \Big) \leqslant \tau_i \|L\|, \\
I_3 &= \sum_{k=n}^{-1} \int_{k\tau_i}^{(k+1)\tau_i} \Big[\mathrm{e}^{s\delta} - (1-\tau_i\delta)^{-k} \Big] \big| L(\gamma_i(s), V_i(s)) \big|\,\mathrm{d}s \\
&\leqslant \|L\| \Big[\int_{n\tau_i}^0 \mathrm{e}^{s\delta}\,\mathrm{d}s - \tau_i \sum_{k=n}^{-1} (1-\tau_i\delta)^{-k} \Big] \leqslant \tau_i \|L\|.
\end{aligned}$$

We finally obtain

$$I \leqslant R \Big\| \frac{\partial L}{\partial x} \Big\| \frac{\tau_i}{\delta} + 2\tau_i \|L\|,$$

and the claim is proved by letting $\tau_i \to 0$, since $n\tau_i \to t$, $u_i \to u$ uniformly on $\mathbb{R}^d$, and both $\gamma_i \to \gamma_\delta^x$ and $V_i \to \dot\gamma_\delta^x$ uniformly on any compact set of $(-\infty, 0]$.

Part 5. We claim that

$$u(x) - \mathrm{e}^{-t\delta} u(x - tv) \leqslant \int_{-t}^{0} \mathrm{e}^{s\delta} L(x + sv, v)\, \mathrm{d}s, \quad \forall x \in \mathbb{R}^d,\ \forall t \geqslant 0,\ \forall v \in \mathbb{R}^d.$$

We choose as before $n \leqslant 0$ such that $(n-1)\tau_i \leqslant t < n\tau_i$. Let $x_k^i := x - k\tau_i v$, $\forall k \in \{n, \dots, -1, 0\}$. By definition of $u_i = u_{\tau_i,\delta}$, we have

$$u_i(x) \leqslant (1 - \tau_i\delta)^{-n} u_i(x_n^i) + \sum_{k=n}^{-1} (1 - \tau_i\delta)^{-k-1} \tau_i L(x_k^i, v).$$

Then the expression $|\sum_{k=n}^{-1} (1 - \tau_i\delta)^{-k-1} \tau_i L(x_k^i, v) - \int_{n\tau_i}^{0} \mathrm{e}^{s\delta} L(x + sv, v)\, \mathrm{d}s|$ is estimated in the same way as before, and the claim is proved.

Part 6. By approximating any C^2 path piecewise linearly, we obtain that for any $\gamma \in C^2((-\infty, 0], \mathbb{R}^d)$ ending at $\gamma(0) = x$,

$$u(x) - \mathrm{e}^{-t\delta} u(\gamma(-t)) \leqslant \int_{-t}^{0} \mathrm{e}^{s\delta} L(\gamma(s), \dot\gamma(s))\, \mathrm{d}s, \quad \forall x \in \mathbb{R}^d,\ \forall t \geqslant 0,\ \forall v \in \mathbb{R}^d.$$

We have just proved that u is uniquely given by (6), and that γ_δ^x is a C^2 minimizer by the Tonelli–Weierstrass theorem. □

Proof of item (iic) of theorem 20. We first show $u_{\tau,\delta} - u_\delta \leqslant C\frac{\tau}{\delta}$. Thanks to item (i), there exists a constant $C_1 > 0$ such that for every $x \in \mathbb{R}^d$ there exists a $C^{1,1}$ curve $\gamma_\delta^x : (-\infty, 0] \to \mathbb{R}^d$, satisfying $\gamma_\delta^x(0) = x$, $\|\dot\gamma_\delta^x\| \leqslant C_1$ and $\mathrm{Lip}(\dot\gamma_\delta^x) \leqslant C_1$ uniformly on $(-\infty, 0]$, and

$$u_\delta(x) = \int_{-\infty}^{0} \mathrm{e}^{s\delta} L(\gamma_\delta^x(s), \dot\gamma_\delta^x(s))\, \mathrm{d}s.$$

Let $x_{-k} := \gamma_\delta^x(-k\tau)$, $v_{-k} := (x_{-k+1} - x_{-k})/\tau$, for every $k \geqslant 0$. Then

$$u_{\tau,\delta}(x) \leqslant \sum_{k \geqslant 0} (1 - \tau\delta)^k \mathcal{L}_\tau(x_{-k-1}, x_{-k}),$$

$$\begin{aligned}(1 - \tau\delta) u_{\tau,\delta}(x) - u_\delta(x) \leqslant{}& \sum_{k \geqslant 0} \int_{-(k+1)\tau}^{-k\tau} \Big[(1 - \tau\delta)^{k+1} - \mathrm{e}^{s\delta}\Big] L(x_{-k-1}, v_{-k-1}) \\ &+ \sum_{k \geqslant 0} \int_{-(k+1)\tau}^{-k\tau} \mathrm{e}^{s\delta} \big[L(x_{-k-1}, v_{-k-1}) - L(\gamma_\delta(s), \dot\gamma_\delta(s))\big]\, \mathrm{d}s.\end{aligned}$$

For every $s \in [-(k+1)\tau, -k\tau]$,

$$\|\gamma_\delta(s) - x_{-k-1}\| \leqslant C_1\tau, \quad \|\dot\gamma_\delta(s) - v_{-k-1}\| \leqslant C_1\tau,$$
$$|L(x_{-k-1}, v_{-k-1}) - L(\gamma_\delta(s), \dot\gamma_\delta(s))| \leqslant \|DL\|_\infty C_1\tau,$$

(where $\|DL\|_\infty$ is computed by taking the supremum of $\|DL(x, v)\|_\infty$ over $x \in \mathbb{R}^d$ and

$\|v\| \leqslant C_1$). Moreover

$$\sum_{k\geqslant 0}\int_{-(k+1)\tau}^{-k\tau}\left[e^{s\delta}-(1-\tau\delta)^{k+1}\right]\leqslant \frac{1}{\delta}-\frac{\tau(1-\tau\delta)}{\tau\delta}=\tau.$$

Let $\|L\|_\infty$ be the supremum of $L(x,v)$ over $x\in\mathbb{R}^d$ and $\|v\|\leqslant C_1$. Then item (ii) of proposition 11 implies

$$u_{\tau,\delta}(x)-u_\delta(x)\leqslant 2\|L\|_\infty\tau+\|DL\|_\infty C_1\frac{\tau}{\delta}\leqslant\left(2\|L\|_\infty+\|DL\|_\infty C_1\right)\frac{\tau}{\delta}:=C\frac{\tau}{\delta}.$$

We next show $u_{\tau,\delta}-u_\delta\geqslant -C\frac{\tau}{\delta}$. Let $x\in\mathbb{R}^d$ and $\{x_{-k}\}_{k\geqslant 0}$ be a discounted backward calibrated configuration for $\mathcal{L}_\tau$ starting at x, then

$$u_{\tau,\delta}(x)=\sum_{k\geqslant 0}(1-\tau\delta)^k\mathcal{L}_\tau(x_{-k-1},x_{-k}).$$

Let $\gamma:(-\infty,0]\to\mathbb{R}^d$ be the piecewise linear path interpolating the points x_{-k} at the times $-k\tau$. Then, the property (6) implies

$$u_\delta(x)\leqslant\int_{-\infty}^{0}e^{s\delta}L(\gamma(s),\dot\gamma(s))\,ds.$$

Using item (iib) of proposition 11, we notice that for every $s\in[-(k+1)\tau,-k\tau]$,

$$\|\gamma(s)-x_{-k-1}\|\leqslant\|x_{-k}-x_{-k-1}\|\leqslant R\tau,\quad \dot\gamma(s)=(x_{-k}-x_{-k-1})/\tau:=v_{-k-1},$$
$$|L(x_{-k-1},v_{-k-1})-L(\gamma(s),\dot\gamma(s))|\leqslant\left\|\frac{\partial L}{\partial x}\right\|_\infty R\tau,$$

(where $\left\|\frac{\partial L}{\partial x}\right\|_\infty$ is computed by taking the supremum of $\left\|\frac{\partial L}{\partial x}(x,v)\right\|$ over $x\in\mathbb{R}^d$ and $\|v\|\leqslant R$). Let $C_3:=\inf_{x,v\in\mathbb{R}^d}L(x,v)$. Then item (ii) of proposition 11 implies

$$u_{\tau,\delta}(x)-u_\delta(x)\geqslant\left(C_3-\|L\|-\left\|\frac{\partial L}{\partial x}\right\|_\infty R\right)\frac{\tau}{\delta}:=-C\frac{\tau}{\delta}.$$

□

Acknowledgment

We would like to thank A Fathi for giving us access to his preprint [DFIZ16b]. At the time our paper was written, we did not know about the existence of [DFIZ16a]. X Su is supported by both the National Natural Science Foundation of China (grant no. 11301513) and 'the Fundamental Research Funds for the Central Universities'. P Thieullen is supported by ANR WKBHJ ANR-12-BS01-0020.

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