

# LIEB-THIRRING ESTIMATES FOR NON SELF-ADJOINT SCHRÖDINGER OPERATORS

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ABSTRACT. For general non-symmetric operators  $A$ , we prove that the moment of order  $\gamma \geq 1$  of negative real-parts of its eigenvalues is bounded by the moment of order  $\gamma$  of negative eigenvalues of its symmetric part  $H = \frac{1}{2}[A + A^*]$ . As an application, we obtain Lieb-Thirring estimates for non self-adjoint Schrödinger operators. In particular, we recover recent results by Frank, Laptev, Lieb and Seiringer [11]. We also discuss moment of resonances of Schrödinger self-adjoint operators.

## 1. INTRODUCTION

The well known Lieb-Thirring estimates for negative eigenvalues  $\lambda_1, \dots, \lambda_N$  of self-adjoint Schrödinger operators  $-\Delta + V$  say that

$$(1) \quad \sum_{k=1}^N |\lambda_k|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+d/2} dx.$$

Here,  $V_- := \max(0, -V)$  is the negative part of  $V$  and the operator  $-\Delta + V$  is considered on  $L^2(\mathbb{R}^d, dx)$ . If  $\gamma = 0$ , then (1) is the well known Cwickel-Lieb-Rozenblum estimate on the number of negative eigenvalues. The estimate (1) and its analogues are of importance in many problems of mathematical physics. We refer the reader to Lieb [17] for a discussion and applications of (1). In the last years, there is an increasing interest for non-self-adjoint Schrödinger operators. We refer to the review paper of Davies [8]. Abramov, Aslanyan and Davies

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[1] have proved in the one dimensional case (i.e.,  $d = 1$ ) that if  $\lambda \notin \mathbb{R}^+$  is an eigenvalue of the non-self-adjoint Schrödinger operator  $-\Delta + V$  with complex-valued potential  $V$ , then

$$|\lambda| \leq \frac{1}{4} \left( \int_{\mathbb{R}} |V| dx \right)^2.$$

More recently, Frank, Laptev, Lieb and Seiringer [11] proved that if  $\lambda_1, \dots, \lambda_N$  are eigenvalues of  $-\Delta + V$  (with complex-valued potential  $V$ ) such that  $\operatorname{Re} \lambda_j < 0$ , then

$$(2) \quad \sum_j (-\operatorname{Re} \lambda_j)^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} (\operatorname{Re} V)_-^{\gamma+d/2} dx$$

for  $\gamma \geq 1$ . The constant  $L_{\gamma,d}$  is the same as in (1). This gives the analogue of (1) for Schrödinger operators with complex valued potentials. Similar estimates are also obtained for one- and multi-dimensional complex Jacobi matrices by Golinskii-Kupin [12]. We note in passing that (2) for  $\gamma \in [0, 1[$  is an open question.

In this note we prove in an abstract setting that if  $A$  is a given operator on a Hilbert space,  $H := \frac{1}{2}[A + A^*]$  is its symmetric part (see the next section for the precise definitions), and  $\gamma \geq 1$  is any constant, then

$$(3) \quad \sum_j (-\operatorname{Re} \lambda_j)^\gamma \leq \operatorname{Tr}(H)_-^\gamma.$$

Here,  $\operatorname{Tr}(H)_-^\gamma := \sum_j (-\mu_j)^\gamma$  where  $\mu_j$  are the negative eigenvalues of the self-adjoint operator  $H$  and  $\lambda_j$  are eigenvalues of  $A$  with negative real-parts. If  $A = -\Delta + V$ , then  $H = -\Delta + \operatorname{Re} V$  and hence using (1) for the self-adjoint operator  $H$  we obtain (2). We also obtain other estimates for moments of eigenvalues of more general differential operators  $A$ . As a consequence, we obtain estimates for moments of resonances of (self-adjoint) Schrödinger operators.

We mention that (3) can be considered as an extension to infinite dimension of Ky Fan's result (see Bhatia [3], p. 74). It follows from a simple variational lemma (see Lemmas 2 and 3). Such lemma, well known in finite dimensional space (see [3], p. 24), is also used by Fournais-Kachmar [10]. Since the proofs are natural and elementary, we state this lemma in an abstract setting and give all the details of proof.

## 2. MOMENTS OF NEGATIVE EIGENVALUES FOR NON-SELF-ADJOINT OPERATORS

Let  $\mathcal{H}$  be a complex Hilbert space. We denote by  $\langle \cdot, \cdot \rangle$  its scalar product and by  $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$  the corresponding norm.

Let  $\mathfrak{a}$  be a densely defined continuous and closed (non-symmetric) form on  $\mathcal{H}$ . We assume that  $\mathfrak{a}$  is bounded from below. This means that there exists a constant  $\eta$  such

$$\operatorname{Re} \mathfrak{a}(u, u) + \eta \|u\|^2 \geq 0 \text{ for all } u \in D(\mathfrak{a}).$$

We shall denote by  $A$  its associated operator (see for example Chapter 1 in [20] for the definitions). Let  $\mathfrak{h}$  be the symmetric part of  $\mathfrak{a}$ , that is,

$$\mathfrak{h}(u, v) := \frac{1}{2} \left( \mathfrak{a}(u, v) + \overline{\mathfrak{a}(v, u)} \right) \text{ for all } u, v \in D(\mathfrak{h}) = D(\mathfrak{a}).$$

The form  $\mathfrak{h}$  is symmetric and its associated operator  $H$  is self-adjoint.<sup>1</sup> Of course,  $H$  is bounded from below.

It is a classical fact that the spectrum  $\sigma(A)$  is contained in some sector of the complex plane. In general, there is no relationship between the spectrum of the two operators  $A$  and  $H$ . The main result in this section

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<sup>1</sup>Formaly, one writes  $H = \frac{1}{2} [A + A^*]$ .

gives an estimate of the moments of eigenvalues of  $A$  having negative real-parts in terms of the same quantity corresponding to  $H$ .

**Theorem 1.** *i) If the negative spectrum of  $H$  is empty, then  $A$  has no eigenvalue in  $\{z \in \mathbb{C}; \operatorname{Re} z < 0\}$ .*

*ii) Assume now that the negative spectrum of  $H$  is discrete. Let  $\lambda_1, \dots, \lambda_N$  be any finite family of eigenvalues of  $A$  with  $\operatorname{Re} \lambda_j < 0$  for all  $j$  (an eigenvalue with algebraic multiplicity  $k > 1$  might occur  $n$  times with  $n \leq k$ ). Let  $\gamma \geq 1$ . Then*

$$(4) \quad \sum_{k=1}^N (-\operatorname{Re} \lambda_k)^\gamma \leq \operatorname{Tr}(H)_-^\gamma.$$

Recall that the algebraic multiplicity  $m_a(\lambda)$  of an eigenvalue  $\lambda$  of  $A$  is defined by

$$m_a(\lambda) := \sup_{k \in \mathbb{N}} \dim \ker(\lambda I - A)^k,$$

where  $\dim$  denotes the dimension of the Kernel of  $(\lambda I - A)^k$ . The algebraic multiplicity can be infinite. However, a direct consequence of the above theorem is that the algebraic multiplicity of any eigenvalue of  $A$  with negative real-part is finite provided  $\operatorname{Tr}(H)_- < \infty$ . Indeed, fix an eigenvalue  $\lambda$  with negative real-part and let  $k < \infty$  with  $k$  less or equal to  $m_a(\lambda)$ . The estimate (4) applied to the single eigenvalue  $\lambda$  (with  $\gamma = 1$ ) gives

$$k(-\operatorname{Re} \lambda) \leq \operatorname{Tr}(H)_-.$$

Since this holds for every finite  $k \leq m_a(\lambda)$ , one has  $m_a(\lambda)(-\operatorname{Re} \lambda) \leq \operatorname{Tr}(H)_- < \infty$ .

We also recall that for self-adjoint operators, both algebraic and geometric dimensions coincide.

One can also withdraw a similar conclusion for eigenvalues with real-parts less than any fixed value  $t$  under the condition that the spectrum of  $H$  below  $t$  is discrete (just replace  $A$  by  $A - tI$ ,  $H$  is then replaced by  $H - tI$ ).

Another consequence is that the co-dimension of range of  $(\bar{\lambda}I - A)$  is finite for every  $\lambda$  in the residual spectrum of  $A$  with  $\text{Re } \lambda < 0$ . The obvious reason is that by duality,  $\bar{\lambda}$  turns to be in the point spectrum of its adjoint  $A^*$ . The symmetric part of  $A^*$  is also  $H$ . We then apply the previous observations to  $A^*$ .

The proof of the above theorem is based on the following variational lemmas for the sum of eigenvalues of a self-adjoint operator. As mentioned in the introduction, these are infinite dimensional versions of Ky Fan's maximum principle [3].

**Lemma 2.** *Let  $(H, D(H))$  be a self-adjoint operator in a Hilbert space  $\mathcal{H}$  (with scalar product  $\langle \cdot, \cdot \rangle$ ). Suppose the spectrum of  $H$  is discrete on  $] - \infty, E[$ , where  $E$  is any fixed real number.*

*Let  $E_1 \leq E_2 \leq \dots \leq E_N$  be the first  $N$  eigenvalues of  $H$  (in  $] - \infty, E[$ ) repeated according to their multiplicities (eigenvalues that are equal to  $E_N$  are repeated  $k$  times with  $k \leq m_a(E_N)$ ).*

*Then we have:*

$$(5) \quad \sum_{n=1}^N E_n = \inf \sum_{n=1}^N \langle H u_n, u_n \rangle$$

*where the infimum is taken over all orthonormal sets  $\{u_1, \dots, u_N\} \subset D(H)$ .*

*Proof.* Let  $\{e_j\}_{1 \leq j \leq N}$  be an orthonormal family of eigenvectors associated to the eigenvalues  $\{E_j\}_{1 \leq j \leq N}$ . Then for any orthonormal family

$\{u_k\}_{1 \leq k \leq N}$  we have:

$$(6) \quad u_k = \sum_{j=1}^N \langle u_k, e_j \rangle e_j + r_k,$$

with  $r_k \in \text{Vect}\{e_1, \dots, e_N\}^\perp$  (the orthogonal subspace to  $e_1, \dots, e_N$ ). Due to the orthonormal properties, we have:

$$(7) \quad 1 = \|u_k\|^2 = \sum_{j=1}^N |\langle u_k, e_j \rangle|^2 + \|r_k\|^2,$$

and

$$(8) \quad \langle H u_k, u_k \rangle = \sum_{j=1}^N E_j |\langle u_k, e_j \rangle|^2 + \langle H r_k, r_k \rangle.$$

By the spectral theorem (applied to the part of  $H$  on  $\text{Vect}\{e_1, \dots, e_N\}^\perp$ ), we have:  $\langle H r_k, r_k \rangle \geq \Sigma_N \|r_k\|^2$  with

$$\Sigma_N \geq E_N.$$

The value  $\Sigma_N$  is just the bottom of the spectrum of the part of  $H$  on  $\text{Vect}\{e_1, \dots, e_N\}^\perp$ . Therefore,

$$(9) \quad \sum_{k=1}^N \langle H u_k, u_k \rangle \geq \sum_{j=1}^N E_j \sum_{k=1}^N |\langle u_k, e_j \rangle|^2 + \Sigma_N \sum_{k=1}^N \|r_k\|^2.$$

According to (7), we have:

$$(10) \quad \sum_{k=1}^N \|r_k\|^2 = \sum_{k=1}^N \left( 1 - \sum_{j=1}^N |\langle u_k, e_j \rangle|^2 \right) = \sum_{j=1}^N \left( 1 - \sum_{k=1}^N |\langle u_k, e_j \rangle|^2 \right).$$

On the other hand, exploiting the fact that  $\{u_k\}$  is an orthonormal family and that  $\|e_j\| = 1$ , we have

$$1 - \sum_{k=1}^N |\langle u_k, e_j \rangle|^2 = \|e_j\|^2 - \sum_{k=1}^N |\langle u_k, e_j \rangle|^2 \geq 0.$$

Consequently, combining (9) with (10), and using that for any  $1 \leq j \leq N$ ,  $\Sigma_N \geq E_j$ , we obtain:

$$\begin{aligned} \sum_{k=1}^N \langle H u_k, u_k \rangle &\geq \sum_{j=1}^N E_j \sum_{k=1}^N |\langle u_k, e_j \rangle|^2 + \sum_{j=1}^N E_j \left( 1 - \sum_{k=1}^N |\langle u_k, e_j \rangle|^2 \right) \\ &= \sum_{j=1}^N E_j. \end{aligned}$$

Clearly, the equality holds for  $(u_1, \dots, u_N) = (e_1, \dots, e_N)$  and (5) follows.  $\square$

The following lemma is a variation of the previous one. Suppose we have  $N$  vectors  $u_j$  and  $N'$  eigenvalues  $E_1 \leq E_2 \leq \dots \leq E_{N'}$  (with  $N' \leq N$ ). We denote again by  $e_1, \dots, e_{N'}$  the corresponding orthonormal family of corresponding eigenvectors. Let  $\Sigma_{N'}$  be the infimum of the spectrum of the part of  $H$  on  $Vect\{e_1, \dots, e_{N'}\}^\perp$ . Of course

$$\Sigma_{N'} \geq E_{N'}.$$

Now we have

**Lemma 3.** *Let  $H$  be a self-adjoint operator in  $\mathcal{H}$  and  $E_1 \leq E_2 \leq \dots \leq E_{N'}$  be the  $N'$  first eigenvalues of  $H$ . Let  $u_1, \dots, u_N$  be an orthonormal set such that  $u_k \in D(H)$  for each  $k$ . Then, for  $N \geq N'$  we have*

$$\sum_{k=1}^N \langle H u_k, u_k \rangle \geq E_1 + \dots + E_{N'} + (N - N') \Sigma_{N'}.$$

The proof is similar to that of the previous lemma. It suffices to replace (6) by

$$u_k = \sum_{j=1}^{N'} \langle u_k, e_j \rangle e_j + r_k,$$

and argue as before.

**Remark.** In the previous lemmas, it is not necessary to have  $u_k \in D(H)$ . It is enough to have  $u_k$  in the domain of the quadratic form of  $H$  (the later coincides with  $D(\sqrt{H})$ ). In that case  $\langle Hu_k, u_k \rangle$  is understood in the quadratic form sense and equals  $\langle \sqrt{H}u_k, \sqrt{H}u_k \rangle$ .

*Proof of Theorem 1.* i) If the negative spectrum of  $H$  is empty, then  $H$  is a non negative operator. Consequently each eigenvalue of  $A$ ,  $\lambda_j$  associated to an eigenvector  $u_j$  satisfies:

$$\operatorname{Re} \lambda_j = \operatorname{Re} \langle Au_j, u_j \rangle = \langle Hu_j, u_j \rangle \geq 0.$$

ii) Assume now that  $\lambda_1, \dots, \lambda_N$  are eigenvalues of  $A$  such that  $\operatorname{Re} \lambda_j < 0$  for  $j = 1, \dots, N$ .

1) We consider first the case  $\gamma = 1$ . Following [11], if  $\lambda_j$  has algebraic multiplicity  $k$ , then by the upper triangular representation one finds an orthonormal family  $v_1, \dots, v_k$  such that

$$Av_l = \lambda_j v_l + \sum_{k < l} \alpha_{kl} v_k.$$

Using this for each  $\lambda_j$ , we obtain an orthonormal family  $\{u_1, \dots, u_N\}$  which satisfies the above property. Taking the scalar product with  $u_j$  yields

$$(11) \quad \langle Au_j, u_j \rangle = \lambda_j \text{ for } j = 1, \dots, N.$$

Taking the sum, we obtain

$$(12) \quad \sum_{n=1}^N \operatorname{Re} \lambda_n = \sum_{n=1}^N \operatorname{Re} \langle Au_n, u_n \rangle = \sum_{n=1}^N \langle Hu_n, u_n \rangle.$$

Here  $\langle Hu_n, u_n \rangle$  is understood in the quadratic form sense because it is not clear whether  $u_n \in D(H)$  (however  $u_n \in D(\mathfrak{a}) = D(\mathfrak{h})$ ). Now we



apply the previous lemmas. For this, we proceed in two steps.

Assume that  $H$  has only a finite number of negative eigenvalues. Denote these negative eigenvalues by  $\mu_1 \leq \dots \leq \mu_M$  (all them are repeated according to their multiplicities). If  $M \leq N$ , we apply Lemma 3 with  $N' = M$ . Note that  $\Sigma_M \geq 0$  because  $H$  has only eigenvalues below 0 (by assumption) and these eigenvalues are  $\mu_1 \leq \dots \leq \mu_M$ . Hence

$$\begin{aligned} \sum_{n=1}^N \operatorname{Re} \lambda_n &= \sum_{n=1}^N \langle H u_n, u_n \rangle \\ &\geq \mu_1 + \dots + \mu_M \\ &= -\operatorname{Tr}(H)_-. \end{aligned}$$

If  $M > N$ , we apply Lemma 3 with  $N' = N$  and obtain

$$\begin{aligned} \sum_{n=1}^N \langle H u_n, u_n \rangle &\geq \mu_1 + \dots + \mu_{N-1} + \mu_N \\ &\geq \mu_1 + \dots + \mu_{N-1} + \mu_N + \mu_{N+1} + \dots + \mu_M \\ &= -\operatorname{Tr}(H)_-. \end{aligned}$$

Assume now that  $H$  has infinite number of eigenvalues below 0. We choose  $M > N$  and let  $\mu_1 \leq \dots \leq \mu_M$  be the  $M$  first eigenvalues. Now we proceed as above by applying Lemma 3 with  $N' = N$  and obtain

$$\sum_{n=1}^N \langle H u_n, u_n \rangle \geq \mu_1 + \dots + \mu_N \geq -\operatorname{Tr}(H)_-.$$

Thus, in all cases, we have

$$(13) \quad \sum_{n=1}^N \operatorname{Re} \lambda_n \geq -\operatorname{Tr}(H)_-,$$

which proves the theorem when  $\gamma = 1$ .

2) The case  $\gamma > 1$  follows from the previous case. The proof uses an idea of Aizenman-Lieb [2]. It was also used in [11]. We follow the same

arguments as in the proof of Lemma 1 of the later references. Denote again  $f_- := \max(-f, 0)$ . There exists a constant  $C_\gamma$  such that

$$C_\gamma s_-^\gamma = \int_0^\infty t^{\gamma-2} (s+t)_- dt.$$

Therefore, applying the case  $\gamma = 1$  to the operators  $A + tI$  and  $H + tI$ , we obtain

$$\begin{aligned} C_\gamma \sum_{n=1}^N (-\operatorname{Re} \lambda_n)^\gamma &= \int_0^\infty t^{\gamma-2} \sum_{n=1}^N (\operatorname{Re} \lambda_n + t)_- dt \\ &\leq - \int_0^\infty t^{\gamma-2} \operatorname{Tr}(H + tI)_- dt \\ &= C_\gamma \operatorname{Tr}(H)_-^\gamma. \end{aligned}$$

This proves the theorem.  $\square$

In general settings the above result does not hold for  $\gamma < 1$ . The following elementary example was communicated to us by J.F. Bony. We thank him for fruitful discussion.

Consider on  $H := \ell_2(\mathbb{N})$  the finite rank operator  $A$  defined as follows. Fix  $n$  large enough and let

$$Au(j) := \begin{cases} -u(j) - 2u(j+1) - \dots - 2u(n), & 1 \leq j < n \\ -u(n), & j = n \\ 0, & j > n \end{cases}$$

Clearly,  $-1$  is the only non zero eigenvalue of  $A$  and its algebraic multiplicity is  $n$ . The symmetric part  $H$  of  $A$  has only  $-n$  as a negative eigenvalue (it has multiplicity 1). Therefore,  $\sum_{j=1}^n 1^\gamma = n$ ,  $\operatorname{Tr}(H)_-^\gamma = n^\gamma$  and the inequality  $n \leq Cn^\gamma$  cannot hold for any constant  $C$  (independently of  $n$  when  $\gamma < 1$ ).

## 3. APPLICATION TO NON SELF-ADJOINT SCHRÖDINGER OPERATORS

In this section we consider non self-adjoint Schrödinger operators

$$P := -\Delta + i\left(a(x)\cdot\nabla + \nabla\cdot a(x)\right) + V$$

where  $V$  is a complex-valued potential and  $a$  is a complex-valued vector field.

We first describe how  $P$  is defined. Consider the symmetric non-negative sesquilinear form

$$\mathbf{e}_0(u, v) := \int_{\mathbb{R}^d} (i\nabla + \operatorname{Re} a) u \overline{(i\nabla + \operatorname{Re} a) v} dx + \int_{\mathbb{R}^d} (\operatorname{Re} V)_+ u \bar{v} dx,$$

defined on the space  $C_c^\infty$  of  $C^\infty$  functions with compact support. For  $\operatorname{Re} a \in L^2_{loc}$  and  $(\operatorname{Re} V)_+ \in L^1_{loc}$ , this form is closable (see [21]). We denote again by  $\mathbf{e}_0$  its closure.<sup>2</sup> Consider now the (non-symmetric) sesquilinear form  $\mathbf{e} := \mathbf{e}_0 + \mathbf{e}_1$  where

$$\begin{aligned} \mathbf{e}_1(u, v) := & \int_{\mathbb{R}^d} \{ \operatorname{Im} a(x) u \nabla \bar{v} - \operatorname{Im} a(x) \nabla u \bar{v} \} dx + \\ & \int_{\mathbb{R}^d} \{ -|\operatorname{Re} a(x)|^2 - (\operatorname{Re} V)_- + i \operatorname{Im} V \} u \bar{v} dx. \end{aligned}$$

We assume that there exist two constants  $\beta \in [0, 1[$  and  $c_\beta \in \mathbb{R}$  such that for every  $u \in D(\mathbf{e}_0)$

$$(14) \quad |\mathbf{e}_1(u, u)| \leq \beta \mathbf{e}_0(u, u) + c_\beta \int_{\mathbb{R}^d} |u|^2 dx$$

(in particular, this holds if  $\operatorname{Im} a, (\operatorname{Re} V)_-, \operatorname{Im} V \in L^\infty(\mathbb{R}^d)$ ). By the well known KLMN theorem (see for example [14] p. 320 or [20], p. 12), the form  $\mathbf{e}$ , with domain  $D(\mathbf{e}) = D(\mathbf{e}_0)$ , is well defined as is closed. One can then associate with  $\mathbf{e}$  an operator. Formally, this operator is given

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<sup>2</sup>the operator associated with this form is the magnetic Schrödinger operator  $(i\nabla + \operatorname{Re} a)^2 + (\operatorname{Re} V)_+$ .

by  $P$  above. In the sequel, we assume that  $\operatorname{Re} a \in L^2_{loc}$ ,  $(\operatorname{Re} V)_+ \in L^1_{loc}$  and (14) are satisfied and  $P$  will be the associated operator with  $\mathbf{e}$ .

We want to study Lieb-Thirring estimates for  $P$  (for moments of eigenvalues having negative real-parts). The next theorem was proved recently in [11] for potential perturbations. Our proof is easier and applies in many situations (we can consider for example operators with boundary conditions on domains or Schrödinger operators on some manifolds).

For real-valued potentials  $W$  and real vector field  $b$ , the well-known Lieb-Thirring estimates for negative eigenvalues  $\mu_j$  of Schrödinger (self-adjoint) operators  $(i\nabla + b)^2 + W$  say that

$$(15) \quad \operatorname{Tr}((i\nabla + b)^2 + W)_-^\gamma := \sum_{\mu_j < 0} (-\lambda_j)^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} W(x)_-^{\gamma+d/2} dx,$$

where  $L_{\gamma,d}$  is a positive constant. See [22], [16] for details. Here  $\gamma \geq 1/2$  if  $d = 1$ ,  $\gamma > 0$  if  $d = 2$  and  $\gamma \geq 0$  for  $d \geq 3$ . For information concerning the best value of the constant  $L_{\gamma,d}$  see [15], [9]. In the sequel, we shall refer to  $L_{\gamma,d}$  as the best possible constant for which (15) holds. For complex-valued potentials  $V$  and complex-valued vector fields  $a$ , we introduce the following family of self-adjoint Schrödinger operators:

$$(16) \quad H(\alpha) := (i\nabla + b(\alpha))^2 - |b(\alpha)|^2 + W(\alpha); \quad \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$$

with

$$b(\alpha) := \frac{1}{\sin \alpha} \operatorname{Re} (e^{-i(\alpha - \frac{\pi}{2})} a) = \operatorname{Re} a - (\cot \alpha) \operatorname{Im} a$$

and

$$W(\alpha) := \frac{1}{\sin \alpha} \operatorname{Re} (e^{-i(\alpha - \frac{\pi}{2})} V) = \operatorname{Re} V - (\cot \alpha) \operatorname{Im} V.$$

We have

**Theorem 4.** *i) If  $\alpha \in ]0, \frac{\pi}{2}]$  (respectively,  $-\alpha \in ]0, \frac{\pi}{2}]$ ) is such that  $H(\alpha)$  is non negative, then*

$$\sigma(P) \subset e^{i[\alpha-\pi, \alpha]} \mathbb{R}^+ \quad (\text{respectively, } \sigma(P) \subset -e^{i[\alpha-\pi, \alpha]} \mathbb{R}^+).$$

*ii) Fix  $\alpha \in ]0, \frac{\pi}{2}]$  (respectively,  $-\alpha \in ]0, \frac{\pi}{2}]$ ) such that*

$$\left( W(|\alpha|) - |b(|\alpha|)|^2 \right)_- \in L^{\gamma+d/2}(\mathbb{R}^d).$$

*Let  $\lambda_1, \dots, \lambda_N$  be any finite family of eigenvalues of  $P$  contained outside the sector  $e^{i[\alpha-\pi, \alpha]} \mathbb{R}^+$  (respectively,  $-e^{i[\alpha-\pi, \alpha]} \mathbb{R}^+$ ) for all  $j$  (an eigenvalue with algebraic multiplicity  $k > 1$  might occur  $n$  times with  $n \leq k$ ).*

*Then for  $\gamma \geq 1$ , we have*

$$\sum_{k=1}^N (-\operatorname{Re} \lambda_k + (\cot \alpha) \operatorname{Im} \lambda_k)^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} [W(|\alpha|) - |b(|\alpha|)|^2]_-^{\gamma+d/2} dx,$$

*where  $L_{\gamma,d}$  is the best possible value for which (15) holds.*

*Proof.* We apply Theorem 1 to the operator  $A := \pm e^{-i(\alpha-\frac{\pi}{2})} P$  with  $\pm\alpha \in ]0, \frac{\pi}{2}]$  fixed. The real part of  $A$  (in the sense of quadratic forms, see the beginning of the previous section) is

$$\begin{aligned} \operatorname{Re} A &= \pm(\sin \alpha \operatorname{Re} P - \cos \alpha \operatorname{Im} P) \\ &= -|\sin \alpha| \Delta + i|\sin \alpha| \left( \operatorname{Re} a(x) \cdot \nabla + \nabla \cdot \operatorname{Re} a(x) \right) + |\sin \alpha| \operatorname{Re} V \\ &\quad \mp i(\cos \alpha) \left( \operatorname{Im} a(x) \cdot \nabla + \nabla \cdot \operatorname{Im} a(x) \right) \mp (\cos \alpha) \operatorname{Im} V. \\ &= |\sin \alpha| H(\alpha). \end{aligned}$$

Observing that eigenvalues of  $A$  are eigenvalues of  $P$  times  $\pm e^{-i(\alpha-\frac{\pi}{2})}$ , and that

$$\operatorname{Re} \left( \pm e^{-i(\alpha-\frac{\pi}{2})} \lambda \right) \geq 0 \Leftrightarrow \lambda \in \pm e^{i[\alpha-\pi, \alpha]} \mathbb{R}^+,$$

we deduce Theorem 4 from Theorem 1 and (15). In fact, for the proof of ii), we have:

$$(17) \quad \sum_k \left( \operatorname{Re} (\pm e^{-i(\alpha - \frac{\pi}{2})} \lambda_k) \right)_-^\gamma \leq |\sin \alpha|^\gamma \operatorname{Tr}(H(\alpha))_-^\gamma.$$

Then, dividing both sides by  $|\sin \alpha|^\gamma$ , yields:

$$\sum_k (\operatorname{Re} \lambda_k - (\cot \alpha) \operatorname{Im} \lambda_k)_-^\gamma \leq \operatorname{Tr}(H(\alpha))_-^\gamma.$$

and Theorem 4 follows then from (15). □

We also have the following estimates for  $\sum |\lambda_k|^\gamma$ .

**Corollary 5.** *Under the assumptions and notation of Theorem 4 ii), we have for  $\gamma \geq 1$ ,  $\varepsilon \in ]0, \frac{\pi}{2}]$  and  $\alpha \in ]0, \frac{\pi}{2}]$ :*

$$\sum_{\lambda_k \in e^{i[\alpha + \varepsilon, \alpha - \varepsilon + \pi]} \mathbb{R}^+} |\lambda_k|^\gamma \leq \left( \frac{|\sin \alpha|}{\sin \varepsilon} \right)^\gamma L_{\gamma, d} \int_{\mathbb{R}^d} [W(|\alpha|) - |b(|\alpha|)|^2]_-^{\gamma + d/2} dx.$$

If  $\alpha \in [-\frac{\pi}{2}, 0[$ , then the same estimate holds for the sum over  $\lambda_k \in -e^{i[\alpha + \varepsilon, \alpha - \varepsilon + \pi]} \mathbb{R}^+$ .

Note that for  $\alpha \in ]0, \frac{\pi}{2}]$  such that  $(W(\alpha) - |b(\alpha)|^2)_- \in L^{\gamma + d/2}(\mathbb{R}^d)$ , the sum of the above estimates for  $\alpha$  and for  $-\alpha$  yields estimate for eigenvalues outside of the sector  $e^{i[-\alpha - \varepsilon, \alpha + \varepsilon]} \mathbb{R}^+$ :

$$\sum_{\lambda_k \in e^{i[\alpha + \varepsilon, 2\pi - \alpha - \varepsilon]} \mathbb{R}^+} |\lambda_k|^\gamma \leq 2 \left( \frac{|\sin \alpha|}{\sin \varepsilon} \right)^\gamma L_{\gamma, d} \int_{\mathbb{R}^d} [W(|\alpha|) - |b(|\alpha|)|^2]_-^{\gamma + d/2} dx.$$

*Proof.* Assume that  $\alpha \in ]0, \frac{\pi}{2}]$ . Let  $\alpha_k \in [\alpha + \varepsilon, \alpha - \varepsilon + \pi]$  be the argument of the eigenvalue  $\lambda_k \in \pm e^{i[\alpha + \varepsilon, \alpha - \varepsilon + \pi]} \mathbb{R}^+$ . Since  $\lambda_k = e^{i\alpha_k} |\lambda_k|$ , then

$$\operatorname{Re} (-e^{-i(\alpha - \frac{\pi}{2})} \lambda_k) = |\lambda_k| \operatorname{Re} (-e^{-i(\alpha - \alpha_k - \frac{\pi}{2})}) = -|\lambda_k| \sin(\alpha - \alpha_k).$$

Therefore,

$$-\operatorname{Re} (e^{-i(\alpha - \frac{\pi}{2})} \lambda_k) \geq |\lambda_k| \sin \varepsilon.$$

Inserting this inequality in (17), we obtain

$$\sum_{\lambda_k \in e^{i[\alpha+\varepsilon, \alpha-\varepsilon+\pi]} \mathbb{R}^+} |\lambda_k|^\gamma \leq \left( \frac{|\sin \alpha|}{\sin \varepsilon} \right)^\gamma \text{Tr}(H(\alpha))_-^\gamma.$$

The assertion in Corollary 5 follows then from (15).

The arguments are similar if  $\alpha \in [-\frac{\pi}{2}, 0[$ .  $\square$

Of course, taking only one eigenvalue  $\lambda \in e^{i[\alpha+\varepsilon, \alpha-\varepsilon+\pi]} \mathbb{R}^+$  of  $P$ , one has

$$|\lambda| \leq \left( \frac{|\sin \alpha|}{\sin \varepsilon} \right)^\gamma L_{1,d} \int_{\mathbb{R}^d} [W(|\alpha|) - |b(|\alpha|)|^2]_-^{1+d/2} dx.$$

For  $P = -\Delta + V$ , a result with a better constant is obtained in [1] in the case of dimension  $d = 1$ . Indeed, Theorem 4 in [1] says in this case that

$$|\lambda| \leq \frac{1}{4} \left( \int_{\mathbb{R}} |V(x)| dx \right)^2.$$

The previous results can be applied to estimate the moments of resonances for Schrödinger operators. Consider a potential  $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$  such that for every  $x \in \mathbb{R}^d$ ,  $z \rightarrow V(zx)$  has an analytic extension to a neighbourhood of the sector of angle  $\theta/2$  (for a fixed  $\theta \in ]0, \pi[$ ). Assume also that this analytic extension is relatively compact with respect to  $-\Delta$ . Then, the resonances of  $-h^2\Delta + V(x)$  ( $h > 0$ ) in the sector  $S_\theta := e^{i]0, \theta]} \mathbb{R}^+$ ,  $\theta \in [0, \pi]$  are the eigenvalues of the non-self-adjoint operator

$$P_\theta := -e^{-i\theta} h^2 \Delta + V(e^{i\theta/2} x).$$

See for instance [6], [13] for a general introduction to the theory of resonances.

First, observe that

$$\sigma(P_\theta) = h^2 e^{-i\theta} \sigma(h^{-2} e^{i\theta} P_\theta) = h^2 e^{-i\theta} \sigma(-\Delta + M),$$

where  $M(x) = h^{-2}e^{i\theta}V(e^{i\theta/2}x)$ . It follows from assertion i) of Theorem 4 (which we apply to  $-\Delta + M$ ) that if the operator

$$H_\theta(\alpha) := -\Delta + \frac{h^{-2}}{\sin \alpha} \operatorname{Re} \left( e^{-i(\alpha-\theta-\frac{\pi}{2})} V(e^{i\theta/2} \cdot) \right)$$

is non-negative, then there are no resonances in the sector  $S_{\alpha-\theta}$ . Related results on localization of resonances are studied in several settings. Localisation results which rely on numerical range are given in [1]. A lot of results, often related to some dynamical assumptions, are obtained by microlocal arguments (see for instance [4], [23], [18] and [19]). For the magnetic Schrödinger operator, we also mention [5] where localization of resonances is obtained by perturbation methods.

If  $H_\theta(\alpha)$  is not non-negative, we can apply Corollary 5 to obtain an estimate for moments of resonances. Let  $w_j \in \sigma(P_\theta) \cap S_\theta$  be resonances of  $-h^2\Delta + V(x)$ . We apply Corollary 5 (to  $-\Delta + M$ ) and obtain

$$\sum_{w_j \in e^{i[\alpha-\theta+\varepsilon, 0]} \mathbb{R}^+} |h^{-2}e^{i\theta}w_j|^\gamma \leq \left( \frac{|\sin \alpha|}{\sin \varepsilon} \right)^\gamma L_{\gamma,d} \int_{\mathbb{R}^d} \left[ \frac{h^{-2}}{\sin \alpha} \operatorname{Re} \left( e^{-i(\alpha-\theta-\frac{\pi}{2})} V(e^{i\theta/2}x) \right) \right]_-^{\gamma+d/2} dx.$$

Consequently, for any  $h > 0$ ,  $\alpha \in ]0, \theta[$ ,  $\varepsilon \in ]0, \theta - \alpha[$ , we have:

$$\sum_{w_j \in e^{i[\alpha-\theta+\varepsilon, 0]} \mathbb{R}^+} |w_j|^\gamma \leq \frac{h^{-d} L_{\gamma,d}}{(\sin \varepsilon)^\gamma (\sin \alpha)^{d/2}} \int_{\mathbb{R}^d} \left[ \operatorname{Im} \left( e^{i(\theta-\alpha)} V(e^{i\theta/2}x) \right) \right]_+^{\gamma+d/2} dx.$$

In particular, we have proved the following proposition.

**Proposition 6.** *Suppose that  $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$  and satisfies the above analyticity and relative compactness (with respect to  $-\Delta$ ) assumptions.*



Then for  $\gamma \geq 1$  and  $\varphi \in [0, \theta[$ , the resonances of  $-\Delta + V$  in  $e^{i[-\varphi, 0]}\mathbb{R}^+$  satisfy:

$$\sum_{w_j \in e^{i[-\varphi, 0]}\mathbb{R}^+} |w_j|^\gamma \leq \frac{h^{-d} L_{\gamma, d}}{(\sin \varepsilon)^\gamma (\sin(\theta - \varphi - \varepsilon))^{d/2}} \times \int_{\mathbb{R}^d} \left[ \operatorname{Im} \left( e^{i(\varphi + \varepsilon)} V(e^{i\theta/2} x) \right) \right]_+^{\gamma + d/2} dx.$$

for all  $\varepsilon \in ]0, \theta - \varphi[$ .

**Remark.** The above estimate give the well known upper bound  $O(h^{-d})$  for the number of resonances in a compact domain (see [24]). Here, we have an explicit coefficient of  $h^{-d}$ .

On the other hand it is clear that Corollary 5 allows to extend Proposition 6 to first order perturbations.

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