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LIPSCHITZ SUB-ACTIONS FOR LOCALLY MAXIMAL HYPERBOLIC SETS OF A C^1 MAP

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Abstract. Livšic theorem asserts that, for Anosov diffeomorphisms, a Lipschitz observable is a coboundary if all its Birkhoff sums on every periodic orbits are equal to zero. The transfer function is then Lipschitz. We prove a positive Livšic theorem which asserts that a Lipschitz observable is bounded from below by a coboundary if and only if all its Birkhoff sums on periodic orbits are non negative. The new result is that the coboundary can be chosen Lipschitz with a uniform control on the Lipschitz norm. In addition our result holds true for possibly non invertible and not transitive C^1 maps. We actually prove the main result in the setting of locally maximal hyperbolic sets for general C^1 map. The construction of the coboundary uses a new notion of the Lax-Oleinik operator that is a standard tool in the discrete Aubry-Mather theory.

1. **Introduction and main results.** A C^r dynamical system, $r \geq 1$, is a couple (M, f) where M is a C^r manifold of dimension $d_M \geq 2$, without boundary, not necessarily compact, and $f : M \to M$ is a C^r map, not necessarily injective nor transitive. The tangent bundle TM is assumed to be equipped with a Finsler norm $\|\cdot\|$ depending C^{r-1} with respect to the base point. A topological dynamical system is a couple (M, f) where M is a metric space and $f : M \to M$ is a continuous map. We recall several standard definitions. The theory of Anosov systems is well explained in Hasselblatt, Katok [15], or in Bonatti, Diaz, Viana [1].

Definition 1.1. Let (M, f) be a C^r dynamical system and $\Lambda \subseteq M$ be a compact set strongly invariant by f, $f(\Lambda) = \Lambda$. Let $d_M = d^u + d^s$, $d^u \geq 1$, $d^s \geq 1$, (d^u and d^s denote the dimensions of the unstable and stable vector spaces respectively).

i. Λ is said to be hyperbolic if there exist constants $\lambda^s < 0 < \lambda^u$, $C_\Lambda \geq 1$, and a continuous equivariant splitting over Λ, that is
   (a) $\forall\, x \in \Lambda,\ T_xM = E^u_\Lambda(x) \oplus E^s_\Lambda(x)$,

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(b) the two maps

$$\begin{cases} \Lambda & \to & \mathrm{Grass}(TM, d^u) \\ x & \mapsto & E^u_\Lambda(x) \end{cases} \qquad \begin{cases} \Lambda & \to & \mathrm{Grass}(TM, d^s) \\ x & \mapsto & E^s_\Lambda(x) \end{cases}$$

are C^0,

(c) the tangent map is hyperbolic in the following sense

$$\forall\, x \in \Lambda,\ T_x f(E^u(x)) = E^u(f(x)),\ T_x f(E^s(x)) \subseteq E^s(f(x)),$$

$$\forall x \in \Lambda,\ \forall n \geq 0,\ \begin{cases} \forall v \in E^s_\Lambda(x),\ \|T_x f^n(v)\| \leq C_\Lambda\, e^{n\lambda^s} \|v\|, \\ \forall v \in E^u_\Lambda(x),\ \|T_x f^n(v)\| \geq C_\Lambda^{-1}\, e^{n\lambda^u} \|v\|. \end{cases}$$

ii. Λ is said to be *locally maximal* if there exists an open neighborhood U of Λ of compact closure such that

$$\bigcap_{n\in\mathbb{Z}} f^n(\bar{U}) = \Lambda.$$

We also consider a Lipschitz continuous observable $\phi : U \to \mathbb{R}$. We want to understand the structure of the orbits that minimize the Birkhoff averages of ϕ. We recall several standard definitions.

Definition 1.2. Let (M, f) be a topological dynamical system, $\Lambda \subseteq M$ be an f-invariant compact set, $U \supseteq \Lambda$ be an open neighborhood of Λ, and $\phi : U \to \mathbb{R}$ be a continuous function.

i. The *ergodic minimizing value* of ϕ restricted to Λ is the quantity

$$\bar{\phi}_\Lambda := \lim_{n\to+\infty} \frac{1}{n} \inf_{x\in\Lambda} \sum_{k=0}^{n-1} \phi \circ f^k(x). \tag{1.1}$$

ii. A continuous function $u : U \to \mathbb{R}$ is said to be a *subaction* if

$$\forall\, x \in U \cap f^{-1}(U),\ \phi(x) - \bar{\phi}_\Lambda \geq u \circ f(x) - u(x). \tag{1.2}$$

iii. A function ψ of the form $\psi = u \circ f - u$ for some u is called a *coboundary.*

iv. The Lipschitz constant of ϕ is the number

$$\mathrm{Lip}(\phi) := \sup_{x,y\, \in U,\ x\neq y} \frac{|\phi(y) - \phi(x)|}{d(x, y)},$$

where $d(\cdot, \cdot)$ is the distance associated to the Finsler norm.

The first main result is the following. We would remark that the new result here is the fact that u is Lipschitz continuous, improving the known Hölder regularity.

Theorem 1.3. *Let (M, f) be a C^1 dynamical system, $\Lambda \subseteq M$ be a locally maximal compact hyperbolic set, $\phi : M \to \mathbb{R}$ be a Lipschitz continuous function, and $\bar{\phi}_\Lambda$ be the ergodic minimizing value of ϕ restricted to Λ. Then there exist an open set Ω_{AS} containing Λ and a Lipschitz continuous function $u : M \to \mathbb{R}$ such that*

$$\forall\, x \in \Omega_{AS}, \quad \phi(x) - \bar{\phi}_\Lambda \geq u \circ f(x) - u(x).$$

Moreover, $\mathrm{Lip}(u) \leq K_\Lambda \mathrm{Lip}(\phi)$ *for some constant K_Λ depending only on the hyperbolicity of f on Λ.*

The constant K_Λ is semi-explicit

$$K_\Lambda = \max\left(\frac{(N_{AS} + 1)\mathrm{diam}(\Omega_{AS})}{\varepsilon_{AS}},\ K_{AS} \frac{1 + \exp(-\lambda_{AS})}{1 - \exp(-\lambda_{AS})} \right),$$

where

$$\Omega_{AS} = \{x \in M : d(x,\Lambda) < \epsilon_{AS}\}$$

and ε_{AS}, K_{AS}, λ_{AS} are constants of the shadowing lemma defined in Theorem 1.5, and N_{AS} denotes a covering number of Ω_{AS} by balls of radius $\epsilon_{AS}/2$.

The positive Livšic theorem becomes then a simple corollary of the Theorem 1.3 by taking $\bar\phi_\Lambda \geq 0$. Notice that the dynamical systems (Λ, f) we are studying possess a dense set of periodic orbits (see Corollary 6.4.19 in [15]).

Corollary 1.4. *Let (M, f) be a C^1 dynamical system, $\Lambda \subseteq M$ be a locally maximal compact hyperbolic set, and $\phi : M \to \mathbb{R}$ be a Lipschitz continuous function. Assume the Birkhoff sum of ϕ on every periodic orbit on Λ is non negative. Then there exist an open neighborhood Ω of Λ, a Lipschitz continuous function $u : M \to \mathbb{R}$, such that*

$$\forall\, x \in \Omega,\ \ \phi(x) - u \circ f(x) + u(x) \geq 0.$$

The proof of Theorem 1.3 depends on a new version of the shadowing lemma. We recall that a sequence $(x_i)_{0\leq i\leq n}$ of points of M is said to be an ϵ-pseudo orbit (with respect to the dynamics f) if

$$\forall\, i \in [\![0, n-1]\!], \quad d(f(x_i), x_{i+1}) \leq \epsilon.$$

The sequence is said to be a periodic ϵ-pseudo orbit if $x_n = x_0$.

Theorem 1.5 (Improved Anosov shadowing lemma)**.** *Let (M, f) be a C^1 dynamical system and $\Lambda \subseteq M$ be an f-invariant compact hyperbolic set. Then there exist constants $\epsilon_{AS} > 0$, $K_{AS} \geq 1$, and $\lambda_{AS} > 0$, such that for every $n \geq 1$, for every ϵ_{AS}-pseudo orbit $(x_i)_{0\leq i\leq n}$ in the neighborhood $\Omega_{AS} = \{x \in M : d(x,\Lambda) < \epsilon_{AS}\}$, there exists a point $y \in M$ such that*

$$\forall\, i \in [\![0, n]\!],\ d(x_i, f^i(y)) \leq K_{AS} \sum_{k=1}^{n} d(f(x_{k-1}), x_k)\, \exp(-\lambda_{AS}|k-i|), \tag{1.3}$$

$$\sum_{i=0}^{n} d(x_i, f^i(y)) \leq K_{AS} \sum_{k=1}^{n} d(f(x_{k-1}), x_k). \tag{1.4}$$

Both equations (1.3) and (1.4) are new for two reasons: the map f is not necessarily invertible (the proof could actually be extended in infinite dimension), the distance between the pseudo orbit $(x_i)_{i=0}^n$ and the shadowing orbit $(f^i(y))_{i=0}^n$ is not bounded by the number of jumps n (an estimate that the standard Anosov shadowing lemma would give) but by the sum of the errors $d(f(x_{i-1}), x_i)$.

In order to obtain a periodic shadowing point in Λ, we assume in addition in the next corollary that Λ is locally maximal.

Corollary 1.6 (Anosov periodic shadowing lemma)**.** *Let (M, f) be a C^1 dynamical system and $\Lambda \subseteq M$ be a locally maximal compact hyperbolic set. Then there exists a constant $K_{APS} \geq 1$ such that for every $n \geq 1$, for every periodic ϵ_{AS}-pseudo orbit $(x_i)_{0\leq i\leq n}$ of the neighborhood $\Omega_{AS} := \{x \in M : d(x,\Lambda) < \epsilon_{AS}\}$, there exists a periodic point $p \in \Lambda$ of period n such that*

$$\sum_{i=1}^{n} d(x_i, f^i(p)) \leq K_{APS} \sum_{k=1}^{n} d(f(x_{k-1}), x_k), \tag{1.5}$$

where $K_{APS} = K_{AS}\frac{1+\exp(-\lambda_{AS})}{1-\exp(-\lambda_{AS})}$, and ϵ_{AS}, K_{AS}, λ_{AS} are the constants given in Theorem 1.5.

Notice that the standard shadowing lemma would give the estimate

$$\max_{0\le i\le n-1} d(x_i, f^i(p)) \le K_{APS} \max_{0\le k\le n-1} d(f(x_k), x_{k+1}). \tag{1.6}$$

We conclude the introduction by comparing our results with other results related to the positive Livšic theorem.

- The existence of a Lipschitz subaction is the first unavoidable step for proving Contreras' theorem [5] claiming that, for a generic observable, the Mather set is a unique periodic orbit. The proof in [5] was nevertheless done only for one-sided subshifts or expanding maps. A multidimensional version of Contreras' theorem is worth considering without coding.
- Weaker versions of Theorem 1.3 were known. The regularity of the subaction in [20], [21], and [19] is only Hölder. In Bousch's article [4], the setting is more abstract. The existence of a C-Lipschitz subaction is proved under the condition (1.1) similar to our "discrete positive Livšic criterion with distortion C" (Definition 3.2). Our main contribution is twofolds: we emphasize the role of the Lax-Oleinik operator and the role of the ergodic minimizing value in section 3; we mainly show in sections 2 and 4 how to compute the constant $C = K_\Lambda \mathrm{Lip}(\phi)$ with respect to the Lipschitz norm of ϕ for some constant K_Λ depending only on the hyperbolic set, compared to (3.2) in [4] where the supremum of ϕ is used.
- Huang, Lian, Ma, Xu, and Zhang quote Bousch's result in [17, Appendix A] and obtain an integrated version $\frac{1}{N}\sum_{k=0}^{N-1}[\phi - \bar\phi] \ge u_N \circ f^N - u_N$ for some large integer $N \ge 1$ and some u_N Lipschitz. The size of N and the Lipschitz size of u is not clearly explained. We show it is true for $N = 1$ and gives a precise estimate of the Lipschitz norm of the subaction in terms of the Lipschitz norm of the observable.
- The improved Anosov shadowing lemma may be used in other contexts. As we do not assume f to be invertible, the lemma is also true in infinite dimension where the tangent map admits an equivariant splitting with a finite dimensional unstable direction and a possibly infinite dimensional stable direction that could contain the kernel of the tangent map. This abstract setting could be applied for instance in the study of compact attractors for the Navier-Stockes equation. A review of the dynamical aspects of these equations is developed in [22].
- We introduce in section 3 a notion of calibrated subactions for maps, that is stronger than the notion of subaction (Definition 1.2). Calibrated subactions or weak KAM solutions have been introduced in the continuous setting for Lagrangian dynamics by Fathi [6], and in the discrete setting for twist maps in [12]. The main advantage of our construction is that it enables us to construct backward calibrated orbits and obtain, both numerically and theoretically, the Aubry set (defined in Definition 11 of [20]) as α-limit sets of these orbits. We leave as a question the fact that the Aubry set could be obtained as $\bigcap\{\phi - \bar\phi = u \circ f - u\}$ over all subactions u.
- We highlight the notion of "discrete positive Livšic criterion" (Definition 3.2) because it implies the existence of a Lipschitz subaction even in the case the dynamics is not hyperbolic. As the referee suggested, for instance, the proof in section 3 could be used for showing the existence of a weak KAM solution in Aubry-Mather theory. In this framework $M = \mathbb{T}^d \times \mathbb{R}^d$, $E : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a ferromagnetic generating function, $f : M \to M$ is the twist map associated to E. Adapting the proof in section 3, it is not difficult to obtain a Lipschitz

calibrated subaction as in Definition 3.1, defined on the whole set M, also referred as weak KAM solution.

- We intend to extend Theorem 1.3 in the continuous setting for Anosov flows, see [25]. The main technical estimate of the present paper, Theorem 2.1, is used again in [25] for the Poincaré maps.

The plan of the proof is the following. We revisit the *Anosov shadowing lemma* in section 2. We extend in section 3 to any C^1 maps the techniques of construction of a coboundary in [12], valid only for twist maps, by introducing a new *Lax-Oleinik operator*, Definition 3.1, and by showing under the assumption of *positive Livšic criterion* the existence of calibrated subactions, Proposition 3.3. We then check in section 4 that a locally maximal hyperbolic set satisfies the positive Livšic criterion and prove the main result. The proof of Theorem 1.5 requires a precise description of the notions of *adapted local hyperbolic maps and graph transforms with respect to a family of adapted charts.* We revisit these notions in Appendix A for non invertible hyperbolic maps.

2. **An improved shadowing lemma for maps.** We show in this section an improved version of the shadowing lemma, Theorem 1.5 that will be needed in the next section to check the existence of a fixed point of the Lax-Oleinik operator.

The heart of the proof is done through the notion of adapted local charts. In appendix A, we recall the notion of *adapted local dynamics* in which the dynamics is observed through the iteration of a sequence of maps which are uniformly hyperbolic with respect to a family of norms that are adapted to the unstable/stable splitting and the constants of hyperbolicity.

The following Theorem 2.1 is the technical counterpart of Theorem 1.5. We consider a sequence of local hyperbolic maps as described more rigorously in Appendix A

$$f_i : B_i(\rho) \to \mathbb{R}^d,\ B_i(\rho) \subset \mathbb{R}^d = E^u_i \oplus E^s_i = E^u_{i+1} \oplus E^s_{i+1},\ A_i = T_0 f_i,$$

where $E^{u/s}_i$ are the unstable/stable vector spaces, A_i is the tangent map of f_i at the origin which is assumed to be uniformly hyperbolic with respect to an adapted norm $\| \cdot \|_i$ and the constants of hyperbolicity $(\sigma^s, \sigma^u, \eta, \rho)$. The constants $\sigma^s < 1 < \sigma^u$ represent the contraction term and the expansion term along respectively the stable and unstable direction. The constant $\eta > 0$ represents the size of the perturbation of the non linear term $f_i(v) - f_i(0) - A_i v$. The constant $\rho > 0$ represents the size of the domain of definition of f_i; $B_i(\rho)$ is the ball of radius ρ for the adapted norm $\| \cdot \|_i$, and $\|f_i(0)\|_i \le \epsilon(\rho)$ is the size of the shadowing constant with $\epsilon(\rho) \ll \rho$.

As previously said, the maps f_i are not supposed to be invertible. In particular that hypothesis will prevent us to use the backward graph transform along the stable direction. The forward graph transform along the unstable direction is though well defined and recalled in Appendix A.3.

Theorem 2.1 (Adapted Anosov shadowing lemma)**.** *Let* $(f_i, A_i, E^{u/s}_i, \| \cdot \|_i)_{i=0}^{n-1}$ *be a family of adapted local hyperbolic maps and* $(\sigma^u, \sigma^s, \eta, \rho)$ *be a set of hyperbolic constants as in Definition A.1. Assume the stronger estimate (compare to* (A.1)*)*

$$\eta < \min\Big(\frac{(1-\sigma^s)^2}{12}, \frac{\sigma^u - 1}{6}\Big).$$

Define λ_Γ *and* K_Γ *by,*

$$\exp(-\lambda_\Gamma) := \max\Big(\frac{\sigma^s+3\eta}{1-3\eta}, \frac{1}{\sigma^u-3\eta}\Big), \quad K_\Gamma := \frac{7}{\big(1-\exp(-\lambda_\Gamma)\big)^2}.$$

Let $(q_i)_{i=0}^n$ be a "pseudo sequence" of points in the sense

$$\forall\, i \in [\![0,n-1]\!], \quad q_i \in B_i\Big(\frac{\rho}{2}\Big) \ \text{ and } \ f_i(q_i) \in B_{i+1}\Big(\frac{\rho}{2}\Big).$$

Then there exists a "true sequence" of points $(p_i)_{i=0}^n$, $p_i \in B_i(\rho)$, such that

i. $\forall\, i \in [\![0,n-1]\!]$, $f_i(p_i) = p_{i+1}$, *(the true orbit),*

ii. $\forall\, i \in [\![0,n]\!]$, $\|q_i - p_i\|_i \le K_\Gamma \sum_{k=1}^n \|f_{k-1}(q_{k-1}) - q_k\|_k \exp(-\lambda_\Gamma|k-i|)$,

iii. $\sum_{i=0}^n \|q_i - p_i\|_i \le K_\Gamma \sum_{k=1}^n \|f_{k-1}(q_{k-1}) - q_k\|_k$,

iv. $\max_{0\le i\le n} \|q_i - p_i\|_i \le K_\Gamma \max_{1\le k\le n} \|f_{k-1}(q_{k-1}) - q_k\|_k$.

Moreover assume $(f_i, A_i, E_i^{u/s}, \|\cdot\|_i)_{i\in\mathbb{Z}}$ is n-periodic in the sense

$$f_{i+n} = f_i, \ A_{i+n} = A_i, \ E_{i+n}^{u/s} = E_i^{u/s}, \ \|\cdot\|_{i+n} = \|\cdot\|_i,$$

assume in addition that $(q_i)_{i\in\mathbb{Z}}$ is a periodic pseudo sequence in the following sense

$$\forall\, i \in \mathbb{Z}, \ q_{i+n} = q_i, \ q_i \in B_i\Big(\frac{\rho}{2}\Big), \ f_{i-1}(q_{i-1}) \in B_i\Big(\frac{\rho}{2}\Big).$$

Then there exists a periodic true sequence $(p_i)_{i\in\mathbb{Z}}$ satisfying

v. $\forall\, i \in \mathbb{Z}$, $f_i(p_i) = p_{i+1}$, $p_{i+n} = p_i$,

vi. $\sum_{i=0}^{n-1} \|q_i - p_i\|_i \le \tilde{K}_\Gamma \sum_{k=1}^n \|f_{k-1}(q_{k-1}) - q_k\|_k$,

with $\tilde{K}_\Gamma := K_\Gamma(1+\exp(-\lambda_\Gamma))/(1-\exp(-\lambda_\Gamma))$.

Notice that the items ii and iii are the technical counterparts of the estimates (1.3) and (1.4). The main difficulty of the proof comes from the fact that f may not be injective and that the backward graph transform does not exist anymore. We use as an alternative the backward invariance of the stable cones as recalled in A.7.

For the reader's convenience, before going into the details of the proof, we sketch the main argument, by pointing out the following steps.

- In Step 1, we construct a grid of points $Q_i(j,k)$ and prove item i;
- The proof of item ii is divided into Steps 2-4, and the proof of items iii and iv follows readily from item ii;
- In Step 5, we show the existence of a periodic orbit and finish the proof of items v and vi.

Proof. Let P_i^u, P_i^s be the projections onto E_i^u, E_i^s respectively. Let

$$\alpha = \frac{6\eta}{\sigma^u - \sigma^s}, \quad \delta_i = \|f_{i-1}(q_{i-1}) - q_i\|_i, \tag{2.1}$$

where α is the a priori slope of the unstable graphs given in (A.2). Let $\mathcal{C}_i^u$ and $\mathcal{C}_i^s$ be the unstable and stable cone of angle α as in Definition A.6.

Step 1. We construct by induction a grid of points

$$Q_i(j,k) \in B_i(\rho) \ \text{ for } \ i \in [\![0,n]\!], \quad j \in [\![0,n-i]\!], \quad \text{and} \quad k \in [\![0,i]\!]$$

in the following way (see Figure 1):

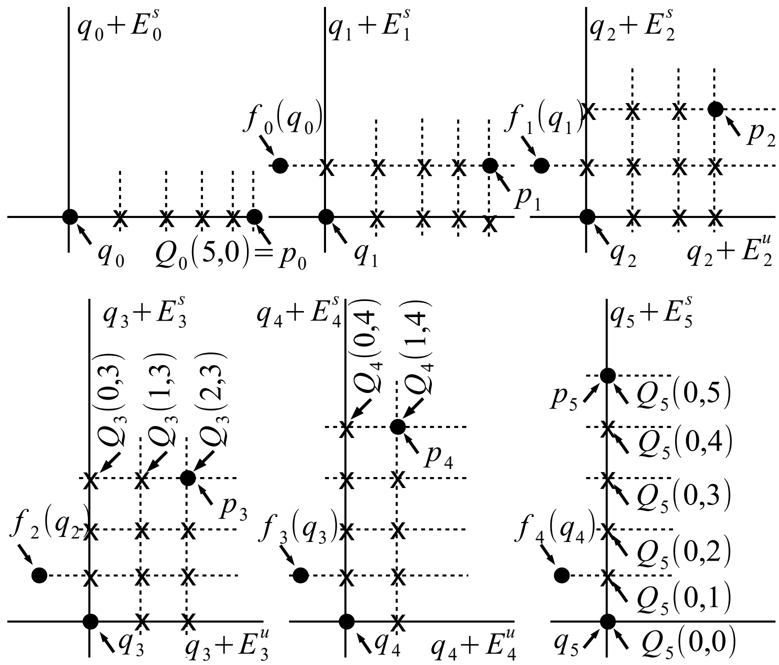

FIGURE 1. A schematic description of the grid $Q_i(j,k)$ for $n=5$. The horizontal axis is the unstable direction attached at each q_i, the vertical axis is the stable direction. The dashed "horizontal lines" are obtained by iteration of the horizontal axes by the forward graph transform; they are graphs of small slope α. We highlight the positions of the two points q_i and $f_{i-1}(q_{i-1})$ at each index i to show that they must be close. The points $Q_i(0,k)$, $k\in[\![0,i]\!]$, are obtained by intersecting the vertical axis with these dashed horizontal lines. The other points are obtained recursively, starting at $i=n$, by taking the preimages by f_{i-1} of the dashed "vertically aligned" points at index i except those on the horizontal axis. These new points are pushed by f_{i-1}^{-1}, down and to the right of the previously defined points $Q_{i-1}(0,k)$. The representation as vertical dashed lines and the relative positions of the points $Q_i(j,k)$ are only a convenient way to index the grid as a product (j,k) in $[\![0,n-i]\!]\times[\![0,i]\!]$. The points $p_i=Q_i(n-i,i)$ we are looking for are located at the upper right corner of the grid. By definition $f_{i-1}(p_{i-1})=p_i$.

(a) For all $i\in[\![0,n]\!]$, let $G_{i,0}:B^u_i(\rho)\to B^s_i(\rho)$ be the horizontal graph passing through the point q_i,

$$\forall\, v\in B^u_i(\rho),\ G_{i,0}(v)=P^s_i q_i.$$

For all $i\in[\![1,n]\!]$ and $k\in[\![1,i]\!]$, let $G_{i,k}:B^u_i(\rho)\to B^s_i(\rho)$ be the graph obtained by the graph transform of $G_{i-k,0}$ (see Proposition A.3 and equation (A.3)), iterated k times,

$$G_{i,k}=(\mathcal{T})^u_{i-1}\circ\cdots\circ(\mathcal{T})^u_{i-k}(G_{i-k,0}).$$

Notice that $\|G_{i,k}(0)\|_i \le \rho/2$ and $\mathrm{Lip}(G_{i,k}) \le \alpha$, see (A.2).

(b) For all $i \in [\![0,n]\!]$ and $k \in [\![0,i]\!]$, let $Q_i(0,k)$ be the point on $\mathrm{Graph}(G_{i,k})$ whose unstable projection is $P_i^u q_i$, or more precisely,

$$Q_i(0,k) = P_i^u q_i + G_{i,k}(P_i^u q_i).$$

(c) We then define recursively the other points starting at $i=n$. Assume that the points $Q_i(j,k)$, $i \ge 1$, have been defined for all $j \in [\![0,n-i]\!]$ and $k \in [\![0,i]\!]$. Let $j \in [\![1,n-i+1]\!]$ and $k \in [\![0,i-1]\!]$. As $Q_i(j-1,k+1) \in \mathrm{Graph}(G_{i,k+1})$, there exists a unique point $Q_{i-1}(j,k)$ on $\mathrm{Graph}(G_{i-1,k})$ such that

$$f_{i-1}(Q_{i-1}(j,k)) = Q_i(j-1,k+1).$$

For $j=0$, the points $Q_{i-1}(0,k)$ have been defined in item (b).

We will then choose $p_i = Q_i(n-i,i)$. By construction

$$\forall\, i \in [\![1,n]\!],\ f_{i-1}(p_{i-1}) = p_i,$$

and item i is proved.

Step 2. Let $h_{i,j} := \|P_i^s\big[Q_i(j,0) - Q_i(j,i)\big]\|_i$. We claim that, for all $i \in [\![1,n]\!]$,

$$h_{i,0} \le \Big[(1+\alpha) + \frac{\alpha}{1-\alpha^2}\frac{\sigma^s+3\eta}{\sigma^u-3\eta}\Big]\delta_i + \frac{\sigma^s+3\eta}{1-\alpha^2}\,h_{i-1,0}. \tag{2.2}$$

The quantity $h_{i,0}$ corresponds to the length between $q_i = Q_i(0,0)$ and the furthest point $Q_i(0,i)$ above q_i on the vertical axis. We decompose this quantity into two lengths $Q_i(0,0) - Q_i(0,1)$ and $Q_i(0,1) - Q_i(0,i)$. We will also use the quantity $h_{i-1,1}$ that corresponds to the length between the vertically aligned points $Q_{i-1}(1,0)$ and $Q_{i-1}(1,i-1)$, located next to the vertical axis at index $i-1$ and sent by f_{i-1} to the points $Q_i(0,1)$ and $Q_i(0,i)$.

Proposition A.3 with slope $\alpha = 6\eta/(\sigma^u - \sigma^s)$ for the unstable graphs shows that

$$\begin{aligned}
&\|P_i^s\big[Q_i(0,0) - Q_i(0,1)\big]\|_i \\
&\le \|P_i^s\big[q_i - f_{i-1}(q_{i-1})\big]\|_i + \|P_i^s\big[f_{i-1}(q_{i-1}) - Q_i(0,1)\big]\|_i \\
&\le \delta_i + \alpha\|P_i^u\big[f_{i-1}(q_{i-1}) - Q_i(0,1)\big]\|_i \\
&\le \delta_i + \alpha\|P_i^u\big[f_{i-1}(q_{i-1}) - q_i\big]\|_i \\
&\le (1+\alpha)\delta_i.
\end{aligned} \tag{2.3}$$

By forward induction, using (A.4) in Lemma A.7, we justify the vocabulary "horizontally aligned points",

$$\begin{aligned}
&Q_{i-1}(j,k) - Q_{i-1}(j',k) \in \mathcal{C}_{i-1}^u \ \Rightarrow\ Q_i(j-1,k+1) - Q_i(j'-1,k+1) \in \mathcal{C}_i^u, \\
&\|P_{i-1}^u\big[Q_{i-1}(j,k) - Q_{i-1}(j',k)\big]\|_{i-1} \\
&\qquad\qquad \le \frac{1}{\sigma^u-3\eta}\|P_i^u\big[Q_i(j-1,k+1) - Q_i(j'-1,k+1)\big]\|_i.
\end{aligned}$$

In particular, taking $k=1$, $j=0$, and $j'=1$, one obtains with the convention $Q_i(-1,1) = f_{i-1}(q_{i-1})$,

$$\begin{aligned}
\|Q_{i-1}(0,0) - Q_{i-1}(1,0)\|_{i-1} &\le \frac{1}{\sigma^u-3\eta}\|P_i^u\big[f_{i-1}(q_{i-1}) - Q_i(0,1)\big]\|_i \\
&\le \frac{1}{\sigma^u-3\eta}\|P_i^u\big[f_{i-1}(q_{i-1}) - q_i\big]\|_i \\
&\le \frac{1}{\sigma^u-3\eta}\delta_i.
\end{aligned} \tag{2.4}$$

By backward induction, using (A.5) in Lemma A.7, we justify the vocabulary "vertically aligned points",

$$\begin{aligned} &Q_i(j,k) - Q_i(j,k') \in \mathcal{C}_i^s \;\Rightarrow\; Q_{i-1}(j+1,k-1) - Q_{i-1}(j+1,k'-1) \in \mathcal{C}_{i-1}^s, \\ &\|P^s\big[Q_i(j,k) - Q_i(j,k')\big]\|_i \\ &\qquad\qquad \leq (\sigma^s + 3\eta)\|P^s\big[Q_{i-1}(j+1,k-1) - Q_{i-1}(j+1,k'-1)\big]\|_{i-1}. \end{aligned}$$

In particular, taking $j=0$, $k=0$, and $k'=i$, and using (2.3), one obtains

$$\begin{aligned} h_{i,0} &= \|P_i^s\big[Q_i(0,0) - Q_i(0,i)\big]\|_i \\ &\leq \|P_i^s\big[Q_i(0,0) - Q_i(0,1)\big]\|_i + \|P_i^s\big[Q_i(0,1) - Q_i(0,i)\big]\|_i \\ &\leq (1+\alpha)\delta_i + (\sigma^s + 3\eta)h_{i-1,1}. \end{aligned} \tag{2.5}$$

We estimate $h_{i-1,1}$ using a path passing through the vertical axis

$$Q_{i-1,}(1,0) \to Q_{i-1}(0,0) \to Q_{i-1}(0,i-1) \to Q_{i-1}(1,i-1).$$

We obtain

$$\begin{aligned} h_{i-1,1} \leq\ & \|P_{i-1}^s\big[Q_{i-1}(1,0) - Q_{i-1}(0,0)\big]\|_{i-1} \\ & + \|P_{i-1}^s\big[Q_{i-1}(0,0) - Q_{i-1}(0,i-1)\big]\|_{i-1} \\ & + \|P_{i-1}^s\big[Q_{i-1}(0,i-1) - Q_{i-1}(1,i-1)\big]\|_{i-1} \\ \leq\ & h_{i-1,0} + \alpha\|P_{i-1}^u\big[Q_{i-1}(0,i-1) - Q_{i-1}(1,i-1)\big]\|_{i-1}. \end{aligned} \tag{2.6}$$

The last inequality is obtained using $P_{i-1}^s\big[Q_{i-1}(1,0) - Q_{i-1}(0,0)\big] = 0$ and the fact that the top horizontally aligned branch $Q_{i-1}(0,i-1) - Q_{i-1}(1,i-1)$ belongs to the cone $\mathcal{C}_{i-1}^u$. The top branch is estimated using the path

$$Q_{i-1}(0,i-1) \to Q_{i-1}(0,0) \to Q_{i-1}(1,0) \to Q_{i-1}(1,i-1).$$

We obtain

$$\begin{aligned} &\|P_{i-1}^u\big[Q_{i-1}(0,i-1) - Q_{i-1}(1,i-1)\big]\|_{i-1} \\ &\qquad \leq \|P_{i-1}^u\big[Q_{i-1}(0,i-1) - Q_{i-1}(0,0)\big]\|_{i-1} \\ &\qquad\quad + \|P_{i-1}^u\big[Q_{i-1}(0,0) - Q_{i-1}(1,0)\big]\|_{i-1} \\ &\qquad\quad + \|P_{i-1}^u\big[Q_{i-1}(1,0) - Q_{i-1}(1,i-1)\big]\|_{i-1} \\ &\qquad \leq \frac{1}{\sigma^u - 3\eta}\delta_i + \alpha h_{i-1,1}. \end{aligned} \tag{2.7}$$

The last inequality is obtained using $P_{i-1}^u\big[Q_{i-1}(0,i-1) - Q_{i-1}(0,0)\big] = 0$ for the first term, (2.4) for the second term, the fact that $Q_{i-1}(1,0) - Q_{i-1}(1,i-1)$ belongs to the cone $\mathcal{C}_{i-1}^s$ for the third term, and the estimate

$$\|P_{i-1}^u\big[Q_{i-1}(1,0) - Q_{i-1}(1,i-1)\big]\|_{i-1} \leq \alpha\|P_{i-1}^s\big[Q_{i-1}(1,0) - Q_{i-1}(1,i-1)\big]\|_{i-1}.$$

Combining (2.6) and (2.7), we obtain

$$\begin{aligned} h_{i-1,1} &\leq h_{i-1,0} + \frac{\alpha}{\sigma^u - 3\eta}\delta_i + \alpha^2 h_{i-1,1}, \\ &\leq \frac{1}{1-\alpha^2}h_{i-1,0} + \frac{\alpha}{(1-\alpha^2)(\sigma^u - 3\eta)}\delta_i. \end{aligned} \tag{2.8}$$

Using (2.5) and (2.8) one obtains

$$h_{i,0} \leq (1+\alpha)\delta_i + (\sigma^s + 3\eta)h_{i-1,1}$$

$$\leq \Big[(1+\alpha) + \frac{\alpha}{1-\alpha^2}\frac{\sigma^s+3\eta}{\sigma^u-3\eta}\Big]\delta_i + \frac{\sigma^s+3\eta}{1-\alpha^2}\,h_{i-1,0},$$

which proves the claim of Step 2.

Step 3. We claim that, for every $i \in [\![0, n-1]\!]$,

$$\|P_i^u\big[Q_i(0,i) - Q_i(1,i)\big]\|_i \leq \frac{\delta_{i+1}}{(1-\alpha^2)(\sigma^u-3\eta)} + \frac{\alpha}{1-\alpha^2}h_{i,0}. \tag{2.9}$$

The estimate (2.9) follows readily from (2.7) and (2.8) as

$$\|P_i^u\big[Q_i(0,i) - Q_i(1,i)\big]\|_i \leq \frac{1}{\sigma^u-3\eta}\delta_{i+1} + \alpha h_{i,1}$$
$$h_{i,1} \leq \frac{1}{1-\alpha^2}h_{i,0} + \frac{\alpha}{(1-\alpha^2)(\sigma^u-3\eta)}\delta_{i+1}.$$

Step 4. We simplify the previous inequalities

$$\frac{\sigma^s+3\eta}{\sigma^u-3\eta} \leq 1,\ \ \alpha \leq \frac{1}{2},\ \ (1+\alpha) + \frac{\alpha}{1-\alpha^2}\frac{\sigma^s+3\eta}{\sigma^u-3\eta} \leq \frac{13}{6}. \tag{2.10}$$

Then for every $i \in [\![0, n-1]\!]$, using the fact that $Q_i(k,i) - Q_i(k+1,i)$ belongs to the cone $\mathcal{C}_i^u$ and the estimate (2.9), one obtains

$$\begin{aligned}
\|P_i^u\big[Q_i(0,i) - Q_i(n-i,i)\big]\|_i &\leq \sum_{k=0}^{n-i-1} \|P_i^u\big[Q_i(k,i) - Q_i(k+1,i)\big]\|_i \\
&\leq \sum_{k=0}^{n-i-1} \Big(\frac{1}{\sigma^u-3\eta}\Big)^k \|P_{i+k}^u\big[Q_{i+k}(0,i+k) - Q_{i+k}(1,i+k)\big]\|_{i+k} \\
&\leq \sum_{k=0}^{n-i-1} \Big(\frac{1}{\sigma^u-3\eta}\Big)^k \Big(\frac{\delta_{i+k+1}}{(1-\alpha^2)(\sigma^u-3\eta)} + \frac{\alpha}{1-\alpha^2}h_{i+k,0}\Big).
\end{aligned} \tag{2.11}$$

Using $\|P_i^s\big[Q_i(0,i) - Q_i(n-i,i)\big]\|_i \leq \alpha \|P_i^u\big[Q_i(0,i) - Q_i(n-i,i)\big]\|_i$, the estimate (2.11) becomes for every $i \in [\![0, n]\!]$,

$$\begin{aligned}
\|Q_i(0,i) - Q_i(n-i,i)\|_i &\leq (1+\alpha)\|P_i^u\big[Q_i(0,i) - Q_i(n-i,i)\big]\|_i \\
&\leq \frac{1}{1-\alpha}\sum_{k=i+1}^{n}\Big(\frac{1}{\sigma^u-3\eta}\Big)^{k-i}\delta_k \\
&\qquad + \frac{\alpha}{1-\alpha}\sum_{k=i}^{n-1}\Big(\frac{1}{\sigma^u-3\eta}\Big)^{k-i}h_{k,0}.
\end{aligned} \tag{2.12}$$

Using (2.5), $\|P_i^s\big[Q_i(0,0) - Q_i(0,i)\big]\|_i = \|Q_i(0,0) - Q_i(0,i)\|_i$, and $h_{0,0} = 0$, one obtains

$$h_{i,0} = \|Q_i(0,0) - Q_i(0,i)\|_i \leq \frac{13}{6}\sum_{k=1}^{i}\Big(\frac{\sigma^s+3\eta}{1-\alpha^2}\Big)^{i-k}\delta_k. \tag{2.13}$$

As $12\eta \leq (1-\sigma_s)^2 \leq (\sigma_u - \sigma_s)^2$, we have $\alpha^2 \leq 3\eta$. Let

$$\sigma_\Gamma := \max\Big(\frac{\sigma^s+3\eta}{1-\alpha^2}, \frac{1}{\sigma^u-3\eta}\Big) \leq \exp(-\lambda_\Gamma).$$

Combining (2.12) and (2.13), we obtain

$$\|Q_i(0,0) - Q_i(n-i,i)\|_i \le \frac{13}{3}\sum_{k=1}^{n}\sigma_\Gamma^{|k-i|}\delta_k + \sum_{k=i}^{n-1}\sigma_\Gamma^{k-i}h_{k,0}, \tag{2.14}$$

$$\sum_{k=i}^{n}\sigma_\Gamma^{k-i}h_{k,0} \le \frac{13}{6}\sum_{k=i}^{n}\sigma_\Gamma^{k-i}\sum_{\ell=1}^{k}\sigma_\Gamma^{k-\ell}\delta_\ell = \frac{13}{6}\sum_{\ell=1}^{n}\sigma_\Gamma^{|\ell-i|}\Big(\sum_{k\ge\max(i,\ell)}\frac{\sigma_\Gamma^{k-i}\sigma_\Gamma^{k-\ell}}{\sigma_\Gamma^{|\ell-i|}}\Big)\delta_\ell. \tag{2.15}$$

In both cases, $k \ge i \ge \ell$ or $k \ge \ell \ge i$,

$$\frac{\sigma_\Gamma^{k-i}\sigma_\Gamma^{k-\ell}}{\sigma_\Gamma^{|\ell-i|}} = \sigma_\Gamma^{2(k-i)} \quad \text{or} \quad \frac{\sigma_\Gamma^{k-i}\sigma_\Gamma^{k-\ell}}{\sigma_\Gamma^{|\ell-i|}} = \sigma_\Gamma^{2(k-\ell)}.$$

Equation (2.15) becomes

$$\sum_{k=i}^{n}\sigma_\Gamma^{k-i}h_{k,0} \le \frac{13}{6}\frac{1}{1-\sigma_\Gamma^2}\sum_{\ell=1}^{n}\sigma_\Gamma^{|\ell-i|}\delta_\ell. \tag{2.16}$$

We obtain item ii by adding (2.14) and (2.16): for every $i \in [\![0,n]\!]$,

$$\|p_i - q_i\|_i = \|Q_i(0,0) - Q_i(n-i,i)\|_i \le \frac{13}{2}\frac{1}{1-\sigma_\Gamma^2}\sum_{\ell=1}^{n}\sigma_\Gamma^{|\ell-i|}\delta_\ell.$$

Items iii and iv follow from

$$\forall\, \ell \in [\![1,n]\!],\ \sum_{i=0}^{n}\sigma_\Gamma^{|\ell-i|} \le 1 + \frac{2\sigma_\Gamma}{1-\sigma_\Gamma} = \frac{1+\sigma_\Gamma}{1-\sigma_\Gamma}.$$

Step 5. Consider now a periodic sequence $(q_j)_{j\in\mathbb{Z}}$. For every integer $s \ge 1$, consider the restriction of that sequence over $[\![-sn, sn]\!]$ and apply item ii with a shift in the indices $i = j + sn$. There exists a sequence $(p_j^s)_{j=-sn}^{sn}$ such that, for every $j \in [\![-sn, sn-1]\!]$, $f_j(p_j^s) = p_{j+1}^s$, and

$$\begin{aligned}\|p_j^s - q_j\|_j &\le K_\Gamma \sum_{k=-sn+1}^{sn}\|f_{k-1}(q_{k-1}) - q_k\|_k \exp(-\lambda_\Gamma|k-j|) \\ &\le K_\Gamma \sum_{l=1}^{n}\|f_{l-1}(q_{l-1}) - q_l\|_l \sum_{h=-s}^{s-1}\exp(-\lambda_\Gamma|l+hn-j|).\end{aligned} \tag{2.17}$$

Adding (2.17) over $j \in [\![0, n-1[\![$, one obtains

$$\begin{aligned}\sum_{j=0}^{n-1}\|p_j^s - q_j\|_j &\le K_\Gamma \sum_{l=1}^{n}\|f_{l-1}(q_{l-1}) - q_l\|_l \sum_{j=1}^{n}\sum_{h=-s-1}^{s-1}\exp(-\lambda_\Gamma|j+hn-l|) \\ &\le K_\Gamma \sum_{l=1}^{n}\|f_{l-1}(q_{l-1}) - q_l\|_l \sum_{k=-(s-1)n}^{(s+1)n-1}\exp(-\lambda_\Gamma|l-k|).\end{aligned} \tag{2.18}$$

By compactness of the balls $B_j(\frac{\varrho}{2})$ one can extract a subsequence over the index s of $(p_j^s)_{j=-sn}^{sn}$ converging for every $j \in \mathbb{Z}$ to a sequence $(p_j)_{j\in\mathbb{Z}}$. In particular we have for every $j \in \mathbb{Z}$, $f_j(p_j) = p_{j+1}$. Notice that

$$\sum_{k=-\infty}^{+\infty}\exp(-\lambda_\Gamma|k|) = \frac{1+\exp(-\lambda_\Gamma)}{1-\exp(-\lambda_\Gamma)}.$$

The estimate (2.17) becomes

$$\|p_j - q_j\|_j \leq K_\Gamma \frac{1+\exp(-\lambda_\Gamma)}{1-\exp(-\lambda_\Gamma)} \sum_{l=1}^{n} \|f_{l-1}(q_{l-1}) - q_l\|_l.$$

The estimate (2.18) becomes

$$\sum_{j=0}^{n-1} \|p_j - q_j\|_j \leq K_\Gamma \frac{1+\exp(-\lambda_\Gamma)}{1-\exp(-\lambda_\Gamma)} \sum_{l=1}^{n} \|f_{l-1}(q_{l-1}) - q_l\|_l,$$

Define $\tilde{p}_j := p_{j+n}$. As $\|\tilde{p}_j - p_j\|_j$ is uniformly bounded in j and both sequences satisfy $f_j(\tilde{p}_j) = \tilde{p}_{j+1}$, $f_j(p_j) = p_{j+1}$, for every $j \in \mathbb{Z}$, the cone property given in Lemma A.7 implies $\tilde{p}_j = p_j$ for every $j \in \mathbb{Z}$ and therefore $(p_j)_{j\in\mathbb{Z}}$ is a periodic sequence, $p_{j+n} = p_j$ for every $j \in \mathbb{Z}$. □

The proofs of Theorem 1.5 and Corollary 1.6 are standard and consist in rewriting a pseudo orbit under the dynamics of f as a pseudo orbit in a family of adapted local charts.

Proof of Theorem 1.5 and Corollary 1.6. The proof follows from items ii, iii, vi of Theorem 2.1, and from the precise description of the notion of a family of local charts as described in Definition A.4. □

3. **The discrete Lax-Oleinik operator.** We extend the definition of the Lax-Oleinik operator (usually defined for Hamiltonian dynamics [9] or for discrete twist maps [12]) for general maps (bijective or not) and show how it produces a particular subaction (item ii of Definition 1.2) that we call a calibrated subaction.

Definition 3.1 (Discrete Lax-Oleinik operator)**.** Let (M, f) be a topological dynamical system, $\Lambda \subseteq M$ be a compact f-invariant subset, $\Omega \supset \Lambda$ be an open neighborhood of Λ of compact closure, and $\phi \in C^0(\bar{\Omega}, \mathbb{R})$. Let $C \geq 0$ and $\bar{\phi}_\Lambda$ be the ergodic minimizing value of the restriction of ϕ to Λ, see (1.1).

i. The *Discrete Lax-Oleinik operator* is the nonlinear operator T acting on the space of functions $u : \bar{\Omega} \to \mathbb{R}$ defined by

$$\forall x \in \bar{\Omega},\ T[u](x) := \inf_{x' \in \bar{\Omega}} \big\{ u(x') + \phi(x') - \bar{\phi}_\Lambda + Cd(f(x'), x) \big\}. \tag{3.1}$$

ii. A *calibrated subaction of the Lax-Oleinik operator* is a continuous function $u : \bar{\Omega} \to \mathbb{R}$ solution of the equation

$$T[u] = u. \tag{3.2}$$

Item ii implies readily that a calibrated subaction is a particular subaction

$$\forall\, x \in \Omega \cap f^{-1}(\Omega),\ u \circ f(x) = T[u] \circ f(x) \leq u(x) + \phi(x) - \bar{\phi}_\Lambda.$$

The Lax-Oleinik operator is a fundamental tool for studying the set of minimizing configurations in ergodic optimization (Thermodynamic formalism) or discrete Lagrangian dynamics (Aubry-Mather theory, weak KAM theory). It appears for the first time without any name for expanding maps in Bousch [2, Lemma A], [3, Theorem 1] and under the name Lax-Oleinik operator for continuous Lagrangian dynamics in Fathi [6, 7]. Then Gomes [13, Theorem 3.3] understood the connection between the continuous and the discrete versions of the two operators. Garibaldi, Lopes [11] extended the definition of the Lax-Oleinik operator for more general expanding maps. Garibaldi, Thieullen [12] adapted the notion of the Lax-Oleinik operator for interaction models (or generating functions, or generalized Frenkel-Kontorova models). Su, Thieullen [23] studied the discounted Lax-Oleinik operator and its convergence to the non-discounted one. Complete reviews on Ergodic Optimization are given in Garibaldi [10] and Jenkinson [18].

A calibrated subaction is in some sense an optimal subaction. For expanding endomorphisms or one-sided subshifts of finite type, the theory is well developed, see for instance Definition 3.A in Garibaldi [10]. Unfortunately the standard definition requires the existence of many inverse branches. Definition 3.1 is new though its use is implicit in the proof of [4, Proof of Lemma 1.1]. The extended Lax-Oleinik operator has the further advantage that it may be used for invertible dynamics.

Definition 3.2 (Discrete positive Livšic criterion)**.** Let $(M, f, \phi, \Lambda, \Omega, C)$ be as in Definition 3.1. We say that ϕ satisfies the *discrete positive Livšic criterion on* Ω *with distortion constant* C if

$$\inf_{n\geq 1} \inf_{(x_0,x_1,\ldots,x_n)\in\bar\Omega^{n+1}} \sum_{i=0}^{n-1} \big(\phi(x_i) - \bar\phi_\Lambda + Cd(f(x_i), x_{i+1})\big) > -\infty. \tag{3.3}$$

The discrete positive Livšic criterion is the key ingredient of the proof of the existence of a calibrated subaction with a controlled Lipschitz constant. Here $\mathrm{Lip}(\phi)$, $\mathrm{Lip}(u)$, denote the Lipschitz constant of ϕ and u restricted on $\bar\Omega$ respectively.

Proposition 3.3. *Let $(M, f, \phi, \Lambda, \Omega, C)$ be as in Definition 3.1. Assume that ϕ satisfies the discrete positive Livšic criterion on Ω with distortion C. Then*

i. the Lax-Oleinik operator admits a C^0 calibrated subaction,
ii. every C^0 calibrated subaction u is Lipschitz with $\mathrm{Lip}(u) \leq C$.

Notice that conversely the discrete positive Livšic criterion is satisfied whenever ϕ admits a Lipschitz subaction u with $\mathrm{Lip}(u) \leq C$. When $C = 0$ and the infimum in (3.3) is taken over true orbits instead of all sequences, there always exists a lower semi-continuous subaction (1.2) as it is discussed in [24].

We recall without proof some basic facts of the Lax-Oleinik operator.

Lemma 3.4. *Let T be the Lax-Oleinik operator as in Definition 3.1. Then*

i. if $u_1 \leq u_2$ then $T[u_1] \leq T[u_2]$,
ii. for every constant $c \in \mathbb{R}$, $T[u + c] = T[u] + c$,
iii. for every sequence of functions $(u_n)_{n\geq 0}$ bounded from below,

$$T[\inf_{n\geq 0} u_n] = \inf_{n\geq 0} T[u_n].$$

The proof of Proposition 3.3 is well known in weak KAM theory, see [6, 8, 9]. We give the proof for the convenience of the reader.

Proof of Proposition 3.3. Define

$$\forall\, x, y \in \bar\Omega,\ \ E(x, y) := \phi(x) - \bar\phi_\Lambda + Cd(f(x), y),$$

and

$$I := \inf_{n\geq 1} \inf_{(x_0,x_1,\ldots,x_n)\in\bar\Omega^{n+1}} \sum_{i=0}^{n-1} E(x_i, x_{i+1}).$$

Part 1. We show that $T[u]$ is C-Lipschitz whenever u is continuous. Indeed if $x', y' \in \bar\Omega$ are given,

$$T[u](x') = u(x) + E(x, x'), \quad \text{for some } x \in \bar\Omega,$$

$$T[u](y') \le u(y) + E(y,y'), \quad \text{for every } y \in \bar\Omega.$$

Then by choosing $y = x$ in the previous inequality, we obtain

$$T[u](y') - T[u](x') \le E(x,y') - E(x,x') = C\big[d(f(x),y') - d(f(x),x')\big] \le Cd(y',x').$$

Part 2. Let $v := \inf_{n\ge0} T^n[0]$. Notice that $v \le 0$. We show that v is C-Lipschitz and satisfies $T[v] \ge v$. Indeed we first have

$$\forall n \ge 1,\ \forall x' \in \bar\Omega,\ T^n[0](x') = \inf_{x_0,\ldots,x_n=x'} \sum_{i=0}^{n-1} E(x_i,x_{i+1}) \ge I.$$

In particular v is bounded from bellow by I. Moreover v is C-Lipschitz since $T^n[0]$ is C-Lipschitz thanks to part 1. Finally we have

$$T[v] = T[\inf_{n\ge0} T^n[0]] = \inf_{n\ge0} T^{n+1}[v] \ge v.$$

Part 3. Let $u := \sup_{n\ge0} T^n[v] = \lim_{n\to+\infty} T^n[v]$. We show that u is a C-Lipschitz calibrated subaction. We already know from parts 1 and 2 that $T^n[v]$ is C-Lipschitz for every $n \ge 0$. Using the definition of $\bar\phi_\Lambda$, we know that, for every $n \ge 1$ there exists $x \in \Lambda$ such that $\sum_{i=0}^{n-1}\big(\phi\circ f^i(x) - \bar\phi_\Lambda\big) \le 0$, and using the fact that $T^n[v]$ is C-Lipschitz, we have

$$T^n[v](f^n(x)) \le v(x) + \sum_{i=0}^{n-1} E(f^i(x), f^{i+1}(x)) = v(x) + \sum_{k=0}^{n-1}(\phi\circ f^k(x) - \bar\phi_\Lambda) \le 0,$$
$$T^n[v](x') \le Cd(x', f^n(x)) \le C\operatorname{diam}(\bar\Omega), \quad \forall x' \in \bar\Omega.$$

In particular u is bounded from above. As $T[v] \ge v$, we also have $T[u] \ge u$. We next show $T[u] \le u$. Let $x' \in \bar\Omega$. For every $n \ge 1$, $T[T^n[v]] = T^{n+1}[v] \le u$, there exists $x_n \in \bar\Omega$ such that

$$T^n[v](x_n) + E(x_n,x') \le u(x').$$

By compactness of $\bar\Omega$, $(x_n)_{n\ge1}$ admits a converging subsequence (denoted the same way) to some $x_\infty \in \bar\Omega$. Thanks to the uniform Lipschitz constant of the sequence $(T^n[v])_{n\ge1}$ and the fact that $\lim_{n\to+\infty} T^n[v] = u$, we obtain,

$$\forall\, x' \in \bar\Omega,\ T[u](x') = \inf_{x\in\bar\Omega}\{u(x) + E(x,x')\} \le u(x_\infty) + E(x_\infty,x') \le u(x').$$

We have proved $T[u] = u$ and u is C-Lipschitz. □

4. **The discrete positive Livšic criterion.** Let (M,f) be a C^1 dynamical system, $\Lambda \subseteq M$ be a locally maximal hyperbolic compact subset, and $\phi : M \to \mathbb{R}$ be a Lipschitz continuous function. A calibrated subaction u (3.2) is in particular a subaction (1.2)

$$\forall x \in \bar\Omega, \quad u\circ f(x) - u(x) \le \phi(x) - \bar\phi_\Lambda.$$

Theorem 1.3 is therefore a consequence of Proposition 3.3 provided we prove that f satisfies the discrete positive Livšic criterion (3.3).

Proposition 4.1. *Let (M,f,Λ,ϕ) be as in Theorem 1.3. Define*

$$C = \max\Big(\frac{(N_{AS}+1)\operatorname{diam}(\Omega_{AS})}{\varepsilon_{AS}},\ K_{APS}\Big)\operatorname{Lip}(\phi),$$
$$\Omega_{AS} = \{x \in M : d(x,\Lambda) < \epsilon_{AS}\},$$

where ϵ_{AS}, K_{APS} have been defined in Theorem 1.5 and Corollary 1.6, N_{AS} is a covering number of Ω_{AS} by balls of radius $\epsilon_{AS}/2$. Then ϕ satisfies the discrete positive Livšic criterion on Ω_{AS} with distortion C.

For a true orbit instead of a pseudo orbit, the positive Livšic criterion amounts to bounding from below the normalized Birkhoff sum $\frac{1}{n}\sum_{i=0}^{n-1}\big(\phi\circ f^i(x)-\bar\phi\big)$. As we saw in [24], this is equivalent to the existence of a bounded lower semi-continuous subaction. To obtain a better regularity of the subaction we need the stronger criterion (3.3).

We first start by proving two intermediate lemmas, Lemma 4.2 for periodic pseudo-orbits, and Lemma 4.4 for pseudo-orbits.

Lemma 4.2. *Let $C \ge K_{APS}\mathrm{Lip}(\phi)$. Then for every periodic ϵ_{AS}-pseudo orbit $(x_i)_{i=0}^n$ of Ω_{AS},*

$$\sum_{i=0}^{n-1}\big(\phi(x_i)-\bar\phi_\Lambda + Cd(f(x_i),x_{i+1})\big)\ge 0.$$

Proof. Corollary 1.6 tells us that there exists a periodic orbit $p\in\Lambda$, $f^n(p)=p$, such that

$$\sum_{i=0}^{n-1} d(x_i,f^i(p)) \le K_{APS}\sum_{i=0}^{n-1} d(f(x_i),x_{i+1}).$$

Then

$$\begin{aligned}
&\sum_{i=0}^{n-1}\big(\phi(x_i)-\bar\phi_\Lambda + Cd(f(x_i),x_{i+1})\big)\\
&\quad\ge \sum_{i=0}^{n-1}\big(\phi\circ f^i(p)-\bar\phi_\Lambda\big)+\sum_{i=0}^{n-1}\big(\phi(x_i)-\phi\circ f^i(p)+Cd(f(x_i),x_{i+1})\big)\\
&\quad\ge \sum_{i=0}^{n-1}\big(\phi\circ f^i(p)-\bar\phi_\Lambda\big)+\sum_{i=0}^{n-1}\big(-\mathrm{Lip}(\phi)d(x_i,f^i(p))+Cd(f(x_i),x_{i+1})\big)\\
&\quad\ge \sum_{i=0}^{n-1}\big(\phi\circ f^i(p)-\bar\phi_\Lambda\big)\ge 0. \qquad\square
\end{aligned}$$

Lemma 4.3. *Let $N_{AS}\ge 1$ be the smallest number of balls of radius $\epsilon_{AS}/2$ that can cover Ω_{AS}. Let $(x_i)_{i=0}^n$ be a sequence of points of Ω_{AS}. Then there exists $r\in[\![1,N_{AS}]\!]$ and times $0=\tau_0<\tau_1<\cdots<\tau_r=n$ such that,*

i. $\forall\, k\in[\![1,r-1]\!]$, $\forall\, l\in[\![0,k-1]\!]$, $\forall\, j\in[\![\tau_k,n-1]\!]$, $d(x_j,x_{\tau_l})\ge\epsilon_{AS}$,
ii. $\forall\, k\in[\![1,r-1]\!]$, *if* $\tau_k\ge\tau_{k-1}+2$ *then* $d(x_{\tau_k-1},x_{\tau_{k-1}})<\epsilon_{AS}$,
iii. *either* $d(x_{\tau_r-1},x_{\tau_{r-1}})<\epsilon_{AS}$ *or* $d(x_{\tau_r},x_{\tau_{r-1}})<\epsilon_{AS}$.

Proof. We construct by induction the sequence τ_k. Assume we have constructed $\tau_k<n$. Define

$$T:=\{j\in[\![\tau_k+1,n]\!]: d(x_j,x_{\tau_k})<\epsilon_{AS}\}.$$

If $T=\emptyset$, choose $\tau_{k+1}=\tau_k+1$; if $T\ne\emptyset$ and $\max(T)<n$ then $\tau_{k+1}=\max(T)+1$, $d(x_{\tau_{k+1}-1},x_{\tau_k})<\epsilon_{AS}$ and for every $j\ge\tau_{k+1}$, $d(x_j,x_{\tau_k})\ge\epsilon_{AS}$; if $\max(T)=n$ then $\tau_{k+1}=n$. Since $(x_{\tau_k})_{k=0}^{r-1}$ are ϵ apart, $r\le N_{AS}$. $\square$

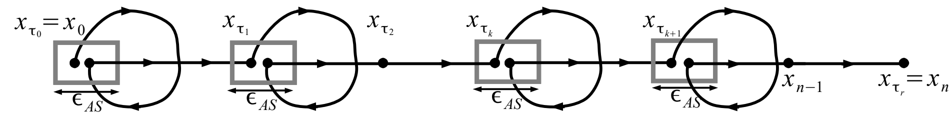

FIGURE 2. The schematic r returns of Lemma 4.3. The orbit starts at time $\tau_0 = 0$ and waits until the last return time to the ball $B_0 = B(x_{\tau_0}, \epsilon_{AS})$. Either there is no return time, then $\tau_1 = 1$; or $x_n \in B_0$, then $r = 1$ and $\tau_1 = n$; or $x_n \notin B_0$, then $\tau_1 \geq 2$, $\tau_1 - 1$ is the last return and $x_j \notin B_0$ for every $j \geq \tau_1$. The orbit restarts at τ_1, let $B_1 = B(x_{\tau_1}, \epsilon_{AS})$, wait until the last return to B_1, and so on.

Lemma 4.4. *Let $C = K_{APS}\mathrm{Lip}(\phi)$ and N_{AS} be the smallest number of balls of radius $\epsilon_{AS}/2$ that can cover Ω_{AS}. Let $\delta_{AS} := N_{AS}\,\mathrm{diam}(\Omega_{AS})$. Then for every ϵ_{AS}-pseudo orbit $(x_i)_{i=0}^n$ of Ω_{AS},*

$$\sum_{i=0}^{n-1} \big(\phi(x_i) - \bar\phi_\Lambda + Cd(f(x_i), x_{i+1})\big) \geq -\mathrm{Lip}(\phi)\delta_{AS}.$$

Proof. We split the pseudo orbit $(x_i)_{i=0}^{n-1}$ into $r \leq N_{AS}$ segments of the form $(x_i)_{i=\tau_k}^{\tau_{k+1}-1}$ according to Lemma 4.3, for $0 \leq k \leq r-1$ with $0 = \tau_0 < \tau_1 < \cdots < \tau_r = n$. To simplify the notations, denote

$$\phi_i := \phi(x_i) - \bar\phi_\Lambda + Cd(f(x_i), x_{i+1}).$$

Notice that for every $i \in [\![0, n-1]\!]$

$$\phi_i \geq \phi(x_i) - \bar\phi_\Lambda = \lim_{n\to+\infty} \frac{1}{n} \sup_{x\in\Lambda} \sum_{k=0}^{n-1} \Big(\phi(x_i) - \phi\circ f^k(x)\Big) \geq -\mathrm{Lip}(\phi)\mathrm{diam}(\Omega_{AS}).$$

If $\tau_{k+1} \geq \tau_k + 2$ and $k \in [\![0, r-1[\![$ then $d(x_{\tau_k}, x_{\tau_{k+1}-1}) < \epsilon_{AS}$, $(x_i)_{i=\tau_k}^{\tau_{k+1}-1}$ is a periodic pseudo orbit as in Lemma 4.2 and

$$\sum_{i=\tau_k}^{\tau_{k+1}-2} \phi_i \geq 0, \quad \sum_{i=\tau_k}^{\tau_{k+1}-1} \phi_i \geq -\mathrm{Lip}(\phi)\mathrm{diam}(\Omega_{AS}).$$

If $\tau_r \geq \tau_{r-1} + 2$ then either $(x_i)_{i=\tau_{r-1}}^{\tau_r-1}$ or $(x_i)_{i=\tau_{r-1}}^{\tau_r}$ is a periodic pseudo orbit. In both cases we have

$$\sum_{i=\tau_{r-1}}^{\tau_r-1} \phi_i \geq -\mathrm{Lip}(\phi)\mathrm{diam}(\Omega_{AS}).$$

If $\tau_{k+1} = \tau_k + 1$ then

$$\sum_{i=\tau_k}^{\tau_{k+1}-1} \phi_i = \phi_{\tau_k} \geq -\mathrm{Lip}(\phi)\mathrm{diam}(\Omega_{AS}).$$

By adding these inequalities for $k \in [\![0, r-1]\!]$, we have

$$\sum_{i=\tau_0}^{\tau_r-1} \phi_i \geq -\mathrm{Lip}(\phi)N_{AS}\mathrm{diam}(\Omega_{AS}).$$ ☐

We recall that K_{APS}, ϵ_{AS}, have been defined in Theorem 1.5, Corollary 1.6, and N_{AS}, δ_{AS}, in Lemma 4.4.

Proof of Proposition 4.1. Let $(x_i)_{i=0}^n$ be a sequence of points of Ω_{AS}. We split the sequence into disjoint segments $(x_i)_{i=\tau_k}^{\tau_{k+1}-1}$, $0=\tau_0<\tau_1<\cdots<\tau_k<\tau_{k+1}<\cdots<\tau_r=n$, having one of the following form.

Segment of the first kind: $\tau_{k+1}=\tau_k+1$ and $d(f(x_{\tau_k}),x_{\tau_{k+1}})\geq\epsilon_{AS}$. Then

$$\phi(x_{\tau_k})-\bar\phi_\Lambda\geq-\mathrm{Lip}(\phi)\mathrm{diam}(\Omega_{AS}),\quad d(f(x_{\tau_k}),x_{\tau_k+1})\geq\epsilon_{AS}.$$

By choosing $C\geq\mathrm{Lip}(\phi)\mathrm{diam}(\Omega_{AS})/\epsilon_{AS}$, we obtain

$$\phi(x_{\tau_k})-\bar\phi_\Lambda+Cd(f(x_{\tau_k}),x_{\tau_k+1})\geq 0.$$

Segment of the second kind: $\tau_{k+1}\geq\tau_k+2$ and

$$\begin{cases}\forall\,\tau_k\leq i\leq\tau_{k+1}-2,\ d(f(x_i),x_{i+1})<\epsilon_{AS},\\ d(f(x_{\tau_{k+1}-1}),x_{\tau_{k+1}})\geq\epsilon_{AS}.\end{cases}$$

Then $(x_i)_{i=\tau_k}^{\tau_{k+1}-1}$ is a pseudo orbit. By using Lemma 4.4 and $C\geq K_{APS}\mathrm{Lip}(\phi)$, we have

$$\sum_{i=\tau_k}^{\tau_{k+1}-2}\big(\phi(x_i)-\bar\phi_\Lambda+Cd(f(x_i),x_{i+1})\big)\geq-\mathrm{Lip}(\phi)\delta_{AS},$$

$$\phi(x_{\tau_{k+1}-1})-\bar\phi_\Lambda+Cd(f(x_{\tau_{k+1}-1}),x_{\tau_{k+1}})\geq-\mathrm{Lip}(\phi)\mathrm{diam}(\Omega_{AS})+C\epsilon_{AS}.$$

By choosing $C\geq\mathrm{Lip}(\phi)(\delta_{AS}+\mathrm{diam}(\Omega_{AS}))/\epsilon_{AS}$, we obtain

$$\sum_{i=\tau_k}^{\tau_{k+1}-1}\big(\phi(x_i)-\bar\phi_\Lambda+Cd(f(x_i),x_{i+1})\big)\geq 0.$$

Segment of the third kind: if it exists, this segment is the last one and $(x_i)_{i=\tau_{r-1}}^{\tau_r}$ is a pseudo orbit. By using again Lemma 4.4

$$\sum_{i=\tau_{r-1}}^{\tau_r-1}\big(\phi(x_i)-\bar\phi_\Lambda+Cd(f(x_i),x_{i+1})\big)\geq-\mathrm{Lip}(\phi)\delta_{AS}.$$

□

Proof of Theorem 1.3. The proof readily follows from the conclusions of Propositions 3.3 and 4.1. □

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Appendix A. Local hyperbolic dynamics. We recall in this section the local theory of hyperbolic dynamics. The dynamics is obtained by iterating a sequence of (non linear) maps defined locally and close to uniformly hyperbolic linear maps that may be non invertible. The notion of adapted local charts is defined in A.3. In these charts the expansion along the unstable direction, or the contraction along the stable direction, is realized at the first iteration, instead of after some number of iterations. It is a standard notion that can be extended in different directions, see for instance, Hasselblatt, Katok [15] or Gourmelon [14]. We will not give any proof here.

It will be important to keep in mind that we are considering maps that may not be invertible. These maps are seen as perturbations of their tangent maps. We only assume that the tangent maps are invertible along the unstable direction. They may have a kernel belonging to the stable direction. In particular, the following description is also valid for quasi-compact maps in infinite dimension.

A.1. **Adapted local hyperbolic map.** A local hyperbolic map is a Lipschitz perturbation of a hyperbolic linear map that could be non injective. The constants $(\sigma^s, \sigma^u, \eta, \rho)$ that appear in the following definition are used in the proof of Theorem 2.1.

Definition A.1 (Adapted local hyperbolic map)**.** Let $(\sigma^s, \sigma^u, \eta, \rho)$ be positive real numbers called *constants of hyperbolicity*. Let $\mathbb{R}^d = E^u \oplus E^s$ and $\mathbb{R}^d = \tilde E^u \oplus \tilde E^s$ be two Banach spaces equipped with two norms $|\cdot|$ and $\|\cdot\|$ respectively. Let $P^u : \mathbb{R}^d \to E^u$ and $P^s : \mathbb{R}^d \to E^s$ be the two linear projectors associated with the splitting $\mathbb{R}^d = E^u \oplus E^s$ and similarly $\tilde P^u : \mathbb{R}^d \to \tilde E^u$ and $\tilde P^s : \mathbb{R} \to \tilde E^s$ be the two projectors associated with $\mathbb{R}^d = \tilde E^u \oplus \tilde E^s$. Let $B(\rho)$, $B^u(\rho)$, $B^s(\rho)$ be the balls of radius ρ on each E, E^u, E^s respectively, with respect to the norm $|\cdot|$. Let $\tilde B(\rho)$, $\tilde B^u(\rho)$, $\tilde B^s(\rho)$ be the corresponding balls with respect to the norm $\|\cdot\|$. We assume that both norms are *sup norm adapted to the splitting* in the sense,

$$\begin{cases} \forall v, w \in E^u \times E^s, \ |v+w| = \max(|v|, |w|), \\ \forall v, w \in \tilde E^u \times \tilde E^s, \ \|v+w\| = \max(\|v\|, \|w\|). \end{cases}$$

In particular $B(\rho) = B^u(\rho) \times B^s(\rho)$, $\tilde B(\rho) = \tilde B^u(\rho) \times \tilde B^s(\rho)$. We also assume

$$\begin{gathered} \sigma^u > 1 > \sigma^s, \quad \eta < \min\Big(\frac{\sigma^u - 1}{6}, \frac{1-\sigma^s}{6}\Big) \\ \epsilon(\rho) := \rho \min\Big(\frac{\sigma^u - 1}{2}, \frac{1-\sigma^s}{8}\Big). \end{gathered} \tag{A.1}$$

An *adapted local hyperbolic map with respect to the two norms and the constants of hyperbolicity* is a set of data $(f, A, E^{u/s}, \tilde E^{u/s}, |\cdot|, \|\cdot\|)$ such that:

i. $f : B(\rho) \to \mathbb{R}^d$ is a Lipschitz map,
ii. $A : \mathbb{R}^d \to \mathbb{R}^d$ is a linear map which may not be invertible and is defined into block matrices

$$A = \begin{bmatrix} A^u & D^u \\ D^s & A^s \end{bmatrix}, \quad \begin{cases} (v,w) \in E^u \times E^s, \\ A(v+w) = \tilde v + \tilde w, \end{cases} \Rightarrow \begin{cases} \tilde v = A^u v + D^u w \in \tilde E^u, \\ \tilde w = D^s v + A^s w \in \tilde E^s, \end{cases}$$

that satisfies

$$\begin{cases} \forall\, v \in E^u, \ \|A^u v\| \geq \sigma^u \|v\|, \\ \forall\, w \in E^s, \ \|A^s w\| \leq \sigma^s \|w\|, \end{cases} \quad \text{and} \quad \begin{cases} \|D^u\| \leq \eta, \quad \mathrm{Lip}(f - A) \leq \eta, \\ \|D^s\| \leq \eta, \quad \|f(0)\| \leq \epsilon(\rho), \end{cases}$$

where the Lip constant is computed using the two norms $|\cdot|$ and $\|\cdot\|$.

The constant σ^u is called the *expanding constant*, σ^s is called the *contracting constant*, $\ker(A^s)$ could be non trivial. The constant ρ represents a uniform size of local charts. The constant $\epsilon(\rho)$ represents the error in a pseudo-orbit. The constant η represents a deviation from the linear map and should be thought of as small compared to the gaps $\sigma^u - 1$ and $1 - \sigma^s$. Notice that $\epsilon(\rho)$ is independent of η. The map $f : B(\rho) \to \mathbb{R}^d$ should be considered as a perturbation of its linear part A.

A.2. **Adapted local graph transform.** The graph transform is a perturbation technique of a hyperbolic linear map. A hyperbolic linear map preserves a splitting into an unstable vector space on which the linear map is expanding, and a stable vector space on which the linear map is contracting. It is standard to show that a Lipschitz map close to a hyperbolic linear map also preserves similar objects that are Lipschitz graphs tangent to the unstable direction. We recall that the operator A may have a non trivial kernel and that we don't assume f to be invertible.

Definition A.2. Let $(\sigma^u, \sigma^s, \eta, \rho)$, $\mathbb{R}^d = E^u \oplus E^s = \tilde{E}^u \oplus \tilde{E}^s$ be as in Definition A.1. We denote by $\mathcal{G}^u$ the set of Lipschitz graphs over the unstable direction E^u with controlled Lipschitz constant and height. More precisely, let

$$\mathcal{G}^u = \Big\{ [G : B^u(\rho) \to B^s(\rho)] : \mathrm{Lip}(G) \le \frac{6\eta}{\sigma^u - \sigma^s},\ |G(0)| \le \frac{\rho}{2} \Big\}, \tag{A.2}$$

and similarly $\tilde{\mathcal{G}}^u$ the set of Lipschitz graphs using the norm $\|\cdot\|$. The graph of $G \in \mathcal{G}^u$ is by definition the subset of $B(\rho)$:

$$\mathrm{Graph}(G) := \{v + G(v) : v \in B^u(\rho)\}.$$

Notice that, thanks to (A.1), $\mathrm{Lip}(G) \le \frac{1}{2}$ for every $G \in \mathcal{G}^u$. Notice also that the Lipschitz constant of G goes to zero as f becomes more and more linear, as $\eta \to 0$, independently of the location of $f(0)$ controlled by $\epsilon(\rho)$ depending only on $(\sigma^u, \sigma^s, \rho)$.

Proposition A.3 (Forward local graph transform)**.** *Let $(\sigma^u, \sigma^s, \eta, \rho, \epsilon)$, $\mathbb{R}^d = E^u \oplus E^s = \tilde{E}^u \oplus \tilde{E}^s$, and (A, f) be as defined in Definition A.1. Then*

i. For every graph $G \in \mathcal{G}^u$ there exists a unique graph $\tilde{G} \in \tilde{\mathcal{G}}^u$ such that

$$\begin{cases} \forall\, \tilde{v} \in \tilde{B}^u(\rho),\ \exists!\, v \in B^u(\rho),\ \tilde{v} = \tilde{P}^u f(v + G(v)), \\ \tilde{G}(\tilde{v}) = \tilde{P}^s f(v + G(v)). \end{cases}$$

ii. For every $G_1, G_2 \in \mathcal{G}^u$ and $\tilde{G}_1, \tilde{G}_2$ the corresponding graphs,

$$\|\tilde{G}_1 - \tilde{G}_2\|_\infty \le (\sigma^s + 2\eta)\, |G_1 - G_2|_\infty.$$

iii. The map

$$\mathcal{T}^u := \begin{cases} \mathcal{G}^u \to \tilde{\mathcal{G}}^u, \\ G \mapsto \tilde{G}, \end{cases} \tag{A.3}$$

is called the forward graph transform.

iv. for every $G \in \mathcal{G}^u$, $f(\mathrm{Graph}(G)) \supseteq \mathrm{Graph}(\tilde{G})$,

$$\forall\, q_1, q_2 \in \mathrm{Graph}(G) \cap f^{-1}(\mathrm{Graph}(\tilde{G})), \quad \|f(q_1) - f(q_2)\| \ge (\sigma^u - 3\eta)\, |q_1 - q_2|.$$

For a detailed proof of this proposition we suggest the monography by Hirsch, Pugh, Shub [16]. As we don't assume f to be invertible, the backward graph transform cannot be defined.

A.3. **Adapted local charts.** We consider in this section a C^1 dynamical systems (M, f) on a manifold M of dimension $d \ge 2$ without boundary, $\Lambda \subseteq M$ a hyperbolic f-invariant compact set, and $\Omega \supset \Lambda$ an open neighborhood of Λ of compact closure. Let $\lambda^s < 0 < \lambda^u$, $C_\Lambda \ge 1$, and $T_xM = E^u_\Lambda(x) \oplus E^s_\Lambda(x)$ as in Definition 1.1. We show that we can construct a family of local charts well adapted to the hyperbolicity of Λ. The existence of such a family depends only on the continuity of $x \in \Lambda \mapsto E^u_\Lambda(x) \oplus E^s_\Lambda(x)$ and the C^1 regularity of f.

Definition A.4 (Adapted local charts)**.** Let (M, f) be a C^1 dynamical system, $U \subseteq M$ be an open set, and $\Lambda \subseteq U$ be an f-invariant compact hyperbolic set with constants of hyperbolicity (λ^u, λ^s). *A family of adapted local charts* is a set of data $\Gamma_\Lambda = (\Gamma, E, N, F, A)$ and a set of constants $(\sigma^u, \sigma^s, \eta, \rho)$ satisfying the following properties:

i. The constants $(\sigma^u, \sigma^s, \eta, \rho)$ are chosen so that,

$$\exp(\lambda^s) < \sigma^s < 1 < \sigma^u < \exp(\lambda^u)$$
$$\eta < \min\Big(\frac{\sigma^u - 1}{6}, \frac{1-\sigma^s}{6}\Big), \quad \epsilon(\rho) := \rho \min\Big(\frac{\sigma^u - 1}{2}, \frac{1-\sigma^s}{8}\Big)$$

where λ^u, λ^s are the constants of hyperbolicity of Λ as in Definition 1.1. Notice that $\epsilon(\rho) < \rho/8$.

ii. $\Gamma = (\gamma_x)_{x\in\Lambda}$ is a parametrized family of charts such that for every $x \in \Lambda$, $\gamma_x : B(1) \subset \mathbb{R}^d \to M$ is a diffeomorphism from the unit ball $B(1)$ of $\mathbb{R}^d$ onto an open set in M, $\gamma_x(0) = x$, and such that the C^1 norm of γ_x, γ_x^{-1} is uniformly bounded with respect to x.

iii. $E = (E_x^{u/s})_{x\in\Lambda}$ is a parametrized family of splitting $\mathbb{R}^d = E_x^u \oplus E_x^s$ obtained by pull backward of the corresponding splitting on $T_\Lambda M$ by the tangent map $T_0\gamma_x$ at the origin of $\mathbb{R}^d$,

$$E_x^u = (T_0\gamma_x)^{-1}E_\Lambda^u(x), \quad E_x^s := (T_0\gamma_x)^{-1}E_\Lambda^s(x),$$

and by $\mathrm{Id} = P_x^u + P_x^s$, the corresponding projectors onto E_x^u, E_x^s respectively.

iv. $N := (\|\cdot\|_x)_{x\in\Lambda}$ is a C^0 parametrized family of norms. The *adapted local norm* is a sup norm adapted to the splitting $E_x^u \oplus E_x^s$ that satisfies

$$\forall\, v \in E_x^u,\ w \in E_x^s, \quad \|v + w\|_x = \max(\|v\|_x, \|w\|_x).$$

The ball of radius ρ centered at the origin of $\mathbb{R}^d$ is denoted by $B_x(\rho)$.

v. The constant ρ is chosen so that $\gamma_x(B_x(\rho)) \subset U$ and

$$\forall\, x, y \in \Lambda, \quad \big[f(x) \in \gamma_y(B_y(\rho)) \ \Rightarrow\ f(\gamma_x(B_x(\rho)) \subseteq \gamma_y(B(1))\big].$$

vi. $F := (f_{x,y})_{x,y\in\Lambda}$ is a family of C^1 maps $f_{x,y} : B_x(\rho) \to B(1)$ which is parametrized by couples of points $(x, y) \in \Lambda$ satisfying $f(x) \in \gamma_y(B_y(\rho))$. The *adapted local map* is defined by

$$\forall\, v \in B_x(\rho),\ f_{x,y}(v) := \gamma_y^{-1} \circ f \circ \gamma_x(v).$$

vii. $A := (A_{x,y})_{x,y\in\Lambda}$ is the family of tangent maps $A_{x,y} : \mathbb{R}^d \to \mathbb{R}^d$ of $f_{x,y}$ at the origin, that is parametrized by the couples of points $x, y \in \Lambda$ satisfying $f(x) \in \gamma_y(B_y(\rho))$. Let

$$A_{x,y} := Df_{x,y}(0),$$

where $Df_{x,y}(0)$ denotes the differential map of $v \mapsto f_{x,y}(v)$ at $v = 0$.

viii. For every $x, y \in \Lambda$ satisfying $f(x) \in \gamma_y(B_y(\epsilon))$, the set of data

$$(f_{x,y}, A_{x,y}, E_x^{u/s}, E_y^{u/s}, \|\cdot\|_x, \|\cdot\|_y)$$

is an adapted local hyperbolic map with respect to the constant of hyperbolicity $(\sigma^u, \sigma^s, \eta, \rho)$ as in Definition A.1. We have

$$A_{x,y} = \begin{bmatrix} P_y^u A_{x,y} P_x^u & P_y^u A_{x,y} P_x^s \\ P_y^s A_{x,y} P_x^u & P_y^s A_{x,y} P_x^s \end{bmatrix},$$

$$\begin{cases} \forall\, v \in E^u_x, \ \|A_{x,y} v\|_y \geq \sigma^u \|v\|_x, \\ \forall\, v \in E^s_x, \ \|A_{x,y} v\|_y \leq \sigma^s \|v\|_x, \end{cases}, \quad \begin{cases} \|P^s_y A_{x,y} P^u_x\|_{x,y} \leq \eta, \\ \|P^u_y A_{x,y} P^s_x\|_{x,y} \leq \eta, \end{cases}$$

$$\begin{cases} \|f_{x,y}(0)\|_y \leq \epsilon(\rho), \\ \forall v \in B_x(\rho), \ \|Df_{x,y}(v) - A_{x,y}\|_{x,y} \leq \eta, \end{cases}$$

where $\|\cdot\|_{x,y}$ denotes the matrix norm computed according to the two adapted local norms $\|\cdot\|_x$ and $\|\cdot\|_y$.

The existence of a family of adapted local norms is one of the central results in the Definition A.4. We don't repeat the proof here.

Definition A.5 (Admissible transitions for maps)**.** Let Γ_Λ be a family of adapted local charts as given in Definition A.4. Let $x, y \in \Lambda$. We say that $x \overset{\Gamma_\Lambda}{\to} y$ is a Γ_Λ-admissible transition if

$$f(x) \in \gamma_y(B_y(\epsilon(\rho))) \quad (\Leftrightarrow \quad f_{x,y}(0) \in B_y(\epsilon(\rho)) \ ).$$

A sequence $(x_i)_{i=0}^n$ of points of Λ is said to be Γ_Λ-admissible if $x_i \overset{\Gamma_\Lambda}{\to} x_{i+1}$ for every $0 \leq i < n$.

A.4. **Adapted local unstable cones.**

Definition A.6 (Unstable/stable cones)**.** Let $\mathbb{R}^d = E^u \oplus E^s$ be a splitting equipped with a norm $|\cdot|$. Let $\alpha \in (0,1)$

i. The *unstable cone of angle* α is the set

$$\mathcal{C}^u(\alpha) := \big\{ w \in \mathbb{R}^d : |P^s w| \leq \alpha |P^u w| \big\}.$$

ii. The *stable cone of angle* α is the set

$$\mathcal{C}^s(\alpha) := \big\{ w \in \mathbb{R}^d : |P^u w| \leq \alpha |P^s w| \big\}.$$

Notice that the unstable cone $\mathcal{C}^u(\alpha)$ contains the unstable vector space E^u.

Lemma A.7 (Equivariance of unstable/stable cones)**.** *We consider the notations of Definition A.1, where $(\sigma^u, \sigma^s, \rho, \eta, \epsilon)$ are some positive constants, $\mathbb{R}^d = E^u \oplus E^s$ and $\mathbb{R}^d = \tilde{E}^u \oplus \tilde{E}^s$ are two vector spaces with norms $|\cdot|$ and $\|\cdot\|$ respectively, and (A, f) is an adapted local hyperbolic map. Let*

$$\alpha \in \Big(\frac{6\eta}{\sigma^u - \sigma^s}, 1 \Big) \quad \textit{and} \quad \beta := \frac{\alpha \sigma^s + 3\eta}{\sigma^u - 3\eta}.$$

Then $\beta \leq \alpha$ and, for every $a, b \in B(\rho) = B^u(\rho) + B^s(\rho)$,

i. if $b - a \in \mathcal{C}^u(\alpha)$, then $f(b) - f(a) \in \tilde{\mathcal{C}}^u(\beta)$ and

$$\|\tilde{P}^u (f(b) - f(a))\| \geq (\sigma^u - 3\eta) |P^u(b - a)|, \tag{A.4}$$

ii. if $f(b) - f(a) \in \tilde{\mathcal{C}}^s(\alpha)$, then $b - a \in \mathcal{C}^s(\beta)$ and

$$\|\tilde{P}^s (f(b) - f(a))\| \leq (\sigma^s + 3\eta) |P^s(b - a)|. \tag{A.5}$$

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