Perturbation of a magnetic Schrödinger operator near an embedded infinite-multiplicity eigenvalue

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Abstract. In this paper we consider a 3D magnetic Schrödinger operator having infinitely many eigenvalues of infinite multiplicity, embedded in the continuous spectrum. We perturb this operator by a relatively compact potential and analyse the transition of these eigenvalues into a "cloud" of resonances. Several different approaches are employed. First we consider resonances as eigenvalues of a non-selfadjoint operator by using analytic distortion. Then we study the dynamical aspect of the resonances and finally we study the behavior of the spectral shift function near the infinite-multiplicity eigenvalues.

1. Introduction

We consider a 3D Schrödinger operator H_0 with constant magnetic field $\mathbf{B} = (0,0,b)$, $b > 0$, and with electric field $\mathbf{E} = -(0,0,v_0')$ where v_0 is a scalar potential depending only on the variable x_3. This operator, introduced by Astaburuaga-Briet-Bruneau-Fernández-Raikov [**4**], has infinitely many eigenvalues of infinite multiplicity, embedded in its continuous spectrum. These eigenvalues have the form $2bq + \lambda$, $q \in \mathbb{Z}_+ := \{0,1,2,\ldots\}$, where $2bq$, $q \in \mathbb{Z}_+$, are the Landau levels, i.e. the infinite-multiplicity eigenvalues of the (shifted) 2D Landau Hamiltonian, and λ is a simple eigenvalue of the 1D operator $-\frac{d^2}{dx^2} + v_0(x)$. We introduce the perturbed operator $H = H_0 + V$ where V is a H_0-compact multiplier by a real function, and study the transition of the eigenvalues $2bq+\lambda$, $q \in \mathbb{Z}_+$, into a "cloud" of resonances near $2bq + \lambda$.

We analyze such phenomena using several different approaches. First, we define resonances via analytic distortion as developed by Hunziker [**22**] (see also Aguilar-Combes [**2**] for the analytic dilation). For the Schrödinger operator with constant magnetic field, and potentials which are analytic near the real axis, it is standard to dilate the variable along the magnetic field (see [**5**], [**40**], [**4**]). For potentials

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which are analytic near the real axis only outside a compact set, it is also possible to distort the variable along the magnetic field and to define resonances near the real axis. Here a difficulty to justify this definition (i.e. to show the independence with respect to the distortion) comes from the infinite multiplicity of $2bq + \lambda$ as eigenvalue of H_0. This problem is overcome in [**26**] by introducing appropriate determinant and considering resonances as zeroes of this determinant rather than as poles of a resolvent. With this definition, for v_0 and V of a sufficiently rapid decay at infinity, we have an upper bound of the number of resonances in a ring centered at $2bq + \lambda$ with radii r and $2r$ tending to 0. This estimate is expressed via the eigenvalue counting function for compact Berezin-Toeplitz operators whose asymptotics is well-known (see Raikov [**33**], Raikov-Warzel [**36**]). However, upper bounds do not imply accumulation of resonances. In order to perform this analysis, we assume that the perturbation V is axisymmetric. In this case, as $\varkappa \searrow 0$, we have infinitely many resonances of $H_0 + \varkappa V$, which converge to $2bq + \lambda$. We obtain an asymptotic expansion as $\varkappa \to 0$ of each of these resonances in the spirit of the Fermi Golden Rule (see e.g. [**37**, Section XII.6]).

Our second approach consists of considering the dynamical aspect of the resonances. For resonances defined by analytic distortion, we estimate the time decay of the resonance states. A similar relation between the small-coupling-constant asymptotics of the resonance, and the exponential time decay of the resonance states has been established by Herbst [**21**] in the case of the Stark Hamiltonian, and later by other authors in the case of various quantum Hamiltonians (see e.g. [**39**], [**19**], [**3**]). Cancelling the analyticity assumption, we can use the time dependent methods developed in [**38**] and [**13**], and, above all, the recent article by Cattaneo, Graf and Hunziker [**12**], where the dynamic estimates of the resonance states are based on appropriate Mourre estimates [**29**], [**23**]. Applying a general abstract result of [**12**], we formulate a theorem on the dynamics of the resonance states for axisymmetric V satisfying no analyticity assumptions.

As a last approach, we analyze the behavior of the Spectral Shift Function (SSF) near $2bq+\lambda$ canceling the restriction that V is axisymmetric. For potentials having analytic continuations near the real axis outside a compact set, we have the so-called Breit-Wigner approximation which state that near a resonance $w = a - ib$, the derivative of the SSF behaves like the harmonic measure: $E \mapsto \frac{b}{\pi((E-a)^2+b^2)}$ which tends to the Dirac measure $\delta(E-a)$ as $b \searrow 0$. Such result was obtained by J.F.Bony, Bruneau and Raikov [**7**] near the Landau levels for the 3D Schrödinger operator with constant magnetic field (corresponding to $v_0 = 0$). In the semi-classical regime such representations were studied by Petkov-Zworski [**30, 31**], J.F.Bony-Sjöstrand [**8**], Bruneau-Petkov [**11**], Dimassi [**14**] and Dimassi-Petkov [**15, 16**] for the Schrödinger operator and by Khochman [**25**] for the Dirac operator.

It is natural to think that accumulation of resonances near $2bq+\lambda$ will produce blow up of the SSF. Accumulation of resonances is not yet well understood, but we can directly analyze the singularities of the SSF. We do not need analyticity assumption on V and v_0, but we suppose that the perturbation V is of definite sign. As in the work of Fernández-Raikov [**18**] (see also Briet-Raikov-Soccorsi [**9**] and Raikov [**34, 35**]), using a representation of the SSF due to Pushnitski [**32**], we show that the leading term of this singularity can be expressed via the eigenvalue counting function for compact Berezin-Toeplitz operators. Using the well-known results on the spectral asymptotics for such operators (see [**33**], [**36**]), we obtain explicitly the main asymptotic term of the SSF as the energy approaches the fixed point $2bq + \lambda$ for several classes of perturbations with prescribed decay rate with respect to the variables on the plane perpendicular to the magnetic field.

The paper is organized as follows. In Section 2 we introduce our model. Section 3 is devoted to the resonances defined by analytic distortion. The dynamical aspect of the resonances is considered in Section 4 while the analysis of the SSF near the points $2bq+\lambda$ is given in Section 5. This article is essentially a survey paper of the article [**4**] and the preprint [**26**].

2. Magnetic Schrödinger operator

We consider a 3D Schrödinger operator subject to an electromagnetic field $(\mathbf{E},\mathbf{B})$ with electric component $\mathbf{E} = -(0,0,v_0')$ where v_0 is a scalar potential depending only on the variable x_3, and magnetic component $\mathbf{B} = (0,0,b)$ where b is a positive constant. In $L^2(\mathbb{R}^3) \approx L^2(\mathbb{R}^2_{x_1,x_2}) \otimes L^2(\mathbb{R}_{x_3})$, it is given by:

$$H_0 := H_{0,\perp} \otimes I_{\|} + I_{\perp} \otimes H_{0,\|}$$

where $I_{\|}$ and $I_{\perp}$ are the identity operators in $L^2(\mathbb{R}_{x_3})$ and $L^2(\mathbb{R}^2_{x_1,x_2})$ respectively,

$$H_{0,\perp} := \left(i\frac{\partial}{\partial x_1} - \frac{bx_2}{2}\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{bx_1}{2}\right)^2 - b, \quad (x_1,x_2) \in \mathbb{R}^2,$$

is the Landau Hamiltonian shifted by the constant b, and

$$H_{0,\|} := -\frac{d^2}{dx_3^2} + v_0, \quad x_3 \in \mathbb{R},$$

with, v_0, the multiplication operator by $v_0 \in L^\infty(\mathbb{R},\mathbb{R})$, satisfying

$$|v_0(x_3)| = O(\langle x_3\rangle^{-m_0}), \quad m_0 > 1, \quad \langle x\rangle = (1+|x|^2)^{\frac{1}{2}}. \tag{2.1}$$

The operator $H_{0,\perp}$ is self-adjoint in $L^2(\mathbb{R}^2)$. Its spectrum is $\sigma(H_{0,\perp}) = \cup_{q=0}^{\infty}\{2bq\}$, and every eigenvalue $2bq$ of $H_{0,\perp}$ has infinite multiplicity (see e.g. [**5**]). Since v_0 is $-\frac{d^2}{dx_3^2}$-compact, the operator $H_{0,\|}$ is essentially self-adjoint in $L^2(\mathbb{R})$ and

$$\sigma_{\text{ess}}(H_{0,\|}) = [0,\infty[.$$

Throughout the article we suppose also that

$$\sigma_{\text{disc}}(H_{0,\|}) = \{\lambda\} \subset]-2b,0[. \tag{2.2}$$

Hence, the operator H_0 is essentially self-adjoint in $L^2(\mathbb{R}^3)$. Moreover, we have $\sigma_{\text{ess}}(H_0) = \cup_{q=0}^{\infty}[2bq,\infty[= [0,\infty[$. Note that $2bq+\lambda$, $q \in \mathbb{Z}_+$ is an eigenvalue of infinite multiplicity of H_0. If $q = 0$, this eigenvalue is isolated, and if $q \geq 1$, it lies on the interval $[0,\infty[$.

We want to study the perturbation of H_0 by the multiplication operator by the potential V. Let us introduce the perturbed operator $H = H_0+V$ with $V \in L^\infty(\mathbb{R}^3)$ satisfying

$$|V(x)| = O(\langle X_\perp\rangle^{-\delta_\perp}\langle x_3\rangle^{-\delta_\|}), \quad X_\perp = (x_1,x_2), \tag{2.3}$$

with $\delta_\perp > 2$ and $\delta_\| > 1$.

On the domain of H_0 the operator H is well defined and $\sigma_{\rm ess}(H) = \sigma_{\rm ess}(H_0) = [0, \infty[$.

In the following sections we will see that the eigenvalue $2bq + \lambda$, $q \in \mathbb{Z}_+$ of infinite multiplicity for the unperturbed operator H_0 can generate infinitely many eigenvalues for $H_0 + V$ when $q = 0$ (see the remarks of Section 5), infinitely many resonances for H_0+V when $q > 0$ (see Sections 3-4), and singularities of the spectral shift function for the pair $(H_0 + V, H_0)$ (see Section 5). When $2bq + \lambda$ is embedded in $\sigma(H_0)$ (i.e. $q > 0$) and V is axisymmetric, we will see that at least one resonance remains an embedded eigenvalue and that we can produce as many resonances with non-zero imaginary part as we wish, provided that the coupling constant is small enough (see the remarks at the end of Section 4). However we do not know if a infinite number of resonances have non-zero imaginary part.

3. Resonances via analytic distortion

In this section, we define the resonances of $H_0 + V$ as the eigenvalues of the associated distorted operator $H_0(\theta) + V(\theta)$. We need to assume that the electric potentials v_0 and V have analytic continuation with respect to x_3 in a sector outside a compact set. Then for axisymmetric potentials V, we obtain asymptotic expansion as $\varkappa$ tends to 0 of an infinite number of resonances of $H_0 + \varkappa V$ which converge to $2bq+\lambda$. This expansion is in the spirit of the Fermi Golden Rule.

We suppose that V and v_0 have holomorphic extensions in the magnetic field direction x_3 in the following neighborhood of $\mathbb{R}\backslash] - R_0, R_0[$:

$$C(\epsilon, R_0) := \{z \in \mathbb{C}, |\operatorname{Im}(z)| \leq \epsilon |\operatorname{Re}(z)|, \quad |\operatorname{Re}(z)| \geq R_0\}, \quad \text{for } 0 < \epsilon < 1.$$

Moreover, we assume that the short range properties (2.3) of V, and (2.1) of v_0, remain valid on $\mathbb{R}^2 \times C(\epsilon, R_0)$ for V and on $C(\epsilon, R_0)$ for v_0.

Let us introduce the one-parameter family of unitary distortions in the magnetic field direction x_3:

$$U_\theta f(x) = J_{\phi_\theta}^{\frac{1}{2}}(x) f(\phi_\theta(x)), \quad \theta \in \mathbb{R}, \quad f \in S(\mathbb{R}),$$

where $\phi_\theta(x) = x + \theta g(x)$, $J_{\phi_\theta}(x) = \det(I + \theta g'(x))$ is the Jacobian of $\phi_\theta(x)$ and $g : \mathbb{R} \longmapsto \mathbb{R}$ is a smooth function satisfying:

$$(\mathbf{A_g}) \begin{cases} \text{(i)} \;\; \sup_{x \in \mathbb{R}} \mid g'(x) \mid << 1, \\ \text{(ii)} \;\; g(x) = 0, \quad \text{on } [-R_0, R_0], \\ \text{(iii)} \;\; g(x) = x, \quad \text{on } \mathbb{R} \setminus [-R_1, R_1], R_1 > R_0. \end{cases}$$

The choice of R_0 in $(\mathbf{A_g})$ depends on the sector $C(\epsilon, R_0)$ where V and v_0 have analytic extension.

For $\theta \in \mathbb{R}$, we define

$$H(\theta) := (I_\perp \otimes U_\theta) H (I_\perp \otimes U_\theta^{-1}) = H_0(\theta) + V(\theta),$$

$$H_0(\theta) = H_{0,\perp} \otimes I_\parallel + I_\perp \otimes H_{0,\parallel}(\theta),$$

$$H_{0,\parallel}(\theta) = U_\theta H_{0,\parallel} U_\theta^{-1} \qquad V(\theta)(x) = V(x_1, x_2, \phi_\theta(x_3)).$$

By assumption on V and v_0, for $D_\epsilon := \{\theta \in \mathbb{C}, \; |\theta| \leq r_\epsilon := \frac{\epsilon}{\sqrt{1+\epsilon^2}}\}$, the families $\Big(H_{0,\parallel}(\theta)\Big)_{\theta \in D_\epsilon}$ and $\Big(H(\theta)\Big)_{\theta \in D_\epsilon}$ form analytic families of type A, in the sense of

Kato (see for instance [**24**, Theorem 4.5.35], [**22**] for the Schrödinger operator or [**25**, Section 3] for the Dirac operator).

Clearly, the difference of the resolvents of $H_{0,\|}(\theta)$ and $-(1+\theta)^{-2}\frac{d^2}{dx_3^2}$ is a compact operator. Hence the essential spectrum of $H_{0,\|}(\theta)$ is $(1+\theta)^{-2}\mathbb{R}_+$. Moreover, since each Landau level is an infinite multiplicity eigenvalue of $H_{0,\perp}$, the essential spectrum of $H(\theta)$ is

$$\sigma_{\rm ess}(H(\theta)) = 2b\mathbb{Z}_+ + \sigma(H_{0,\|}(\theta)).$$

For $\theta_0 \in D_\epsilon$, $\operatorname{Im}\theta_0 > 0$, this allows to define resonances of H in

$$S_{\theta_0} = \bigcup_{q\in\mathbb{N}} \Big(2b]q, q+1[+(1+\theta_0)^{-2}\mathbb{R}_+\Big)$$

as the discrete spectrum of $H(\theta_0)$ in S_{θ_0}. For two values θ_1, $\theta_2 \in D_\epsilon$, $\operatorname{Im}\theta_1 > 0$, $\operatorname{Im}\theta_2 > 0$ the discrete spectrum of $H(\theta_1)$ and of $H(\theta_2)$ coincide on $S_{\theta_1} \cap S_{\theta_2}$ (see [**26**]). This justifies our definition of resonances which is suitable for our purposes if θ_2 is small enough since we consider resonances near the real axis. The multiplicity of a resonance w_0 is defined by

$$\text{mult}(w_0) := \text{rank}\frac{1}{2\pi i}\int_{\Gamma_0}(z - H(\theta_0))^{-1}dz, \tag{3.1}$$

where Γ_0 is a small positively oriented circle centered at w_0. We will denote $\operatorname{Res}(H)$ the set of resonances.

For $\operatorname{Im}\theta > 0$ fixed, let Ω_q be a domain centered at $2bq+\lambda$ such that

$$\sigma_{ess}(H_0(\theta)) \cap \Omega_q = \{2bq+\lambda\}.$$

We have the following upper bound on the number of resonances of H in a small annulus in Ω_q centered at $2bq+\lambda$.

Theorem 3.1. [**26**, Theorem 2.1] *Suppose that V and v_0 satisfy the hypotheses cited above. Then there exist $r_0 > 0$ and $s > 0$, such that for any $0 < r < r_0$,*

$$\#\{z \in \operatorname{Res}(H) \cap \Omega_q; r < |z - 2bq - \lambda| < 2r\} = O(n_+(r/s, p_q W p_q)|\ln r|),$$

where $W = \sup_{x_3 \in C(\epsilon,R_0)} |\langle x_3\rangle^{\delta_\|} V|$, p_q is the orthogonal projection onto $\mathcal{H}_q := \operatorname{Ker}(H_{0,\perp} - 2bq)$ and $n_+(r, p_q W p_q)$ is the counting function of the eigenvalues larger to r of the Toeplitz operator $p_q W p_q$.

Let us mention that asymptotic estimates of the counting function $n_+(r, p_q W p_q)$ as r tends to 0 are known in various cases (see [**33, 36**] and Proposition 5.2 below). In particular, under our assumption we have always $n_+(r, p_q W p_q) = O(r^{-2/\delta_\perp})$ and for V compactly supported, we have $n_+(r, p_q W p_q) = O((\ln|\ln r|)^{-1}|\ln r|)$.

In what follows, we want to study the transition of the eigenvalues $2bq+\lambda$, $q \in \mathbb{Z}_+$, into a "cloud" of resonances which converge to $2bq+\lambda$, for a small perturbation $\varkappa V$, $\varkappa \ll 1$. In order to perform this analysis, we assume that V is axisymmetric so that the operator $H_0 + \varkappa V$ commutes with the x_3-component of the angular-momentum operator L. In this case $H_0 + \varkappa V$ is unitarily equivalent to the orthogonal sum $\oplus_{m\in\mathbb{Z}}(H_0^{(m)} + \varkappa V)$ where $H_0^{(m)}$ has $2bq+\lambda$ as a simple eigenvalue. This allows us to reduce the analysis to a small perturbation of a simple eigenvalue.

More precisely, for a fixed magnetic quantum number m, let us introduce

$$H_{0,\perp}^{(m)} := -\frac{1}{\varrho}\frac{d}{d\varrho}\varrho\frac{d}{d\varrho} + \left(\frac{m}{\varrho} - \frac{b\varrho}{2}\right)^2 - b.$$

The operator $H_{0,\perp}^{(m)}$ is self-adjoint in $L^2(\mathbb{R}_+; \varrho d\varrho)$, and we have $\sigma(H_{0,\perp}^{(m)}) = \cup_{q=m_-}^{\infty}\{2bq\}$ where $m_- = \max\{0, -m\}$ (see e.g. [**5**]). In contrast to the operator $H_{0,\perp}$, every eigenvalue $2bq$ of $H_{0,\perp}^{(m)}$ is simple.

Set

$$H_0^{(m)} := H_{0,\perp}^{(m)} \otimes I_\parallel + \tilde{I}_\perp \otimes H_{0,\parallel}$$

where $\tilde{I}_\perp$ is the identity operator in $L^2(\mathbb{R}_+; \varrho d\varrho)$. Evidently, for any $q \geq m_-$, $2bq+\lambda$ is then a simple eigenvalue of $H_0^{(m)}$. We denote $\Phi_{q,m}$ the associated normalized eigenfunction:

$$H_0^{(m)}\Phi_{q,m} = (2bq+\lambda)\Phi_{q,m}, \quad q \geq m_-. \tag{3.2}$$

Let (ϱ, ϕ, x_3) be the cylindrical coordinates in $\mathbb{R}^3$. The operator $H_0^{(m)}$, $m \in \mathbb{Z}$, is unitarily equivalent to the restriction of H_0 onto $\mathrm{Ker}\,(L-m)$ where

$$L := -i\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) = -i\frac{\partial}{\partial \phi}$$

is the x_3-component of the angular-momentum operator, which commutes with H_0. Moreover, the operator H_0 is unitarily equivalent to the orthogonal sum $\oplus_{m\in\mathbb{Z}}H_0^{(m)}$ under the passage to cylindrical coordinates (ϱ, ϕ, x_3) in $\mathbb{R}^3$, and a subsequent decomposition into a Fourier series with respect to the angular variable ϕ.

For V axisymmetric, that is depending only on the variables (ϱ, x_3), the operator $H_0 + \varkappa V$ is unitarily equivalent to $\oplus_{m\in\mathbb{Z}}(H_0^{(m)} + \varkappa V(\varrho, x_3))$. Moreover, since L commutes with $I_\perp \otimes U_\theta$, the same decomposition holds for the distorted operators. In particular, the analysis of the resonances of $H_0 + \varkappa V$ is reduced to the study of the eigenvalues of $H_0^{(m)}(\theta) + \varkappa V(\theta)$. Let us mention that some eigenvalues of $H_0^{(m)}(\theta) + \varkappa V(\theta)$ are not resonances of $H_0 + \varkappa V$ (in the sense of the above definition), because it could be an eigenvalue of $H_0(\theta) + \varkappa V(\theta)$ with infinite multiplicity.

For the eigenvalues of $H_0^{(m)}(\theta) + \varkappa V(\theta)$, we have:

THEOREM 3.2. *Fix $m \in \mathbb{Z}$, $q > m_-$. Under the assumptions of Theorem 3.1, for V axisymmetric, and for $|\varkappa|$ sufficiently small, the operator $H_0^{(m)} + \varkappa V$ has a resonance $w_{q,m}(\varkappa)$ which obeys the asymptotics*

$$w_{q,m}(\varkappa) = 2bq+\lambda+\varkappa\langle V\Phi_{q,m}, \Phi_{q,m}\rangle - \varkappa^2 F_{q,m}(2bq+\lambda) + O_{q,m,V}(\varkappa^3), \ \varkappa \to 0, \tag{3.3}$$

the eigenfunction $\Phi_{q,m}$ being defined by (3.2). *The quantity $F_{q,m}(2bq+\lambda)$ is the standard term of* the Fermi Golden Rule. *It is given by:*

$$F_{q,m}(2bq+\lambda) = \lim_{\delta\downarrow 0} F_{q,m}(2bq+\lambda+i\delta), \tag{3.4}$$

where

$$F_{q,m}(z) := \langle (H_0^{(m)} - z)^{-1}(I - \mathcal{P}_{q,m})V\Phi_{q,m}, V\Phi_{q,m}\rangle.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{R}_+ \times \mathbb{R}; \varrho d\varrho dx_3)$ and $\mathcal{P}_{q,m}$ is the eigenprojector onto Ker$(H_0^{(m)} - (2bq+\lambda))$, the eigenspace generated by $\Phi_{q,m}$.

PROOF. The proof is as in the analytic case (see Theorem 3.1 of [**4**]). Let us give the main ingredients of the proof. Fix θ such that $\theta_0 > \mathrm{Im}\,\theta \geq 0$. The simple embedded eigenvalue $2bq+\lambda$ of $H_0^{(m)}$ is a simple isolated eigenvalue of $H_0^{(m)}(\theta)$. According to the Kato perturbation theory (see [**24**, Section VIII.2]), for sufficiently small $\varkappa$ there exists a simple eigenvalue $w_{q,m}(\varkappa)$ of $H_0^{(m)}(\theta) + \varkappa V(\theta)$ such that $\lim_{\varkappa \to 0} w_{q,m}(\varkappa) = 2bq + \lambda$. For $z \in \mathbb{C}$ in the resolvent set of the operator $H^{(m)}(\theta) + \varkappa V(\theta)$, put

$$R_{\varkappa,\theta}^{(m)}(z) := (H_0^{(m)}(\theta) + \varkappa V(\theta) - z)^{-1}.$$

Then, for $|\varkappa|$ sufficiently small, the eigenprojector on $\operatorname{Ker}(H_0^{(m)}(\theta) + \varkappa V(\theta) - w_{q,m}(\varkappa))$ is

$$\mathcal{P}_{\varkappa,q,m}(\theta) := -\frac{1}{2\pi i} \int_\Gamma R_{\varkappa,\theta}^{(m)}(z) dz$$

where Γ is a small positively oriented circle centered at $2bq + \lambda$. Moreover, since $w_{q,m}(\varkappa)$ is a simple eigenvalue, we have

$$w_{q,m}(\varkappa) = -\operatorname{Tr}\left(\frac{1}{2\pi i} \int_\Gamma z R_{\varkappa,\theta}^{(m)}(z) dz\right)$$

for Γ and $\varkappa$ as above. Consequently, the asymptotic expansion of $w_{q,m}(\varkappa)$ follows from the resolvent identity: $R_{\varkappa,\theta}^{(m)}(z) =$

$$R_{0,\theta}^{(m)}(z) - \varkappa R_{0,\theta}^{(m)}(z) V(\theta) R_{0,\theta}^{(m)}(z) + \varkappa^2 R_{0,\theta}^{(m)}(z) V(\theta) R_{0,\theta}^{(m)}(z) V(\theta) R_{0,\theta}^{(m)}(z) + O(\varkappa^3).$$

The asymptotic expansion seems depending on θ, but since these quantities are holomorphic with respect to θ and constants for $\theta \in \mathbb{R}$, it is independent of θ. The existence of $F_{q,m}(2bq + \lambda)$ come from the 1D limiting absorption principle (see [**1**]). □

Remarks: (i) Theorem 3.2 implies that generically near $2bq + \lambda$, $q \geq 1$, there are infinitely many resonances of $H_0 + \varkappa V$ with sufficiently small $\varkappa$, namely the resonances of the operators $H_0^{(m)} + \varkappa V$ with $m > -q$.
(ii) Note however that $2bq + \lambda$ is a discrete simple eigenvalue of the operator $H_0^{(-q)}$, and therefore the operator $H_0^{(-q)} + \varkappa V$ has a simple discrete eigenvalue provided that $|\varkappa|$ is small enough. Generically, this eigenvalue is an embedded eigenvalue for the operator $H_0 + \varkappa V$.

4. Dynamical aspect

In this section, we show that the arguments of the previous section gives also an estimate on the time decay of the resonances states. We mention also the time dependent approach ([**38**], [**13**], [**12**]) developed in [**4**] for our model, where no analyticity is required.

Proposition 4.1. Under the assumptions of Theorem 3.2 there exists a function $g \in C_0^\infty(\mathbb{R};\mathbb{R})$ such that $g = 1$ near $2bq + \lambda$, and

$$\langle e^{-i(H_0^{(m)} + \varkappa V)t} g(H_0^{(m)} + \varkappa V) \Phi_{q,m}, \Phi_{q,m} \rangle = a(\varkappa) e^{-iw_{q,m}(\varkappa)t} + b(\varkappa, t), \quad t \geq 0, \tag{4.1}$$

with a and b satisfying the asymptotic estimates

$$|a(\varkappa) - 1| = O(\varkappa^2),$$

$$|b(\varkappa, t)| = O(\varkappa^2 (1 + t)^{-n}), \quad \forall n \in \mathbb{Z}_+, \tag{4.2}$$

as $\varkappa \to 0$ uniformly with respect to $t \geq 0$.

PROOF. For the detailed proof, we refer to the Proposition 3.1 of [**4**]. It first uses the Helffer-Sjöstrand formula,

$$e^{-i(H_0^{(m)}+\varkappa V)t}\, g(H_0^{(m)}+\varkappa V)\,\mathcal{P}_{q,m} = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\partial \tilde g}{\partial \bar z}(z)\, e^{-izt}\,(H_0^{(m)}+\varkappa V - z)^{-1}\,\mathcal{P}_{q,m} dxdy \tag{4.3}$$

where $z = x+iy$, $\bar z = x - iy$, $\tilde g$ is a compactly supported, quasi-analytic extension of g, and the convergence of the integral is understood in the operator-norm sense (see e.g. [**17**, Chapter 8]). Then we consider the functions

$$\sigma_\pm(z) := \mathrm{Tr}\,((H_0^{(m)} + \varkappa V - z)^{-1}\,\mathcal{P}_{q,m}), \quad \pm \mathrm{Im} z > 0,$$

As before, we have invariance properties with respect to θ:

$$\sigma_+(z) = \mathrm{Tr}\,(R^{(m)}_{\varkappa,\theta}(z)\,\mathcal{P}_{0,q,m}(\theta)), \quad \mathrm{Im}\, z > 0, \quad \theta_0 > \mathrm{Im}\,\theta > 0, \tag{4.4}$$

and from the asymptotic expansion of the resolvent, we obtain:

$$\sigma_+(z) = \Big(1 + \varkappa^2 r(\varkappa)\Big)(w_{q,m}(\varkappa) - z)^{-1} + \varkappa^2 G_+(\varkappa, z), \tag{4.5}$$

where $r(\varkappa)$ and $G_+(\varkappa, z)$ are uniformly bounded with respect to $|\varkappa|$ small enough. Moreover, $z \mapsto G_+(\varkappa, z)$ is analytic near $2bq + \lambda$.

Similarly,

$$\sigma_-(z) = \Big(1 + \varkappa^2 \overline{r(\varkappa)}\Big)\,(\overline{w_{q,m}(\varkappa)} - z)^{-1} + \varkappa^2 G_-(\varkappa, z), \tag{4.6}$$

where $G_-(\varkappa, z)$ is analytic near $2bq + \lambda$ and uniformly bounded with respect to $|\varkappa|$ small enough. Now, assume that the support of g is such that we can choose $\tilde g$ supported on a neighborhood of $2bq + \lambda$ where the functions $z \mapsto G_\pm(\varkappa, z)$ are holomorphic. Combining (4.3) with the Green formula, we get

$$\mathrm{Tr}\,(e^{-i(H_0^{(m)}+\varkappa V)t}\, g(H_0^{(m)} + \varkappa V)\,\mathcal{P}_{q,m}) = \frac{1}{2\pi i}\int_{\mathbb{R}} g(\mu)\, e^{-i\mu t}\,(\sigma_+(\mu) - \sigma_-(\mu))d\mu. \tag{4.7}$$

Making use of (4.5) – (4.6), we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{R}} g(\mu)\, e^{-i\mu t}\,(\sigma_+(\mu) - \sigma_-(\mu))d\mu &= \frac{\varkappa^2}{2\pi i}\int_{\mathbb{R}} g(\mu)\, e^{-i\mu t}\,(G_+(\varkappa,\mu) - G_-(\varkappa,\mu))d\mu \\ &+ \frac{1+\varkappa^2 r(\varkappa)}{2\pi i}\int_{\mathbb{R}} g(\mu)\, e^{-i\mu t}\,(w_{q,m}(\varkappa) - \mu)^{-1} d\mu \\ &- \frac{1+\varkappa^2 \overline{r(\varkappa)}}{2\pi i}\int_{\mathbb{R}} g(\mu)\, e^{-i\mu t}\,(\overline{w_{q,m}(\varkappa)} - \mu)^{-1} d\mu.\end{aligned}$$

Pick $\varepsilon > 0$ so small that $g(\mu) = 1$ for $\mu \in [2bq + \lambda - 2\varepsilon, 2bq + \lambda + 2\varepsilon]$. Set

$$C_\varepsilon := (-\infty, 2bq + \lambda - \varepsilon] \cup \{2bq + \lambda + \varepsilon e^{it},\, t \in [-\pi, 0]\} \cup [2bq + \lambda + \varepsilon, +\infty),$$
$$g(\mu) := 1, \quad \mu \in C_\varepsilon \setminus \mathbb{R}.$$

Taking into account (4.7), bearing in mind that $\mathrm{Im}\, w_{q,m}(\varkappa) \le 0$, and applying the Cauchy theorem, we easily find that

$$\mathrm{Tr}\,(e^{-i(H_0^{(m)}+\varkappa V)t}\, g(H_0^{(m)}+\varkappa V)\,\mathcal{P}_{q,m}) = (1+\varkappa^2 r(\varkappa))e^{-iw_{q,m}(\varkappa)t} + \varkappa^2 \sum_{j=1,2,3} I_j(t;\varkappa) \tag{4.8}$$

where

$$I_1(t;\varkappa) := \frac{1}{2\pi i}\int_{\mathbb{R}} g(\mu)\, e^{-i\mu t}\,(G_+(\varkappa,\mu) - G_-(\varkappa,\mu))d\mu,$$

$$I_2(t;\varkappa) := \frac{1}{2i\pi}\int_{C_\varepsilon} g(\mu)\, e^{-i\mu t}(r(\varkappa)(w_{q,m}(\varkappa)-\mu)^{-1} - \overline{r(\varkappa)}(\overline{w_{q,m}(\varkappa)}-\mu)^{-1})d\mu,$$

$$I_3(t;\varkappa) := -\frac{\operatorname{Im} w_{q,m}(\varkappa)}{\varkappa^2\pi}\int_{C_\varepsilon} g(\mu)\, e^{-i\mu t}(w_{q,m}(\varkappa)-\mu)^{-1}(\overline{w_{q,m}(\varkappa)}-\mu)^{-1}\, d\mu.$$

Integrating by parts, we find that

$$|I_j(t;\varkappa)| = O((1+t)^{-n}), \quad t>0, \quad j=1,2,3, \quad \forall n\in\mathbb{Z}_+, \tag{4.9}$$

uniformly with respect to $\varkappa$, provided that $|\varkappa|$ is small enough; in the estimate of $I_3(t;\varkappa)$ we have taken into account that by Theorem 3.2 we have $|\operatorname{Im}(w_{q,m}(\varkappa))| = O(\varkappa^2)$. Putting together (4.8) and (4.9), we get (4.1). □

Let us mention that combining Mourre estimates with recent result of Cattaneo, Graf, and Hunziker (see [**12**]), Proposition 4.1 can also be obtained under no analyticity assumptions (see [**4**, Proposition 4.1]). In this case the index n in (4.2) depends of the smoothness and decreasing properties of the potentials.

More precisely, for $v_0 : \mathbb{R}\to\mathbb{R}$ and $V : \mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ set

$$v_j(x_3) := x_3^j v_0^{(j)}(x_3), \quad V_j(\varrho,x_3) = x_3^j\frac{\partial^j V(\varrho,x_3)}{\partial x_3^j}, \quad j\in\mathbb{Z}_+, \tag{4.10}$$

provided that the derivative are well-defined.
For $\nu\geq 5$, $\nu\in\mathbb{Z}_+$, we assume the multipliers by v_j, $j=0,1$, are $-\frac{d^2}{dx_3^2}$-compact, and the multipliers by v_j, $j\leq\nu$, are $-\frac{d^2}{dx_3^2}$-bounded. Moreover, we assume that V_j, $j=0,...,\nu$ are bounded on $\mathbb{R}_+\times\mathbb{R}$ and V tends to 0 at infinity. Then if (2.2) is also satisfied, the finite limit $F_{q,m}(2bq+\lambda)$ exists for $q>m_-$ (see [**12**, Lemma 3.1]), and under *the Fermi Golden Rule* $\mathcal{F}_{q,m,\lambda}$ assumption, that is if

$$\operatorname{Im} F_{q,m}(2bq+\lambda) > 0, \tag{4.11}$$

holds, we have:

THEOREM 4.2. [**4**, Theorem 4.1] *Fix $m\in\mathbb{Z}$, $\nu\in\mathbb{Z}_+$, $\nu\geq 5$. Assume that* (2.2) *holds true and the above assumption on v_0 and V are fulfilled.*
Then if the Fermi Golden Rule $\mathcal{F}_{q,m,\lambda}$ *holds, there exists a function $g\in C_0^\infty(\mathbb{R};\mathbb{R})$ such that $g=1$ near $2bq+\lambda$, and*

$$\langle e^{-i(H_0^{(m)}+\varkappa V)t} g(H_0^{(m)}+\varkappa V)\Phi_{q,m},\Phi_{q,m}\rangle = a(\varkappa)e^{-i\lambda_{q,m}(\varkappa)t} + b(\varkappa,t), \quad t\geq 0, \tag{4.12}$$

where

$$\lambda_{q,m}(\varkappa) = 2bq+\lambda+\varkappa\langle V\Phi_{q,m},\Phi_{q,m}\rangle - \varkappa^2 F_{q,m}(2bq+\lambda) + o_{q,m,V}(\varkappa^2), \quad \varkappa\to 0. \tag{4.13}$$

In particular, we have $\operatorname{Im}\lambda_{q,m}(\varkappa)<0$ for $|\varkappa|$ small enough. Moreover, a and b satisfy the asymptotic estimates

$$|a(\varkappa)-1| = O(\varkappa^2),$$

$$|b(\varkappa,t)| = O(\varkappa^2|\ln|\varkappa||(1+t)^{-(\nu-5)}),$$

$$|b(\varkappa,t)| = O(\varkappa^2(1+t)^{-(\nu-6)}),$$

as $\varkappa\to 0$ uniformly with respect to $t\geq 0$.

Remarks: (i) The Mourre estimate is obtained taking the commutator with the self-adjoint operator:

$$A := \tilde{I}_\perp \otimes \mathcal{A}, \quad \mathcal{A} := -\frac{i}{2}\left(x_3\frac{d}{dx_3} + \frac{d}{dx_3}x_3\right).$$

(ii) For various magnetic quantum Hamiltonians, Mourre estimates can be found in [**20**, Chapter 3].
(iii) In [**4**, Theorem 4.1], the assumptions on V are given in terms of relative boundness. Here we have chosen to simplify these assumptions in order to have the same hypotheses for Theorem 4.2 and its corollary below.
(iv) There is an evident misprint in the last estimate of Theorem 4.1 in the paper version of [**4**]: the exponent at the r.h.s should be $-n+1$ instead of $-(n+1)$. The misprint has been fixed in the last electronic version of ArXiv Preprint 0710.0502.

These dynamical formulas can also be written for the operator $H_0 + \varkappa V$. For $m \in \mathbb{Z}$ and $q \geq m_-$ denote by $\tilde{\Phi}_{q,m} : \mathbb{R}^3 \to \mathbb{C}$ the function written in cylindrical coordinates (ϱ, ϕ, x_3) as $\tilde{\Phi}_{q,m}(\varrho,\phi,x_3) = (2\pi)^{-\frac{1}{2}} e^{im\phi}\Phi_{q,m}(\varrho, x_3)$. As consequence of Theorem 4.2, we have:

Corollary 4.3. Under the assumptions of Theorem 4.2, for every fixed $q \in \mathbb{Z}_+$, and each $m \in \{-q+1, \ldots, 0\} \cup \mathbb{N}$ with $\mathbb{N} := \{1, 2, \ldots\}$, we have

$$\langle e^{-i(H_0+\varkappa V)t} g(H_0 + \varkappa V)\tilde{\Phi}_{q,m}, \tilde{\Phi}_{q,m}\rangle_{L^2(\mathbb{R}^3)} = a(\varkappa)e^{-i\lambda_{q,m}(\varkappa)t} + b(\varkappa, t), \quad t \geq 0,$$

where g, $\lambda_{q,m}(\varkappa)$, a, and b are the same as in Theorem 4.2.

Remarks: (i) Generically $\operatorname{Im} F_{q,m}(2bq + \lambda) > 0$ for all $m \in \mathbb{Z}$, and $q > m_-$. It is justified in [**4**, Section 5] where are given certain classes of perturbations V compatible with the hypotheses of Theorems 3.2, 4.2, for which the Fermi Golden rule $\mathcal{F}_{q,m,\lambda}$ is valid for every $m \in \mathbb{Z}$ and $q > m_-$.
(ii) If $q \geq 1$, then Corollary 4.3 tells us that typically the eigenvalue $2bq + \lambda$ of the operator H_0, which has an infinite multiplicity, generates under the perturbation $\varkappa V$ infinitely many resonances with non-zero imaginary part. Note however that since we have no uniform estimates on the remainders in the asymptotics of $\lambda_{q,m}$ or of $w_{q,m}$, we are only able to prove that for any N for $\varkappa$ sufficiently small, at least N resonances have non-zero imaginary part. Moreover note that $2bq + \lambda$ is a discrete simple eigenvalue of the operator $H_0^{(-q)}$, and therefore the operator $H_0^{(-q)} + \varkappa V$ has a simple discrete eigenvalue provided that $|\varkappa|$ is small enough. Generically, this eigenvalue is an embedded eigenvalue for the operator $H_0 + \varkappa V$.
(iii) If $q = 0$, then λ is an isolated eigenvalue of infinite multiplicity for H_0. By Theorem 5.3 below, in this case there exists an infinite series of discrete eigenvalues of the operator $H_0 + V$ which accumulate at λ, provided that the perturbation V has a definite sign.

5. Singularities of the spectral shift function

We suppose that v_0 satisfies (2.1) and (2.2). We assume moreover that the perturbation $V : \mathbb{R}^3 \to \mathbb{R}$ satisfies (2.3) with $\delta_\perp > 2$ and $\delta_\parallel > 1$. Then the multiplier by V is a relatively trace-class perturbation of H_0. Hence, the spectral shift function (SSF) $\xi(\cdot; H_0 + V, H_0)$ satisfying the Lifshits-Krein trace formula

$$\mathrm{Tr}(f(H_0 + V) - f(H_0)) = \int_{\mathbb{R}} f'(E)\xi(E; H_0 + V, H_0)dE, \quad f \in C_0^\infty(\mathbb{R}),$$

and normalized by the condition $\xi(E; H_0 + V, H_0) = 0$ for $E < \inf \sigma(H_0 + V)$, is well-defined as an element of $L^1(\mathbb{R}; \langle E\rangle^{-2}dE)$ (see [**28**], [**27**]).
If $E < \inf \sigma(H_0)$, then the spectrum of $H_0 + V$ below E could be at most discrete, and for almost every $E < \inf \sigma(H_0)$ we have

$$\xi(E; H_0 + V, H_0) = -\mathrm{rank}\, \mathbf{1}_{]-\infty,E[}(H_0 + V). \tag{5.1}$$

On the other hand, for almost every $E \in \sigma_{\mathrm{ac}}(H_0) = [0, \infty[$, the SSF $\xi(E; H_0 + V, H_0)$ is related to the scattering determinant $\det S(E; H_0 + V, H_0)$ for the pair $(H_0 + V, H_0)$ by *the Birman-Krein* formula

$$\det S(E; H_0 + V, H_0) = e^{-2\pi i\xi(E;H_0+V,H_0)}$$

(see [**6**]).
Under the above assumptions, we know (see [**4**, Proposition 6.1], [**10**, Proposition 2.5]) that the singularities of the SSF $\xi(\cdot; H_0 + V, H_0)$ could be only in $\mathcal{Z} := 2b\mathbb{Z}_+ + \{0, \lambda\}$. Actually, $\xi(\cdot; H_0 + V, H_0)$ is bounded on every compact subset of $\mathbb{R} \setminus \mathcal{Z}$, and is continuous on $\mathbb{R} \setminus (\mathcal{Z} \cup \sigma_p(H_0 + V))$, where $\sigma_p(H_0 + V)$ denotes the set of the eigenvalues of the operator $H_0 + V$.

In this section we give a qualitative result saying that the resonances are poles of the SSF with estimates of the remainder and a quantitative result stating asymptotics behavior of the SSF near $2bq + \lambda$.

For ψ, the normalized real-valued eigenfunction of $H_{0,\parallel}$ associated to λ, put

$$U(X_\perp) := \int_{\mathbb{R}} V(X_\perp, x_3)\psi(x_3)^2 dx_3, \quad X_\perp \in \mathbb{R}^2,$$

and for V having analytic continuation in $\mathbb{R}^2 \times C(\epsilon, R_0)$ (see Section 3), we introduce

$$W(X_\perp) := \sup_{x_3 \in C(\epsilon,R_0)} |\langle x_3\rangle^{\delta_\parallel} V(X_\perp, x_3)|.$$

Clearly, there exists a constant $C > 0$ such that $|U(X_\perp)| \le CW(X_\perp)$.

If the potentials v_0 and V have analytic continuation in $C(\epsilon, R_0)$ as in Section 3, near the energies $2bq + \lambda$ we have the following result for $\xi = \xi(\cdot; H_0 + V, H_0)$.

Let $\tilde\Omega \subset\subset \Omega$ be open relatively compact subsets of $\mathbb{C} \setminus \{0\}$. We assume that these sets are independent of r and that $\tilde\Omega$ is simply connected. Also assume that the intersection between $\tilde\Omega$ and $\mathbb{R}$ is a non-empty interval I.

Theorem 5.1. [**26**, Theorem 2.2] *For $\tilde\Omega \subset\subset \Omega$ and I as above, there exists a function g holomorphic in Ω, such that for $\mu \in 2bq + \lambda + rI$, we have* $\xi'(\mu) =$

$$\sum_{\substack{w \in \mathrm{Res}(H) \cap 2bq+\lambda+r\Omega \\ \mathrm{Im}\, w \neq 0}} \frac{\mathrm{Im} w}{\pi|\mu - w|^2} - \sum_{w \in \mathrm{Res}(H)\cap 2bq+\lambda+rI} \delta(\mu - w) + \frac{1}{\pi r}\mathrm{Im}\, g'(\frac{\mu - 2bq - \lambda}{r}, r)$$

where $g(z, r)$ satisfies the estimate

$$g(z, r) = O\left(n_+(r/s, p_q W p_q)|\ln r| + \tilde n_1(r/s) + \tilde n_2(r/s)\right) = O(|\ln r| r^{-\frac{2}{\delta_\perp}}), \quad s > 0,$$

uniformly with respect to $0 < r < r_0$ *and* $z \in \tilde{\Omega}$, *with* $\tilde{n}_k$, $k = 1,\ 2$, *defined by the following trace norm (for* $k = 1$*) and Hilbert-Schmidt norm (for* $k = 2$*)*

$$\tilde{n}_k(r) := \left\| \frac{p_q W p_q}{r} \mathbf{1}_{[0,r]}(p_q W p_q) \right\|_k^k, \quad r > 0.$$

Let us recall that applying the well known results on the spectral asymptotics for compact Berezin-Toeplitz operators $p_q W p_q$ (see [**33**], [**36**] and [**4**, Corollary 1]), we have:

Proposition 5.2. (i) If $W \in L^\infty(\mathbb{R}^2)$ satisfy $W(X_\perp) \leq C\langle X_\perp\rangle^{-\alpha}$ for some $\alpha > 2$, then for each $q \in \mathbb{Z}_+$:

$$n_+(r, p_q W p_q) + \tilde{n}_k(r) = O(r^{-2/\alpha}).$$

(ii) If $W \in L^\infty(\mathbb{R}^2)$ satisfy

$$\limsup_{|X_\perp| \to \infty} \frac{\ln W(X_\perp)}{|\, X_\perp \,|^{2\beta}} < 0$$

for some $\beta > 0$ (with the convention $\ln(u) = -\infty$ if $u \leq 0$), then for each $q \in \mathbb{Z}_+$:

$$n_+(r, p_q W p_q) = O(\varphi_\beta(r)),$$

$$\tilde{n}_k(r) = o(\varphi_\beta(r)),$$

where, for $0 < r < \frac{1}{e}$,

$$\varphi_\beta(r) := \begin{cases} |\ln r|^{1/\beta} & \text{if} \quad 0 < \beta < 1, \\ |\ln r| & \text{if} \quad \beta = 1, \\ (\ln|\ln r|)^{-1}|\ln r| & \text{if} \quad 1 < \beta < \infty. \end{cases}$$

(iii) If $W \in L^\infty(\mathbb{R}^2)$ is compactly supported, then for each $q \in \mathbb{Z}_+$:

$$n_+(r, p_q W p_q) = O(\varphi_\infty(r)),$$

$$\tilde{n}_k(r) = o(\varphi_\infty(r)),$$

where, for $0 < r < \frac{1}{e}$, $\varphi_\infty(r) := (\ln|\ln r|)^{-1}|\ln r|$.

Since we have no lower bound on the distribution of resonances, the above result implies no quantitative behavior of the SSF. However, without analyticity assumptions, using a representation of the SSF due to Pushnitski [**32**], for $\xi(\cdot; H_0 \pm V, H_0)$, with $V \geq 0$, we have:

Theorem 5.3. [**4**, Theorem 4.1] *For each* $\varepsilon \in (0, 1)$,

$$n_+((1+\varepsilon)\eta; p_q U p_q) + O(1) \leq \pm\xi(2bq + \lambda \pm \eta; H_0 \pm V, H_0) \leq n_+((1-\varepsilon)\eta; p_q U p_q) + O(1),$$

$$\xi(2bq + \lambda \mp \eta; H_0 \pm V, H_0) = O(1),$$

as $\eta \downarrow 0$.

Applying the well known results on the spectral asymptotics for compact Berezin-Toeplitz operators $p_q U p_q$ (see [**33**], [**36**]), we obtain the following:

Corollary 5.4. (i) Suppose that $U \in C^1(\mathbb{R}^2)$, and

$$U(X_\perp) = u_0(X_\perp/|X_\perp|)|X_\perp|^{-\alpha}(1 + o(1)), \quad |X_\perp| \to \infty,$$

$$|\nabla U(X_\perp)| \leq C_1\langle X_\perp\rangle^{-\alpha-1}, \quad X_\perp \in \mathbb{R}^2,$$

where $\alpha > 2$, and u_0 is a continuous function on $\mathbb{S}^1$ which is non-negative and does not vanish identically. Then we have

$$\xi(2bq + \lambda \pm \eta; H_0 \pm V, H_0) = \pm\frac{b}{2\pi}\left|\left\{X_\perp \in \mathbb{R}^2 | U(X_\perp) > \eta\right\}\right| \, (1 + o(1)) =$$

$$\pm\eta^{-2/\alpha}\frac{b}{4\pi}\int_{\mathbb{S}^1} u_0(s)^{2/\alpha} ds\, (1 + o(1)), \quad \eta \downarrow 0,$$

where $|.|$ denotes the Lebesgue measure.
(ii) Let $U \in L^\infty(\mathbb{R}^2)$. Assume that

$$\ln U(X_\perp) = -\mu|X_\perp|^{2\beta}(1 + o(1)), \quad |X_\perp| \to \infty,$$

for some $\beta \in (0, \infty)$, $\mu \in (0, \infty)$. Then we have

$$\xi(2bq + \lambda \pm \eta; H_0 \pm V, H_0) = \pm c_\beta \varphi_\beta(\eta)\, (1 + o(1)), \quad \eta \downarrow 0, \quad \beta \in (0, \infty),$$

where

$$c_\beta = c_\beta(b, \mu) := \begin{cases} \frac{b}{2\mu^{1/\beta}} & \text{if} \quad 0 < \beta < 1, \\ \frac{1}{\ln(1+2\mu/b)} & \text{if} \quad \beta = 1, \\ \frac{\beta}{\beta - 1} & \text{if} \quad 1 < \beta < \infty. \end{cases}$$

(iii) Let $U \in L^\infty(\mathbb{R}^2)$. Assume that the support of U is compact, and that there exists a constant $C > 0$ such that $U \geq C$ on an open non-empty subset of $\mathbb{R}^2$. Then we have

$$\xi(2bq + \lambda \pm \eta; H_0 \pm V, H_0) = \pm\varphi_\infty(\eta)(1 + o(1)), \quad \eta \downarrow 0.$$

Remarks: (i) The threshold behavior of the SSF for various magnetic quantum Hamiltonians has been studied in [**18**] (see also [**34**], [**35**]), and recently in [**9**]. The singularities of the SSF described in Theorem 5.3 and Corollary 5.4 are of somewhat different nature since $2bq + \lambda$ is an infinite-multiplicity eigenvalue, and not a threshold in the continuous spectrum of the unperturbed operator.
(ii) As mentioned above, if $\lambda \in \sigma_{\text{disc}}(H_{0,\|})$, then λ is an isolated eigenvalue of H_0 of infinite multiplicity. According to (5.1), near this eigenvalue, $\xi(\cdot, H_0 + V, H_0)$ is a counting function (it is also given by the Pushnitski's representation of the SSF). Then Theorem 5.3 and Corollary 5.4 imply that the perturbed operator $H_0 - V$ (resp., $H_0 + V$) has an infinite sequence of discrete eigenvalues accumulating to λ from the left (resp., from the right).
(iii) It is conjectured that the singularities of the SSF $\xi(\cdot; H_0 \pm V, H_0)$ at the points $2bq + \lambda$, $q \in \mathbb{Z}_+$, are due to accumulation of resonances to these points. One simple motivation for this conjecture is the fact that if V is axisymmetric, then the eigenvalues of the operators $p_q U p_q$, $q \in \mathbb{Z}_+$, appearing in Theorem 5.3 are equal exactly to the quantities $\langle V\Phi_{q,m}, \Phi_{q,m}\rangle_{L^2(\mathbb{R}_+\times\mathbb{R};\varrho d\varrho dx_3)}$, $m \geq -q$, occurring in (3.3) and (4.13). We leave for a future work the detailed analysis of the relation between the singularities of the SSF at the points $2bq + \lambda$ and the eventual accumulation of resonances at these points.

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