Semiclassical Resonances of Schrödinger operators as zeroes of regularized determinants

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Abstract

We prove that the resonances of long range perturbations of the (semiclassical) Laplacian are the zeroes of natural perturbation determinants. We more precisely obtain factorizations of these determinants of the form $\prod_{w=\text{resonances}}(z-w)\exp(\varphi_p(z,h))$ and give semiclassical bounds on $\partial_z \varphi_p$ as well as a representation of Koplienko's regularized spectral shift function. Here the index $p \geq 1$ depends on the decay rate at infinity of the perturbation.

1 Introduction and results

One of the main purposes of Scattering Theory is the study of selfadjoint operators with absolutely continuous (AC) spectrum. This corresponds physically to extended or delocalized states, by opposition to the localized or confined states which give rise to discrete spectrum. A typical mathematical example of confining system is given by the Laplacian Δ_g (or more general elliptic operators) on a compact riemannian manifold: here, the states (ie the eigenfunctions) are clearly localized by the compactness assumption and the spectrum is a non decreasing sequence of eigenvalues tending to infinity.

Quite naively, Δ_g can be viewed as an infinite dimensional analogue of an hermitian matrix $A = A^*$ on $\mathbb{C}^N$. In that case, the spectrum of A is given by the roots of the characteristic polynomial $\mathrm{Det}(A-z)$. It is elementary to check that, for z in the upper half plane,

$$\mathrm{Det}(A-z) = \exp\left(\partial_s \mathrm{tr}(A-z)^s_{|s=0}\right), \tag{1.1}$$

so $\mathrm{Det}(A-z)$ can be defined as the analytic continuation (with respect to z) of the right hand side of (1.1) to the complex plane. This is an elementary version of the classical definition of determinants via a Zeta function (here $\mathrm{tr}(A-z)^s$), which is used in infinite dimension, typically for elliptic operators on compact manifolds as initially introduced by Ray and Singer [25]. Avoiding any technical point at this stage, we simply recall that such a definition is build from an analytic

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continuation of $s \mapsto \mathrm{tr}(\Delta_g - z)^s$, using that $(\Delta_g - z)^s$ is trace class at least for $\mathrm{Re}(s)$ sufficiently negative, which uses crucially the discreteness of the spectrum of Δ_g.

In this spirit, the first goal of this paper is to realize the *resonances* of Schrödinger operators with AC spectrum, as the zeroes of a determinant defined via a certain Zeta function.

Let us informally recall that, if $H = H_0 + V$ with $H_0 = -\Delta$ on $\mathbb{R}^d$ and V a perturbation tending to 0 at infinity, the resonances are the natural discrete spectral datum of the problem. They can be defined as the poles of some meromorphic continuation of the resolvent of H and thus can be considered as the analogues of the eigenvalues for confining systems. Notice however that, apart from possible real eigenvalues, resonances usually refer to complex poles.

The problem of defining resonances as zeroes of determinants is very natural and has already been considered by several authors, in connection with the important question of their distribution [36, 33, 11, 12, 23, 24, 28, 16, 5, 4, 15]. In these references, various determinants are used such as absolute determinants or relative determinants, determinants of the scattering matrices. In this paper we will basically study relative determinants. The corresponding construction is fairly well known in the *relatively trace class* situation, ie when $(H - z)^{-k} - (H_0 - z)^{-k}$ is of trace class, that is when V decays sufficiently fast at infinity and we refer to [22] for a nice review on this case. The main point in this paper is to consider determinants for slowly decreasing perturbations of long range type. We first recall some well known facts.

When $V = V(x)$ is a potential (or possibly a first order differential operator), a natural candidate for our purpose can be the so called *perturbation determinant* (see [35]) defined by

$$D_p(z) = D_p(H_0, H; z) := \mathrm{Det}_p\left((H - z)(H_0 - z)^{-1}\right) = \mathrm{Det}_p\left(I + V(H_0 - z)^{-1}\right), \tag{1.2}$$

where Det_p is the Fredholm determinant of order p which is defined as follows (see [14, 35] for more details). Given a separable Hilbert space (here $L^2(\mathbb{R}^d)$), one defines the Schatten class of order $p \geq 1$ as the space $\mathbf{S}_p$ of compact operators K whose singular numbers[1] form a sequence in $l^p(\mathbb{N})$ (for $p = \infty$, $\mathbf{S}_\infty$ is the class of compact operators). The most classical examples are $\mathbf{S}_1$, the trace class, and $\mathbf{S}_2$, the Hilbert-Schmidt class. Then, if $K \in \mathbf{S}_p$, the spectrum of K is also in $l^p(\mathbb{N})$ and, if p is an integer, one sets

$$\mathrm{Det}_p(I + K) := \prod_{k\geq 0}(1 + \lambda_k) \exp\left(\sum_{j=1}^{p-1} \frac{(-1)^j}{j}\lambda_k^j\right), \qquad (\lambda_k)_{k\geq 0} = \mathrm{spec}(K), \tag{1.3}$$

where the product is convergent since the Weierstrass function on the right hand side is $1 + \mathcal{O}(\lambda_k^p)$. If V tends to zero with rate $\rho > 0$, ie

$$|V(x)| \leq C\langle x\rangle^{-\rho}, \tag{1.4}$$

it is classical that

$$V(H_0 - z)^{-1} \in \mathbf{S}_p \quad \text{if} \quad \min(2, \rho) > d/p. \tag{1.5}$$

For instance, in dimension $d = 1$ with V of short range, ie $\rho > 1$, $V(H_0 - z)^{-1}$ is trace class and one can define $D_1(H_0, H; z)$, which is essentially the framework of [11, 28]. The Fredholm determinant of order 1 is a rather popular tool for several reasons. For instance, it satisfies the formula

$$\mathrm{Det}_1\left((I + K_1)(I + K_2)\right) = \mathrm{Det}_1\left(I + K_1\right)\mathrm{Det}_1\left(I + K_2\right),$$

[1] ie the spectrum of $|K| := (K^*K)^{1/2}$

as in finite dimension. This formula doesn't hold for $p \geq 2$ (one needs then to add correction factors). Also, formula (1.3) shows that for $p = 1$, we have a 'pure' factorization of the determinant of $I + K$ by its eigenvalues $1 + \lambda_k$. It is nevertheless necessary to consider Fredholm determinants of higher order. Indeed, even for compactly supported potentials, $V(H_0 - z)^{-1}$ is not of trace class in general when $d \geq 2$ (basically $V(H_0 - z)^{-k} \in \mathbf{S}_1$ if $k > d/2$ and $\rho > d$). Furthermore, even for $d = 1$, one also needs to consider $p \neq 1$ to deal with long range potentials, ie when $0 < \rho \leq 1$.

There is in addition a major drawback in the definition (1.2): it is restricted to relatively compact perturbations. In particular, we can not consider V that are second order differential operators.

One can overcome this difficulty by defining relative determinants via relative Zeta functions. This construction was first introduced for relatively trace class perturbations, ie basically for perturbations with coefficients decaying like (1.4) with $\rho > d$ (see [22] for references) and was then extended in [6, 7] to general $\rho > 0$, using an original idea of Koplienko [19]. We recall this construction. Let V be a differential operator of the form

$$V = \sum_{|\alpha| \leq 2} v_\alpha(x) D^\alpha, \qquad D = -i\partial_x,$$

symmetric on $L^2(\mathbb{R}^d)$ such that $-\Delta + V$ is uniformly elliptic, whose coefficients are smooth and satisfy

$$|\partial^\beta v_\alpha(x)| \leq C_\beta \langle x \rangle^{-\rho}, \qquad x \in \mathbb{R}^d, \tag{1.6}$$

for some $\rho > 0$. We shall further on consider semiclassical operators, ie replace D by hD with $h \in (0, 1]$, and all the results quoted here for $h = 1$ will still hold. One defines the so called *regularized spectral shift function* $\xi_p \in \mathcal{S}'(\mathbb{R})$ (see [6, 7]) as the unique distribution vanishing near $-\infty$ such that

$$\langle \xi_p', f \rangle = \operatorname{tr} \left(f(H_0 + V) - \sum_{j=0}^{p-1} \frac{1}{j!} \frac{d^j}{d\varepsilon^j} f(H_0 + \varepsilon V)_{|\varepsilon=0} \right), \tag{1.7}$$

for all Schwartz function f, or more generally $f \in S^{-k}(\mathbb{R})$ (ie $\partial_\lambda^j f(\lambda) = \mathcal{O}(\langle \lambda \rangle^{-k-j})$) with k large enough. For $p = 1$, we recover the well known Kreĭn spectral shift function. For $p \geq 2$, this trace regularization by Taylor's formula is due to Koplienko [19]. We also refer to the recent paper [13] for a general introduction to Koplienko's regularized spectral shift function in connection with determinants. See also [20, 27, 2] in the one dimensional case.

Denoting by $(\cdot - z)^{-s}$ the map $\lambda \mapsto (\lambda - z)^{-s}$, it is shown in [7] that the *regularized Zeta function*,

$$\zeta_p(s, z) := \langle \xi_p', (\cdot - z)^{-s} \rangle, \qquad \operatorname{Im}(z) > 0, \ \operatorname{Re}(s) \gg 1$$

has a meromorphic continuation, with respect to s, which is regular at $s = 0$. This allows to define

$$D_p^\zeta(z) = D_p^\zeta(H_0, H_0 + V; z) := \exp\left(-\partial_s \zeta_p(s, z)_{|s=0}\right),$$

which is holomorphic for $\operatorname{Im}(z) > 0$. The notation D_p^ζ is justified by the fact that

$$D_p^\zeta(H_0, H_0 + V; z) = D_p(H_0, H_0 + V; z), \tag{1.8}$$

when V is a potential (see [7]). In other words, the definitions of the perturbation determinant by Fredholm determinants and regularized Zeta functions coincide if they both make sense. In addition, one proved in [7] that, in the distributions sense,

$$\frac{d}{d\lambda}\arg D_p^{\zeta}(\lambda + i\epsilon) \to -\pi \xi_p'(\lambda), \qquad \epsilon \downarrow 0. \tag{1.9}$$

For this reason, ξ_p is also called *generalized scattering phase* of order p. The above formula is well known for ξ_1 and was initially proved in [17] (see also [3]). See also [19, 20, 13] for $p \geq 2$. Note the parallel with the finite dimensional analogy of the very beginning of this paper: for an hermitian matrix A on $\mathbb{C}^N$ with spectrum $\lambda_1, \dots, \lambda_N$, one easily sees that

$$\frac{d}{d\lambda} \arg \mathrm{Det}(A - \lambda - i\epsilon) \to -\pi \sum_{k=1}^{N} \delta(\lambda - \lambda_k), \qquad \epsilon \downarrow 0,$$

where the right hand side is $-\pi$ times the derivative of the eigenvalue counting function, ie the analogue of the spectral shift function for a discrete spectrum. This also suggests that if the resonances of $H_0 + V$ are indeed the zeroes of (a suitable meromorphic continuation of) $D_p^{\zeta}(z)$, the derivative of $\xi_p(\lambda)$ should involve a function (and/or a measure) with singularities carried by the resonances. Such a result is sometimes referred to as Breit-Weigner formula and is already known for $p = 1$ (see [8] and the references therein). In this paper, we shall prove it for general $p \geq 1$. We will also give semiclassical bounds.

Throughout this paper, we shall use the definition of resonances and some related results given in [30] (see also [31]). The definition is basically taken from the original paper by Sjöstrand-Zworski [32] and the other useful results of [30] come from a simplification of the proof of the trace formula [29]. Before stating the results, we fix the notation and some definitions.

For $0 < \theta_0 < \pi$, $R_0 > 0$ and $\epsilon_0 > 0$, we set

$$\Sigma(\theta_0, R_0, \epsilon_0) := \{ r\omega \ ; \ \omega \in \mathbb{C}^d, \ \mathrm{dist}(\omega, \mathbb{S}^{d-1}) < \epsilon_0, \ r \in e^{i[0,\theta_0]}(R_0, +\infty) \}.$$

Definition 1.1. *Let $\rho > 0$. We define $\mathcal{C}_\rho(\theta_0, R_0, \epsilon_0)$ as the set of smooth functions v on $\mathbb{R}^d$ which have an analytic extension to $\Sigma(\theta_0, R_0, \epsilon_0)$ such that*

$$|v(x)| \leq C \langle x \rangle^{-\rho}, \qquad x \in \Sigma(\theta_0, R_0, \epsilon_0). \tag{1.10}$$

Here $\langle x \rangle = (1 + |x|^2)^{1/2}$. A family $(v_\iota)_{\iota \in I}$ is said to be bounded in $\mathcal{C}_\rho(\theta_0, R_0, \epsilon_0)$ if it is bounded in $C^\infty(\mathbb{R}^d)$ and if the constant C in (1.10) is uniform with respect to $\iota \in I$.

We consider perturbations of $H_0(h) = -h^2\Delta$ by second order differential operators of the form

$$V(h) = \sum_{|\alpha| \leq 2} v_\alpha(x, h)(hD)^\alpha, \tag{1.11}$$

depending on a small parameter $h > 0$. We assume that, for some $h_0 > 0$, the coefficients are such that, for all $|\alpha| \leq 2$,

$$(v_\alpha(., h))_{h \in (0, h_0]} \ \text{ is bounded in } \ \mathcal{C}_\rho(\theta_0, R_0, \epsilon_0), \tag{1.12}$$

and such that, for some $c > 0$,

$$v_\alpha(., h) \text{ doesn't depend on } h \text{ if } |\alpha| = 2, \tag{1.13}$$

$$|\xi|^2 + \sum_{|\alpha| = 2} v_\alpha(x)\xi^\alpha \geq c|\xi|^2, \qquad x \in \mathbb{R}^d, \ \xi \in \mathbb{R}^d. \tag{1.14}$$

We also assume that

$$V(h) \text{ is symmetric on } C_0^\infty(\mathbb{R}^d) . \tag{1.15}$$

These assumptions imply that $H_0(h) + V(h)$ is selfadjoint on $L^2(\mathbb{R}^d)$ with domain $H^2(\mathbb{R}^d)$ the usual Sobolev space.

The assumption (1.12) implies that the coefficients of V must be smooth on $\mathbb{R}^d$. This is mostly for convenience, to simplify the analysis, but we expect that some local singularities could be considered as well, using for instance the black-box formalism of Sjöstrand-Zworski [32]. Notice however that, apart from the special case $p = 1$, we have to consider operators of the form $H_0 + \varepsilon V$ hence with H_0 and V defined on the same space. In particular, the generalization of the present results to perturbations by obstacles (+ long range metrics or potentials) would require a modified approach.

Notation. We shall mostly write H_0, V for $H_0(h)$ and $V(h)$. When no confusion will be possible, V will also denote the family of operators $(V(h))_{0<h\leq h_1}$. Such a family will sometimes be denoted by $(V(h))_{h\ll 1}$ to mean that it is of the form $(V(h))_{0<h\leq h_1}$ for some $h_1 > 0$.

It is convenient to summarize the above properties in the following definition.

Definition 1.2. *We say that $V = (V(h))_{h\in(0,h_1]}$ belongs to $\mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$ if it satisfies (1.12), (1.13), (1.14) and (1.15). A family $(V_\iota)_{\iota\in I} = (V_\iota(h))_{h\in(0,h_1],\iota\in I}$ is bounded in $\mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$ if the families of coefficients $(v_{\alpha,\iota}(., h))_{h\in(0,h_1],\iota\in I}$ are bounded in $\mathcal{C}_\rho(\theta_0, R_0, \epsilon_0)$ for all α and if the constant c in (1.14) can be chosen independently of ι.*

Remark. To state this definition, we have explicitly fixed the range of h, namely $(0, h_1]$, but we will also freely write that $V = (V(h))_{h\ll 1}$ belongs to $\mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$ to mean that, for some h_1 small enough, $(V(h))_{h\in(0,h_1]} \in \mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$. A similar slight abuse of notation will be used for families $(V_\iota)_{\iota\in I} = (V_\iota(h))_{h\ll 1,\iota\in I}$.

Obviously, any $v \in \mathcal{C}_\rho(\theta_0, R_0, \epsilon_0)$ satisfies (1.6). Therefore, using the results of [6], we can define the generalized scattering phase $\xi_p(., h)$ associated to $-h^2\Delta$ and $V(h)$, provided

$$p\rho > d.$$

We can then define the regularized Zeta function $\zeta_p(s, z, h)$ by

$$\zeta_p(s, z, h) = \langle \xi_p'(., h), (. - z)^{-s} \rangle, \qquad \text{Im}(z) > 0, \quad \text{Re}(s) \gg 1.$$

According to [7], $\zeta_p(s, z, h)$ can be continued analytically at $s = 0$ and we can define the relative determinant of order p

$$D_p^\zeta(z, h) := \exp\left(-\partial_s \zeta_p(s, z, h)_{|s=0}\right), \qquad \text{Im}(z) > 0. \tag{1.16}$$

We note that, for more precise purposes, the analytic continuation (in s) of the Zeta function will be reviewed in Section 2.

The determinant $D_p^\zeta(z, h)$ is our candidate to become the 'characteristic polynomial' of the resonances of $H_0 + V$.

We now briefly recall the definition of resonances of [30, 32] (see Section 4 of the present paper for precise statements). Let $\theta_0 \in (0, \pi)$, $\epsilon > 0$ such that $\epsilon < 2\pi - 2\theta_0$ and consider a relatively compact open subset

$$\Omega \Subset e^{i(-2\theta_0,\epsilon)}(0, +\infty) \tag{1.17}$$

which is simply connected and such that

$$\Omega \cap (0, +\infty) \text{ is a non empty interval.} \tag{1.18}$$

The resonances of $H_0 + V$ in Ω are by definition the eigenvalues in $e^{-i[0,2\theta_0)(0,+\infty)} \cap \Omega$ of some non selfadjoint operator $H_0(\theta_0) + V(\theta_0)$ obtained by analytic distortion. We denote by

$$\mathrm{Res}(H_0 + V, \Omega) := \text{set of resonances of } H_0 + V \ \text{ in } \ \Omega,$$

which is a finite set depending on h. We recall here that, for the operators considered in this paper, we have the following Weyl upper bound for the number of resonances in Ω (see for instance [30]),

$$\#\mathrm{Res}(H_0 + V, \Omega) \leq Ch^{-d}, \qquad h \ll 1. \tag{1.19}$$

Note that they are counted with *multiplicity* and that the multiplicity of each resonance is well defined as the rank of a certain projector (see Section 4) which is non orthogonal in general.

Our first result is the following.

Theorem 1.3. *Let $\rho > 0$, $V \in \mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$ and $p > d/\rho$. Then, for all $h \ll 1$, $D_p^\zeta(z, h)$ has an analytic continuation from*

$$\Omega^+ := \Omega \cap e^{i(0,\epsilon)}(0, +\infty) \tag{1.20}$$

to Ω, of the form

$$D_p^\zeta(z, h) = \prod_{w \in \mathrm{Res}(H_0+V,\Omega)} (z - w) \times \exp(\varphi_p(z, h)), \qquad z \in \Omega,$$

where each resonance is repeated according to its multiplicity and the function $z \mapsto \varphi_p(z, h)$ is holomorphic on Ω.

The proof is given in subsection 5.1.

Notice that the function $\varphi_p(z, h)$ is uniquely defined up to a multiple of $2i\pi$ of the form $2ik(h)\pi$. By (1.9), an immediate consequence of Theorem 1.3 is the following Breit-Wigner formula.

Corollary 1.4. *With the notation and assumptions of Theorem 1.3, for all $h \ll 1$ we have*

$$\xi_p'(\lambda, h) = \sum_{w \in \mathrm{Res}(H_0+V,\Omega)\cap\mathbb{R}} \delta(\lambda - w) - \sum_{w \in \mathrm{Res}(H_0+V,\Omega)\setminus\mathbb{R}} \frac{\mathrm{Im}(w)}{\pi|\lambda - w|^2} - \frac{1}{\pi}\mathrm{Im}(\partial_z \varphi_p(\lambda, h)),$$

in the distributions sense on $\Omega \cap (0, +\infty)$.

Here λ is restricted to $(0, +\infty)$, but it is well known that

$$\xi_p'(\lambda, h) = \sum_{w \in \sigma^-(H_0+V)} \delta(\lambda - w), \qquad \lambda \in \Omega \cap (-\infty, 0),$$

where $\sigma^-(H_0 + V) = \sigma(H_0 + V) \cap (-\infty, 0)$ is the set of negative eigenvalues of $H_0 + V$ (see [6] for instance but this is anyway an elementary consequence of the definition (1.7)).

This corollary becomes of real interest if one has estimates on $\partial_z \varphi_p$. This is the purpose of the next results.

Theorem 1.5. *Assume that $V \in \mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$ with $\rho > d/p$ and*

$$p = 1 \qquad or \qquad p = 2.$$

Then any φ_p as in Theorem 1.3 satisfies, for any compact subset $W \Subset \Omega$,

$$|\partial_z \varphi_p(z,h)| \leq C_W h^{-d}, \qquad h \ll 1, \ z \in W. \tag{1.21}$$

This theorem is proved in subsection 5.2. In Section 7, we also prove that a similar result holds for $p \geq 3$ if we assume that V is dilation analytic. However Theorem 1.5 is sharp in general for non globally analytic perturbations as is shown by Theorem 1.6 below.

Fix first

$$W = \{z = re^{-i\theta} \in \mathbb{C} \ ; \ 1 \leq r \leq 4, \ 0 \leq \theta \leq \pi\},$$

and observe that, for $\pi/2 < \theta_0 < \pi$ and all $\epsilon > 0$ small enough, W is clearly contained in a simply connected open set Ω satisfying (1.17) and (1.18). This neighborhood Ω can be chosen close enough to W so that we can define a determination of the square root $z^{1/2}$, with $(re^{-i\theta})^{1/2} = r^{1/2}e^{-i\theta/2}$ on W hence so that

$$\mathrm{Im}(z^{1/2}) \leq 0 \quad \text{on} \quad W.$$

Theorem 1.6. *In dimension $d = 1$, for all $V \in C_0^\infty(\mathbb{R}, \mathbb{R})$, $V \neq 0$, we can find $\delta > 0$ such that,*

$$\limsup_{h \to 0} \sup_{z \in W} |he^{\delta \mathrm{Im}(z^{1/2})/h} \partial_z \varphi_3(z,h)| = +\infty. \tag{1.22}$$

In particular, $|h\partial_z \varphi_3(z,h)|$ can not be bounded on W uniformly with respect to h.

The proof of this theorem is given in Section 6.

We next give a general bound on $\partial_z \varphi_p$ involving the distorted operator $H_0(\theta)$ defined in Section 4 and the semiclassical Sobolev space defined by (3.1). We recall that $H_0(\theta) - z$ is invertible for all $h \ll 1$ and $z \in \Omega$.

Theorem 1.7. *Under the assumption of Theorem 1.3, there exists $N > 0$ such that, for all $W \Subset \Omega$,*

$$|\partial_z \varphi_p(z,h)| \leq C_W h^{-d} \sup_{Z \in \Omega} \Big(1 + ||(H_0(\theta_0) - Z)^{-1}||_{L^2 \to H^{2,0}_{\mathrm{sc}}}\Big)^N, \qquad h \ll 1, \ z \in W.$$

In general, $||(H_0(\theta_0) - Z)^{-1}||_{L^2 \to H^{2,0}_{\mathrm{sc}}}$ is of order $\mathcal{O}(e^{Ch^{-1}})$, locally uniformly with respect to Z (see Proposition 4.7). However, Theorem 1.5 shows that the corresponding exponential upper bound on $\partial_z \varphi_p$ can be much improved if $p = 1, 2$ (and $p \geq 3$ if V is dilation analytic, see Section 7). Note also that Theorem 1.6 can be interpreted as a weak exponential lower bound.

Theorem 1.7 is proved in subsection 5.1.

To motivate the analysis developed in the next sections, let us already show that most of the results above will essentially be reduced to the study of $\zeta_p(k, z, h)$, for some k large enough.

The basic strategy is the following. Using (1.16), we have

$$\partial_z^k \log D_p^\zeta(z,h) = -\partial_z^k \partial_s \zeta_p(s,z,h)_{|s=0}, \qquad k \geq 1, \quad z \in \Omega^+. \tag{1.23}$$

Here and below $\partial_z^k \log g$ stands for $\partial_z^{k-1}(g'/g)$, for any non vanishing holomorphic function g. On the other hand, at least for $k > d/2$, we also have

$$\partial_z^k \partial_s \zeta_p(s,z,h)_{|s=0} = (k-1)! \, \zeta_p(k,z,h), \tag{1.24}$$

as will be proved in Section 2 (see (2.10) and the discussion thereafter) and is formally a consequence of the identity,

$$\partial_z^k \partial_s (\lambda - z)^{-s}_{|s=0} = (k-1)!(\lambda - z)^{-k}. \tag{1.25}$$

Fix then $z_0 \in \Omega^+$. In Section 2 (see Proposition 2.1) we shall also prove that, for all $\nu \geq 0$,

$$|\partial_z^{\nu+1} \partial_s \zeta_p(s, z_0, h)_{|s=0}| \leq C h^{-d}, \qquad h \ll 1. \tag{1.26}$$

In addition, by (1.19), we have, for all $\nu \geq 0$,

$$\sum_{w \in \mathrm{Res}(H_0+V,\Omega)} |z_0 - w|^{-\nu-1} \leq C h^{-d}, \qquad h \ll 1, \tag{1.27}$$

since $|z_0 - w| \gtrsim 1$. These are the essential tools of the reduction given by Proposition 1.8 below. Before stating it and to consider the different possible estimates for $\partial_z \varphi_p$, we introduce the following. Let

$$\mathcal{H}_{\mathrm{hol}}(\Omega, h_1) := \{(\phi(.,h))_{h\in(0,h_1]}\},$$

be the space of h-dependent families of holomorphic functions on Ω. Let $\mathcal{H}(\Omega, h_1)$ be a subspace of $\mathcal{H}_{\mathrm{hol}}(\Omega, h_1)$ such that

$$(h^{-d})_{h\in(0,h_1]} \in \mathcal{H}(\Omega, h_1), \tag{1.28}$$

and which is stable by taking the primitive, ie such that for all $(\phi(.,h))_{h\in(0,h_1]} \in \mathcal{H}_{\mathrm{hol}}(\Omega, h_1)$ and some $z_0 \in \Omega$,

$$(\phi'(.,h))_{h\in(0,h_1]} \in \mathcal{H}(\Omega, h_1) \qquad \Rightarrow \qquad (\phi(.,h) - \phi(z_0,h))_{h\in(0,h_1]} \in \mathcal{H}(\Omega, h_1). \tag{1.29}$$

Note that, if z_0 is such that $|\phi(z_0,h)| \lesssim h^{-d}$, and by using (1.28), one can replace (1.29) by $(\phi'(.,h))_{h\in(0,h_1]} \in \mathcal{H}(\Omega, h_1) \Rightarrow (\phi(.,h))_{h\in(0,h_1]} \in \mathcal{H}(\Omega, h_1)$.

Example. The space $\mathcal{H}_{\mathrm{hol}}(\Omega, h_1)$ itself or the subspace of functions such that, for all $W \Subset \Omega$, $|\phi(z,h)| \leq C_W h^{-d}$ for all $z \in W$ and $h \in (0, h_1]$ satisfy (1.28) and (1.29).

Proposition 1.8. *If we can find $h_1 > 0$ small enough, $k \geq 1$ and $\phi_p \in \mathcal{H}(\Omega, h_1)$ such that*

$$\zeta_p(k, z, h) = \sum_{w \in \mathrm{Res}(H_0+V,\Omega)} \frac{1}{(w-z)^k} + \phi_p(z,h), \qquad z \in \Omega^+,\ h \in (0, h_1], \tag{1.30}$$

then Theorem 1.3 holds true with φ_p such that $\partial_z \varphi_p \in \mathcal{H}(\Omega, h_1)$.

Proof. Setting for simplicity

$$D = D_p^\zeta(z,h), \qquad F = \prod_{w \in \mathrm{Res}(\mathrm{H_0+V},\Omega)} (z - w),$$

which are holomorphic and don't vanish on Ω^+, (1.23), (1.24) and (1.30) give

$$\partial_z^{k-1} \left(\frac{\partial_z D}{D} - \frac{\partial_z F}{F} \right) = -(k-1)!\phi_p, \qquad \text{on } \Omega^+. \tag{1.31}$$

If $k=1$, we therefore obtain

$$\frac{\partial_z D}{D} - \frac{\partial_z F}{F} \in \mathcal{H}(\Omega, h_1), \tag{1.32}$$

which implies easily the result. If $k-1 \geq 1$, we denote by Φ_p the $(k-1)$-th primitive of $-(k-1)!\phi_p$ (ie $\partial_z^{k-1}\Phi_p = -(k-1)!\phi_p$) such that

$$\partial_z^\nu \Phi_p(z_0, h) = \partial_z^\nu \left(\frac{\partial_z D}{D} - \frac{\partial_z F}{F} \right)(z_0, h), \qquad 0 \leq \nu \leq k-2,$$

where z_0 is chosen arbitrarily in Ω^+. The existence and uniqueness of Φ_p is guaranteed by the simple connectedness of Ω. By (1.26) and (1.27), we have

$$|\partial_z^\nu \Phi_p(z_0, h)| \leq C h^{-d},$$

and this implies, together with (1.28) and (1.29), that

$$\phi_p \in \mathcal{H}(\Omega, h_1) \qquad \Rightarrow \qquad \Phi_p \in \mathcal{H}(\Omega, h_1).$$

Thus (1.31) imply that (1.32) holds also if $k-1 \geq 1$ and we get the result. $\square$

2 The Zeta function

In this subsection, we review the construction of the meromorphic continuation of $s \mapsto \zeta_p(s, z, h)$. Although the latter was shown in [7] (for fixed h), we need to review the main lines of the proof in order to prove the identity (1.24) and the estimate (1.26).

We start with general considerations. Using the principal determination of log on $\mathbb{C} \setminus (-\infty, 0]$, we can define $(\lambda - z)^{-s}$ for $s \in \mathbb{C}$, $\lambda \in \mathbb{R}$ and $z \in \mathbb{C} \setminus [\lambda, +\infty)$. One can then check that

$$(\lambda - z)^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} e^{-t(\lambda - z)} dt, \qquad \mathrm{Re}(z) < \lambda, \ \mathrm{Re}(s) > 0, \tag{2.1}$$

since both sides are holomorphic with respect to z and the equality holds for $z \in (-\infty, \lambda)$ by an elementary change of variables in the definition of $\Gamma(s)$. Next, if $u \in \mathcal{S}'(\mathbb{R})$ is a temperate distribution such that, for some $\lambda_0 > 0$,

$$\mathrm{supp}(u) \subset [\lambda_0, +\infty) \tag{2.2}$$

we can consider its Laplace transform $Lu(t) := \langle u, e^{-t\cdot} \rangle$ ($e^{-t\cdot}$ stands for the map $\lambda \mapsto e^{-t\lambda}$), and, for all $\delta > 0$,

$$|Lu(t)| \leq C_\delta e^{-t(\lambda_0 - \delta)}, \qquad t > 0. \tag{2.3}$$

Furthermore, using that $|\langle u, f \rangle| \leq C \sup_{j+k \leq N} \sup_{\lambda \in \mathbb{R}} |\lambda^j \partial_\lambda^k f(\lambda)|$ for some N and all $f \in \mathcal{S}(\mathbb{R})$, $\langle u, f \rangle$ is still well defined if $f(\lambda) = (\lambda - z)^{-s}$ with $\mathrm{Re}(s) > s_0$ large enough and $\mathrm{Re}(z) < \lambda_0$. If in addition, we know that

$$|Lu(t)| \leq C t^{-d/2}, \qquad t \in (0, 1] \tag{2.4}$$

then, one has

$$\langle u, (\cdot - z)^{-s} \rangle = \frac{1}{\Gamma(s)} \int_0^{+\infty} Lu(t) e^{tz} t^{s-1} dt, \qquad \mathrm{Re}(z) < \lambda_0, \ \mathrm{Re}(s) > \max(s_0, d/2).$$

Note that the power $d/2$ could actually be any arbitrary real number but, in the applications below, we shall need only to consider this case. If (2.4) is replaced by the stronger assumption that there is an asymptotic expansion at $t = 0$, namely that, for all $J > 0$,

$$Lu(t) = \sum_{j<J} a_j t^{-d/2+j/2} + t^{-d/2+J/2} b_J(t), \qquad |b_J(t)| \le C, \quad t \in (0,1], \tag{2.5}$$

then we can write, for $\mathrm{Re}(z) < \lambda_0$ and $\mathrm{Re}(s) > \max(s_0, d/2)$,

$$\langle u, (\cdot - z)^{-s} \rangle = I(s,z) + II_J(s,z) + III_J(s,z), \tag{2.6}$$

with

$$\begin{aligned} I(s,z) &= \frac{1}{\Gamma(s)} \int_1^\infty Lu(t) e^{tz} t^{s-1} dt, \\ II_J(s,z) &= \frac{1}{\Gamma(s)} \int_0^1 b_J(t) e^{tz} t^{-d/2+J/2+s-1} dt, \\ III_J(s,z) &= \frac{1}{\Gamma(s)} \sum_{j<J} a_j \int_0^1 e^{tz} t^{-d/2+j/2+s-1} dt. \end{aligned}$$

By choosing $J > d$, both I and II_J are holomorphic close to $s = 0$. Thus, using the fact that $d\Gamma^{-1}(s)/ds = 1$ at $s = 0$ and that $\Gamma^{-1}(s)$ vanishes at 0 one sees that, for all $k \ge 1$,

$$\partial_z^k \partial_s F(s,z)_{|s=0} = \Gamma(k) F(k,z) = (k-1)! F(k,z), \qquad \mathrm{Re}(z) < \lambda_0, \tag{2.7}$$

for $F = I$ and $F = II_J$. The term III_J can be computed explicitly, namely,

$$\Gamma(s) \times III_J(s,z) = \sum_{j=0}^{J-1} a_j \sum_{l \ge 0} \frac{z^l}{l!} \frac{1}{s + j/2 + l - d/2}. \tag{2.8}$$

At $s = 0$, there is at most a simple pole, which corresponds to the terms where $j/2 + l - d/2 = 0$. Thus $III_J(s,z)$ is regular at $s = 0$. This shows the existence of a meromorphic continuation to the complex plane for

$$s \mapsto \langle u, (\cdot - z)^{-s} \rangle =: Z(s,z),$$

which is regular at $s = 0$. Furthermore one has,

$$\partial_z^k \partial_s III_J(s,z)_{|s=0} = (k-1)! III_J(k,z), \qquad k > d/2, \tag{2.9}$$

(with k integer) since this derivative only involves terms with $l > d/2$ in (2.8). Hence, using (2.7), we also have

$$\partial_z^k \partial_s Z(s,z)_{|s=0} = (k-1)! Z(k,z), \qquad \mathrm{Re}(z) < \lambda_0, \quad k > d/2. \tag{2.10}$$

Note that, if u is compactly supported, (2.10) is a direct consequence of the identity (1.25).

When $u = \xi'_p$, the existence of a meromorphic continuation in s for $\zeta_p(s,z,h)$ is a consequence of the existence of an expansion of the form (2.5) proved in [6]. Notice that altering $Lu(t)$ by an analytic function in t will not destroy the form of this expansion. There is no restriction on $\mathrm{Re}(z)$ since, for all $\lambda_0 \in \mathbb{R}$, ξ'_p can be written as the sum of a compactly supported distribution and a temperate distribution supported in $[\lambda_0, +\infty)$ for which (2.5) still holds since the Laplace transform of the compactly supported distribution is analytic in t.

In particular, for $u = \xi'_p$, the relation (2.10) yields (1.24).

We now consider (1.26).

Proposition 2.1. *For all $z_0 \in \Omega^+$ and all integer $\nu \geq 0$, (1.26) holds.*

Proof. We shall see that the result follows from the following two facts: the existence of a semi-norm $||.||_{\mathcal{S}}$ (independent of h) of the Schwartz space $\mathcal{S}(\mathbb{R})$ such that

$$|\langle \xi_p'(h), \psi \rangle| \leq Ch^{-d}||\psi||_{\mathcal{S}}, \qquad \psi \in \mathcal{S}(\mathbb{R}),\ h \in (0, h_0], \tag{2.11}$$

and the existence of an expansion of the form

$$\left\langle \xi_p'(h), e^{-t(.)} \right\rangle \sim t^{-d/2} \sum_{j \geq 0} a_j(h) t^{j/2}, \qquad t \to 0, \qquad \text{with } a_j(h) = \mathcal{O}(h^{-d}). \tag{2.12}$$

The latter means that the difference between the left hand side and the sum truncated at the order M is bounded by $Ch^{-d}t^{(M-d)/2}$, for $t \in (0,1]$ and $h \in (0, h_0]$. Indeed, by writing $\xi_p' = \chi\xi_p' + (1-\chi)\xi_p'$ with $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on a large enough compact set, we may assume that $(1-\chi)\xi_p'$ is supported in $[\lambda_0, +\infty)$ with $\lambda_0 > \mathrm{Re}(z_0)$. Therefore, using (2.11), (2.12) and the discussion prior to Proposition 2.1, we see that $\langle \chi\xi_p'(h), (\cdot - z)^{-s} \rangle$ as well as the terms $I(h), II_J(h), III_J(h)$ corresponding to $u = u(h) = (1-\chi)\xi_p'(h)$ are $\mathcal{O}(h^{-d})$ uniformly with respect to s close to 0 and z close to z_0 which gives the result.

The proof of (2.11) can be found in [6] so we only consider (2.12). For the latter, the main remark is that, for all $\varepsilon \in [0,1]$,

$$-t(H_0 + \varepsilon V) = (ht^{1/2})^2 \Delta - \varepsilon \tilde{V}(h, t^{1/2}, x, ht^{1/2}D)$$

with

$$\tilde{V}(h, t^{1/2}, x, \xi) = \sum_{l=0}^{2} t^{1-\frac{l}{2}} \sum_{|\alpha|=l} v_\alpha(x,h)\xi^\alpha$$

where the v_α are defined by (1.11). By reviewing the proof of Proposition 3.1 in [6] with $ht^{1/2}$ as new semi-classical parameter, we see that, for all M, we have the following expansion

$$\begin{aligned} \mathrm{tr}\left(e^{-t(H_0+V)} - \sum_{j=0}^{p-1} \frac{1}{j!} \frac{d^j}{d\varepsilon^j} e^{-t(H_0+\varepsilon V)}|_{\varepsilon=0} \right) &= \sum_{q<M} (ht^{1/2})^{q-d} d_q(t^{1/2}, h) \\ &\quad + (ht^{1/2})^{M-d} R_M(t^{1/2}, h), \end{aligned}$$

with $R_M(t^{1/2}, h) = \mathcal{O}(1)$ for $h \in (0, h_0]$ and $0 < t \leq 1$. The coefficients $d_q(t^{1/2}, h)$ are smooth at 0 with respect to $t^{1/2}$ and bounded with respect to $h \in (0, h_0]$ as well as their derivatives so (2.12) follows. □

3 Trace class estimates

In the sequel, we shall use the notation $Op_h^w(a)$ for standard h-pseudodifferential operators of the form

$$Op_h^w(a)u(x) = (2\pi)^{-d} \int\int e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, h\xi\right) u(y) d\xi dy, \qquad h \in (0, h_0],$$

with symbols $a \in S^{\mu,\nu}$, $\mu, \nu \in \mathbb{R}$, namely such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle x \rangle^\nu \langle \xi \rangle^{\mu - |\beta|}.$$

We refer for instance to [26, 21, 10] for the proofs of the standard results we shall use below on the analysis of such operators. We equip $S^{\mu,\nu}$ with its standard Fréchet space topology given by the seminorms defined by the best constants $C_{\alpha\beta}$.

We also define the following semiclassical weighted Sobolev spaces

$$H^{s,\sigma}_{\rm sc} := \langle x \rangle^{-\sigma} \langle hD \rangle^{-s} L^2(\mathbb{R}^d), \qquad s, \sigma \in \mathbb{R},$$

equipped with the h-dependent norm

$$||u||_{H^{s,\sigma}_{\rm sc}} := ||\langle hD \rangle^s \langle x \rangle^\sigma u||_{L^2(\mathbb{R}^d)}. \tag{3.1}$$

Notice that

$$H^{s,\sigma}_{\rm sc} \subset H^{s,0}_{\rm sc} \subset L^2(\mathbb{R}^d), \qquad \text{if } s \geq 0, \ \sigma \geq 0.$$

In this section, we will consider h-dependent families of symbols

$$a = (a(h))_{h \in (0,h_0]}, \qquad a(h) \in S^{2,0} \text{ for all } \ h \in (0, h_0].$$

Most of the time, we shall assume the existence of $C > 0$ such that, for all $h \in (0, h_0]$,

$$|a(h, x, \xi)| \geq C^{-1} |\xi|^2, \qquad x \in \mathbb{R}^d, \ |\xi| > C. \tag{3.2}$$

When $a = (a(h))_{h \in (0,h_0]}$ or $b = (b(h))_{h \in (0,h_0]}$, we shall adopt the short notation

$$A = Op^w_h(a(h)), \qquad B = Op^w_h(b(h)),$$

for all $h \in (0, h_0]$.

In the next proposition, $\mathcal{B}$ denotes a subset of $(S^{2,0})^{(0,h_0]}$, namely a set of families $(a(h))_{h \in (0,h_0]}$, uniformly bounded in $S^{2,0}$, ie such that $\{a(h) \ ; \ h \in (0, h_0], a \in \mathcal{B}\}$ is bounded in $S^{2,0}$. We also assume that (3.2) holds for all $a \in \mathcal{B}$, with a constant $C > 0$ independent of a.

Proposition 3.1. *Assume that, for all $a \in \mathcal{B}$ and all $h \in (0, h_0]$,*

$$A : H^{2,0}_{\rm sc} \to L^2(\mathbb{R}^d) \text{ is invertible.}$$

Then, for all $s \geq 0$ and $\sigma \geq 0$, the restriction

$$A_{s,\sigma} = A|_{H^{s+2,\sigma}_{\rm sc}}$$

is bounded from $H^{s+2,\sigma}_{\rm sc}$ to $H^{s,\sigma}_{\rm sc}$ with bounded inverse such that

$$A^{-1}_{s,\sigma} = A^{-1}|_{H^{s,\sigma}_{\rm sc}}. \tag{3.3}$$

Furthermore, there exists $C_{s,\sigma} > 0$ such that, for all $h \in (0, h_0]$ and all $a \in \mathcal{B}$,

$$||A^{-1}_{s,\sigma}||_{H^{s,\sigma}_{\rm sc} \to H^{s+2,\sigma}_{\rm sc}} \leq C_{s,\sigma} \left(1 + ||A^{-1}||_{L^2 \to H^{2,0}_{\rm sc}} \right)^{[\sigma]+1}, \tag{3.4}$$

with $[\sigma]$ the smallest integer $\geq \sigma$.

The equality (3.3) means that we can consider A^{-1} as an operator from $H^{s,\sigma}_{\rm sc}$ into $H^{s+2,\sigma}_{\rm sc}$ and (3.4) gives an estimate on the corresponding norm. Abusing the notation, this proposition will allow us to denote A^{-1} instead of $A^{-1}_{s,\sigma}$ in the sequel.

Proof. The boundedness of $A_{s,\sigma}$ follows from the L^2 boundedness of

$$\langle hD\rangle^s \langle x\rangle^\sigma Op_h^w(a(h))\langle x\rangle^{-\sigma}\langle hD\rangle^{-s-2} =: Op_h^w(b_{s,\sigma}(h))$$

since $b_{s,\sigma}(h)$ so defined belongs to $S^{0,0}$. If $\sigma > 0$, we consider next $\sigma_1 := \sigma/[\sigma] \in [0,1]$. Then, by the resolvent identity,

$$A^{-1}\langle x\rangle^{\sigma_1} = \langle x\rangle^{\sigma_1}A^{-1} - A^{-1}[A,\langle x\rangle^{\sigma_1}]A^{-1}$$

where $[A,\langle x\rangle^{\sigma_1}] = Op_h^w(a_{\sigma_1}(h))$ for some symbol $a_{\sigma_1}(h) \in S^{1,0}$ depending continuously on $a(h)$. Thus

$$\langle x\rangle^{-\sigma_1}A^{-1}\left(1 + [A,\langle x\rangle^{\sigma_1}]A^{-1}\langle x\rangle^{-\sigma_1}\right) = A^{-1}\langle x\rangle^{-\sigma_1}$$

shows that A^{-1} is bounded from H^{0,σ_1} to H^{0,σ_1} with norm controlled, uniformly with respect to $a \in \mathcal{B}$ and $h \in (0,h_0]$, by $||A^{-1}||_{L^2\to H^{2,0}_{\rm sc}}(1 + ||A^{-1}||_{L^2\to H^{2,0}_{\rm sc}})$. By iteration, we obtain that A^{-1} maps continuously $H^{0,2\sigma_1}_{\rm sc}, H^{0,3\sigma_1}_{\rm sc}, \dots, H^{0,[\sigma]\sigma_1}_{\rm sc}$ into themselves and that

$$||A^{-1}||_{H^{0,\sigma}_{\rm sc}\to H^{0,\sigma}_{\rm sc}} \le C||A^{-1}||_{L^2\to H^{2,0}_{\rm sc}}(1 + ||A^{-1}||_{L^2\to H^{2,0}_{\rm sc}})^{[\sigma]}, \tag{3.5}$$

with C independent of h and of $a \in \mathcal{B}$. Using (3.2), we can construct, for all $N \ge 0$, symbols $\tilde{a}_N(h) \in S^{-2,0}$ and $r_N(h) \in S^{-N,0}$, depending continuously on $a(h)$, such that

$$Op_h^w(\tilde{a}_N(h))Op_h^w(a(h)) = 1 + Op_h^w(r_N(h)).$$

Notice that this is not a semiclassical parametrix (that would be the case if we had a remainder of the form $h^N Op_h^w(r_N(h))$) since (3.2) is not an ellipticity condition in the semiclassical sense. This is simply an h-dependent classical parametrix (in the sense of Theorem 18.1.9 of [18]). The symbol $\tilde{a}_N(h)$ is constructed by successive approximations starting from $(1-\chi)(\xi)/a(x,\xi,h)$, with $\chi \in C_0^\infty$ such that $\chi(\xi) = 1$ for $|\xi| \le C$, and then following the usual iterative scheme. We then obtain

$$A^{-1} = Op_h^w(\tilde{a}_N(h)) - Op_h^w(r_N(h))A^{-1}. \tag{3.6}$$

Since $Op_h^w(\tilde{a}_N(h))$ maps $H^{s,\sigma}_{\rm sc}$ into $H^{s+2,\sigma}_{\rm sc}$ and $Op_h^w(r_N(h))$ maps $H^{0,\sigma}_{\rm sc}$ into $H^{N,\sigma}_{\rm sc}$ for all $N \ge 0$, with norms uniformly bounded with respect to a and h, the right hand side of (3.6) is therefore bounded from $H^{s,\sigma}_{\rm sc}$ to $H^{s+2,\sigma}_{\rm sc}$, by choosing $N \ge s+2$ and using (3.5). The result then follows easily. □

In the sequel we shall denote by $\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ the Banach space of linear continuous mapping between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, equipped with the usual norm. We also denote by $\mathcal{L}_{\rm invertible}(\mathcal{H}_1,\mathcal{H}_2)$ the open subset of invertible mappings.

Proposition 3.2. *Let $a = (a(h))_{h\in(0,h_0]}$ be a family of $S^{2,0}$ satisfying (3.2) and let $U \subset \mathbb{C}$ be an open subset. Assume that*

$$A - z : H^{2,0}_{\rm sc} \to L^2(\mathbb{R}^d) \quad \textit{is invertible}$$

for all $z \in U$ and all $h \in (0,h_0]$.
i) Let $b = (b(h))_{h\in(0,h_0]}$ be a family of $S^{2,0}$. Then, for all $h \in (0,h_0]$ and all $z_0 \in U$, there exists $\varepsilon_{h,z_0} > 0$ and a neighborhood $U(z_0) \subset U$ of z_0 such that, for all $s,\sigma \ge 0$, the map

$$(-\varepsilon_{h,z},\varepsilon_{h,z}) \times U(z_0) \ni (\varepsilon, z) \mapsto (A + \varepsilon B - z)^{-1} \in \mathcal{L}(H^{s,\sigma}_{\rm sc}, H^{s+2,\sigma}_{\rm sc}) \tag{3.7}$$

is well defined and smooth. In addition

$$\frac{d^j}{d\varepsilon^j}(A+\varepsilon B - z)^{-1} = (-1)^j j!(A+\varepsilon B - z)^{-1}\left(B(A+\varepsilon B - z)^{-1}\right)^j. \tag{3.8}$$

ii) Assume that, for all $h \in (0,h_0]$, we have a sequence $(a_n(h))_{n\in\mathbb{N}}$ converging to $a(h)$ in $S^{2,0}$. Then, for all $h \in (0,h_0]$ and all relatively compact subset $U_0 \Subset U$, there exists $n_{h,U_0} \in \mathbb{N}$ such that,

$$A_n - z : H^{2,0}_{\rm sc} \to L^2(\mathbb{R}^d), \qquad z \in U_0, \ n \geq n_{h,U_0}, \tag{3.9}$$

is invertible, and, for all $s, \sigma \geq 0$,

$$||(A_n - z)^{-1} - (A - z)^{-1}||_{H^{s,\sigma}_{\rm sc} \to H^{s+2,\sigma}_{\rm sc}} \to 0, \qquad n \to \infty, \tag{3.10}$$

uniformly on U_0.

Proof. Fix $h \in (0,h_0]$. Since B is bounded from $H^{2,0}_{\rm sc}$ to $L^2(\mathbb{R}^d)$, for ε small enough and z close enough to z_0, $A + \varepsilon B - z$ is invertible. It is then also invertible as a bounded operator from $H^{s+2,\sigma}_{\rm sc}$ to $H^{s,\sigma}_{\rm sc}$ by Proposition 3.1. Since the map $T \mapsto T^{-1}$ is C^1 from $\mathcal{L}_{\rm invertible}(H^{s+2,\sigma}_{\rm sc}, H^{s,\sigma}_{\rm sc})$ to $\mathcal{L}(H^{s,\sigma}_{\rm sc}, H^{s+2,\sigma}_{\rm sc})$, (3.7) is C^1 with derivative given by (3.8) with $j = 1$. The result then follows by induction. Let us now prove *ii)*. Let $z_0 \in U$. Since $A - z_0$ is invertible and by convergence of A_n to A, there exists $n_{h,z_0} > 0$ and $\delta_{z_0,h} > 0$ such that $A_n - z$ is invertible for $n \geq n_{z_0,h}$ and $|z - z_0| < \delta_{z_0,h}$. By compactness, U_0 can be covered by finitely many balls of the form $\{|z - z_j| < \delta_{z_j,h}\}$ and thus $A_n - z$ is invertible for all $z \in U_0$ and $n \geq n_{h,U_0} := \max_j n_{h,z_j}$. The balls can be chosen such that

$$\sup_{n \geq n_{h,z_j}} \ \sup_{|z - z_j| < \delta_{h,z_j}} ||(A_n - z)^{-1}||_{H^{s,\sigma}_{\rm sc} \to H^{s+2,\sigma}_{\rm sc}} < +\infty$$

so the norms $||(A_n - z)^{-1}||_{H^{s,\sigma}_{\rm sc} \to H^{s+2,\sigma}_{\rm sc}}$ are uniformly bounded with respect to $n \geq n_{h,U_0}$ and $z \in U_0$. Then (3.10) follows from the resolvent identity. □

For $k \geq 1$ integer, to be fixed further on, we set

$$f_z^k(\lambda) = (\lambda - z)^{-k}.$$

Proposition 3.3. *Let $U \subset \mathbb{C}$ an open subset and $a = (a(h))_{h\in(0,h_0]}$ be a family of $S^{2,0}$ satisfying (3.2). Let $b = (b(h))_{h\in(0,h_0]}$ be a family of $S^{m,\mu}$ with $m \leq 2$ and $\mu < 0$. Assume that, for all $h \in (0,h_0]$ and all $z \in U$,*

$$A - z : H^{2,0}_{\rm sc} \to L^2(\mathbb{R}^d)$$

is invertible.
i) Let $j \geq 1$. Then, $\frac{d^j}{d\varepsilon^j} f_z^k(A + \varepsilon B)_{|\varepsilon=0}$ is well defined and is a linear combination of

$$(A - z)^{-k_1} B (A - z)^{-k_2} \cdots B (A - z)^{-k_{j+1}}, \qquad k_1 + \cdots + k_{j+1} = k + j \tag{3.11}$$

with $k_1, \ldots, k_{j+1} \geq 1$. Furthermore, if

$$j(m-2) - 2k < -d \qquad \text{and} \qquad j\mu < -d, \tag{3.12}$$

each operator of the form (3.11) is of trace class in $L^2(\mathbb{R}^d)$.
ii) Assume in addition that, for all $h \in (0,h_0]$ and all $z \in U$,

$$A + B - z : H^{2,0}_{\rm sc} \to L^2(\mathbb{R}^d)$$

is invertible. Then

$$f_z^k(A + B) - f_z^k(A) - \sum_{j=1}^{p-1} \frac{1}{j!} \frac{d^j}{d\varepsilon^j} f_z^k(A + \varepsilon B)_{|\varepsilon=0} \tag{3.13}$$

is well defined and is a linear combination of

$$(A+B-z)^{-k_1}B(A-z)^{-k_2}\cdots B(A-z)^{-k_{p+1}}, \qquad k_1+\cdots+k_{p+1}=k+p \tag{3.14}$$

with $k_1,\dots,k_{p+1}\geq 1$. *If*

$$p(m-2)-2k<-d \qquad \textit{and} \qquad p\mu<-d \tag{3.15}$$

then each operator of the form (3.14) is trace class on $L^2(\mathbb{R}^d)$.

First recall that from the standard estimate

$$||\langle x\rangle^{-s}\langle hD\rangle^{-\sigma}||_{\mathrm{tr}} \leq Ch^{-d}, \qquad h\in(0,h_0],$$

we have:

Lemma 3.4. *For all* $s>d$ *and* $\sigma>d$*, the injection* $H^{s,\sigma}_{\mathrm{sc}}\hookrightarrow L^2(\mathbb{R}^d)$ *is trace class with norm* $\mathcal{O}(h^{-d})$.

Proof of Proposition 3.3. That $\frac{d^j}{d\varepsilon^j}f_z^k(A+\varepsilon B)_{|\varepsilon=0}$ is well defined follows directly from Proposition 3.2 *i)*, as well as its expression for $k=1$ which is given by (3.8). The formula for $k\geq 2$ is obtained by applying ∂_z^{k-1} to (3.8), using

$$(k-1)!(\lambda-z)^{-k}=\partial_z^{k-1}(\lambda-z)^{-1}, \tag{3.16}$$

and the smoothness of (3.7). By Proposition 3.1, each operator of the form (3.11) is bounded from $L^2(\mathbb{R}^d)$ to $H_{\mathrm{sc}}^{j(2-m)+2k,-j\mu}$ thus is trace class by (3.12) and Lemma 3.4. This completes the proof of *i)*. The proof of *ii)* is completely similar once observed that, for $k=1$, (3.13) equals

$$(-1)^p(A+B-z)^{-1}\left(B(A-z)^{-1}\right)^p \ ,$$

which is obtained using (3.8). □

Conclusion. Under the assumptions of Proposition 3.3 *ii)*, the following expression is well defined:

$$T_p^k(A,B,z):=\mathrm{tr}\left(f_z^k(A+B)-f_z^k(A)-\sum_{j=1}^{p-1}\frac{1}{j!}\frac{d^j}{d\varepsilon^j}f_z^k(A+\varepsilon B)_{|\varepsilon=0}\right), \tag{3.17}$$

(with the usual convention that $\sum_{j=1}^{p-1}\equiv 0$ if $p=1$) provided that (3.15) holds, thus in particular for

$$k>d/2 \qquad \text{and} \qquad p\mu<-d.$$

If in addition $(a(h))_{h\in(0,h_0]}\in\mathcal{B}$ as in Proposition 3.1, we have the following bound,

$$\left|T_p^k(A,B,z)\right|\leq Ch^{-d}\left(1+||(A-z)^{-1}||_{L^2\to H^{2,0}_{\mathrm{sc}}}+||(A+B-z)^{-1}||_{L^2\to H^{2,0}_{\mathrm{sc}}}\right)^N, \tag{3.18}$$

for some $C,N>0$ independent of $h\in(0,h_0]$ and $z\in U$, using (3.4), (3.14) and Lemma 3.4.

4 Resonances

4.1 The analytic distortion method

In this subsection, we recall the definition of resonances by the analytic distortion method after Sjöstrand-Zworski. We also collect additional results that will be necessary for our applications.

We first recall the definition of a maximal totally real manifold $\Gamma \subset \mathbb{C}^d$ parametrized by $\kappa : \mathbb{R}^d \to \mathbb{C}^d$. By this it is meant that $\kappa : \mathbb{R}^d \to \kappa(\mathbb{R}^d) = \Gamma$ is a diffeomorphism (between real manifolds) such that

$$T_{\kappa(x)}\Gamma \cap i(T_{\kappa(x)}\Gamma) = \{0\}, \qquad x \in \mathbb{R}^d.$$

Equivalently this means that, for all x, $(\partial_1\kappa(x), \ldots, \partial_d\kappa(x), i\partial_1\kappa(x), \ldots, i\partial_d\kappa(x))$ is a basis of $\mathbb{C}^d$ viewed as a real vector space, or that $(\partial_1\kappa(x), \ldots, \partial_d\kappa(x))$ is a basis of $\mathbb{C}^d$ as a complex vector space, so the fact that Γ is totally real simply means that

$$\det\left(\frac{\partial\kappa(x)}{\partial x}\right) \neq 0, \qquad x \in \mathbb{R}^d. \tag{4.1}$$

Then, to any differential operator

$$P = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha,$$

with coefficients that are smooth on $\mathbb{R}^d$ and holomorphic in some neighborhood of $\Gamma \cap \left(\mathbb{C}^d \setminus \mathbb{R}^d\right)$ (typically a sector of the form $\Sigma(\theta_0, R_0, \epsilon_0)$), we can associate the operator

$$\mathcal{A}_\kappa P := \sum_{|\alpha|\leq m} a_\alpha(\kappa(x)) \left(({}^t\partial_x\kappa(x))^{-1} D\right)^\alpha. \tag{4.2}$$

The analytic distortion method is as follows. Given $R_1 > 0$ and $\epsilon_1 > 0$, we can find a non decreasing smooth function $\phi : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\begin{aligned} \phi(t) &= 0 \qquad t \leq R_1, && (4.3)\\ \phi(t) &= 1 \qquad t \gg 1, && (4.4)\\ 0 \leq t\theta\phi'(t) &\leq \epsilon_1, \qquad t > 0,\ \theta \in [0, \pi], && (4.5) \end{aligned}$$

and the latter condition implies, by possibly considering ϕ associated with a smaller ϵ_1, that we can additionally assume

$$0 \leq \arg(1 + it\theta\phi'(t)) \leq \epsilon_1, \qquad t > 0,\ \theta \in [0, \pi]. \tag{4.6}$$

We assume in the sequel that, for each $\epsilon_1 > 0$ (small enough) and $R_1 > 0$ (large enough), a function ϕ satisfying (4.3), (4.4), (4.5) and (4.6) has been chosen. Then the function

$$f_\theta(t) = e^{i\phi(t)\theta} t, \qquad t \in \mathbb{R}^+,$$

satisfies

$$f_\theta(t) = t \ \text{ for } \ t \leq R_1, \qquad f_\theta(t) = e^{i\theta} t \ \text{ for } \ t \gg 1, \qquad \partial_t f_\theta \neq 0 \tag{4.7}$$

$$0 \leq \arg(f_\theta(t)) \leq \theta, \qquad \arg(f_\theta(t)) \leq \arg(\partial_t f_\theta(t)) \leq \arg(f_\theta(t)) + \epsilon_1. \tag{4.8}$$

Using this function, we can now define $\kappa_\theta : \mathbb{R}^d \to \mathbb{C}^d$ and Γ_θ by

$$\kappa_\theta(x) = f_\theta(|x|)\frac{x}{|x|} = e^{i\theta\phi(|x|)}x, \qquad \Gamma_\theta = \kappa_\theta(\mathbb{R}^d). \tag{4.9}$$

Notice that,

$$\partial_x \kappa_\theta(x) = e^{i\theta\phi(|x|)}\left(I_d + i\theta|x|\phi'(|x|)\frac{x\otimes x}{|x|^2}\right), \tag{4.10}$$

thus (4.1) holds, at least for ϵ_1 small enough. Now, if P is a differential operator whose coefficients can be continued analytically to $\Sigma(\theta_0, R_0, \epsilon_0)$, by choosing ϵ_1 small enough and

$$R_1 > R_0, \qquad 0 \le \theta \le \theta_0,$$

we can define the following differential operator on $\mathbb{R}^d$

$$P(\theta) := \mathcal{A}_{\kappa_\theta} P, \tag{4.11}$$

with $\mathcal{A}_{\kappa_\theta}$ defined by (4.2) and (4.9).

Remark. The reader should keep in mind that operators of the form $P(\theta)$ depend not only on θ (and $h \in (0, h_0]$ below) but also on the parameters R_1 and ϵ_1 (and also on the choice of the function ϕ), although this dependence is omitted in the notation.

Definition 4.1. *Let $V \in \mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$. The pair $(R_1, \epsilon_1) \in \mathbb{R}^2_+$ is said to be Fredholm admissible for $H_0 + V$ if, for all $\theta \in [0, \theta_0]$, the following hold:*
i) for all $h \ll 1$ and all $z \in \mathbb{C} \setminus e^{-2i\theta}[0, +\infty)$,

$$H_0(\theta) + V(\theta) - z : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \ \text{ is a Fredholm operator of index } 0,$$

ii) the principal symbol, in the classical sense, $p_\theta^{\rm cl}$ of $H_0(\theta) + V(\theta)$ is elliptic, ie for some $C \ge 1$

$$|p_\theta^{\rm cl}(x, \xi)| \ge C^{-1}|\xi|^2, \qquad (x, \xi) \in \mathbb{R}^{2d}.$$

Here $H_0(\theta)$ and $V(\theta)$ are defined by (4.11) with κ_θ given by (4.9).

Proposition 4.2. *Let $(V_\iota)_{\iota\in I}$ be bounded family of $\mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$. We can find $\overline{R}_1 > 0$, $\overline{\epsilon}_1 > 0$ and $\overline{C} > 0$ such that, for all $\iota \in I$, any $(R_1, \epsilon_1) \in [\overline{R}_1, +\infty) \times (0, \overline{\epsilon}_1]$ is Fredholm admissible for $H_0 + V_\iota$, with constant $\overline{C}$ in ii). More explicitly*

$$|p_{\iota,\theta}^{\rm cl}(x, \xi)| \ge \overline{C}^{-1}|\xi|^2, \tag{4.12}$$

uniformly with respect to $\epsilon_1 \in (0, \overline{\epsilon_1}]$, $R_1 \ge \overline{R_1}$, $\theta \in [0, \theta_0]$ and $\iota \in I$. In addition, we may also assume that, for all $\theta \in [0, \theta_0]$,

$$-2\theta - 3\epsilon_1 \le \arg\left(p_{\iota,\theta}^{\rm cl}(x, \xi)\right) \le \epsilon_1, \qquad x \in \mathbb{R}^d, \ \xi \in \mathbb{R}^d \setminus 0, \tag{4.13}$$

uniformly with respect to $\iota \in I$.

Proposition 4.2 is proved, for a single V, in the lecture notes [31, Lemma 7.3] in the more general framework of *black box perturbations*. Its extension to a bounded family of $\mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$ involves no new argument and we therefore omit the proof. The reason for considering a bounded family in $\mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$ is that we shall approximate $V \in \mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$ by a sequence $V_n \in \mathcal{V}_{\bar{d}}(\theta_0, R_0, \epsilon_0)$, with $\bar{d} > d$, and use a certain deformation along $\kappa_\theta(\mathbb{R}^d)$. It will be important that κ_θ (which depends on ϵ_1 and R_1) can be chosen independently of n.

The Fredholm admissibility is important to define the resonances as we shall see below. In the case of a single V, the first part of Proposition 4.2 simply states that this condition is fulfilled for $H_0 + V$. The additional uniform estimates (4.12) and (4.13) will be useful later on to prove some resolvent estimates.

The definition of resonances relies on the following theorem.

Theorem 4.3. *([32, 29, 31]) Let $0 < \theta_0 < \pi$ and $V \in \mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$. Assume that we are given $R_1 > 0$ and $\epsilon_1 > 0$ which are Fredholm admissible. Then, for all $h \ll 1$ and all $z \in \Omega$, we have*
(i) $z \in \sigma(H_0(\theta) + V(\theta))$ if and only if $\ker(H_0(\theta) + V(\theta) - z) \neq 0$.
(ii) For all $0 \leq \theta_1 \leq \theta_2 \leq \theta_0$, if $z \in \mathbb{C} \setminus e^{-2i[\theta_1,\theta_2]}[0, +\infty)$ then

$$\dim \ker(H_0(\theta_1) + V(\theta_1) - z) = \dim \ker(H_0(\theta_2) + V(\theta_2) - z).$$

The first statement is an immediate consequence of the fact that the operator has a zero index. The second one requires a non trivial analytic deformation result, which uses the analyticity of the coefficients of V near infinity.

Let us recall the main consequence of Theorem 4.3.

First, if $0 \leq \theta \leq \theta_0 < \pi$ and $0 < \epsilon < 2\pi - 2\theta_0$, then for all $h \ll 1$ and all $z \in e^{i(0,\epsilon)}(0, +\infty)$,

$$H_0(\theta) + V(\theta) - z : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \text{ is an isomorphism.} \tag{4.14}$$

Furthermore, by analytic Fredholm theory, one can show that the spectrum of $H_0(\theta) + V(\theta)$ is discrete in $\mathbb{C}\setminus e^{-2i\theta}[0, +\infty)$. The part *(ii)* guarantees that, if $\theta' > \theta$, the eigenvalues of $H_0(\theta)+V(\theta)$ and $H_0(\theta') + V(\theta')$ coincide on $e^{-2i[0,\theta)}(0, +\infty)$ and this makes the following definition natural.

Definition 4.4. *Given Ω satisfying (1.17), the set of resonances of $H_0 + V$ in Ω is*

$$\mathrm{Res}(H_0 + V, \Omega) = \Omega \cap \sigma(H_0(\theta_0) + V(\theta_0)) \cap e^{-i[0,2\theta_0)}(0, +\infty).$$

Recall that $\mathrm{Res}(H_0 + V, \Omega)$ is finite (for each h).

By analytic Fredholm theory again, for any $w \in \mathrm{Res}(H_0 + V, \Omega)$, the operator

$$\Pi_{\theta,w} = \frac{i}{2\pi} \int_{\gamma(w)} (H_0(\theta) + V(\theta) - z)^{-1} dz \tag{4.15}$$

is of finite rank, if $\gamma(w)$ a small enough contour enclosing w and this allows to state the following definition.

Definition 4.5. *The multiplicity of w is the rank of $\Pi_{\theta,w}$.*

This definition is independent of θ in the sense that we get the same rank if θ is replaced by some larger θ' (smaller than θ_0).

We conclude this subsection with the following elementary resolvent estimates.

Proposition 4.6. *Let Ω be satisfying (1.17) and let $\Omega^+_\delta := \Omega^+ \cap \{\mathrm{Im}(z) \geq \delta\}$ (see (1.20)) with δ small enough to be non empty. Let $(V_\iota)_{\iota \in I}$ be a bounded family of $\mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$. Then, for all $\epsilon_1 > 0$ small enough, we can choose $R_1 > 0$ as large as we want such that*

$$||(H_0(\theta_0) + V_\iota(\theta_0) - z)^{-1}||_{L^2 \to H^{2,0}_{\mathrm{sc}}} \lesssim 1, \qquad h \ll 1, \ z \in \Omega^+_\delta, \ \iota \in I \ . \tag{4.16}$$

Proof. Denote by $p_\iota(x,\xi,h)$ the full Weyl symbol of H_0+V_ι, which is then real on $\mathbb{R}^{2d}$ and of the form

$$p_\iota(x,\xi,h)=p_\iota^{\rm cl}(x,\xi)+a_\iota(x,\xi,h),$$

with $a_\iota(.,h)$ polynomial of degree ≤ 1 in ξ with coefficients bounded in $\mathcal{C}_\rho(\theta_0,R_0,\epsilon_0)$. Setting

$$p_{\iota,\theta_0}(x,\xi,h)=p_\iota\Big(\kappa_{\theta_0}(x),({}^t\partial_x\kappa_{\theta_0}(x))^{-1}\xi,h\Big),$$

we then have

$$H_0(\theta_0)+V_\iota(\theta_0)=Op_h^w(p_{\iota,\theta_0})+hOp_h^w(b_{\iota,\theta_0}(h))$$

for some symbol $b_{\iota,\theta_0}(h)$ which, for fixed ϵ_1 and R_1, is bounded in $S^{1,0}$ as h and ι vary. We thus only need to show that, for $\epsilon_1>0$ small enough and $R_1>0$ large enough,

$$|p_{\iota,\theta_0}(x,\xi,h)-z|\gtrsim 1,\qquad h\ll 1,\ z\in\Omega_\delta^+,\ \iota\in I. \tag{4.17}$$

The result then follows from the standard construction of a semiclassical parametrix, yielding the invertibility of $H_0(\theta_0)+V_\iota(\theta_0)-z$ for h small enough (uniformly with respect to z and ι) as well as the bound (4.16). Let us prove (4.17). Using (4.12), we can choose $C_0>0$ large enough, independent of $0<\epsilon_1\leq\overline{\epsilon}_1$, $R_1\geq\overline{R}_1$, $x\in\mathbb{R}^d$, $h\ll 1$ and $\iota\in I$ such that

$$|p_{\iota,\theta_0}(x,\xi,h)|\geq 1+\max_{\overline{\Omega}}|z|,\qquad |\xi|\geq C_0,$$

since $|(p_{\iota,\theta_0}-p_{\iota,\theta_0}^{\rm cl})(x,\xi,h)|\lesssim\langle\xi\rangle$, uniformly with respect to h,ι,ϵ_1,R_1. Using (4.13), if $\epsilon_1>0$ and $\delta'>0$ are small enough, we also have

$$|p_{\iota,\theta_0}^{\rm cl}(x,\xi)-z|\geq\delta',\qquad x,\xi\in\mathbb{R}^d,\ z\in\Omega_\delta^+.$$

Then, once such ϵ_1 and δ' have been chosen, we have, for all R_1 large enough,

$$|a_\iota\Big(\kappa_{\theta_0}(x),({}^t\partial_x\kappa_{\theta_0}(x))^{-1}\xi,h\Big)|\leq\frac{\delta'}{2},\qquad |x|\geq R_1,\qquad |\xi|\leq C_0,$$

since the coefficients of a_ι decay like $\langle x\rangle^{-\rho}$ in $\Sigma(\theta_0,R_0,\epsilon_0)$ uniformly with respect to h and ι. It is then straightforward to check that (4.17) holds since p_{ι,θ_0} is real for $|x|\leq R_1$. □

In the next proposition, we prove an exponential bound for the resolvent of $H_0(\theta)$. The latter can be used with Theorem 1.7 to obtain an exponential upper bound on $\partial_z\varphi_p(z,h)$, when $p\geq 3$. Let us recall that, since $H_0=-h^2\Delta$ has no resonances away from 0, $(H_0(\theta)-z)^{-1}$ is well defined for all $z\in\Omega$ (see [32]).

For simplicity, we only consider the case where $\theta_0<\pi/2$ and $d\geq 3$.

Proposition 4.7. *Assume that $\theta_0<\pi/2$ and that $d\geq 3$. Let Ω be a simply connected open set satisfying (1.17). Then, if ϵ (in (1.17)) and ϵ_1 (in (4.5)) are small enough, we have*

$$||(H_0(\theta_0)-z)^{-1}||_{L^2\to H^{2,0}_{\rm sc}}\lesssim e^{Ch^{-1}},\qquad h\ll 1,\ z\in\Omega\ . \tag{4.18}$$

Proof. By (4.9) and (4.10), the coefficients of $H_0(\theta)$ are holomorphic with respect to θ in a small neighborhood of $[0,\theta_0]$ and thus so is

$$\theta\mapsto\big(v,(H_0(\theta)-z)^{-1}u\big)\,, \tag{4.19}$$

for θ in a complex neighborhood of $[0, \theta_0]$ and for all $u, v \in C_0^\infty(\mathbb{R}^d)$, $z \in \Omega$ and $h \in (0, 1]$. On the other hand, for $i\theta \in \mathbb{R}$ small,

$$H_0(\theta) = U_\theta \, H_0 \, U_\theta^{-1},$$

with $U_\theta : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ the isomorphism defined by $U_\theta(u)(x) = u(\kappa_\theta(x))$. Since U_θ maps $H^2(\mathbb{R}^d)$ into itself, we then have

$$(H_0(\theta) - z)^{-1} = U_\theta (H_0 - z)^{-1} U_\theta^{-1}, \qquad z \in \Omega^+,$$

and if we denote by $\mathcal{R}(x - y, z, h)$ the Schwartz kernel of $(H_0 - z)^{-1}$ we can rewrite (4.19) as

$$\int_{\mathbb{R}^{2d}} \mathcal{R}(\kappa_\theta(x) - \kappa_\theta(y), z, h) u(y) \overline{v(x)} \det(\kappa_\theta(y)) \, dxdy \tag{4.20}$$

for $i\theta \in \mathbb{R}$ small and $z \in \Omega^+$. Let us recall that, for $\mathrm{Im}(z^{1/2}) > 0$,

$$\mathcal{R}(x - y, z, h) = \frac{i}{4h^2} \left(\frac{z^{1/2}}{2\pi h |x - y|} \right)^{\frac{d}{2} - 1} H^1_{\frac{d}{2} - 1}(z^{1/2} |x - y| / h),$$

where the Hankel function $H^1_\nu(Z)$ (with $\nu = \frac{d}{2} - 1$) is given by

$$H^1_\nu(Z) = \left(\frac{2}{\pi Z} \right)^{1/2} \frac{e^{i(Z - \frac{\nu}{2}\pi - \frac{\pi}{4})}}{\Gamma(\nu + \frac{1}{2})} \int_0^{+\infty} e^{-s} \left(s(1 + isZ^{-1}/2) \right)^{\nu - \frac{1}{2}} ds,$$

using everywhere the determination of the square root defined on $\mathbb{C} \setminus (-\infty, 0]$ taking its values in $e^{i(-\pi/2, \pi/2)}(0, +\infty)$ (see for instance section VII.7.2 of [34]). The function H^1_ν is holomorphic for $Z \in e^{i(-\pi/2, \pi/2)}(0, +\infty)$, with the following rough bound, for all $0 < \delta < \pi/2$,

$$|H^1_\nu(Z)| \leq C_\delta |Z|^{-1/2} e^{|\mathrm{Im}(Z)|} \max\left(1, |Z|^{\frac{1}{2} - \nu}\right), \qquad \arg(Z) \in (\delta - \pi/2, \pi/2 - \delta). \tag{4.21}$$

Independently, by writing $\varphi(x) = \phi(|x|)$, we have

$$\kappa_\theta(x) - \kappa_\theta(y) = (x - y) \int_0^1 e^{i\theta\varphi(y + t(x - y))} \Big(i\theta \nabla\varphi(y + t(x - y)) \otimes (y + t(x - y)) + 1 \Big) dt,$$

where $|\theta \nabla\varphi(X) \otimes X| \lesssim \epsilon_1$ by (4.5) and $0 \leq \varphi(X) \leq 1$. Therefore, if ϵ_1 and ϵ are small enough, there exists $\delta > 0$ small enough such that

$$z^{1/2} |\kappa_\theta(x) - \kappa_\theta(y)| := \left(z \langle \kappa_\theta(x) - \kappa_\theta(y), \kappa_\theta(x) - \kappa_\theta(y) \rangle \right)^{1/2} \in e^{i(\delta - \pi/2, \pi/2 - \delta)}(0, +\infty),$$

for $x \neq y$, $x, y \in \mathbb{R}^d$, $z \in \Omega$ and θ in a neighborhood of $[0, \theta_0]$. Furthermore, the modulus of $|\kappa_\theta(x) - \kappa_\theta(y)| / |x - y|$ is bounded from above and from below. This allows to continue (4.20) analytically with respect to $\theta \in [0, \theta_0]$ and then with respect to $z \in \Omega$. Using (4.21) and the Schur Lemma, we deduce that, for any $\chi \in C_0^\infty(\mathbb{R}^d)$,

$$||\chi (H_0(\theta_0) - z)^{-1} \chi||_{L^2 \to L^2} \lesssim e^{Ch^{-1}}, \qquad z \in \Omega.$$

This easily implies a similar $L^2 \to L^2$ bound on the whole resolvent using the elementary estimate

$$||(e^{-2i\theta_0} H_0 - z)^{-1}||_{L^2 \to H^{2,0}_{\mathrm{sc}}} \lesssim 1, \qquad z \in \Omega,$$

and two applications of the resolvent identity yielding

$$\begin{aligned} (H_0(\theta_0) - z)^{-1} \;&=\; (e^{-2i\theta_0} H_0 - z)^{-1} - (e^{-2i\theta_0} H_0 - z)^{-1} V_0 (e^{-2i\theta_0} H_0 - z)^{-1} \\ &\quad + (e^{-2i\theta_0} H_0 - z)^{-1} V_0 (H_0(\theta_0) - z)^{-1} V_0 (e^{-2i\theta_0} H_0 - z)^{-1}, \end{aligned}$$

where $V_0 := H_0(\theta_0) - e^{-2i\theta_0} H_0$ is a compactly supported differential operator of order 2. The $L^2 \to H^{2,0}_{\mathrm{sc}}$ bound then follows from the $L^2 \to L^2$ one by the resolvent identity between $z_0 \in \Omega^+_\delta$ and z, using (4.16). $\square$

4.2 A deformation result

We recall first the following result.

Proposition 4.8 (Sjöstrand [30])**.** *Let $\overline{d} > d$ and $V \in \mathcal{V}_{\overline{d}}(\theta_0, R_0, \epsilon_0)$. Let $R_1 > 0$ and $\epsilon_1 > 0$ be Fredholm admissible for H_0 and $H_0 + V$. Then, if $k > d/2 + 1$,*

$$\operatorname{tr}\left((H_0 + V - z)^{-k} - (H_0 - z)^{-k}\right) = \operatorname{tr}\left((H_0(\theta) + V(\theta) - z)^{-k} - (H_0(\theta) - z)^{-k}\right),$$

for all $\theta \in [0, \theta_0]$ and all $z \in \Omega^+$.

In the next proposition, we simply state that the above invariance of the trace by analytic distortion still holds for the regularized traces of the form (3.17).

Proposition 4.9. *Let $p \in \mathbb{N}$ and $\rho > 0$ such that $\rho > d/p$. Let $V \in \mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$. Then, if ϵ_1 is small enough, R_1 is large enough and $k > d/2 + 1$, we have*

$$T_p^k\left(H_0, V, z\right) = T_p^k\left(H_0(\theta), V(\theta), z\right),$$

for all $\theta \in [0, \theta_0]$ and all $z \in \Omega^+$.

As the reader may guess, this proposition is a fairly elementary consequence of Proposition 4.8, approximating V by a sequence $V_n \in \mathcal{V}_{\bar{d}}(\theta_0, R_0', \epsilon_0')$ with $\bar{d} > d$.

Lemma 4.10. *Let $V \in \mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$. Let $\bar{d} > d$. We can find $R_0' > R_0$, $0 < \epsilon_0' \leq \epsilon_0$ and a sequence $(V_n)_{n\geq 1} \in \mathcal{V}_{\bar{d}}(\theta_0, R_0', \epsilon_0')$, bounded in $\mathcal{V}_\rho(\theta_0, R_0', \epsilon_0')$ such that, for all $\rho' < \rho$ and all $s, \sigma \in \mathbb{R}$,*

$$||V_n - V||_{H^{s,\sigma}_{\mathrm{sc}} \to H^{s-2,\sigma+\rho'}_{\mathrm{sc}}} \to 0, \qquad n \to \infty, \tag{4.22}$$

for all $h \ll 1$.

Proof. Choose first a determination of $Z \mapsto Z^{1/4}$ for $Z \in \mathbb{C} \setminus e^{2i\theta_0'}[0, +\infty)$, with $\theta_0 < \theta_0' < \pi$. We may assume that it is positive on $\mathbb{R}^+$. Choose also $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leq \chi \leq 1$, $\chi(x) \equiv 1$ for $|x| \leq R_0'/2$, and $\chi(x) = 0$ for $|x| \geq R_0'$. We then define

$$\chi_n(x) = \chi(x) + (1 - \chi(x)) \exp\left(-(x^2)^{1/4}/n\right), \qquad n \geq 1,$$

with $x^2 = x_1^2 + \cdots + x_d^2$, and

$$V_n = \chi_n V \chi_n.$$

If R_0' is large enough, the coefficients of V_n are then such that (1.13), (1.14) and (1.15) hold, with c independent of n in (1.14), and (4.22) is elementary. Furthermore, if ϵ_0' is small enough $x \mapsto \exp\left(-(x^2)^{1/4}/n\right)$ has an analytic continuation to $\Sigma(\theta_0, R_0', \epsilon_0')$ where it is uniformly bounded with respect to $n \geq 1$. Therefore $(V_n)_{n\geq 1}$ is bounded in $\mathcal{V}_\rho(\theta_0, R_0', \epsilon_0')$. Also, it clearly belongs to $\mathcal{V}_{\bar{d}}(\theta_0, R_0', \epsilon_0')$ since, if $x = te^{i\theta}\omega$ with $t \gg 1$, ω close to $\mathbb{S}^{d-1}$ and $\theta \in [0, \theta_0]$, we then have $\operatorname{Re}\left((x^2)^{1/4}\right) \gtrsim t^{1/2}\cos(\theta/2) \gtrsim t^{1/2}$. □

Proof of Proposition 4.9. By Proposition 4.2, for all R_1 large enough and all ϵ_1 small enough, (R_1, ϵ_1) is Fredholm admissible for εV_n and εV, for all $n \geq 1$ and $\varepsilon \in [0, 1]$. Using Proposition 4.8 with R_0' and ϵ_0', we then have

$$\operatorname{tr}\left((H_0 + \varepsilon V_n - z)^{-k} - (H_0 - z)^{-k}\right) = \operatorname{tr}\left((H_0(\theta) + \varepsilon V_n(\theta) - z)^{-k} - (H_0(\theta) - z)^{-k}\right)$$

and the latter can be differentiated with respect to ε using Proposition 3.2 since the operators inside the trace are smooth with respect to ε, in the trace norm. This is easily seen, for instance for the left hand side, by writing the operator inside the trace as a linear combination of operators of the form

$$(H_0 + \varepsilon V_n - z)^{-k_1} \varepsilon V_n (H_0 - z)^{-k_2}, \qquad k_1 + k_2 = k + 1.$$

Therefore,

$$T_p^k (H_0, V_n, z) = T_p^k (H_0(\theta), V_n(\theta), z)$$

gives the result by letting n go to ∞, using (4.22) with ρ' such that $p\rho' > d$, Propositions 3.2 and 3.3. $\square$

4.3 The main tool of Sjöstrand's trace formula

Proposition 4.11. *Let Ω be an open subset satisfying (1.17) with $0 < \theta_0 < \pi$ and $0 < \epsilon < 2\pi - 2\theta$. Let $V \in \mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$ with $\rho > 0$. Then, we can fix h_1, ϵ_1 small enough and R_1 large enough such that there exists a family of finite rank operators $(K_\varepsilon(\theta_0))_{0<h\leq h_1, \varepsilon \in [0,1]}$ with the following properties:*

$$\text{rank}(K_\varepsilon(\theta_0)) \lesssim h^{-d}, \tag{4.23}$$

$$||(H_0(\theta_0) + \varepsilon V(\theta_0) + K_\varepsilon(\theta_0) - z)^{-1}||_{L^2 \to H^{2,0}_{\text{sc}}} \lesssim 1, \tag{4.24}$$

for all $h \in (0, h_1]$, $z \in \Omega$, $\varepsilon \in [0, 1]$. For all $N, s, \sigma \in \mathbb{R}$ and $k \in \mathbb{N}$

$$||\partial_\varepsilon^k K_\varepsilon(\theta_0)||_{H^{s,\sigma}_{\text{sc}} \to H^{N,N}_{\text{sc}}} \lesssim 1, \qquad h \in (0, h_1], \ \varepsilon \in [0, 1]. \tag{4.25}$$

In addition, there exists $\chi \in C_0^\infty(\mathbb{R}^d)$, independent of h and ε, such that $K_\varepsilon(\theta_0) = \chi K_\varepsilon(\theta_0)\chi$.

Note that (4.25) and Lemma 3.4 imply that

$$||\partial_\varepsilon^k K_\varepsilon(\theta_0)||_{\text{tr}} \lesssim h^{-d}, \qquad h \in (0, h_1], \ \varepsilon \in [0, 1]. \tag{4.26}$$

This proposition is essentially proved in [30, 31]. We however recall the main argument of the proof to emphasize the dependence on ε which was not considered in those references.

Lemma 4.12. *For all $\epsilon_1 > 0$ such that $2\pi - 2\theta_0 - 4\epsilon_1 > \epsilon$ and $\epsilon_1 < \epsilon$, and for all $C \gg 1$, we can construct a smooth function $F : D_F \to \mathbb{C}$, with D_F a neighborhood of $e^{i[-2\theta_0 - 4\epsilon_1, \epsilon]}[0, +\infty)$, such that*

$$F(Z) = Z, \quad \text{for } Z \text{ such that } |Z| \notin [C^{-1}, C] \quad \text{or with argument close to } -2\theta_0, \tag{4.27}$$

and

$$|F(Z) - z| \gtrsim 1, \qquad Z \in D_F, \quad z \in \Omega. \tag{4.28}$$

Proof. We can define a function $\arg(Z)$ smooth on $e^{i(-2\theta_0 - 4\epsilon_1', \epsilon')}(0, +\infty)$, with ϵ_1' and ϵ' slightly larger that ϵ_1 and ϵ respectively, such that

$$Z = |Z| \exp(i\arg(Z)), \qquad \arg(Z) \in (-2\theta_0 - 4\epsilon_1', \epsilon').$$

Observe next that, for some $\theta < \theta_0$ and $r_2 > r_1 > 0$,

$$\Omega \subset \{z \in \mathbb{C} \ ; \ r_1 \leq |z| \leq r_2, \ \ -2\theta \leq \arg(z) \leq \epsilon\}. \tag{4.29}$$

We next take C large enough so that $C^{-1} < r_1 < r_2 < C$ and choose $\psi \in C_0^\infty(C^{-1}, C)$ such that $\psi \equiv 1$ near $[r_1, r_2]$. For δ small enough, we also choose $\Theta \in C^\infty(\mathbb{R})$ non decreasing such that

$$\Theta(\alpha) = \begin{cases} \text{const.} \geq -2\theta_0 - 2\delta, & \text{if } \alpha < -2\theta_0 - 2\delta \\ \alpha, & \text{if } |-2\theta_0 - \alpha| \leq \delta \ . \\ \text{const.} \leq -2\theta_0 + 2\delta, & \text{if } \alpha > -2\theta_0 + 2\delta \end{cases}$$

We choose δ such that the sector defined by $-2\theta_0 - 2\delta \leq \arg(Z) \leq -2\theta_0 + 2\delta$ doesn't meet the sector $-2\theta \leq \arg(Z) \leq \epsilon$. We then set

$$F(Z) = |Z| \exp\Big(-2i\Theta(\arg(Z))\psi(|Z|) + i(1 - \psi(|Z|))\arg(Z)\Big).$$

It is clearly smooth where $\arg(Z)$ is defined hence in the sector $e^{i(-2\theta_0 - 4\epsilon_1', \epsilon')}(0, +\infty)$. We have $F(Z) = Z$ for for $|Z| \leq C^{-1}$ and $|Z| \geq C$ so F is smooth near 0. Since $\Theta(\arg(Z)) = \arg(Z)$ if $\arg(Z)$ is close to $-2\theta_0$, we have (4.27). Furthermore, for Z in the right hand side of (4.29), we have $F(Z) - z \neq 0$ otherwise we should have $|z| = |Z| \in [r_1, r_2]$ and then $z = F(Z) = |z| \exp(-2i\Theta(\arg(Z)))$ which is impossible by the choice of δ. This is sufficient to prove (4.28) since $|F(Z)| \to \infty$ as $|Z| \to \infty$. □

Proof of Proposition 4.11. We choose first ϵ_1 small enough and R_1 large enough to ensure that (4.12) and (4.13) hold. We also assume that ϵ_1 satisfies the condition of Lemma 4.12. The full Weyl symbol of $H_0(\theta_0) + \varepsilon V(\theta_0)$ is of the form

$$p_{\varepsilon,\theta_0}(x, \xi, h) + h b_{\varepsilon,\theta_0}(x, \xi, h)$$

with b_{ε,θ_0} polynomial of degree 1 in ξ, and with

$$\begin{aligned} p_{\varepsilon,\theta_0}(x, \xi, h) \quad &= \quad p_\varepsilon^{\text{cl}}\left(\kappa_{\theta_0}(x), {}^t\kappa_{\theta_0}'(x)\xi, h\right) + a_\varepsilon\left(\kappa_{\theta_0}(x), {}^t\kappa_{\theta_0}'(x)\xi, h\right), \\ &=: \quad p_{\varepsilon,\theta_0}^{\text{cl}}(x, \xi) + a_\varepsilon\left(\kappa_{\theta_0}(x), {}^t\kappa_{\theta_0}'(x)\xi, h\right), \end{aligned}$$

where $p_\varepsilon^{\text{cl}}$ is the classical principal symbol and $a_\varepsilon(.,.,h)$ a polynomial of degree 1 in ξ with coefficients in $\mathcal{C}_\rho(\theta_0, R_0, \epsilon_0)$, bounded with respect to $h \in (0, h_0]$ and $\varepsilon \in [0, 1]$. All these symbols are affine (hence smooth) with respect to ε. We then claim that, by possibly increasing R_1, we may also assume that

$$p_{\varepsilon,\theta_0}(x, \xi, h) \in D_F, \tag{4.30}$$

for all $h \ll 1$, $(x, \xi) \in \mathbb{R}^{2d}$ and $\varepsilon \in [0, 1]$. Note first that, with no loss of generality in Lemma 4.12, we may assume that D_F is constructed for $\pi/2 < \theta_0 < \pi$ so that D_F is also a neighborhood of $\mathbb{R}$. Then, for $|x| \leq R_1$, $p_{\varepsilon,\theta_0}(x, \xi, h)$ is real hence belongs to D_F. On the other hand, there exists C_V such that

$$|a_\varepsilon\left(\kappa_{\theta_0}(x), {}^t\kappa_{\theta_0}'(x)\xi, h\right)| \leq C_V R_1^{-\rho}\langle\xi\rangle,$$

for all $R_1 \gg 1$, $|x| \geq R_1$, $\xi \in \mathbb{R}^d$, $h \in (0, h_0]$ and $\varepsilon \in [0, 1]$. Thus, using (4.13) with $p_{\iota,\theta_0}^{\text{cl}} = p_{\varepsilon,\theta_0}^{\text{cl}}$, we see that for any neighborhood of $e^{i[-2\theta_0 - 4\epsilon_1, \epsilon]}[0, +\infty)$, we can choose R_1 large enough such that $p_{\varepsilon,\theta_0}(x, \xi, h)$ belongs to this neighborhood for $|x| \geq R_1$. This implies (4.30) which then shows that $F \circ p_{\varepsilon,\theta_0}$ is smooth on $\mathbb{R}^{2d}$. Actually, we have

$$\psi_{\varepsilon,\theta_0} := F(p_{\varepsilon,\theta_0}) - p_{\varepsilon,\theta_0} \in C_0^\infty(\mathbb{R}^{2d}), \tag{4.31}$$

and, more precisely, $\psi_{\varepsilon,\theta_0}$ is bounded in C_0^∞ as ε and h vary. Indeed, by (4.12), $|p_{\varepsilon,\theta_0}(x,\xi,h)| \to \infty$ as $|\xi| \to \infty$ and, on the other hand, for ξ in a compact set, $p_{\varepsilon,\theta_0}(x,\xi,h) \to e^{-2i\theta_0}|\xi|^2$ as $|x| \to \infty$. Using (4.27), this gives (4.31).

To construct $K_\varepsilon(\theta_0)$, we recall the following point. For all $\Psi \in C_0^\infty(\mathbb{R}^{2d})$, we may write

$$Op_h^w(\Psi) = K(h) + R(h),$$

with $K(h)$ of finite rank, $\mathrm{rank}(K(h)) \lesssim h^{-d}$, and for all $N \geq 0$,

$$||R(h)||_{H_{\mathrm{sc}}^{-N,-N} \to H_{\mathrm{sc}}^{N,N}} \leq Ch^N, \qquad h \ll 1.$$

In addition, for some fixed $\chi \in C_0^\infty(\mathbb{R}^d)$,

$$K(h) = \chi K(h) \chi.$$

Let us now choose $\Psi \in C_0^\infty(\mathbb{R}^{2d})$ such that $\Psi \equiv 1$ near a compact set (independent of h and ε) containing the support of $\psi_{\varepsilon,\theta_0}$. We then have

$$Op_h^w(\psi_{\varepsilon,\theta_0}) = K(h) Op_h^w(\psi_{\varepsilon,\theta_0}) K(h) + R_{\varepsilon,\theta_0}(h)$$

with, for all $N \geq 0$,

$$||R_{\varepsilon,\theta_0}(h)||_{H_{\mathrm{sc}}^{-N,-N} \to H_{\mathrm{sc}}^{N,N}} \leq Ch^N, \qquad h \ll 1, \ \varepsilon \in [0,1],$$

using that $Op_h^w(\psi_{\varepsilon,\theta_0}) = Op_h^w(\Psi) Op_h^w(\psi_{\varepsilon,\theta_0}) Op_h^w(\Psi) + \mathcal{O}(h^\infty)$ by pseudodifferential calculus. We then set

$$K_\varepsilon(\theta_0) := K(h) Op_h^w(\psi_{\varepsilon,\theta_0}) K(h).$$

It satisfies (4.23), (4.25) and has a Schwartz kernel supported in a fixed compact set. To get (4.24), we simply observe that

$$H_0(\theta_0) + \varepsilon V(\theta_0) + K_\varepsilon(\theta_0) - z = Op_h^w(F(p_{\varepsilon,\theta_0}) - z) + hT_\varepsilon(\theta_0),$$

with $||T_\varepsilon(\theta_0)||_{H_{\mathrm{sc}}^{2,0} \to L^2} \lesssim 1$ as $h \ll 1$ and $\varepsilon \in [0,1]$. By (4.28), $Op_h^w(F(p_{\varepsilon,\theta_0}) - z)$ is invertible for h small enough (uniformly with respect to ε and $z \in \Omega$) and so is $Op_h^w(F(p_{\varepsilon,\theta_0}) - z) + hT_\varepsilon(\theta_0)$ by an elementary perturbation argument. □

Using the notation of Sjöstrand-Zworski [32], we now set

$$\widehat{H_\varepsilon(\theta_0)} = H_0(\theta_0) + \varepsilon V(\theta_0) + K_\varepsilon(\theta_0), \tag{4.32}$$

and

$$\widetilde{K}_\varepsilon(\theta_0, z) = -K_\varepsilon(\theta_0)(\widehat{H_\varepsilon(\theta_0)} - z)^{-1}, \tag{4.33}$$

or, equivalently,

$$1 + \widetilde{K}_\varepsilon(\theta_0, z) = (H_0(\theta_0) + \varepsilon V(\theta_0) - z)(\widehat{H_\varepsilon(\theta_0)} - z)^{-1} \tag{4.34}$$

for all $z \in \Omega \setminus \mathrm{Res}(H_0 + \varepsilon V, \Omega)$. We then have (see [31])

$$\begin{aligned} \mathrm{tr}\left((H_0(\theta_0) + \varepsilon V(\theta_0) - z)^{-1} - (\widehat{H_\varepsilon(\theta_0)} - z)^{-1}\right) &= -\mathrm{tr}\left((1 + \widetilde{K}_\varepsilon(\theta_0, z))^{-1} \partial_z \widetilde{K}_\varepsilon(\theta_0, z)\right) \\ &= -\partial_z \log \det{}_1 \left(1 + \widetilde{K}_\varepsilon(\theta_0, z)\right). \end{aligned} \tag{4.35}$$

Remark that the zeroes of $\det_1(1 + \widetilde{K}_\varepsilon(\theta_0, z))$ are contained in the set of resonances since, if z is not a resonance, (4.34) is invertible. Actually, the zeroes of $\det_1(1 + \widetilde{K}_\varepsilon(\theta_0, z))$ in Ω are exactly the resonances of $H_0 + \varepsilon V$ in Ω with the same multiplicities (see Definition 4.5). More precisely we recall the following result (see [31]).

Proposition 4.13. *If $w \in \mathrm{Res}(H_0+V,\Omega)$, there exists a holomorphic function $G_w(z)$, for z close to w, such that $G_w(w) \neq 0$ and*

$$\det{}_1\left(1+\widetilde{K}_1(\theta_0,z)\right) = (z-w)^{m(w)}G_w(z), \tag{4.36}$$

where $m(w)$ is the multiplicity of the resonance.

Proof. Let $l(w)$ be the multiplicity of w as zero of $\det_1\left(1+\widetilde{K}_1(\theta_0,z)\right)$ given by

$$l(w) = \frac{1}{2i\pi}\int_\gamma \partial_z \log\det{}_1\left(1+\widetilde{K}_1(\theta_0,z)\right)dz, \tag{4.37}$$

with γ a small positively oriented circle centered at w. According to (4.35), we have

$$\begin{aligned} l(w) &= \frac{i}{2\pi}\int_\gamma \mathrm{tr}\left((H_0(\theta_0)+V(\theta_0)-z)^{-1}K_1(\theta_0)(\widehat{H_1(\theta_0)}-z)^{-1}\right)dz \\ &= \frac{i}{2\pi}\mathrm{tr}\left(\int_\gamma (H_0(\theta_0)+V(\theta_0)-z)^{-1}-(\widehat{H_1(\theta_0)}-z)^{-1}dz\right). \end{aligned}$$

By construction of $\widehat{H_1(\theta_0)}$, the resolvent $(\widehat{H_1(\theta_0)}-z)^{-1}$ is holomorphic near w and its integral on γ vanishes. It follows that

$$l(w) = \mathrm{tr}\left(\frac{i}{2\pi}\int_\gamma (H_0(\theta_0)+V(\theta_0)-z)^{-1}dz\right) = \mathrm{tr}(\Pi_{\theta_0,w}),$$

where $\Pi_{\theta_0,w}$, defined by (4.15), is a projector which (by definition of the multiplicity $m(w)$) satisfies

$$\mathrm{tr}(\Pi_{\theta_0,w}) = \mathrm{rank}(\Pi_{\theta_0,w}) = m(w).$$

This conclude the proof of Proposition 4.13. □

Therefore, the multiplicities of the resonances as zeroes of $\det_1\left(1+\widetilde{K}_1(\theta_0,z)\right)$ or as given by Definition 4.5 coincide and we have the factorization

$$\det{}_1\left(1+\widetilde{K}_1(\theta_0,z)\right) = \prod_{w\in\mathrm{Res(H_0+V,\Omega)}}(z-w)G_1(z,h) \tag{4.38}$$

where, for each $h \in (0,h_1]$, $G_1(.,h)$ is a non vanishing holomorphic function on Ω.

We now recall a beautiful result due to Sjöstrand which is a crucial consequence of Proposition 4.11.

Proposition 4.14 ([30])**.** *There exists $\varphi^{G_1}(.,h)$ holomorphic on Ω such that*

$$G_1(z,h) = \exp\left(\varphi^{G_1}(z,h)\right), \qquad h \ll 1,\ z\in\Omega,$$

and, for all $W \Subset \Omega$

$$|\partial_z\varphi^{G_1}(z,h)| \le C_W h^{-d}, \qquad h\ll 1,\ z\in W.$$

An immediate consequence of (4.38) and Proposition 4.14 is that, for all $W \Subset \Omega$,

$$\left|\partial_z^k \log\det{}_1\left(1+\widetilde{K}_1(\theta_0,z)\right) - \sum_{w\in\mathrm{Res(H_0+V,\Omega)}} \frac{(k-1)!}{(w-z)^k}\right| \le C_W h^{-d}, \tag{4.39}$$

for $h \ll 1$ and $z \in W$. The same result applied with $V \equiv 0$, using that H_0 has no resonances, shows that

$$\left|\partial_z^k \log\det{}_1\left(1+\widetilde{K}_0(\theta_0,z)\right)\right| \le C_W h^{-d}, \tag{4.40}$$

for $h \ll 1$ and $z \in W$.

Another useful consequence of the absence of resonance for H_0 is the following. Since H_0 has no resonances, $H_0(\theta_0) - z$ is invertible for all $h \ll 1$ and all z in a neighborhood of $\overline{\Omega}$. Therefore, for all $h \ll 1$, there exists ε_h, such that $H_0(\theta_0) + \varepsilon V(\theta_0) - z$ is invertible for $|\varepsilon| < \varepsilon_h$ and $z \in \Omega$. Thus, by (4.34), the function

$$G_\varepsilon(z,h) := \det{}_1\left(1+\widetilde{K}_\varepsilon(\theta_0,z)\right), \qquad z\in\Omega,\ \varepsilon\in(-\varepsilon_h,\varepsilon_h), \tag{4.41}$$

is holomorphic and doesn't vanish. This allows to choose a branch of its logarithm which we denote by $\mathrm{Log}_h\ G_\varepsilon(z,h)$, to stress on the h dependence of such a choice.

Proposition 4.15. *The branch* $\mathrm{Log}_h\ G_\varepsilon(z,h)$ *can be chosen such that, given a fixed* $z_0 \in \Omega_\delta^+$, *we have, for all* $j \ge 0$, $l \ge 1$,

$$\left|\frac{\partial^l}{\partial z^l}\frac{\partial^j}{\partial\varepsilon^j}\mathrm{Log}_h\ G_\varepsilon(z_0,h)_{|\varepsilon=0}\right| \lesssim h^{-d}.$$

Proof. According to (4.40), $G_0(z,h) = \exp(\varphi^{G_0}(z,h))$ with $|\partial_z\varphi^{G_0}(z,h)| \lesssim h^{-d}$. On the other hand, for all $h \ll 1$, we can find $\varepsilon(z_0,h) > 0$ such that

$$\left|\frac{G_\varepsilon(z_0,h)}{G_0(z_0,h)} - 1\right| \le 1/2, \qquad |\varepsilon| \le \varepsilon(z_0,h)$$

thus we can set

$$\mathrm{Log}_h\ G_\varepsilon(z_0,h) = \varphi^{G_0}(z_0,h) + \log\left(\frac{G_\varepsilon(z_0,h)}{G_0(z_0,h)}\right) \tag{4.42}$$

where log is the principal determination of the logarithm on $\mathbb{C}\setminus(-\infty,0]$. We can then define $\mathrm{Log}_h\ G_\varepsilon(z,h)$ as the unique primitive of $\partial_z G_\varepsilon(z,h)/G_\varepsilon(z,h)$ coinciding with the right hand side of (4.42) at $z = z_0$. The smoothness with respect to z and ε (close to 0) is then clear. The bounds on the derivatives at $z = z_0$ and $\varepsilon = 0$ are obtained by applying $\partial_\varepsilon^k\partial_z^{l-1}$ to (4.35), using Proposition 4.11 and (4.16). ◻

Regarding the behavior of $\partial_\varepsilon^j\mathrm{Log}_h\ G_\varepsilon(z,h)_{|\epsilon=0}$ for $z \in \Omega$, we have the following result.

Proposition 4.16. *For all* $j \ge 0$, $l \ge 1$, *there exists* $N_{j,l} \in \mathbb{N}$ *such that, for all compact subset* $W \Subset \Omega$,

$$\left|\partial_\epsilon^j\partial_z^l\mathrm{Log}_h\ G_\varepsilon(z,h)_{|\epsilon=0}\right| \le C_W h^{-d}\sup_{Z\in\Omega}\left(1+||(H_0(\theta_0)-Z)^{-1}||_{L^2\to H^{2,0}_{\mathrm{sc}}}\right)^{N_{j,l}}, \qquad h\ll 1,\ z\in W.$$

Proof. By writing $\mathrm{Log}_h\ G_\varepsilon(z,h)_{|\epsilon=0}$ as the sum of $\mathrm{Log}_h\ G_\varepsilon(z_0,h)_{|\epsilon=0}$ and the integral of its derivative over a path joining z_0 to z, the result follows from (4.35), (4.41), Proposition 4.11. and Proposition 4.15. ◻

5 Proofs of Theorems 1.3, 1.5 and 1.7

5.1 The general case

Using the notation (3.17), we have, for $k > d/2$,

$$\zeta_p(k,z,h) = T_p^k(H_0, V, z), \qquad h \ll 1, \quad z \in \Omega^+, \tag{5.1}$$

and, by Proposition 4.9, we also have, if $k > d/2 + 1$ which we now assume,

$$T_p^k(H_0, V, z) = T_p^k(H_0(\theta_0), V(\theta_0), z), \qquad h \ll 1, \quad z \in \Omega^+. \tag{5.2}$$

To analyze the right hand side of (5.2), we consider first

$$\widehat{T}_p^k(\theta_0, z, h) := tr\left((\widehat{H_1(\theta_0)} - z)^{-k} - \sum_{j=0}^{p-1} \frac{1}{j!} \frac{d^j}{d\varepsilon^j} (\widehat{H_\varepsilon(\theta_0)} - z)^{-k}{}_{|\varepsilon=0} \right),$$

where $\widehat{H_\varepsilon(\theta_0)}$ is defined by (4.32).

Lemma 5.1. *For all $h \ll 1$, the function $\widehat{T}_p^k(\theta_0, z, h)$ is well defined, has an holomorphic continuation from Ω^+ to Ω and, for all $W \Subset \Omega$,*

$$|\widehat{T}_p^k(\theta_0, z, h)| \le C_W h^{-d}, \qquad h \ll 1, \ z \in W.$$

Proof. Write first that

$$\frac{d}{d\varepsilon}(\widehat{H_\varepsilon(\theta_0)} - z)^{-1} = -(\widehat{H_\varepsilon(\theta_0)} - z)^{-1}\left(V(\theta_0) + \partial_\epsilon K_\varepsilon(\theta_0)\right)(\widehat{H_\varepsilon(\theta_0)} - z)^{-1}. \tag{5.3}$$

Then, an elementary induction shows that the operator

$$\frac{d^j}{d\varepsilon^j}(\widehat{H_\varepsilon(\theta_0)} - z)^{-1} - j!\left(-(\widehat{H_\varepsilon(\theta_0)} - z)^{-1} V(\theta_0)\right)^j (\widehat{H_\varepsilon(\theta_0)} - z)^{-1}$$

is a linear combination of holomorphic finite rank operators with trace norm of order h^{-d}, for all j. This formula for $j = p$ combined with Taylor's formula and Proposition 4.11 shows that the operator

$$(\widehat{H_1(\theta_0)} - z)^{-1} - \sum_{j=0}^{p-1} \frac{1}{j!}\frac{d^j}{d\varepsilon^j}(\widehat{H_\varepsilon(\theta_0)} - z)^{-1}_{|\varepsilon=0} + p\int_0^1 (\varepsilon - 1)^{p-1}\left((\widehat{H_\varepsilon(\theta_0)} - z)^{-1} V(\theta_0)\right)^p (\widehat{H_\varepsilon(\theta_0} - z)^{-1} d\varepsilon$$

is a linear combination of holomorphic trace class operators with norm $\mathcal{O}(h^{-d})$, locally uniformly on compact subsets of Ω. Using (3.16), Proposition 3.3 and (4.24), the k-th derivative of the operator in the integral above is trace class, holomorphic on Ω and with trace norm $\mathcal{O}(h^{-d})$, locally uniformly with respect to z. The result follows. $\square$

Using (3.16) and (5.2), we obtain

$$T_p^k(H_0, V, z) = \widehat{T}_p^k(\theta_0, z, h) + \frac{1}{(k-1)!}\partial_z^{k-1} A(z, h), \qquad h \ll 1, \ z \in \Omega^+, \tag{5.4}$$

where

$$A(z,h) = \operatorname{tr}\Big((H_0(\theta_0)+V(\theta_0)-z)^{-1} - (\widehat{H_1(\theta_0)}-z)^{-1}\Big)$$
$$-\sum_{j=0}^{p-1}\frac{1}{j!}\frac{d^j}{d\varepsilon^j}\operatorname{tr}\Big((H_0(\theta_0)+\varepsilon V(\theta_0)-z)^{-1} - (\widehat{H_\varepsilon(\theta_0)}-z)^{-1}\Big)_{|\varepsilon=0},$$

that is

$$-A(z,h) = \partial_z \log\ \det{}_1\Big(1+\widetilde{K}_1(\theta_0,z)\Big) - \partial_z\sum_{j=0}^{p-1}\frac{1}{j!}\frac{d^j}{d\varepsilon^j}\ \operatorname{Log}_h G_\varepsilon(z,h)_{|\varepsilon=0}. \tag{5.5}$$

by (4.35), (4.41) and the notation of Propositions 4.15 and 4.16.

Proof of Theorems 1.3 and 1.7. By (5.1), (5.2), (5.4), (5.5) and (4.38) we have an expression of the form (1.30) with

$$\phi_p(z,h) = \widehat{T}_p^k(\theta_0,z,h) - \frac{1}{(k-1)!}\partial_z^k\Big(\varphi^{G_1}(z,h) - \sum_{j=0}^{p-1}\frac{1}{j!}\frac{d^j}{d\varepsilon^j}\ \operatorname{Log}_h G_\varepsilon(z,h)_{|\varepsilon=0}\Big) \tag{5.6}$$

which is holomorphic on Ω. This proves Theorem 1.3 using Proposition 1.8 with $\mathcal{H}(\Omega,h_1)$ the set of families of holomorphic functions on Ω.

To prove Theorem 1.7, we simply additionally note that, by Proposition 4.14 and Proposition 4.16, we can find $N>0$ such that, for all $W \Subset \Omega$,

$$|\phi_p(z,h)| \le C_W h^{-d}\sup_{Z\in\Omega}\Big(1+||(H_0(\theta_0)-Z)^{-1}||_{L^2\to H^{2,0}_{\rm sc}}\Big)^N, \qquad h\ll 1,\ z\in W. \tag{5.7}$$

Then, Proposition 1.8 gives the result using the space $\mathcal{H}(\Omega,h_1)$ of families of holomorphic functions locally bounded by (a constant times) $h^{-d}\sup_{Z\in\Omega}\Big(1+||(H_0(\theta_0)-Z)^{-1}||_{L^2\to H^{2,0}_{\rm sc}}\Big)^N$. Note that it satisfies (1.28) and (1.29). □

5.2 Proof of Theorem 1.5

In this subsection, $\mathcal{H}(\Omega,h_1)$ denotes the space of families of holomorphic functions $(\phi(.,h))_{h\in(0,h_1]}$ such that, for all $W\Subset\Omega$, $|\phi(z,h)|\le C_W h^{-d}$, for $z\in W$ and $h\in(0,h_1]$.

For $p=1$, the result can be considered as essentially a consequence of [8]. For completeness, we give the proof. In that case, ϕ_1 (given by (5.6) with $p=1$) belongs to $\mathcal{H}(\Omega,h_1)$ according to Lemma 5.1, Proposition 4.14 and (4.40). The result follows then from Proposition 1.8.

In the case $p=2$, (5.6) gives

$$\phi_2(z,h) - \widehat{T}_2^k(\theta_0,z,h) + \frac{1}{(k-1)!}\partial_z^k\varphi^{G_1}(z,h) =$$

$$\frac{1}{(k-1)!}\partial_z^k\ \operatorname{Log}_h G_0(z,h)_{|\varepsilon=0} + \operatorname{tr}\Big(\frac{d}{d\varepsilon}(\widehat{H_\varepsilon(\theta_0)}-z)^{-k}_{|\varepsilon=0} - \frac{d}{d\varepsilon}(H_0(\theta_0)+\varepsilon V(\theta_0)-z)^{-k}_{|\varepsilon=0}\Big). \tag{5.8}$$

By Lemma 5.1, Proposition 4.14 and (4.40), it remains to study the second term of (5.8). We first remark that this term can be written as the sum of

$$\operatorname{tr}\left(\frac{d}{d\varepsilon}(\widehat{H_0(\theta_0)}+\varepsilon V(\theta_0)-z)^{-k}_{|\varepsilon=0}-\frac{d}{d\varepsilon}(H_0(\theta_0)+\varepsilon V(\theta_0)-z)^{-k}_{|\varepsilon=0}\right) \tag{5.9}$$

and

$$-\partial_z^{k-1}\operatorname{tr}\left((\widehat{H_0(\theta_0)}-z)^{-1}\partial_\varepsilon K_\varepsilon(\theta_0)_{|\varepsilon=0}(\widehat{H_0(\theta_0)}-z)^{-1}\right)/(k-1)!,$$

using (4.32) and (5.3). This last expression clearly belongs to $\mathcal{H}(\Omega,h_1)$ by Proposition 4.11 and we are left with the study of (5.9).

For that purpose, we use the approximation V_n of V introduced in Lemma 4.10. Using (3.8), Lemma 3.4 and an elementary cyclicity argument, we can write

$$\operatorname{tr}\left(\frac{d}{d\varepsilon}(\widehat{H_0(\theta)}+\varepsilon V_n(\theta)-z)^{-k}_{|\varepsilon=0}\right)=-k\operatorname{tr}\left(V_n(\theta)(\widehat{H_0(\theta)}-z)^{-k-1}\right). \tag{5.10}$$

Writing $\frac{d}{d\varepsilon}(H_0(\theta)+\varepsilon V_n(\theta)-z)^{-k}$ as the derivative of $(H_0(\theta)+\varepsilon V_n(\theta)-z)^{-k}-(H_0(\theta)-z)^{-k}$ with respect to ε and using Proposition 4.8, we obtain similarly

$$\operatorname{tr}\left(\frac{d}{d\varepsilon}(H_0(\theta)+\varepsilon V_n(\theta)-z)^{-k}_{|\varepsilon=0}\right)=-k\operatorname{tr}\left(V_n(H_0-z)^{-k-1}\right). \tag{5.11}$$

Substracting $-k\operatorname{tr}\left(V_n(\theta)(e^{-2i\theta}H_0-z)^{-k-1}\right)$ to (5.10) and (5.11) and then letting $n\to\infty$ using Proposition 3.2, (5.9) can thus be written as the sum of

$$-k\operatorname{tr}\left(V(\theta)\big((\widehat{H_0(\theta)}-z)^{-k-1}-(e^{-2i\theta}H_0-z)^{-k-1}\big)\right) \tag{5.12}$$

and

$$\lim_{n\to\infty}k\operatorname{tr}\left(V_n(H_0-z)^{-k-1}-V_n(\theta)(e^{-2i\theta}H_0-z)^{-k-1}\right). \tag{5.13}$$

Proposition 5.2. *(5.12) belongs to $\mathcal{H}(\Omega,h_1)$.*

Proof. By the resolvent identity, (4.4) and Proposition 4.11, we have

$$(\widehat{H_0(\theta)}-z)^{-1}-(e^{-2i\theta}H_0-z)^{-1}=(\widehat{H_0(\theta)}-z)^{-1}B(h)(e^{-2i\theta}H_0-z)^{-1},$$

with $B(h)=Op_h^w(b(h))$ for some family $(b(h))_{h\ll 1}$ bounded in $S^{2,-N}$ for all N. Using (3.16), the operator $V(\theta)\big((\widehat{H_0(\theta)}-z)^{-k-1}-(e^{-2i\theta}H_0-z)^{-k-1}\big)$ is therefore a linear combination of operators of the form

$$V(\theta)(\widehat{H_0(\theta)}-z)^{-k_1-1}B(h)(e^{-2i\theta}H_0-z)^{-k_2-1},\qquad k_1+k_2=k.$$

By (3.4), (4.24) and Lemma 3.4, each such operator has a trace norm of order h^{-d}, uniformly with respect to $z\in\Omega$, so the result follows. □

Proposition 5.3. *(5.13) belongs to $\mathcal{H}(\Omega,h_1)$.*

Proof. The operators $V_n(H_0-z)^{-k-1}$ and $V_n(\theta)(e^{-2i\theta}H_0-z)^{-k-1}$ are both trace class so we compute their traces separately. By writing

$$V_n(\theta)=\sum_{|\alpha|\leq 2}v_{n,\alpha,\theta}(x,h)(hD)^\alpha,$$

we first have

$$\operatorname{tr}\left(V_n(\theta)(e^{-2i\theta}H_0-z)^{-k-1}\right)=(2\pi h)^{-d}\iint_{\mathbb{R}^{2d}}\sum_{|\alpha|\le 2}v_{n,\alpha,\theta}(x,h)\xi^\alpha(e^{-2i\theta}\xi^2-z)^{-k-1}dxd\xi. \quad (5.14)$$

This holds also for $\theta=0$ which gives an expression for $\operatorname{tr}\left(V_n(H_0-z)^{-k-1}\right)$. In the latter case, deforming $\mathbb{R}^d_\xi$ into $e^{-i\theta}\mathbb{R}^d_\xi$, we get

$$\operatorname{tr}\left(V_n(H_0-z)^{-k-1}\right)=(2\pi h)^{-d}\iint_{\mathbb{R}^{2d}}\sum_{|\alpha|\le 2}v_{n,\alpha,0}(x,h)(e^{-i\theta}\xi)^\alpha(e^{-2i\theta}\xi^2-z)^{-k-1}e^{-id\theta}d\xi dx,$$

and the last integral can be rewritten as

$$(2\pi h)^{-d}\iint_{\mathbb{R}^{2d}}\sum_{|\alpha|\le 2}v_{n,\alpha,0}(\kappa_\theta(x),h)(e^{-i\theta}\xi)^\alpha(e^{-2i\theta}\xi^2-z)^{-k-1}e^{-id\theta}d\xi\ \det(\partial_x\kappa_\theta(x))dx. \quad (5.15)$$

To justify this last deformation, one simply notices that $\int v_{n,\alpha,0}(\kappa_\theta(x),h)\ \det(\partial_x\kappa_\theta(x))dx$ depends holomorphically on θ and that it is constant for $i\theta$ real and close to zero since κ_θ is then a diffeomorphism from $\mathbb{R}^d$ to itself. Now for $|x|\ge R$ large enough, (independent of n), we have $\kappa_\theta(x)=e^{i\theta}x$ and

$$v_{n,\alpha,0}(\kappa_\theta(x),h)e^{-i|\alpha|\theta}=v_{n,\alpha,\theta}(x,h),\qquad e^{-id\theta}\ \det(\partial_x\kappa_\theta(x))=1.$$

Therefore, if we set

$$c_{n,\alpha,\theta}(x,h)=v_{n,\alpha,0}(\kappa_\theta(x),h)e^{-i|\alpha|\theta}e^{-id\theta}\ \det(\partial_x\kappa_\theta(x))-v_{n,\alpha,\theta}(x,h)$$

which is compactly supported, we have

$$(5.13)=\lim_{n\to\infty}k\sum_{|\alpha|\le 2}(2\pi h)^{-d}\int_{\mathbb{R}^d}\xi^\alpha(e^{-2i\theta}\xi^2-z)^{-k-1}d\xi\times\int_{|x|\le R}c_{n,\alpha,\theta}(x,h)dx,$$

which is easily seen to belong to $\mathcal{H}(\Omega,h_1)$. □

The conclusion follows then from (5.8), Propositions 5.2, 5.3 and 1.8. □

6 A counter example for $p=3$

In this section, we prove Theorem 1.6. We consider $H_0=-h^2\frac{d^2}{dx^2}$ on $L^2(\mathbb{R})$ and V a compactly supported bounded potential. In that case $V(H_0-z)^{-1}$ is in the trace class for all $z\notin[0,+\infty)$ hence in any Schatten class $\mathbf{S}_p$. For trace class operators $K\in\mathbf{S}_1$, the formula (1.3) can be written

$$\operatorname{Det}_p(I+K)=\operatorname{Det}_1(I+K)\exp\left(\sum_{j=1}^{p-1}\frac{(-1)^j}{j}\operatorname{tr}(K^j)\right).$$

We therefore obtain

$$D_3(H_0,H_0+V;z,h)=D_2(H_0,H_0+V;z,h)e^{\frac{1}{2}\phi(z,h)} \quad (6.1)$$

where

$$\phi(z,h) = \operatorname{tr}\left(V(H_0-z)^{-1}V(H_0-z)^{-1}\right).$$

For $z=k^2$ with $\operatorname{Im}(k)>0$, the integral kernel of $(H_0-z)^{-1}$ is $ie^{ik|x-x'|/h}/(2hk)$ and $\phi(z,h)$ can be computed explicitly, namely

$$\begin{aligned}\phi(k^2,h) &= \frac{-1}{(2hk)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} V(x)V(x')e^{2ih^{-1}k|x-x'|}dxdx', \\ &= \frac{-1}{(2hk)^2}\int_{\mathbb{R}} \widetilde{V}(y)e^{2ih^{-1}k|y|}dy, \end{aligned} \tag{6.2}$$

with

$$\widetilde{V}(y) = \int_{\mathbb{R}} V(x)V(x-y)dx. \tag{6.3}$$

Setting

$$\widetilde{V}^+_{\mathrm{ev}}(y) = \mathbf{1}_{[0,+\infty)}\left(\widetilde{V}(y)+\widetilde{V}(-y)\right),$$

we have

$$\phi(k^2,h) = -\frac{2\pi}{(2kh)^2}(\mathcal{F}_{\mathrm{inv}}\widetilde{V}^+_{\mathrm{ev}})(2kh^{-1}), \tag{6.4}$$

where $\mathcal{F}_{\mathrm{inv}}$ is the usual inverse Fourier transform

$$\mathcal{F}_{\mathrm{inv}}g(\xi) = \frac{1}{2\pi}\int e^{ix\xi}g(x)dx.$$

For example, for the characteristic function $V(x)=\chi_a(x):=\mathbf{1}_{[-a,a]}(x)$, we have

$$\widetilde{V}(y) = \begin{cases}(2a-y)_+ & \text{if } y\geq 0,\\ (2a+y)_+ & \text{if } y<0,\end{cases}$$

where $(t)_+=\max(t,0)$. After elementary computations, we also obtain in this explicit case

$$\phi(k^2,h) = \frac{-ia}{2k^3h} + \frac{1}{8k^4}(e^{4iah^{-1}k}-1).$$

For $k=z^{1/2}$ with $\operatorname{Im}(k)<0$, which makes sense at least close to 1, this examples shows that

$$|\partial_z\phi(z,h)| \gtrsim \exp\left(a|\operatorname{Im}(k)|/h\right), \qquad h\ll 1.$$

This proves that the logarithmic derivative of the corrective factor in (6.1) can indeed blow up exponentially, which is a strong form of the estimate (1.22).

This elementary striking example doesn't however fit in our framework since V is not smooth. In particular, it can not be used directly to prove Theorem 1.6. For the latter proof, we need the following lemma.

Lemma 6.1. *Let $g\in L^\infty(\mathbb{R},\mathbb{R})$ be supported in $[0,b]$, $b>0$, but in no smaller interval. Setting, for all $0<b'<b$ and $h\in(0,1]$,*

$$s_{b'}(h) := \sup_{\substack{1\leq|\xi|\leq 2\\ \operatorname{Im}(\xi)<0,\ \operatorname{Re}(\xi)\geq 0}} |e^{b'\operatorname{Im}(\xi)/h}(\mathcal{F}_{\mathrm{inv}}g)(\xi/h)|,$$

we have

$$\limsup_{h\to 0} s_{b'}(h) = +\infty.$$

Proof. We clearly have

$$|(\mathcal{F}_{\text{inv}}g)(\xi)| \leq \frac{b}{2\pi}||g||_\infty e^{b|\text{Im}(\xi)|}, \qquad \xi \in \mathbb{C},\ \text{Im}(\xi) \leq 0,$$

and $(\mathcal{F}_{\text{inv}}g)$ is bounded for $\text{Im}(\xi) > 0$. Fix $0 < b' < b$. By the Paley-Wiener Theorem, we have

$$\sup_{\text{Im}(\xi)<0} |e^{-b'|\text{Im}(\xi)|}(\mathcal{F}_{\text{inv}}g)(\xi)| = +\infty, \tag{6.5}$$

otherwise g should be supported in $[0, b']$ which is excluded. Furthermore, since g is real valued, we have

$$|(\mathcal{F}_{\text{inv}}g)(\text{Re}(\xi) + i\text{Im}(\xi))| = |(\mathcal{F}_{\text{inv}}g)(-\text{Re}(\xi) + i\text{Im}(\xi))|,$$

so the supremum in (6.5) can be taken over $\text{Re}(\xi) \geq 0$ and $\text{Im}(\xi) < 0$. Then, using the local boundedness of $(\mathcal{F}_{\text{inv}}g)$ and by writing the set $\{\xi \mid \text{Im}(\xi) < 0,\ \text{Re}(\xi) \geq 0\}$ as

$$\{\xi \mid \text{Im}(\xi) < 0,\ \text{Re}(\xi) \geq 0,\ |\xi| < 1\} \sqcup_{k\geq 0} \{\xi \mid \text{Im}(\xi) < 0,\ \text{Re}(\xi) \geq 0,\ 2^k \leq |\xi| < 2^{k+1}\},$$

we have

$$\limsup_{k\to+\infty} \sup_{\substack{2^k \leq |\xi| < 2^{k+1}, \\ \text{Im}(\xi)<0,\ \text{Re}(\xi)\geq 0}} |e^{-b'|\text{Im}(\xi)|}(\mathcal{F}_{\text{inv}}g)(\xi)| = +\infty,$$

and the result follows. □

Proof of Theorem 1.6. Fix $V \in C_0^\infty(\mathbb{R}, \mathbb{R})$ with $V \neq 0$. By Theorem 1.3, we can write

$$D_2^\zeta(H_0, H_0 + V; z, h) = \prod_{w\in\text{Res}(H_0+V,\Omega)} (z - w) \times \exp(\varphi_2(z, h))$$

where, by Theorem 1.5,

$$|\partial_z\varphi_2(z, h)| \lesssim h^{-1}, \qquad z \in W. \tag{6.6}$$

On the other hand, by (1.8), $D_p^\zeta(H_0, H_0 + V; z, h)$ can be replaced by the definition (1.2) using Fredholm determinants. Thus, by (6.1), we have

$$D_3^\zeta(H_0, H_0 + V; z, h) = \prod_{w\in\text{Res}(H_0+V,\Omega)} (z - w) \times e^{\varphi_3(z,h)}$$

where

$$\varphi_3(z, h) := \varphi_2(z, h) + \frac{\phi(z, h)}{2},$$

with ϕ given by (6.4). In particular we have

$$(\partial_z\phi)(z, h) = -\frac{\pi}{2z^{3/2}h^3}(\partial_\xi\mathcal{F}_{\text{inv}}\widetilde{V}_{\text{ev}}^+)(2z^{1/2}h^{-1}) + \frac{\pi}{2z^2h^2}(\mathcal{F}_{\text{inv}}\widetilde{V}_{\text{ev}}^+)(2z^{1/2}h^{-1}) = \frac{\pi}{2z^2h^2}f(2z^{1/2}h^{-1}),$$

where

$$f(\xi) := (\mathcal{F}_{\text{inv}}\widetilde{V}_{\text{ev}}^+)(\xi) - \frac{1}{2}\xi\partial_\xi(\mathcal{F}_{\text{inv}}\widetilde{V}_{\text{ev}}^+)(\xi) = (\mathcal{F}_{\text{inv}}g)(\xi),$$

with

$$g(x) := \mathbf{1}_{[0,+\infty)}\Big(\frac{3}{2}\left(\widetilde{V}(x) + \widetilde{V}(-x)\right) + \frac{1}{2}x\partial_x\left(\widetilde{V}(x) + \widetilde{V}(-x)\right)\Big).$$

Since $V \neq 0$, we have $\widetilde{V}(0) = \int V^2 > 0$ so g is supported in an interval $[0, b]$, $b > 0$, and no smaller one. We then obtain (1.22) with $\delta = b/4$, first remarking that, by (6.6),

$$|e^{\delta \mathrm{Im}(z^{1/2})/h} h \partial_z \varphi_2(z, h)| \lesssim 1,$$

secondly that

$$|e^{\delta \mathrm{Im}(z^{1/2})/h} h \partial_z \phi(z, h)| \quad \gtrsim \quad |h^{-1} e^{2\delta \mathrm{Im}(\xi)/h'} (\mathcal{F}_{\mathrm{inv}} g)(\xi/h')|, \qquad \xi = z^{1/2}, \quad h' = h/2,$$

and finally using Lemma 6.1 with $b' = b/2$. □

7 Analytic perturbations

In this section, we briefly prove a result similar to Theorem 1.5 for $p \geq 3$ in the more restrictive situation of analytic perturbations. Namely, we consider V with coefficients analytic close to $x = 0$ (uniformly bounded with respect to h) and such that $V \in \mathcal{V}_\rho(\theta_0, R_0, \epsilon_0)$, for any $R_0 > 0$. We denote by $\mathcal{V}_\rho(\theta_0, 0, \epsilon_0)$ the set of such perturbations V and we assume that $0 < \theta_0 < \pi/2$. Here $\rho > 0$ is arbitrary.

In the following lemma, we first check that we can approximate such operators by fast decaying ones. To avoid any confusion with $\langle x \rangle = (1 + |x_1|^2 + \cdots + |x_d|^2)^{1/2}$, we set

$$\langle\langle x \rangle\rangle = (1 + x_1^2 + \cdots + x_d^2)^{1/2}, \qquad \text{for } x \in \mathbb{C}^d \text{ such that } 1 + x_1^2 + \cdots + x_d^2 \notin (-\infty, 0],$$

using the principal determination of the square root mapping $\mathbb{C} \setminus (-\infty, 0]$ into $e^{i(-\pi/2, \pi/2)}(0, +\infty)$.

Lemma 7.1. *Let*

$$\chi_n(x) = \exp\left(-\langle\langle x \rangle\rangle / n\right), \qquad n \gg 1, \; x \in \mathbb{R}^d.$$

If ϵ_0 is small enough, then, for $n \geq n_0$ large enough,

$$V_n := \chi_n V \chi_n$$

belongs to $\mathcal{V}_{\overline{d}}(\theta_0, 0, \epsilon_0)$ for all $\overline{d} > d$, the sequence $(V_n)_{n \geq n_0}$ is bounded in $\mathcal{V}_\rho(\theta_0, 0, \epsilon_0)$ and, for all $\rho' < \rho$ and all $s, \sigma \in \mathbb{R}$,

$$||V_n - V||_{H^{s,\sigma}_{\mathrm{sc}} \to H^{s-2,\sigma+\rho'}_{\mathrm{sc}}} \to 0, \qquad n \to \infty, \tag{7.7}$$

for all $h \ll 1$.

Proof. The proof is similar to the one of Lemma 4.10 (and anyway fairly elementary). The only new point to check is that the coefficients of V_n belong to $\mathcal{V}_{\overline{d}}(\theta_0, 0, \epsilon_0)$ and are bounded in $\mathcal{V}_\rho(\theta_0, 0, \epsilon_0)$. Indeed, for $r = e^{i\theta} t$, with $t > 0$ and $\theta \in [0, \theta_0]$, and for ω such that $\mathrm{dist}_{\mathbb{C}^d}(\omega, \mathbb{S}^{d-1}) < \epsilon_0$, we first note that, if ϵ_0 is small enough, $r^2 \omega^2 \notin (-\infty, 0]$. Furthermore, if t is large, $1 + r^2 \omega^2 = t^2 e^{2i\theta} (1 + o(1))$, thus

$$\mathrm{Re} \langle\langle r\omega \rangle\rangle \gtrsim t \cos(\theta).$$

It is then easy to check that, for all α, $\partial^\alpha \chi_n$ is bounded on $\Sigma(\theta_0, 0, \epsilon_0)$, uniformly with respect to $n \geq 1$. Since the coefficients of V_n are linear combinations of products of coefficients of V by $\chi_n \partial_x^\alpha \chi_n$, we see that $(V_n)_{n \geq 1}$ is bounded in $\mathcal{V}_\rho(\theta_0, 0, \epsilon_0)$. It also clearly belongs to $\in \mathcal{V}_{\overline{d}}(\theta_0, 0, \epsilon_0)$. □

We next give an elementary deformation result along $e^{i\theta}\mathbb{R}^d$. Let us denote

$$V_{\rm dil}(\theta) := \sum_{|\alpha|\leq 2} v_\alpha(e^{i\theta}x, h)(e^{-i\theta}hD)^\alpha,$$

if $V = \sum_{|\alpha|\leq 2} v_\alpha(x,h)(hD)^\alpha$, that is (4.2) with $\kappa(x) = e^{i\theta}x$ and $P = V$. For $i\theta \in \mathbb{R}$, we also have

$$V_{\rm dil}(\theta) = U_{\rm dil}(i\theta)VU_{\rm dil}(i\theta)^*,$$

where $U_{\rm dil}(t)$ is the generator of dilations introduced for similar purposes in [1]

$$U_{\rm dil}(t)u(x) = e^{td/2}u(e^t x).$$

Lemma 7.2. *Let $k > d/2+1$. For all $n \gg 1$, $\theta \in [0,\theta_0]$, $z \in \Omega^+$ and $j \geq 1$,*

$$\mathrm{tr}\left(\frac{d^j}{d\varepsilon^j}(H_0+\varepsilon V_n - z)^{-k}_{|\varepsilon=0}\right) = \mathrm{tr}\left(\frac{d^j}{d\varepsilon^j}(e^{-2i\theta}H_0+\varepsilon V_{n,\rm dil}(\theta) - z)^{-k}_{|\varepsilon=0}\right). \tag{7.8}$$

Proof. For $i\theta \in \mathbb{R}$, the result is obvious since the right hand side of (7.8) reads

$$\mathrm{tr}\left(\frac{d^j}{d\varepsilon^j}U_{\rm dil}(i\theta)(H_0+\varepsilon V_n - z)^{-k}U_{\rm dil}(i\theta)^*_{|\varepsilon=0}\right).$$

On the other hand, $\theta \mapsto V_{n,\rm dil}(\theta)$ is holomorphic from $(0,\theta_0)+i(-1,0)$ to $\mathcal{L}(H^{s+2,\sigma}_{\rm sc}, H^{s,\sigma+\overline{d}}_{\rm sc})$, for all $s\in\mathbb{N}, \sigma\in\mathbb{R}$ and $\overline{d} > d$. It is also continuous for $\theta \in [0,\theta_0]+i[-1,0]$. Since $e^{-2i\theta}H_0 - z$ is invertible, Proposition 3.2 proves the existence of the resolvent $(e^{-i2\theta}H_0 + \varepsilon V_{n,\rm dil}(\theta) - z)^{-1}$ for ε small enough (depending on h but this harmless for we shall eventually set $\varepsilon = 0$). It is then holomorphic for $\theta \in (0,\theta_0)+i(-1,0)$ and continuous for $\theta\in[0,\theta_0]+i[-1,0]$, with values in $\mathcal{L}(H^{s,\sigma}_{\rm sc}, H^{s+2,\sigma}_{\rm sc})$. Therefore the expression of the right hand side of (7.8) given by Proposition 3.3 is holomorphic with respect to $\theta \in (0,\theta_0)+i(-1,0)$, continuous on $[0,\theta_0]+i[-1,0]$ and constant on $i[-1,0]$ hence constant in $[0,\theta_0]+i[-1,0]$ by analytic continuation. This completes the proof. □

Next, using Propositions 3.1, 3.2, 3.3, Lemma 7.1 and the notation (3.17), we can write, for each $z\in\Omega^+$,

$$\zeta_p(k,z,h) = \lim_{n\to\infty} T^k_p(H_0, V_n, z),$$

that is the limit of

$$\mathrm{tr}\left((H_0+V_n-z)^{-k} - (H_0-z)^{-k}\right) - \sum_{j=1}^{p-1}\frac{1}{j!}\mathrm{tr}\left(\frac{d^j}{d\varepsilon^j}(H_0+\varepsilon V_n - z)^{-k}_{|\varepsilon=0}\right),$$

or, by Lemma 7.2, the limit of

$$\mathrm{tr}\left((H_0+V_n-z)^{-k} - (H_0-z)^{-k}\right) - \sum_{j=1}^{p-1}\frac{1}{j!}\mathrm{tr}\left(\frac{d^j}{d\varepsilon^j}(e^{-2i\theta_0}H_0+\varepsilon V_{n,\rm dil}(\theta_0) - z)^{-k}_{|\varepsilon=0}\right).$$

Observing that Proposition 4.11 can be extended to the sequence V_n (ie that the corresponding finite rank operators $K_n(\theta_0)$ converge as $n\to+\infty$), this limit is the sum of

$$\mathrm{tr}\left((\widehat{H_1(\theta_0)}-z)^{-k} - (\widehat{H_0(\theta_0)}-z)^{-k} - \sum_{j=1}^{p-1}\frac{1}{j!}\frac{d^j}{d\varepsilon^j}(e^{-2i\theta_0}H_0+\varepsilon V_{\rm dil}(\theta_0) - z)^{-k}_{|\varepsilon=0}\right), \tag{7.9}$$

and of

$$-\frac{\partial_z^{k-1}}{(k-1)!}\Big(\partial_z \log \det{}_1\Big(1+\widetilde{K}_1(\theta_0,z)\Big) - \partial_z \log \det{}_1\Big(1+\widetilde{K}_0(\theta_0,z)\Big)\Big) = \sum_{w\in \mathrm{Res}(H_0+V,\Omega)} \frac{1}{(w-z)^k} + \phi(z,h),$$

with $\phi(z,h)$ holomorphic on Ω and $\mathcal{O}(h^{-d})$ locally uniformly. This follows from (4.32), (4.35), (4.38), Proposition 4.14 and from the absence of resonances for H_0. The operator inside the trace in (7.9) is trace class because it is the sum of of

$$(\widehat{H_1(\theta_0)}-z)^{-k} - \sum_{j=0}^{p-1}\frac{1}{j!}\frac{d^j}{d\varepsilon^j}(e^{-2i\theta_0}H_0+\varepsilon V_{\mathrm{dil}}(\theta_0)-z)^{-k}_{|\varepsilon=0}, \tag{7.10}$$

and of

$$(\widehat{H_0(\theta_0)}-z)^{-k} - (e^{-2i\theta_0}H_0-z)^{-k}$$

which is $\mathcal{O}(h^{-d})$ in the trace class for $z\in\Omega$ by Propositions 3.1, 3.3 (recall that $\widehat{H_0(\theta_0)} - e^{-2i\theta_0}H_0$ is compactly supported) and 4.11, using the elementary bound $||(e^{-2i\theta_0}H_0-z)^{-1}||_{L^2\to H^{2,0}_{\mathrm{sc}}} \lesssim 1$. Setting

$$\widehat{V}(\theta_0) := \widehat{H_1(\theta_0)} - (e^{-2i\theta_0}H_0 + V_{\mathrm{dil}}(\theta_0))$$

which is compactly supported, (7.10) is the sum of the trace class operators

$$\frac{1}{j!}\frac{d^j}{d\varepsilon^j}\Big((e^{-2i\theta_0}H_0+\varepsilon V_{\mathrm{dil}}(\theta_0)+\varepsilon\widehat{V}(\theta_0)-z)^{-k} - (e^{-2i\theta_0}H_0+\varepsilon V_{\mathrm{dil}}(\theta_0)-z)^{-k}\Big)_{|\varepsilon=0}$$

and of

$$(e^{-2i\theta_0}H_0+V_{\mathrm{dil}}(\theta_0)+\widehat{V}(\theta_0)-z)^{-k} - \sum_{j=0}^{p-1}\frac{1}{j!}\frac{d^j}{d\varepsilon^j}\Big(e^{-2i\theta_0}H_0+\varepsilon V_{\mathrm{dil}}(\theta_0)+\varepsilon\widehat{V}(\theta_0)-z\Big)^{-k}_{|\varepsilon=0},$$

which are all of order h^{-d} in the trace class, locally uniformly with respect to $z\in\Omega$, by Proposition 3.3, (3.18), Proposition 4.11 and again the estimate $||(e^{-2i\theta_0}H_0-z)^{-1}||_{L^2\to H^{2,0}_{\mathrm{sc}}} \lesssim 1$.

Using Proposition 1.8, we obtain the following theorem.

Theorem 7.3. *Let $\rho>0$ and $p\in\mathbb{N}$ such that $p\rho>d$. Let $\Omega \Subset e^{-i(2\theta_0,\epsilon)}(0,+\infty)$ be a simply connected open subset with $0<\theta_0<\pi/2$, $\epsilon>0$ small enough and satisfying (1.18). Then, if $V\in\mathcal{V}_\rho(\theta_0,0,\epsilon_0)$, any φ_p as in Theorem 1.3 satisfies, for all $W\Subset\Omega$,*

$$|\partial_z\varphi_p(z,h)| \le C_W h^{-d}, \qquad z\in W,\ h\ll 1.$$

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References

[1] J. Aguilar, J.-M. Combes, *A class of analytic perturbations for one-body Schrödinger Hamiltonians*, Comm. Math. Phys. 22, 269–279 (1971).

[2] S. M. Belov, A. V. Rybkin, *Higher order trace formulas of the Buslaev-Faddeev type for the half line Schrödinger operator with long range potentials*, J. Math. Phys. 44, 7, 2748-2761 (2003).

[3] M. Sh. Birman, M. G. Krein, *On the theory of wave operators and scattering operators*, Sov. Math. Dokl. 3, 740-744 (1962).

[4] J.-F. Bony, V. Bruneau and G. Raikov, *Resonances and spectral shift function near the landau levels*, Ann. Inst. Fourier 57, no. 2, 629-671 (2007).

[5] D. Borthwick, C. Judge, P.A. Perry, *Determinants of Laplacians and isopolar metrics on surfaces of infinite area*, Duke Math. J. 118, no. 1, 61-102 (2003).

[6] J.-M. Bouclet, *Trace formulae for relatively Hilbert-Schmidt perturbations*, Asymp. Analysis 32, no. 3-4, 257-291 (2002).

[7] ________, *Spectral distributions for long range perturbations*, J. Funct. Anal. 212, no. 2, 431-471 (2004).

[8] V. Bruneau, V. Petkov, *Meromorphic continuation of the spectral shift function*, Duke Math. J. vol. 116, No. 3, 389-430 (2003).

[9] G. Carron, *Déterminant relatif et la fonction Xi,* Amer. J. Math, vol. 124 n. 2, 307-352 (2002).

[10] M. Dimassi, J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society, Lecture Notes Series 268, Cambridge University Press (1999).

[11] R. Froese, *Asymptotic distribution of resonances in one dimension*, J. Diff. Equa. 137, 251-272 (1997).

[12] ________, *Upper bounds for the resonance counting function of Schrödinger operators in odd dimensions* Canad. J. Math. 50 no. 3, 538-546 (1998). Correction in Canad. J. Math. 53 , no. 4, 756-757 (2001).

[13] F. Gesztesy, A. Pushnitski, B. Simon, *On the Koplienko spectral shift function, I. Basics*, Journal of Mathematical Physics, Analysis, Geometry, to appear.

[14] I. C. Gohberg, M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators,* Translations of Mathematical Monographs, Vol. 18. AMS. Providence, R.I. (1969).

[15] C. Guillarmou, *Generalized Krein formula and determinants for even dimensional Poincaré-Einstein manifolds*, preprint, arXiv:math/0512173.

[16] L. Guillopé, M. Zworski, *Scattering asymptotics for Riemann surfaces*, Ann. of Math. (2) 145, n. 3, 597-660 (1997).

[17] M. G. Krein, *Perturbation determinants and a formula for the trace of unitary and selfadjoint operators*, Soviet Math. Dokl. 3, 707-710 (1962).

[18] L. Hörmander, *The Analysis of Linear PDO III*, Springer Verlag (1985).

[19] L. S. Koplienko, *Trace formula for non trace class perturbations*, Siberian Math. J. 25, 735-743 (1984).

[20] ________, *Regularized spectral shift function for one dimensional Schrödinger operator with slowly decreasing potential*, Siberian Math. J. 26, 365-369 (1985).

[21] A. Martinez, *An introduction to semiclassical and microlocal analysis*, Universitext, Springer-Verlag, New-York (2002).

[22] W. Müller, *Relative zeta functions, relative determinants and scattering theory* Comm. Math. Phys. 192, no. 2, 309-347 (1998).

[23] V. Petkov, M. Zworski, *Breit-Wigner approximation and the distribution of resonances*, Comm. Math. Phys. 204, no. 2, 329-351 (1999). Erratum: Comm. Math. Phys. 214, no. 3, 733-735 (2000).

[24] ________, *Semi-classical estimates on the scattering determinant*, Ann. Henri Poincaré 2, no. 4, 675-711 (2001).

[25] D. B. Ray, I. M. Singer, *R-Torsion and the Laplacian on Riemannian manifolds*, Adv. in Math. 7, 145-210 (1971).

[26] D. Robert, *Autour de l'approximation semi-classique*, Progress in mathematics, 68, Birkhaüser, Boston (1987).

[27] A. V. Rybkin, *On a trace formula of the Buslaev-Faddeev type for a long range potential*, J. Math. Phys 40, 3, 1334-1343 (1999).

[28] B. Simon, *Resonances in one dimension and Fredholm determinants*, J. Funct. Anal. 178, 396-420 (2000).

[29] J. Sjöstrand, *A trace formula and review of some estimates for resonances*, p. 377-437 in Microlocal Analysis and Spectral Theory, NATO ASI Series C, vol. 490, Kluwer (1997).

[30] ________, *Resonances for bottles and trace formulae*, Math. Nach. 221, 95-149 (2001).

[31] ________, *Lectures on resonances*, available at: www.math.polytechnique. fr/∼sjoestrand/.

[32] J. Sjöstrand, M. Zworski, *Complex scaling and the distribution of scattering poles*, J. Amer. Math. Soc. 4, n. 4 729-769 (1991).

[33] G. Vodev, *Sharp bounds on the number of scattering poles for perturbations of the Laplacian*, Comm. Math. Phys. 146, 205-216 (1992).

[34] G.N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, Cambridge (1966).

[35] D. Yafaev, *Mathematical Scattering Theory*, AMS, Providence, RI (1992).

[36] M. Zworski, *Sharp polynomial bounds on the number of scattering poles*, Duke Math. J. 59, no. 2, 311-323 (1989).