Some Quasi-Periodic and almost periodic solutions in coupled map lattices and flows

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Closely related work with D. Blazevski, G. Huguet, M. Jiang, P. Martín

Basic set up. Formulation of final result

- We consider a lattice $\mathbb{Z}^d$ (or a more general network) of Hamiltonian systems. (analytic)

Each of the single site systems contains

  - A positive measure of KAM tori, nondegenerate.
  - One hyperbolic fixed point

- We couple different sites by a local interaction which is also Hamiltonian (analytic)

Some examples

Klein-Gordon model

$$\ddot{q}_j = -\nabla[V(q_j) + \varepsilon \sum_{|i-j|=1} W^1(q_i - q_j) + \varepsilon \sum_{|i-j|=2} W^2(q_i - q_j) + \cdots]$$

Very often $V(q) = \cos(q)$, $W^1(t) = \frac{1}{2}t^2$, $W^2(t) = \frac{1}{4}t^2$.

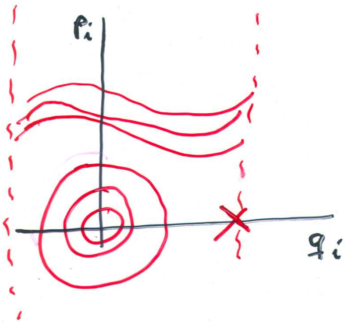
XY-Heisenberg models of spin waves

$$W^1(\eta) = \sin(\eta) \quad V(q) = B\sin(q)$$

.

These models appear also as discretization of non-linear wave equations

Dynamics of each of the sites



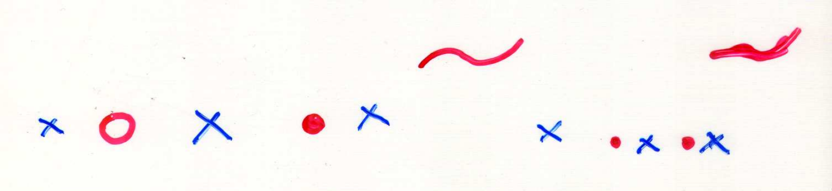
- Positive Lebesgue measure of frequency $\omega \in \Omega$, nondegenerate.

NOTE: KAM tori of the system may have different topology.

Dynamics of each of the sites

$$H_0(x) = \sum_{i \in \mathbb{Z}^d} H^0(x_i)$$

In the full systems we can intersperse tori with fixed points



All of those are *"whiskered"* KAM tori.

We construct solutions of the form of cluster of sites oscillating with frequencies ω_i but separated by positions separated by wide swatches of systems that are close to the hyperbolic fixed point.

- We will also obtain that the tori produced have decay properties.

That is, the effect of one site on another far appart is very small

Roughly, we start by choosing ω (satisfying some mild conditions) then, we start placing the oscillations.

If we place them far enough apart, we get approximate solutions of an invariance equation. We will show that they persist in the full system.

For simplicity we will formulate the theorem for maps.

The proofs can be easily adapted.

Theorem A

There is a set of frequencies $\Omega(\varepsilon)$*,* $\varepsilon < \varepsilon^*$

$$|\Omega(\varepsilon)|/|\Omega| \to 1 \quad as \varepsilon \to 0$$

so that we can find one breather localized at one site.

Theorem B

Given any probability measure on $\Omega(\varepsilon)$ equivalent to Lebesgue.

In a set of full measure in

$$\Omega(\varepsilon)^{\mathbb{N}}$$

we can find a breather with frequency $\boldsymbol{\omega}$.

Theorem A does not need that the system is translation invariant but assumes that the local properties are uniformly smooth. (e.g. random media). Theorem B uses translation invariance as formulated here, but there is a version weakening it.

No smallness condition, no loss of measure of in Theorem B.

We can be significantly more precise — even if more technical —

Both theorems will be deduced from a a more technical theorem.

If we have an approximation solution of an ivariance equation, which satisfies some appropriate non-degeneracy conditions

$\Longrightarrow$

There exists a true solution nearby.

(Furthermore, it is the only solution in this neighborhood up to change of origin in the torus.)

This is, in particular, a theorem on persistence of whiskered KAM tori.

- It has an *a posteriori* format. Approximate solutions lead to true solutions.
- The method does not rely on transformation theory.
- The proof can be supplemented by algorithmic details to become very efficient numerical algorithms. (Developed and implemented with G. Huguet and Y. Sire).
- A finite dimensional version of the proof has been published by (E. Fontich, R.L. and Y. Sire in Jour. Diff. Eq. (2009), ERA (2009)).

In this lecture, we will emphasize more the infinite dimensional aspectes.

There are reasons to prefer the a-priori format.

- In large dimensional systems, nost of the KAM behavior seems to happen in tori that cannot be continued to the uncoupled system. Phys. Rev. Let. **85** (9) 1859-1862 (2000) mparc 00-75:

  How relevant are KAM tori in chains of coupled oscillators?
- In systems such as molecules we can have very weak hyperbolicity. generated by the coupling inside the molecule (somewhat stronger than the intramolecule coupling). We can

  get a wedge of parameters (accumulating at zero) where the result applies

• It is widely believed (see the lecture this morning), that the coupling to radiation destroys invariance.

• There are theorems (Soffer-Weinstein, Pyke, Sigal, Comech-Komech) showing that solutions with this form cannot happen in hyperbolic PDES ("damping by radiation").

We think that this damping by radiation mechanism also applies in other regions of phase space. These systems share properties with PDE's.

Of course, the sytstem also changes the spectrum of radiation!.

• There are other invariant objects which also can be studied by similar formalisms.

E. Fontich, R. L. P. Martín (Jour. Diff. Eq. (2010) and MP_ARC 10-75, 10-76) consider also hyperbolic sets and their manifolds. Show decay properties and develop a structural stability.

Invariant measures in some of these sets were already considered in the literature (M. Jiang, Y. Pesin, etc.).

Note that these infinite dimensional stystems have uncountably many hyperbolic orbits.

Hence, it is impossible to find a topology that makes them compact (good for measure theory), or C^1 (good for geometry).

One has to compromise and choose carefully topologies for different parts of the arguments. See M. Jiang, R. L. (2001).

- The main idea is that the couplings are smooth and decay “fast enough” with the distance.

One of the key ideas is how to quantify the decay of the couplings.

We will obtain also that the invariant objects produced also can be written as local objects and some non-local corrections which decay fast.

$$H_\varepsilon(\mathbf{x}) = \sum_i H^0(x_i) + \varepsilon H^1(\mathbf{x})$$

$$\|\partial_j \partial_i H^1(\mathbf{x})\|_\rho \leq \Gamma(i-j)$$

$$\| \ \|_\rho \ \text{ an analytic norm}$$

$$\Gamma \ \text{ a decay function}$$

$$\left[\begin{array}{l} \Gamma \geq 0 \\ \sum_i \Gamma(i) = 1 \\ \sum_j \Gamma(i-j)\Gamma(j-k) \leq \Gamma(i-k) \end{array} \right.$$

Even if H_ε is a formal sum, the derivatives are bona-fide functions.

Example 1

$\Gamma(i) = ce^{-\beta|i|}$ not a decay function.

$\Gamma(i) = c_{da}|i|^{-d+\theta}e^{-\alpha|i|}$ is a decay function for $\theta > 0$, $\alpha \geq 0$.

Physical interpretation

$\partial_j \partial_i H^1 \equiv$ effect of particle j on particle i

$\|\partial_j \partial_i H^1\| \equiv$ bound on the effect of j on i

$\sum_j \partial_k \partial_j H^1 \partial_j \partial_i H^i \equiv$ effect of k on i through affecting j

For matrices (may be ∞ matrices), define

$$\|A\|_\Gamma = \sup_{i,j} |A_{ij}| \Gamma(i-j)^{-1}$$

Hence,

$$|A_{ij}| \le \|A\| \Gamma(i-j)$$

$$\begin{aligned}(A\,B)_{ij}| &\le \sum_k A_{i_k} B_{kj} \\ &\le \|A\|_\Gamma\ \|B\|_\Gamma \sum_k \Gamma(i-k)\Gamma(k-j) \\ &\le \|A\|_\Gamma\ \|B\|_\Gamma \Gamma(i-k)\end{aligned}$$

Banach algebra.

Decay functions were introduced in M. Jiang, R.L. Comm. Math. Phys. (2000) to study dependence on parameters of SRB measures. Note that since sup-norms are used. We get pathologies inherited from ℓ^∞. For example, there are *"observables at infinity"*, i.e. nontrivial functionals that have zero partial derivatives, non-duality, no smooth functions with bounded support, etc.

- We can choose any frequency which satisfies an ∞-dimensional Diophantine condition

$$
|\sum_{|i|\le R} \omega_i \cdot k_i| \ge c_R \Big(\sum_{|i|\le R} |k_i| \Big)^{-\tau_R}
$$

The theorem will be deduced from another theorem on persistence of whiskered tori with decay properties.

Consider mappings on $\mathcal{P} \equiv (\mathbb{R}^n \times \mathbb{T}^n)^{\mathbb{Z}^d}$ (endowed with ℓ^∞)

f such that

- f analytic
- $(Df(\mathbf{x})\eta)_i = \sum \frac{\partial f_i}{\partial x_j} \eta_j$

(The derivative is given by the Jacobian matrix)

$$\left\| \frac{\partial f_i}{\partial x_j} \right\|_\rho \leq c\Gamma(i-j)$$

$\| \ \|_\rho =$ sup in a complex neighborhood of size 0

In ℓ^∞, it is non-trivial to assume that the derivative is given by the Jacobian matrix.

Example 2

$$A\eta = \lim_{i \to \infty} \eta_i$$

is a non-trivial linear operator (Hann-Banach) whose matrix is zero.

We consider embeddings

$$K : (\mathbb{T}^n)^N \to \mathcal{P}$$

which are "centered around $\{c_i\}_{i=1}^N$"

$$\left\| \frac{\partial K_i}{\partial \theta_j} \right\|_\rho \leq C\Gamma(i - c_j)$$

$$\|K_i\|_\rho \leq C \max_k \left(\Gamma(i - c_k)\right)$$

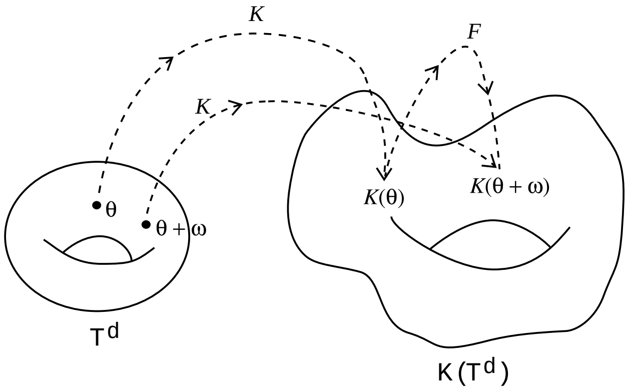
Symplectic geometry in infinite dimensional spaces has, in general many pathologies.

Turns out one can develop a nice theory for maps with decay and embeddings with decay.

The pull back of a formal symplectic form under embeddings with decay, is a well defined finite dimensional form.

These K give an invariant torus when

$$f \circ K(\theta) = K(\theta + \omega) \qquad \star$$



Theorem

f analytic, decay Γ exact symplectic

K : an embedding centered at c_i, decay Γ

- *$Df \circ K(\theta)$ admits an approximate invariant splitting which is hyperbolic*

$$d_\Gamma \left([Df \circ K(\theta)] E^{s,u,c}(\theta) \, , \, E^{s,u,c}_{(\theta+\omega)} \right) \leq \varepsilon$$

$$\left| Df \circ K(\theta)|_{E^s_\theta} \right|_\Gamma \leq \lambda < 1$$

$$\left| Df^{-1} \circ K(\theta)|_{E^u_\theta} \right|_\Gamma \leq \lambda < 1$$

$$\left| Df \circ K(\theta)|_{E^c_\theta} \right|_\Gamma \leq \eta$$

$$\left| Df^{-1} \circ K(\theta)|_{E^c_{(\theta)}} \right|_\Gamma \leq \eta$$

$$\lambda\eta < 1$$

[Actually what is assumed is a bit more general]

* The splittings have decay properties. Denote by π the projections.

$$\|\pi^{s,u,c}\|_{\rho,\Gamma} < \infty$$

The embeddings of the model spaces are also centered around **c**

* The center direction is $2Nd$ dimensional
* $Df \circ K$ is non-degenerate in the center direction (explicit expression to be given later)
* ω is $\sigma - \tau$ Diophantine $\omega \in (\mathbb{R}^n)^u$
* $\|f \circ K - K \circ T_\omega\|_{\mathbf{c},\Gamma,\rho} \leq \varepsilon$
* $\varepsilon \leq \varepsilon^* \, (\sigma, \tau, Nd$ non-degeneracy, $\rho - \rho')$

$\Longrightarrow$ There exists an $\widetilde{K}$ solving exactly

$$\|K - \widetilde{K}\|_{\mathbf{c},\Gamma,\rho'} \leq C(\ )\varepsilon$$

Moreover, if $\exists$ another $\widetilde{\widetilde{K}}$ solution satisfying the above, $\exists\ \eta$

$$\widetilde{K} = \widetilde{\widetilde{K}} \circ T_\eta$$

- Note that the tori are
  - → "very" hyperbolic
  - → do not tend to zero "ground state" at ∞
  - → frequencies remain bounded.

- Related work on quasi-periodic orbits in lattice by many people

  Chierchia-Perfetti
  Wayne, Frohlich, Spencer
  Poeschel
  Kuksin
  Chen, Guo, Yi, Viveros
  - - -

Sketch of proof

(1) Approximately invariant bundles $\Longrightarrow$ exactly invariant bundles + regularity + estimates

Use the same method of stability of bundles in literature

- Invariant bundle $\equiv$ graph of a linear bundle map.
- Invriance equation can be manipulated into a fixed point of a contraction.

The Banach algebra properties of decay functions make it work in pretty much the same way than *some* finite dimensional proofs.

Note for experts:

- One has to use essentially that the notion on the base is a rotation.

  In general theory of NHIM, one only gets finite differentiability

- To apply KAM theorem, one needs quantitative estimates.

  (This requires one more derivative.)

- Effective algorithms follow another route.

(2) The Newton method

(I) $$Df \circ K - K \circ T_\omega = E$$

(N) $$Df \circ K\Delta - \Delta \circ T_\omega = -E$$

$$\Updownarrow$$

$$\pi^s Df \circ K\Delta - \pi^s \Delta \circ T_\omega = \pi^s E$$

$$\pi^u Df \circ K\Delta - \pi^u \Delta \circ T_\omega = \pi^u E$$

$$\pi^c Df \circ K\Delta - \pi^c \Delta \circ T_\omega = \pi^c E$$

Using invariance of splittings one can move proyections inside the Df.

$$(Df \circ K)|_{E^s} \Delta^s - \Delta^s \circ T_\omega = -E^s$$

$$Df \circ K|_{E^s} \circ T_{-\omega} \Delta^s + E^s = \Delta^s$$

The equation for Δ^s can be solved iterating

Similar manipulations work for Δ^u and we can get a fixed point.

No symplectic geometry used yet. Nor Diophantine properties.
□ □ □

One can use similar arguments to get a theory of hyperbolic sets with decay, invariant manifolds

- Afraimovich, Bunimovich, Pesin, Sinai
- Jiang, R.L.
- Fontich, Martin, R.L

The center equation is more delicate, requires geometry and small divisors.

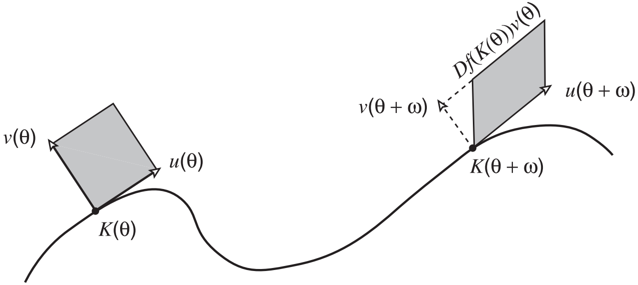
But it is *finite dimensional*!

Taking derivates of the invariance equation

$$Df \circ K \cdot DK = DK \circ T_\omega + DE$$

geometrically, DK “almost” invariant

$$\Longrightarrow R_k DK(\sigma) \subset E^c_\theta$$



$Df(K(\theta))v(\theta)$
$v(\theta + \omega)$
$u(\theta + \omega)$
$v(\theta)$
$u(\theta)$
$K(\theta + \omega)$
$K(\theta)$

We write

$$v(\theta) = J^{-1} \circ K \, DK(\theta) N(\theta)$$

where N is chosen so that

$$\Omega(v(\theta) DK(\theta)) = \mathrm{Id}$$

$$\Updownarrow$$

$$N = [DK(\theta) DK(\theta)]^{-1}$$

Write

$$M(\theta) = [\ \underset{\substack{\uparrow \\ \text{juxtapose it}}}{DK(0)}\ , v(\theta)]$$

Then, we have:

$$Df \circ K\, M(\theta) = M(\theta + \omega) \begin{pmatrix} \mathrm{Id} & A(\theta) \\ 0 & \mathrm{Id} \end{pmatrix} + O(E)$$

The range of M is in E^c (up to an small error).

We can also show that the torus is approximately lagrangian so that

$$\Omega(DK, v(\theta)) = 0$$

Counting dimensions

$$E^c = \text{Range } M \ .$$

(Some extra details needed to really estimate the distance).

(3) Recovering induction assumptions

- The invariant subspaces for K, approximately invariant for $K + \Delta$
- The non-degeneracy conditions, invariance do not change much

(4) Convergence is standard once we have all the appropriate definitions in place.

Going from technical lemma to main theorem

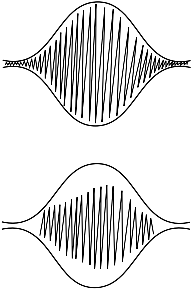
Theorem A

- For ε small enough breathers with a finite number of sites are quasi-invariant

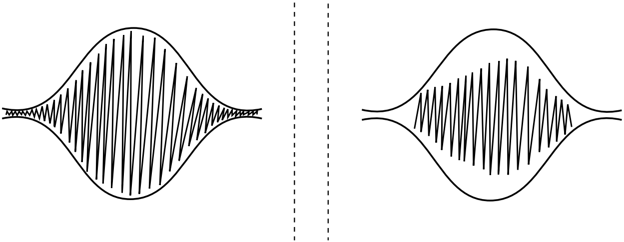
  $\Longrightarrow$ invariant nearby

Theorem B.

Consider two localized solutions.



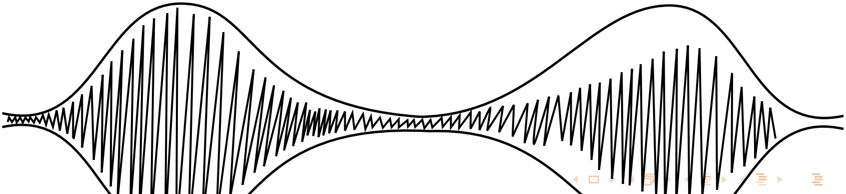
If $\widetilde{\Gamma} \ll \Gamma$



It is very approximately a breather (in $\widetilde{\Gamma}$) sense.

$\Longrightarrow$

$\exists$ a true $\widetilde{\Gamma}$ breather close by to it.



To get theorem B, we chose the one-site breathers, we choose the sequence of decay functions.

Then, we can apply the main theorem by placing them succesively far appart. If you repeat the process with decreasing Γ, you get a limit in Γ^∞

(if the convergence is fast enough — or just coordinate wise, one recovers that the limiting object satisfies the invariance

Some extensions and some questions

• (G. Huguet, R. L, Y. Sire) *Efficient algorithms*

The proof suggests that one can obtain fast algorithms. One needs to do several tricks

- Use systematically FFT
- Compute not the splittings but the projections of the splittings
- Fast solutions of cohomology equations in the hyperbolic case.

The final product is an algorithm that, to perform a Newton method on a discretization on N points requires $O(N \ln(N))$ operations, $O(N)$ storage.

It is numerically stable.

Preprint in MP_ARC # 09-02

• (D. Blazevski, R. L.) *Existence of whiskers with decay properties. Melnikov theory*

One can show that there are stable and unstable manifolds which are analytic and have decay properties.

One can also develop a perturbation theory for them and the beginnings of *“Melnikov theory*. The symplectic geometry is very tricky in infinite dimensions.

This suggests that there are some mechanisms to transport energy.

• *Radiation damping?*

In PDE's it is well established that in systems that radiate, the energy goes to infinity.

Can one show that this happens near the solutions in the minima of the potential.

This is the regime studied by many people (J. Geng, J. Viveros, Y. Yi...) in finite coupled map lattices.

• (P. Martín, R. Ramirez-Ros) *Normally hyperbolic manifolds with decay*

• (A. Haro, R.L.) *Study the phenomena that happen at resonances*

The *Secondary* tori dominate in volume as the number of degrees of freedom grows.

(Phys Rev.Let (2000). Detailed quantitative conjectures).

It seems that one can prove existence of many of these secondary tori.

”There is lots of room in infinite dimensional space”