Magnetic Quantum Hamiltonians and Berezin-Toeplitz Operators

Spectral analysis

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Chapter 1

Introduction

Since the emergence of the first models of quantum physics given by W. Heisenberg, M. Born, E. Schrödinger and P. Dirac, a century ago, the spectral analysis of quantum Hamiltonians has played an important role in the description of physical phenomena. The rise of Mathematical Physics, the foundations of which are well described in Reed-Simon's books [162, 163, 164, 165], was notably accelerated by the development of microlocal analysis. This allowed, in particular, a sharp investigation of the Schrödinger operator $-\Delta + V$ for a large class of electric potentials V. Taking a magnetic field into account generally does not present any new important difficulties as long as the associated magnetic potential goes to zero at infinity. But the physical reality requires the study of the Schrödinger operator with stronger magnetic fields, starting with constant magnetic fields (that is linear magnetic potentials). One of the first articles that gives an overview of the specificities of the magnetic Schrödinger operators is the Avron-Herbst-Simon's paper [7] where, in comparison with Schrödinger operators without magnetic field, the "well-behaved" and "better-behaved" properties are discussed. The study of the distribution of the eigenvalues of magnetic Hamiltonians has then been the subject of numerous works. Berezin-Toeplitz operators (with symbol being the perturbation of a reference magnetic Hamiltonian) are natural effective operators that have played an important role in the asymptotic study of the spectra of magnetic quantum Hamiltonians. First obtained for eigenvalues (pure point spectrum), these asymptotics were then studied on the continuous spectrum (via the Spectral Shift Function) as well as for the distribution of resonances (scattering poles).

The aim of this book is to provide an overview of these results to which the authors have contributed. Unfortunately the Covid-19 did not let Georgi Raikov bring this book project to fruition. In March 2021, Georgi Raikov, instigator of this book, succumbed to Covid-19 when we had written most of the three Chapters 2, 3 and 4.

Let us now describe the contents of these chapters as well as a fifth chapter which completes the description of spectral results for the magnetic quantum Hamiltonians.

In Chapter 2, we first recall the canonical Halmiltonian formulation of the equations of motion of classical particles in electromagnetic fields, and give examples of trajectories for constant magnetic fields. Then, based on the concepts of quantization, we introduce the Weyl and the magnetic Weyl quantizations, as well as the Hamiltonians discussed in this book: the magnetic Schrödinger operator, the magnetic Pauli operator and the magnetic Dirac operator. We describe the spectrum of these operators in the particular case of constant magnetic field. For non constant magnetic fields, including the general class of *admissible non-constant magnetic fields*, the bottom of the spectrum of the Pauli hamiltonian is discussed. In particular, the kernel of the 2D Pauli hamiltonian can be of infinite multiplicity or any fixed dimension.

Chapter 3 is devoted to the introduction and the spectral study of Berezin-Toeplitz operators who play an essential role in the spectral analysis of the magnetic quantum Hamiltonians considered in this book. Along with generalized anti-Wick pseudodifferential operators, these operators, are described in the general context of contravariant symbol operators. These operators are studied in general holomorphic spaces and then in weighted Fock-Segal-Bargmann spaces. For appropriate Gaussian weights, these spaces, of infinite dimension, coincide with the eigenspaces of the 2D Schrödinger operator with a constant magnetic field. The associated eigenvalues, of infinite multiplicity, are the so-called *Landau levels*. Boundness and compactness properties are given and dependence on the Landau level is considered.

Chapter 4 has a central role in the book because it shows the interaction between chapters 2 and 3, and the results given there will be fondamental for Chapter 5. First, we specify the meaning that we give to the notion of *asymptotics of the semi-classical type* by pointing out the specificities of the magnetic frame. Then eigenvalue asymptotics for Berezin-Toeplitz operators are given for different type of symbols. These asymptotics are of semi-classical type for symbols of power-like decay and not of semi-classical nature for compactly supported symbols. For symbols having intermediate behaviors (exponential or gaussian decaying) the asymptotic order can be semi-classical but not the coefficient. From these asymptotics for Berezin-Toeplitz operators, we deduce the first results of spectral asymptotics for perturbations of the 2D magnetic Schrödinger operator $H_S(A,0)$ with constant magnetic field $b \neq 0$, and of the 2D magnetic Pauli operator $H_P(A,0)$ with admissible b of non-zero mean value b_0. The generic situation is that under the considered perturbations, the essential spectrum is preserved and the eigenvalues of the unperturbed operators ($H_S(A,0)$ or $H_P(A,0)$) become accumulation points of the spectrum of the perturbed operators. Moreover the distribution of the eigenvalues near each accumulation point (in general, a Landau level) is governed by that of a Berezin-Toeplitz operator (close to zero). Most of the perturbations are electric potentials, but magnetic, geometric and obstacle perturbations are also considered. Even if principles of semi-classical analysis appear, no semi-classical limit is considered. For an overview of the many results on asymptotics with respect to the Planck's constant - mainly on bounded domains - we refer to [161]. This Chapter 4 also includes a controllability result deduced from the investigation of Berezin-Toeplitz operators with compactly supported symbols. To conclude Chapter 4 on magnetic Hamiltonian in the plane, we exploit results of Chapter 3 on the Landau level's dependence of the norm of Berezin-Toeplitz operators in order to prove a clustering phenomena near each Landau level, when q, the Landau level number, tends to infinity. For electric potentials V which decrease more or less rapidly to infinity, as q tends to infinity, we study the rate at which the discrete eigenvalues of $H_S(A,V)$ approach the qth Landau level as well as the distribution of eigenvalues within this qth cluster.

The eigenvalue asymptotics for Berezin-Toeplitz operators find also applications to 3D magnetic operators and to magnetic fibered models. For these operators which have continuous spectra, the phenomena of spectral accumulations result in singularities of the Spectral Shift Function or accumulations of resonances. In Chapter 5 we give an overview on investigations realized on *Spectral Shift Function* (SSF) and *resonances* for quantum magnetic Hamiltonians. After the introduction of the notions of spectral shift functions and resonances, we study their properties near Landau levels for perturbations of 3D magnetic Hamiltonians by potentials of fixed sign or by an obstacle. We also discuss 2D magnetic Schrödinger operators in the half-plane and in a strip. These hamiltonians are fibered operators with non-empty continuous spectrum. For electric perturbations of these operators, we give results on the SSF near thresholds (Landau levels or other extrema of the band functions) but several questions remain open.

Chapter 2

Magnetic Quantum Hamiltonians

Abstract: This chapter is dedicated to the introduction of the Hamiltonians discussed in this book. First, in Section 2.1 we recall the properties of motion of classical particles in electromagnetic fields, in particular for constant magnetic fields. Then after having given the general concept of the quantization (in Section 5.4.2) and more particularly introduced the Weyl quantization (in Section 2.3), in Section 5.2.2, we investigate the magnetic momentum together with the magnetic Weyl quantization. The magnetic Schrödinger operator, then the magnetic Pauli and Dirac operators are successively introduced in Sections 2.5 and 2.6. The particular case of the constant magnetic field is discussed in Section 2.7, while a more general class of variable magnetic fields is treated in Section 2.8. For the Pauli hamiltonian, depending on the magnetic field, several possible spectral structures are given.

2.1 Classical magnetic systems

Let us begin our story with the equations of motion of a three-dimensional classical non-relativistic particle in an electromagnetic field $(\mathbf{E},\mathbf{B})$. Here $\mathbf{E}:\mathbb{R}^3\to\mathbb{R}^3$ is the electric component of the field, and $\mathbf{B}:\mathbb{R}^3\to\mathbb{R}^3$ is its magnetic component. We assume that $(\mathbf{E},\mathbf{B})$ is stationary, i.e. independent of time t. Moreover, we suppose that the medium is uniform and isotropic, so that the electric permeability and the magnetic permittivity are scalar constants. By the Maxwell equations,

$$\operatorname{curl}\mathbf{E}=0,\quad \operatorname{div}\mathbf{B}=0. \tag{2.1.1}$$

Let $\mathbf{x}=(x_1,x_2,x_3)$ be the coordinates of the particle. Then the equations of its motion can be written in Newtonian form

$$m\ddot{\mathbf{x}}=\frac{e}{c}\,\dot{\mathbf{x}}\times\mathbf{B}+e\mathbf{E} \tag{2.1.2}$$

(see [114]). Here m is the mass of the particle, e is its charge, c is the speed of light in vacuum, $\dot{\mathbf{x}}=\frac{\partial\mathbf{x}}{\partial t}$ is the velocity of the particle, and $\ddot{\mathbf{x}}=\frac{\partial^2\mathbf{x}}{\partial t^2}$ is its acceleration. The first

term $\frac{e}{c}\,\dot{\mathbf{x}} \times \mathbf{B}$ appearing at the right-hand side of (2.1.2) is the Lorentz force, while the second term $e\mathbf{E}$ is the electrostatic force. In the sequel we assume $e = 1$ and $c = 1$.
In order to pass to the Lagrangian version of (2.1.2), we introduce the electromagnetic potential (A, V) with $A : \mathbb{R}^3 \to \mathbb{R}^3$, $V : \mathbb{R}^3 \to \mathbb{R}$, satisfying $\mathbf{B} = \text{curl}\, A$, $\mathbf{E} = -\nabla V$, which is coherent with (2.1.1), and define the Lagrangian function

$$L(\mathbf{x}, \dot{\mathbf{x}}) := \frac{m|\dot{\mathbf{x}}|^2}{2} + A(\mathbf{x}) \cdot \dot{\mathbf{x}} - V(\mathbf{x}). \tag{2.1.3}$$

Evidently, the expression (2.1.3) for L admits an immediate generalization to an arbitrary dimension $n \geq 1$; in this case $\mathbf{x} \in \mathbb{R}^n$, $A : \mathbb{R}^n \to \mathbb{R}^n$, and $V : \mathbb{R}^n \to \mathbb{R}$. The equations of motion of the n-dimensional particle can be written in their Lagrangian form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0, \quad j = 1, \ldots, n, \tag{2.1.4}$$

corresponding to the Newtonian equation

$$m\ddot{\mathbf{x}} = B\dot{\mathbf{x}} + \mathbf{E} \tag{2.1.5}$$

where as before $\mathbf{E} = -\nabla V$ while $B = \{B_{jk}(\mathbf{x})\}_{j,k=1}^n$ is *the magnetic-field tensor* defined by

$$B_{jk}(\mathbf{x}) := \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}, \quad j, k = 1, \ldots, n. \tag{2.1.6}$$

Note that if $n = 1$, then $B = 0$.
We may identify the magnetic potential A with the 1-differential form

$$A := \sum_{j=1}^{n} A_j\, dx_j,$$

and the tensor B with the 2-form

$$B := dA = \sum_{1 \leq j < k \leq n} B_{jk}\, dx_j \wedge dx_k.$$

Moreover, if $n = 3$, we may identify the tensor B with the magnetic-field vector $\mathbf{B} = (B_1, B_2, B_3)$ by using the cyclic permutations

$$B_1 = B_{23}, \quad B_2 = B_{31}, \quad B_3 = B_{12}. \tag{2.1.7}$$

Thus, if $n = 3$, then (2.1.5) coincides with (2.1.2).
In many situations, it is useful to have explicit formulae for magnetic potentials A which generate a given magnetic field B. Below, we give such formulae for dimensions $n = 2, 3$. If $n = 3$, we may define

$$A(\mathbf{x}) := \left(\int_0^1 \mathbf{B}(s\mathbf{x})s\,ds \right) \times \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3. \tag{2.1.8}$$

If $n = 2$, we set $b := B_{12}$ and

$$A_1(\mathbf{x}) := x_2\left(\int_0^1 b(s\mathbf{x})sds\right), \quad A_2(\mathbf{x}) := -x_1\left(\int_0^1 b(s\mathbf{x})sds\right), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2. \tag{2.1.9}$$

The magnetic potential A which generates a given magnetic field is not unique: the potentials A and $A + \nabla\phi$ with $\phi : \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, generate the same B. In this case we will say that the magnetic potentials are *gauge equivalent*. In this context, the magnetic potential defined in (2.1.8) is said to be in the Poincaré (or transversal) gauge.

Let us consider now the explicit example of the motion of a classical particle in constant magnetic field, and zero electric potential (i.e. $V = 0$). We will consider dimensions $n = 2, 3$, and assume $m = 1$.

Let at first $n = 2$. We suppose that $B_{12} = 1$. Then the coordinates $(x, y) \in \mathbb{R}^2$ of the particle satisfy the equations

$$\begin{cases} \ddot{x} = \dot{y}, \\ \ddot{y} = -\dot{x}, \end{cases} \tag{2.1.10}$$

and the initial conditions

$$\begin{cases} x(0) = x_0,\ \dot{x}(0) = x_1, \\ y(0) = y_0,\ \dot{y}(0) = y_1. \end{cases} \tag{2.1.11}$$

The solution of Cauchy problem (2.1.10)–(2.1.11) is given by

$$\begin{cases} x(t) = x_0 + x_1 \sin t + y_1(1 - \cos t), \\ y(t) = y_0 + y_1 \sin t - x_1(1 - \cos t), \end{cases} \tag{2.1.12}$$

i.e. unless $\dot{x}(0) = \dot{y}(0) = 0$, the trajectory of the particle is a circle of radius $\sqrt{x_1^2 + y_1^2}$ centered at the point $\left(x_0 + y_1, y_0 - x_1\right)$. In particular, the trajectories of a 2D particle in a constant magnetic field are closed in contrast to the free particle whose trajectories are straight lines unless $\dot{\mathbf{x}}(0) = 0$.

If $n = 3$, we suppose $B_{12} = 1$, $B_{23} = B_{31} = 0$, i.e. $\mathbf{B} = (0, 0, 1)$. Then the first two coordinates (x, y) in the plane perpendicular to the magnetic field $\mathbf{B}$ satisfy (2.1.10)–(2.1.11), and their dynamics is given by (2.1.12). The third coordinate z along $\mathbf{B}$ satisfies the free-motion equation $\ddot{z} = 0$, and the initial conditions $z(0) = z_0$, $\dot{z}(0) = z_1$. Therefore,

$$z(t) = z_0 + z_1 t. \tag{2.1.13}$$

Combining (2.1.12) and (2.1.13), we find that generically the three-dimensional particle moves along a helix whose axis is parallel to the magnetic field. Hence, roughly speaking, a 3D dimensional particle disposes of a unique direction to escape to infinity, that of the magnetic field, in contrast to the free particle which may move along any straight line in $\mathbb{R}^3$.

These differences in the behavior of the classical particle in the presence and in the absence of a (constant) magnetic field provide a useful heuristic intuition which may help to better understand various purely magnetic effects in classical but also in quantum physics.

More examples of trajectories of classical particles moving in electromagnetic fields can be found in [101, Chapter 6, Appendix F].

In order to pass to the canonical Hamiltonian formulation of the equations of motion of our n-dimensional particle, we introduce the generalized momentum $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$, with $\xi_j := \frac{\partial L}{\partial \dot{x}_j}$, $j = 1, \dots, n$, write $\dot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{x}, \boldsymbol{\xi})$ as a function of $\mathbf{x}$ and $\boldsymbol{\xi}$, and compose *the Hamiltonian function*

$$\mathscr{H}(\mathbf{x}, \boldsymbol{\xi}) := \boldsymbol{\xi} \cdot \dot{\mathbf{x}}(\mathbf{x}, \boldsymbol{\xi}) - L(\mathbf{x}, \dot{\mathbf{x}}(\mathbf{x}, \boldsymbol{\xi})).$$

Taking into account (2.1.3), we get $\boldsymbol{\xi} = m\dot{\mathbf{x}} + A(\mathbf{x})$ or $\dot{\mathbf{x}} = (\boldsymbol{\xi} - A(\mathbf{x}))/m$. Therefore,

$$\mathscr{H}(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{2m}|\boldsymbol{\xi} - A(\mathbf{x})|^2 + V(\mathbf{x}), \quad (\mathbf{x}, \boldsymbol{\xi}) \in T^*\mathbb{R}^n = \mathbb{R}^{2n}. \tag{2.1.14}$$

The Hamiltonian function $\mathscr{H}$ determines the canonical system of Hamiltonian equations

$$\dot{x}_j = \frac{\partial \mathscr{H}}{\partial \xi_j}, \quad \dot{\xi}_j = -\frac{\partial \mathscr{H}}{\partial x_j}, \quad j = 1, \dots, n, \tag{2.1.15}$$

which is equivalent to (2.1.4) (see [5, Section 15]). System (2.1.15) can be re-written in the form

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\xi}} \end{pmatrix} = (J^T)^{-1} \nabla \mathscr{H} = J \nabla \mathscr{H}, \tag{2.1.16}$$

where $J = (J^T)^{-1} := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, and I_n is the unit $n \times n$-matrix. Introduce the canonical[1] symplectic bi-linear form

$$\boldsymbol{\sigma}((\mathbf{x}, \boldsymbol{\xi}), (\mathbf{x}', \boldsymbol{\xi}')) := \boldsymbol{\xi} \cdot \mathbf{x}' - \mathbf{x} \cdot \boldsymbol{\xi}', \quad (\mathbf{x}, \boldsymbol{\xi}), (\mathbf{x}', \boldsymbol{\xi}') \in \mathbb{R}^{2n}. \tag{2.1.17}$$

In other words, if $\mathbf{w}, \mathbf{w}' \in \mathbb{R}^{2n}$, then

$$\boldsymbol{\sigma}(\mathbf{w}, \mathbf{w}') = J\mathbf{w} \cdot \mathbf{w}'.$$

Thus, the canonical symplectic 2-form is

$$\omega := \sum_{j=1}^{n} dx_j \wedge d\xi_j. \tag{2.1.18}$$

The co-tangent bundle $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ equipped with the symplectic form ω is a symplectic manifold (see [5, Section 37]).
For $\mathscr{F}, \mathscr{G} \in C^1(\mathbb{R}^{2n})$ introduce the canonical Poisson bracket

$$\{\mathscr{F}, \mathscr{G}\} := \nabla \mathscr{F} \cdot J \nabla \mathscr{G} = \sum_{j=1}^{n} \left(\frac{\partial \mathscr{F}}{\partial x_j} \frac{\partial \mathscr{G}}{\partial \xi_j} - \frac{\partial \mathscr{F}}{\partial \xi_j} \frac{\partial \mathscr{G}}{\partial x_j} \right). \tag{2.1.19}$$

In particular,

$$\{x_j, x_k\} = 0, \quad \{\xi_j, \xi_k\} = 0, \quad \{x_j, \xi_k\} = \delta_{jk}, \quad j, k = 1, \dots, n. \tag{2.1.20}$$

[1] One of the reasons to call various objects "canonical" is to distinguish them from their "magnetic" counterparts.

Let $(\mathbf{x}(t), \boldsymbol{\xi}(t))$ be the solution of (2.1.16) related to fixed initial data. Then the Hamiltonian system (2.1.16) implies that the dynamics of $\mathscr{F}(t) := \mathscr{F}(\mathbf{x}(t), \boldsymbol{\xi}(t))$ is governed by the equation

$$\frac{d\mathscr{F}}{dt} = \{\mathscr{F}, \mathscr{H}\}. \tag{2.1.21}$$

The Hamiltonian equations of motion (2.1.16) can be also written in terms of another Hamiltonian function, namely

$$\tilde{\mathscr{H}}(\mathbf{x}, \boldsymbol{\xi}) := \frac{1}{2m}|\boldsymbol{\xi}|^2 + V(\mathbf{x}), \tag{2.1.22}$$

which is independent of the magnetic field. Note that $\mathscr{H}$ is transformed into $\tilde{\mathscr{H}}$ under the change of variables

$$\mathbf{x} \mapsto \mathbf{x}, \quad \boldsymbol{\xi} \mapsto \boldsymbol{\xi} + A(\mathbf{x}). \tag{2.1.23}$$

In other words, we have

$$\mathscr{H}(\mathbf{x}, \boldsymbol{\xi} + A(\mathbf{x})) = \tilde{\mathscr{H}}(\mathbf{x}, \boldsymbol{\xi}), \quad (\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^{2n}.$$

Under the same change of variables, the canonical symplectic form ω is transformed into *the magnetic symplectic form*

$$\omega_B := \omega - B = \sum_{j=1}^{n} dx_j \wedge d\xi_j - \sum_{1 \le j < k \le n} B_{jk}(\mathbf{x})\, dx_j \wedge dx_k, \tag{2.1.24}$$

which depends on B and corresponds to the magnetic symplectic bi-linear form

$$\boldsymbol{\sigma}_B(\mathbf{w}, \mathbf{w}') = J_B(\mathbf{x})\mathbf{w} \cdot \mathbf{w}', \quad \mathbf{w}, \mathbf{w}' \in \mathbb{R}^{2n},$$

with

$$J_B(\mathbf{x}) := \begin{pmatrix} -B(\mathbf{x}) & I_n \\ -I_n & 0 \end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^n.$$

Then the equations of motion, analogous to (2.1.16), can be written as

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\xi}} \end{pmatrix} = (J_B^T)^{-1} \nabla \widetilde{\mathscr{H}} = K_B \nabla \widetilde{\mathscr{H}}, \tag{2.1.25}$$

where $K_B := (J_B^T)^{-1} = \begin{pmatrix} 0 & I_n \\ -I_n & B \end{pmatrix}$, and the Hamiltonian function $\widetilde{\mathscr{H}}$ is defined in (2.1.22) (for more details, see [126, Subsection 2.10]). Thus, the passage from the couple $(\omega, \mathscr{H})$ to $(\omega_B, \tilde{\mathscr{H}})$ is related to non-degenerate coordinate change (2.1.23) in the underlying symplectic manifold.

Similarly to (2.1.19), introduce the magnetic Poisson bracket

$$\{\mathscr{F}, \mathscr{G}\}_B := \nabla\mathscr{F} \cdot K_B \nabla\mathscr{G} = \{\mathscr{F}, \mathscr{G}\} + \sum_{j,k=1}^{n} B_{jk}(\mathbf{x}) \frac{\partial \mathscr{F}}{\partial \xi_j} \frac{\partial \mathscr{G}}{\partial \xi_k} \tag{2.1.26}$$

with $\mathscr{F}, \mathscr{G} \in C^1(\mathbb{R}^{2n})$. In particular, by analogy with (2.1.20), we have

$$\left\{x_j, x_k\right\}_B = 0, \quad \left\{\xi_j, \xi_k\right\}_B = B_{jk}(\mathbf{x}), \quad \left\{x_j, \xi_k\right\}_B = \delta_{jk}, \quad j, k = 1, \ldots, n. \tag{2.1.27}$$

Let now $(\tilde{\mathbf{x}}(t), \tilde{\boldsymbol{\xi}}(t))$ be the solution of the equations of motion (2.1.25), corresponding to fixed initial data. Then the dynamics of $\widetilde{\mathscr{F}}(t) := \mathscr{F}(\tilde{\mathbf{x}}(t), \tilde{\boldsymbol{\xi}}(t))$ is governed by the equation

$$\frac{d\widetilde{\mathscr{F}}}{dt} = \left\{ \mathscr{F}, \widetilde{\mathscr{H}} \right\}_B . \tag{2.1.28}$$

2.2 The concept of quantization

According to the axioms of the quantum mechanics, the state of a physical system is described by a non-zero element ψ of a Hilbert space $\mathfrak{H}$ equipped with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Proportional elements $\psi \in \mathfrak{H}$ determine the same physical state, so that we may assume $\|\psi\| = 1$. Further, the quantum physical observables are represented by linear operators, self-adjoint in $\mathfrak{H}$. If the state of a quantum system is determined by $\psi \in \mathfrak{H}$, and a given quantum observable is represented by the operator $F = F^*$, then the probability that the value of this observable is in a given Borel set $\mathscr{J} \subset \mathbb{R}$, equals

$$\langle \mathbb{1}_{\mathscr{J}}(F)\psi, \psi \rangle = \| \mathbb{1}_{\mathscr{J}}(F)\psi \|^2.$$

Here and in the sequel $\mathbb{1}_S$ denotes the characteristic function of the set S, so that $\mathbb{1}_{\mathscr{J}}(F)$ is just the spectral projection of the operator F associated with $\mathscr{J}$. In particular, if $\mathscr{J}$ does not intersect with the spectrum $\sigma(F)$ of F, then the probability that the value of the observable represented by F, is in the set $\mathscr{J}$, equals zero. A similar probabilistic interpretation is valid also for the quantity $\|\mathbb{1}_{\mathscr{J}}(F_1, \ldots, F_n))\psi\|^2$ where $(F_1, \ldots, F_n)$ is a finite family of commuting operators, self-adjoint in $\mathfrak{H}$, and $\mathscr{J}$ is now a Borel set in $\mathbb{R}^n$ (see e.g. [13, Section 1.1, Postulate 2]). This interpretation is one of the main reasons for which the spectral theory of self-adjoint operators plays a fundamental role in quantum physics. For further references, we introduce here our notations of the spectral components: $\sigma_{ess}(F)$ denotes the essential spectrum of the operator F, $\sigma_{disc}(F)$, is the discrete one, $\sigma_c(F)$ is the continuous one, $\sigma_{ac}(F)$ is the absolutely continuous (a.c.) one, and $\sigma_{sc}(F)$ is the singular continuous (s.c.) spectrum.

In the Schrödinger formalism, the dynamics of ψ is determined by the non-stationary Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \tag{2.2.1}$$

where $2\pi\hbar > 0$ is *the Planck constant*, while H is *the quantum Hamiltonian*, i.e. the self-adjoint operator corresponding to the energy of the system. If we assume that H is independent of time, then the solution of (2.2.1) is $\psi(t) = U_\hbar(t)\psi(0)$ where

$$U_\hbar(t) := e^{-\frac{it}{\hbar}H}, \quad t \in \mathbb{R},$$

is the unitary group defined by the spectral theorem for self-adjoint operators.

In the Heisenberg formalism, the time evolution of the quantum observable F is given by

$$F(t) = U_\hbar^*(t)F(0)U_\hbar(t), \quad t \in \mathbb{R},$$

and, hence,

$$\frac{dF}{dt} = \frac{1}{i\hbar}[F, H] \tag{2.2.2}$$

where $[F,H] := FH - HF$ is *the commutator* of the operators F and H. Equation (2.2.2) can be interpreted as the quantum analogue of (2.1.21).
Let us now pass to the concept of *quantization*. As we saw in the previous section, the physical observables are described by the classical mechanics as sufficiently regular functions defined on a given symplectic manifold. Then the quantization $\mathfrak{Q}$ is a procedure which maps classical observables into quantum ones. In other words, the quantization $\mathfrak{Q}$ could be interpreted as a mapping which puts into correspondence to the function $\mathscr{F}$ defined on the symplectic manifold, the operator $F = \mathfrak{Q}(\mathscr{F})$, self-adjoint in the Hilbert space $\mathfrak{H}$.
In the case where the symplectic manifolds is the phase space $T^*\mathbb{R}^n = \mathbb{R}^{2n}$, equipped with the canonical symplectic form (2.1.18), the mapping $\mathfrak{Q}$ should satisfy the following minimal set of axioms (see e.g. [87, Subsection 3.7] and the references cited there):

Axiom 1. $\mathfrak{Q}$ is linear, and $\mathfrak{Q}(1) = I$, where I is the identity $\mathfrak{H}$.

In order to formulate our next axiom we need the concept of a canonical system, or, in brief, a C-system (see [20, Chapter 12]). Assume that the operators a_j, b_j, $j = 1, \dots, n$, are symmetric operators with common domain $\mathscr{D}$, dense in $\mathfrak{H}$, invariant with respect to the action of these operators. Suppose that:
(i) For $u \in \mathscr{D}$ and $j, k = 1, \dots, n$, we have

$$(a_j b_k - b_k a_j)u = i\hbar\delta_{jk}u, \quad (a_j a_k - a_k a_j)u = 0, \quad (b_j b_k - b_k b_j)u = 0.$$

(ii) The symmetric operator

$$\sum_{j=1}^{n} \left(a_j^2 + b_j^2\right)$$

with domain $\mathscr{D}$ is essentially self-adjoint in $\mathfrak{H}$.
Then we say that the operators a_j, b_j, $j = 1, \dots, n$, form a C_0-system. If the closures $\alpha_j := \bar{a}_j$ and $\beta_j := \bar{b}_j$ are self-adjoint operators, we say that α_j, β_j, $j = 1, \dots, n$, form a C-system. Note that the operators α_j, β_j, $j = 1, \dots, n$, form a C-system if and only if the corresponding unitary groups satisfy the Weyl canonical relations

$$e^{is\alpha_j} e^{it\beta_k} = e^{-ist\hbar\delta_{jk}} e^{it\beta_k} e^{is\alpha_j},$$

$$e^{is\alpha_j} e^{it\alpha_k} = e^{it\alpha_k} e^{is\alpha_j}, \quad e^{is\beta_j} e^{it\beta_k} = e^{it\beta_k} e^{is\beta_j}, \quad j, k = 1, \dots, n, \quad s, t \in \mathbb{R},$$

(see [20, Chapter 12, Subsection 5.4]). However, if we drop condition (ii) in the definition of a C_0-system, then this equivalence does not hold true any more (see the counterexamples in [162, Section VIII.5] and [88, Section 12.2]).

Axiom 2. The operators $\mathfrak{Q}(x_j)$, $\mathfrak{Q}(\xi_j)$, $j = 1, \dots, n$, form a C-system. In particular, on the common operator core $\mathscr{D}$ of $\mathfrak{Q}(x_j)$, $\mathfrak{Q}(\xi_j)$, we have

$$[\mathfrak{Q}(x_j), \mathfrak{Q}(x_k)] = 0, \quad [\mathfrak{Q}(\xi_j), \mathfrak{Q}(\xi_k)] = 0,$$

$$[\mathfrak{Q}(x_j), \mathfrak{Q}(\xi_k)] = i\delta_{jk}\,\hbar\mathfrak{Q}(1) = i\hbar\,\delta_{jk}\,I, \quad j, k = 1, \dots, n. \tag{2.2.3}$$

Axiom 3. The Hilbert space $\mathfrak{H}$ is irreducible under the action of the operators $\mathfrak{Q}(x_j)$ and $\mathfrak{Q}(\xi_j)$, $j = 1, \dots, n$.

By the Stone – von Neumann theorem (see e.g. [162, Theorem VIII.14]), Axioms 1, 2, and 3 imply that, up to unitary equivalence, we have $\mathfrak{H} = L^2(\mathbb{R}^n)$, and

$$(\mathfrak{Q}(x_j)u)(x) = x_j u(x), \quad (\mathfrak{Q}(\xi_j)u)(x) = -i\hbar \frac{\partial u}{\partial x_j}(x), \quad x \in \mathbb{R}^n, \quad j = 1, \dots, n. \tag{2.2.4}$$

Moreover, the operators $\mathfrak{Q}(x_j)$ and $\mathfrak{Q}(\xi_j)$, $j = 1, \dots, n$, are essentially self-adjoint on the *Schwartz class* $\mathscr{S}(\mathbb{R}^n)$ which is invariant under their action, and, in accordance with the definition of a C-system, the operator

$$\sum_{j=1}^{n} \mathfrak{Q}(x_j)^2 + \mathfrak{Q}(\xi_j)^2 = -\Delta + |\mathbf{x}|^2$$

is also essentially self-adjoint on $\mathscr{S}(\mathbb{R}^n)$.

Let us mention two more axioms which are sometimes imposed on the quantization scheme. First, if the classical observables are allowed to be complex-valued with the purpose to develop a rigorous mathematical theory of the quantization procedure, then the quantum observables are supposed to be closed linear, not necessarily self-adjoint, operators. However, in this case one imposes the following

Axiom 4: If $\mathscr{F}$ is real-valued, then $\mathfrak{Q}(\mathscr{F})$ is self-adjoint.

Finally, in some cases one uses the following postulate:

Axiom 5: Let $\mathscr{F} : T^*\mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{C}$ be sufficiently regular functions. Then we have

$$\mathfrak{Q}(g \circ \mathscr{F}) = g(\mathfrak{Q}(\mathscr{F})).$$

Note that Axiom 5, known as *the von Neumann rule*, usually is not satisfied for all admissible $\mathscr{F}$ and g. Hence, it should be treated not as an axiom but rather as a desirable property which holds true at least for all admissible $\mathscr{F}$ which depend only on the coordinates $\mathbf{x} \in \mathbb{R}^n$, or only on the momenta $\boldsymbol{\xi} \in \mathbb{R}^n$.

Let us comment in more detail on Axiom 2. We have stated above its "minimal" version concerning only the operators $\mathfrak{Q}(x_j)$ and $\mathfrak{Q}(\xi_j)$, $j = 1, \dots, n$. Its "maximal" version can be formulated as:

Axiom 2′: We have

$$\mathfrak{Q}(\{\mathscr{F}, \mathscr{G}\}) = \frac{1}{i\hbar}[\mathfrak{Q}(\mathscr{F}), \mathfrak{Q}(\mathscr{G})],$$

for *any* sufficiently regular functions $\mathscr{F}, \mathscr{G}$ defined on $T^*\mathbb{R}^n$.

However, the Groenewold - van Hove theorem (see e.g. [88, Section 13.4] implies that there exists no quantization mapping $\mathfrak{Q}$ which satisfies simultaneously Axioms 1,

$2'$, and 3. One possible way to overcome this difficulty is to adopt the semiclassical point of view of the so called deformation quantization where the Poisson bracket is replaced by *the Moyal bracket*

$$\{\{\mathscr{F},\mathscr{G}\}\}_\hbar := \frac{1}{i\hbar}(\mathscr{F}\star\mathscr{G}-\mathscr{G}\star\mathscr{F}). \tag{2.2.5}$$

Here

$$(\mathscr{F}\star\mathscr{G})(\mathbf{w}) := (\pi\hbar)^{-n}\int_{\mathbb{R}^{4n}} e^{\frac{2i}{\hbar}\boldsymbol{\sigma}(\mathbf{w}',\mathbf{w}'')}\mathscr{F}(\mathbf{w}'+\mathbf{w})\mathscr{G}(\mathbf{w}''+\mathbf{w})d\mathbf{w}'d\mathbf{w}'', \quad \mathbf{w}\in\mathbb{R}^{2n}, \tag{2.2.6}$$

is *the Moyal product* called also *the Weyl product*, $\boldsymbol{\sigma}$ being the canonical bi-linear form introduced in (2.1.17). A priori the Moyal product is defined for $\mathscr{F},\mathscr{G}$ in the Schwartz class $\mathscr{S}(\mathbb{R}^{2n})$, and then is extended by continuity to larger classes of functions and distributions. Since, generally speaking the linear operators on a Hilbert space do not commute, the introduction of the non-commutative Moyal product for the classical observables is of a crucial conceptual importance for the quantization procedure. Let us now give an alternative equivalent representation of the Moyal product in terms of the Fourier transforms of $\mathscr{F}$ and $\mathscr{G}$. To this end, we recall the definition of the Fourier transform Φu of $u \in \mathscr{S}(\mathbb{R}^N)$, $N \geq 1$, namely

$$(\Phi u)(\boldsymbol{\xi}) = \hat{u}(\boldsymbol{\xi}) := (2\pi)^{-N/2}\int_{\mathbb{R}^N} e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}u(\mathbf{x})d\mathbf{x}, \quad \boldsymbol{\xi}\in\mathbb{R}^N.$$

As is well known, the inverse operator Φ^{-1} is written as

$$(\Phi^{-1}u)(\mathbf{x}) = \check{u}(\mathbf{x}) := (2\pi)^{-N/2}\int_{\mathbb{R}^N} e^{i\mathbf{x}\cdot\boldsymbol{\xi}}u(\boldsymbol{\xi})d\boldsymbol{\xi}, \quad \mathbf{x}\in\mathbb{R}^N.$$

By duality, Φ and Φ^{-1} extend to the class $\mathscr{S}'(\mathbb{R}^N)$, dual to $\mathscr{S}(\mathbb{R}^N)$. In particular, Φ extends to a unitary operator in $L^2(\mathbb{R}^N)$ so that $\Phi^* = \Phi^{-1}$.

Let $\mathscr{F},\mathscr{G}\in\mathscr{S}(\mathbb{R}^{2n})$. Then we have

$$\widehat{(\mathscr{F}\star\mathscr{G})}(\mathbf{w}) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i\hbar}{2}\boldsymbol{\sigma}(\mathbf{w},\mathbf{w}')}\widehat{\mathscr{F}}(\mathbf{w}-\mathbf{w}')\widehat{\mathscr{G}}(\mathbf{w}')d\mathbf{w}', \quad \mathbf{w}\in\mathbb{R}^{2n}. \tag{2.2.7}$$

In order to see that (2.2.6) and (2.2.7) are equivalent, it suffices to apply the inverse 2n-dimensional Fourier transform to both hand sides of (2.2.7).
Note that (2.2.7) is equivalent to

$$(\mathscr{F}\star\mathscr{G})(\mathbf{x},\boldsymbol{\xi}) = e^{\frac{i\hbar}{2}\left(D_{\mathbf{y}}\cdot D_{\boldsymbol{\xi}}-D_{\mathbf{x}}\cdot D_{\boldsymbol{\eta}}\right)}\mathscr{F}(\mathbf{x},\boldsymbol{\xi})\mathscr{G}(\mathbf{y},\boldsymbol{\eta})_{|\mathbf{y}=\mathbf{x},\boldsymbol{\eta}=\boldsymbol{\xi}} \tag{2.2.8}$$

with $D := -i\nabla$. In particular, we have

$$\mathscr{F}\star\mathscr{G} = \mathscr{F}\mathscr{G} + \frac{i\hbar}{2}\{\mathscr{F},\mathscr{G}\} + \mathscr{O}(\hbar^2), \quad \hbar\downarrow 0. \tag{2.2.9}$$

Putting together (2.2.5) and (2.2.9), we find that

$$\{\{\mathscr{F},\mathscr{G}\}\}_\hbar = \{\mathscr{F},\mathscr{G}\} + \mathscr{O}(\hbar), \quad \hbar \downarrow 0.$$

Hence, the Moyal bracket is a deformation of the Poisson one, i.e. the Poisson bracket is the limit of the Moyal one as $\hbar \downarrow 0$. Accordingly, Axiom $2'$ can be modified to:

Axiom $2''$: If $\mathscr{F}$ and $\mathscr{G}$ are sufficiently regular functions defined on $T^*\mathbb{R}^n$, then

$$\mathfrak{Q}\left(\{\{\mathscr{F},\mathscr{G}\}\}_\hbar\right) = \frac{1}{i\hbar}[\mathfrak{Q}(\mathscr{F}),\mathfrak{Q}(\mathscr{G})]. \tag{2.2.10}$$

2.3 The Weyl quantization

In this section we consider an important quantization mapping $\mathfrak{Q} = \mathrm{Op}_\hbar^w$ called *the Weyl quantization*, which satisfies Axioms 1, $2''$, 3, and 4, for all admissible classical observables as well as the von Neumann rule (Axiom 5) for classical observables which depend only on $\mathbf{x}$ or only on $\boldsymbol{\xi}$. For $\mathscr{F} \in \mathscr{S}(\mathbb{R}^{2n})$, the operator $\mathrm{Op}_\hbar^w(\mathscr{F})$ is defined by

$$(\mathrm{Op}_\hbar^w(\mathscr{F})u)(\mathbf{x})$$
$$:= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \mathscr{F}\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\boldsymbol{\xi}\right) e^{i(\mathbf{x}-\mathbf{x}')\cdot\boldsymbol{\xi}/\hbar} u(\mathbf{x}')d\mathbf{x}'d\boldsymbol{\xi}, \quad u \in L^2(\mathbb{R}^n). \tag{2.3.1}$$

It is easy to check, that $\mathrm{Op}_\hbar^w(\mathscr{F})$ is a continuous mapping from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$, and from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. In the sequel, we will call $\mathrm{Op}_\hbar^w(\mathscr{F})$ a $\hbar$-*pseudo-differential operator* ($\hbar$-ΨDO) with *Weyl symbol* $\mathscr{F}$. Various aspects of the theory of such operators can be found in [96, Section 18.5], [185, Chapter IV], [205, Chapter 7, Section 14], and [63, Chapter 7]). In this chapter we summarize mainly those of their properties which will be used in the sequel. We start with the following elementary fact:

Lemma 2.3.1 *The $\hbar$-ΨDO with Weyl symbol $\mathscr{F}(\mathbf{x},\boldsymbol{\xi})$ coincides with the* 1*-ΨDO with Weyl symbol $\mathscr{F}(\mathbf{x},\hbar\boldsymbol{\xi})$, $(\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^{2n}$.*

Proof. It suffices to change of variables $\boldsymbol{\xi} \mapsto \hbar\boldsymbol{\xi}$ in (2.3.1). □

Set $X := (x_1,\dots,x_n)$ where x_j denotes the multiplier by $x_j \in \mathbb{R}$, $j = 1,\dots,n$, and, as earlier, $D := -i\nabla$. Define the operator-valued function

$$e^{i(\mathbf{q}\cdot X+\mathbf{p}\cdot D)}, \quad \mathbf{q},\mathbf{p} \in \mathbb{R}^n, \tag{2.3.2}$$

where $\mathbf{q}\cdot X + \mathbf{p}\cdot D$ is the closure of the operator, essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. Hence, the operator defined in (2.3.2) is unitary.
Set $\mathrm{Op}^w(\mathscr{F}) := \mathrm{Op}_1^w(\mathscr{F})$.

Proposition 2.3.1 *Let $\mathscr{F} \in \mathscr{S}(\mathbb{R}^{2n})$. Then*

$$\mathrm{Op}^w(\mathscr{F}) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} \widehat{\mathscr{F}}(\mathbf{q},\mathbf{p})\, e^{i(\mathbf{q}\cdot X+\mathbf{p}\cdot D)}\, d\mathbf{q}\, d\mathbf{p}, \tag{2.3.3}$$

the integral being defined in the strong sense.

Proof. It is easy to check that

$$(\mathrm{Op}^{\mathrm{w}}(\mathscr{F})u)(\mathbf{x}) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} \widehat{\mathscr{F}}(\boldsymbol{\xi}, \mathbf{x}' - \mathbf{x}) e^{i\frac{(\mathbf{x}+\mathbf{x}')\cdot\boldsymbol{\xi}}{2}} u(\mathbf{x}')d\mathbf{x}'\, d\boldsymbol{\xi}. \tag{2.3.4}$$

Resolving the Cauchy problem

$$\begin{cases} -i\frac{\partial w}{\partial t}(t,\mathbf{x}) = -i\mathbf{p}\cdot\nabla_{\mathbf{x}} w(t,\mathbf{x}) + \mathbf{q}\cdot\mathbf{x}\, w(t,\mathbf{x}), \\ w(0,\mathbf{x}) = u(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^n, \end{cases} \tag{2.3.5}$$

by the method of characteristics (see e.g. [74, Sections 2.1 and 3.2]), we find that

$$w(t,\mathbf{x}) = \exp\left(i\left(t^2\frac{\mathbf{p}\cdot\mathbf{q}}{2} + t\mathbf{q}\cdot\mathbf{x}\right)\right)u(\mathbf{x}+t\mathbf{p}), \quad t\in\mathbb{R}, \quad \mathbf{x}\in\mathbb{R}^n,$$

and, hence,

$$\left(e^{i(\mathbf{q}\cdot X + \mathbf{p}\cdot D)}u\right)(\mathbf{x}) = w(1,\mathbf{x}) = \exp\left(i\left(\frac{\mathbf{p}\cdot\mathbf{q}}{2} + \mathbf{q}\cdot\mathbf{x}\right)\right)u(\mathbf{x}+\mathbf{p}), \quad \mathbf{x}\in\mathbb{R}^n. \tag{2.3.6}$$

Changing the variables $\boldsymbol{\xi} = \mathbf{q}$ and $\mathbf{x}' = \mathbf{x}+\mathbf{p}$ in (2.3.4), and bearing in mind (2.3.6), we arrive at (2.3.3). □

Proposition 2.3.2 *Let $\mathscr{F},\mathscr{G}\in\mathscr{S}(\mathbb{R}^{2n})$. Then we have*

$$\mathrm{Op}^{\mathrm{w}}_{\hbar}(\mathscr{F}\star\mathscr{G}) = \mathrm{Op}^{\mathrm{w}}_{\hbar}(\mathscr{F})\mathrm{Op}^{\mathrm{w}}_{\hbar}(\mathscr{G}). \tag{2.3.7}$$

Proof. By Lemma 2.3.1 it suffices to prove the proposition for $\hbar = 1$. Write the product $\mathrm{Op}^{\mathrm{w}}_{\hbar}(\mathscr{F})\mathrm{Op}^{\mathrm{w}}_{\hbar}(\mathscr{G})$ using the representation (2.3.3) for the operators $\mathrm{Op}^{\mathrm{w}}_{\hbar}(\mathscr{F})$ and $\mathrm{Op}^{\mathrm{w}}_{\hbar}(\mathscr{G})$. Taking into account (2.2.7), we find that (2.3.7) holds true in the case $\hbar = 1$. □

Corollary 2.3.1 *Under the hypotheses of Proposition 2.3.2, relation* (2.2.10) *holds true, i.e. the quantization $\mathfrak{Q} = \mathrm{Op}^{\mathrm{w}}_{\hbar}$ satisfies Axiom* $2''$ *on* $\mathscr{S}(\mathbb{R}^{2n})$.

Further, we extend the mapping Op^{w} to classes of symbols, essentially larger than $\mathscr{S}(\mathbb{R}^{2n})$. First, we start with such symbols $\mathscr{F}$ that the operators $\mathrm{Op}^{\mathrm{w}}(\mathscr{F})$ remain bounded in $L^2(\mathbb{R}^n)$. It is convenient to describe here also symbols for which $\mathrm{Op}^{\mathrm{w}}(\mathscr{F})$ is compact or even is in certain Schatten-von Neumann classes.

To this end, we need the following notations. Let $\mathfrak{H}_j$, $j = 1,2$, be two (separable) Hilbert spaces. Then, $\mathfrak{B}(\mathfrak{H}_1,\mathfrak{H}_2)$ denotes the Banach space of bounded linear operators $T:\mathfrak{H}_1\to\mathfrak{H}_2$, equipped with the usual operator norm $\|T\|$, and $\mathfrak{S}_\infty(\mathfrak{H}_1,\mathfrak{H}_2)$ is the closed subspace of $\mathfrak{B}(\mathfrak{H}_1,\mathfrak{H}_2)$, consisting of compact operators. Further, for $p\in(0,\infty)$ introduce *the Schatten-Neumann class* $\mathfrak{S}_p(\mathfrak{H}_1,\mathfrak{H}_2)$ of operators $T\in\mathfrak{S}_\infty(\mathfrak{H}_1,\mathfrak{H}_2)$ for which the functional

$$\|T\|_p := \left(\mathrm{Tr}\,(T^*T)^{p/2}\right)^{1/p}$$

is finite. In particular, $\mathfrak{S}_1$ is the trace class, and $\mathfrak{S}_2$ is the Hilbert-Schmidt class. If $p\geq 1$, then $\|\cdot\|_p$ is a norm and $\mathfrak{S}_p$, equipped with this norm, is complete.

Let $T\mathscr{H}_\infty(\mathfrak{H}_1,\mathfrak{H}_2)$. Denote by $\left\{s_j(T)\right\}_{j=1}^{\mathrm{rank}\,T}$ the non-decreasing set of the singular numbers of the operator T, i.e. the square roots of the eigenvalues of the operator T^*T. Then we have

$$s_1(T) = \|T\|, \quad \|T\|_p^p = \sum_{p=1}^{\mathrm{rank},T} s_j(T)^p, \quad p \in (0,\infty).$$

Further, for $p \in (0,\infty)$ introduce the weak Schatten-von Neumann class $\mathfrak{S}_{p,w}(\mathfrak{H}_1,\mathfrak{H}_2)$ of operators $T \in \mathfrak{H}_\infty(\mathfrak{H}_1,\mathfrak{H}_2)$ for which the functional

$$\|T\|_{p,w} := \sup_j j^{1/p} s_j(T),$$

is finite. If $p > 1$, then there exists a norm equivalent to the functional $\|\cdot\|_{p,w}$, and $\mathfrak{S}_{pw}(\mathfrak{H}_1,\mathfrak{H}_2)$ equipped with this norm is complete (see [20, Chapter 11, Section 6, Theorem 2.3]).
If $\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{H}$, we set $\mathfrak{B}(\mathfrak{H}) := \mathfrak{B}(\mathfrak{H},\mathfrak{H})$, etc., and if no confusion is likely, we omit in the notations of the spaces of bounded, compact or Schatten-von Neumann operators the explicit indication of the Hilbert spaces where these operators act.

Let us return to the theory of ΨDOs with Weyl symbols. Note that if, say, $\mathscr{F} \in \mathscr{S}(\mathbb{R}^{2n})$, then the ΨDO $\mathrm{Op}^w(\mathscr{F})$ is an operator with integral kernel

$$K(\mathbf{x},\mathbf{x}') = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathscr{F}\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\boldsymbol{\xi}\right) e^{i(\mathbf{x}-\mathbf{x}')\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}, \quad \mathbf{x},\mathbf{x}' \in \mathbb{R}^n. \tag{2.3.8}$$

On the other hand, a given linear operator T acting in $L^2(\mathbb{R}^n)$ has an integral kernel, say, $K \in \mathscr{S}(\mathbb{R}^{2n})$, then T can be written as a ΨDO with Weyl symbol

$$\mathscr{F}(\mathbf{x},\boldsymbol{\xi}) = \int_{\mathbb{R}^n} e^{-i\mathbf{x}'\cdot\boldsymbol{\xi}} K(\mathbf{x}+\mathbf{x}'/2,\mathbf{x}-\mathbf{x}'/2)\, d\mathbf{x}', \quad (\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^{2n}, \tag{2.3.9}$$

(see e.g. [185, Eq. (23.39)]).
For $u,v \in \mathscr{S}(\mathbb{R}^n)$, define *the Wigner transform* $W_{u,v}$ of the pair (u,v) by

$$W_{u,v}(\mathbf{x},\boldsymbol{\xi}) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\mathbf{x}'\cdot\boldsymbol{\xi}} u(\mathbf{x}+\mathbf{x}'/2)\, \overline{v(\mathbf{x}-\mathbf{x}'/2)}\, d\mathbf{x}', \quad (\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^{2n}. \tag{2.3.10}$$

Then $W_{u,v} \in \mathscr{S}(\mathbb{R}^{2n})$ and we have $W_{u,v} = \overline{W_{v,u}}$. Moreover, the linear combinations of Wigner transforms $W_{u,v}$ with $u,v \in \mathscr{S}(\mathbb{R}^n)$ are dense in $\mathscr{S}(\mathbb{R}^{2n})$. To see this, it suffices to check that we can write the functions

$$\mathbf{x}^\alpha \boldsymbol{\xi}^\beta e^{-(|\mathbf{x}|^2+|\boldsymbol{\xi}|^2)/2}, \quad (\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^{2n}, \tag{2.3.11}$$

with $\alpha,\beta \in \mathbb{Z}_*^n$ as a linear combination of Wigner transforms $W_{u,v}$ with $u,v \in \mathscr{S}(\mathbb{R}^n)$. On the other hand, it is well known that the linear combinations of functions of form (2.3.11) are dense in $\mathscr{S}(\mathbb{R}^{2n})$ (see e.g. [162, Chapter V]).
Further, the Wigner transform extends to $u,v \in cS(\mathbb{R}^n)$ in which case

$$\|W_{u,v}\|^2_{L^2(\mathbb{R}^{2n})} = (2\pi)^{-n} \|u\|^2_{L^2(\mathbb{R}^n)} \|v\|^2_{L^2(\mathbb{R}^n)}.$$

Thus, by (2.3.9), we find that $(2\pi)^n W_{u,v}$ coincides with the Weyl symbol of the operator with integral kernel $u(\mathbf{x})\overline{v(\mathbf{x}')}$, $\mathbf{x},\mathbf{x}' \in \mathbb{R}^n$.
If $\mathscr{F} \in \mathscr{S}(\mathbb{R}^{2n})$ and $u, v \in \mathscr{S}(\mathbb{R}^n)$, then

$$\langle \mathrm{Op}^w(\mathscr{F})u, v\rangle_{L^2(\mathbb{R}^n)} = \langle \mathscr{F}, W_{v,u}\rangle_{L^2(\mathbb{R}^{2n})}. \tag{2.3.12}$$

If $\phi \in \mathscr{S}'(\mathbb{R}^N)$ and $u \in \mathscr{S}(\mathbb{R}^N)$, $N \geq 1$, denote by $(\phi, u)_{\mathscr{S}'(\mathbb{R}^N)}$ the standard pairing between $\mathscr{S}'(\mathbb{R}^N)$ and $\mathscr{S}(\mathbb{R}^N)$; if $\phi \in L^2(\mathbb{R}^N)$, then

$$(\phi, u)_{\mathscr{S}'(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \phi(\mathbf{x})\, u(\mathbf{x})\, d\mathbf{x} = \langle \phi, \bar{u}\rangle_{L^2(\mathbb{R}^N)}.$$

Therefore, if $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^{2n})$, the extension of (2.3.12) defines a linear continuous mapping $\mathrm{Op}^w(\mathscr{F}) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ by

$$(\mathrm{Op}^w(\mathscr{F})u, v)_{\mathscr{S}'(\mathbb{R}^n)} := (\mathscr{F}, W_{u,\bar{v}})_{\mathscr{S}'(\mathbb{R}^{2n})}, \quad u, v \in \mathscr{S}(\mathbb{R}^n), \tag{2.3.13}$$

By the density in $\mathscr{S}(\mathbb{R}^{2n})$ of the linear combinations of Wigner transforms $W_{u,v}$ with $u, v \in \mathscr{S}(\mathbb{R}^n)$, we find that $\mathrm{Op}^w(\mathscr{F}) = 0$ with $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^{2n})$ if and only if $\mathscr{F} = 0$.

Proposition 2.3.3 [96, Lemma 18.6.1] *Let* $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^{2n})$ *with* $\widehat{\mathscr{F}} \in L^1(\mathbb{R}^{2n})$. *Then the operator* $\mathrm{Op}^w(\mathscr{F})$ *defined by* (2.3.13)*, extends uniquely to an operator bounded in* $L^2(\mathbb{R}^n)$. *Moreover,*

$$\|\mathrm{Op}^w(\mathscr{F})\| \leq (2\pi)^{-n} \|\widehat{\mathscr{F}}\|_{L^1(\mathbb{R}^{2n})}. \tag{2.3.14}$$

Note that if $\widehat{\mathscr{F}}, \widehat{\mathscr{G}} \in L^1(\mathbb{R}^{2n})$, then (2.2.7) easily implies that $\widehat{\mathscr{F} \star \mathscr{G}} \in L^1(\mathbb{R}^{2n})$ too, and Proposition 2.3.2 remains valid for such symbols.

Estimate (2.3.14) is the first step in the proof of a more sophisticated upper bound of $\|\mathrm{Op}^w(\mathscr{F})\|$, known usually as *Calderón-Vaillancourt estimate*, contained in Proposition 2.3.4 below. For its statement we need the following notation. Let $\Gamma(\mathbb{R}^{2n})$, $n \geq 1$, denote the set of functions $\mathscr{F} : \mathbb{R}^{2n} \to \mathbb{C}$ such that

$$\|\mathscr{F}\|_{\Gamma(\mathbb{R}^{2n})} := \sup_{\{\alpha,\beta \in \mathbb{Z}_+^n \,|\, |\alpha|,|\beta| \leq [\frac{n}{2}]+1\}} \sup_{(\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^{2n}} |D_{\mathbf{x}}^{\alpha} D_{\boldsymbol{\xi}}^{\beta} \mathscr{F}(\mathbf{x}, \boldsymbol{\xi})| < \infty.$$

Note that $\Gamma(\mathbb{R}^{2n}) \subset \mathscr{S}'(\mathbb{R}^{2n})$.

Proposition 2.3.4 [44], [53], [26, Corollary 2.5 (i)] *There exists a constant* c_0 *such that for any* $\mathscr{F} \in \Gamma(\mathbb{R}^{2n})$*,* $n \geq 1$*, we have*

$$\|\mathrm{Op}^w(\mathscr{F})\| \leq c_0 \|\mathscr{F}\|_{\Gamma(\mathbb{R}^{2n})}.$$

Further, if $\mathscr{F} \in L^2(\mathbb{R}^{2n}) \subset \mathscr{S}'(\mathbb{R}^{2n})$, then, obviously, the integral kernel of the operator $\mathrm{Op}^w(\mathscr{F})$ is in $L^2(\mathbb{R}^{2n})$ so that $\mathrm{Op}^w(\mathscr{F}) \in \mathfrak{S}_2(L^2(\mathbb{R}^n))$, and by the Parseval theorem we have

$$\|\mathrm{Op}^w(\mathscr{F})\|_2^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} |\mathscr{F}(\mathbf{w})|^2\, d\mathbf{w} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} |\widehat{\mathscr{F}}(\mathbf{w})|^2\, d\mathbf{w}. \tag{2.3.15}$$

Interpolating between (2.3.14) and (2.3.15) (see [19, Theorem 3.1]), we find that if $p \in [2,\infty)$, and $p' := \frac{p}{p-1}$ then

$$\|\mathrm{Op}^w(\mathscr{F})\|_p \leq (2\pi)^{-n/p'} \|\widehat{\mathscr{F}}\|_{L^{p'}(\mathbb{R}^{2n})}. \tag{2.3.16}$$

Moreover, if $p \in (0,\infty)$, we have

$$\|\mathrm{Op}^w(\mathscr{F})\|_{p,w} \leq (2\pi)^{-n/p'} \|\widehat{\mathscr{F}}\|_{L^{p'}_w(\mathbb{R}^{2n})}, \tag{2.3.17}$$

where $L^{p'}_w(\mathbb{R}^{2n})$ is *the weak Lebesgue space*. We recall that if M is a space with measure μ, then the μ-measurable function $f : M \to \mathbb{C}$ is in the class $L^q_w(M; d\mu)$, $q \in (0,\infty)$, if the functional

$$\|f\|_{L^q_w(M;d\mu)} := \sup_{t>0} t\,\mu\left(\{\mathbf{x} \in M \,|\, |f(x)| > t\}\right)^{1/q}$$

is finite. If $M \subset \mathbb{R}^N$, $N \geq 1$, and μ is the Lebesgue measure, then we write as usual $L^q_w(M)$ instead of $L^q_w(M; d\mu)$.
Note that (2.3.16) and (2.3.17) are just two simple but efficient estimates of the Schatten-von Neumann norms of Weyl ΨDOs in terms of the Fourier transforms of the symbols, which are convenient for our purposes. At the same, the study of the Schatten-von Neumann properties of Weyl ΨDOs is quite an active research area where many interesting and deep results are available (see, for instance, [43], [200], and the references cited there).

Next, we discuss briefly a possible manner to extend the Weyl pseudo-differential calculus to unbounded symbols and operators. Assume that $f \in C^\infty(\mathbb{R}^N)$, $N \geq 1$, satisfies

$$\left|\left(D^\alpha f\right)(\mathbf{w})\right| \leq C_\alpha \langle \mathbf{w} \rangle^{\gamma - \rho|\alpha|},$$

where, as usual, $\langle \mathbf{w} \rangle := (1+|\mathbf{w}|^2)^{1/2}$, $\mathbf{w} \in \mathbb{R}^N$, $\alpha \in \mathbb{Z}^N_+$, $\gamma \in \mathbb{R}$, $\rho \in [0,1]$, and $C_\alpha \in [0,\infty)$. Then, following [185, Chapter IV], we write $f \in \Gamma^\gamma_\rho(\mathbb{R}^N)$. If $\gamma = \rho = 0$, we also use the notation

$$C^\infty_b(\mathbb{R}^N) := \Gamma^0_0(\mathbb{R}^N),$$

i.e. $C^\infty_b(\mathbb{R}^N)$ is the set of the functions $f \in C^\infty(\mathbb{R}^N)$, bounded with all its derivatives. Note that if $\gamma_1 \geq \gamma_2$ and $\rho_1 \leq \rho_2$, then

$$\Gamma^{\gamma_1}_{\rho_1} \subset \Gamma^{\gamma_2}_{\rho_2}.$$

Moreover, if $\gamma \in \mathbb{R}$, and $\rho \in [0,1]$, then $\Gamma^\gamma_\rho(\mathbb{R}^N) \subset \mathscr{S}'(\mathbb{R}^N)$. Therefore, if $\mathscr{F} \in \Gamma^\gamma_\rho(\mathbb{R}^{2n})$, then the operator $\mathrm{Op}^w(\mathscr{F}) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is well defined by (2.3.13). However, in this particular case we can show also that $\mathrm{Op}^w(\mathscr{F})$ is continuous mapping from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$ (see [185, Subsection 23.2] for the case $\rho \in (0,1]$).
Next, if $\mathscr{F} \in \Gamma^{\gamma_1}_\rho(\mathbb{R}^{2n})$ and $\mathscr{G} \in \Gamma^{\gamma_2}_\rho(\mathbb{R}^{2n})$ with $\gamma_1, \gamma_2 \in \mathbb{R}$ and $\rho \in [0,1]$, then we can extend the Moyal product (2.2.6) with $\hbar = 1$ to the pair $(\mathscr{F}, \mathscr{G})$ by

$$(\mathscr{F} \star \mathscr{G})(\mathbf{w})$$

$$:= \frac{1}{\pi^n} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^{4n}} e^{2i\boldsymbol{\sigma}(\mathbf{w}',\mathbf{w}'')} \chi(\varepsilon \mathbf{w}')\chi(\varepsilon \mathbf{w}'')\mathscr{F}(\mathbf{w}'+\mathbf{w})\mathscr{G}(\mathbf{w}''+\mathbf{w})d\mathbf{w}'d\mathbf{w}'', \quad \mathbf{w} \in \mathbb{R}^{2n},$$

where $\chi \in C_0^\infty(\mathbb{R}^n;[0,1])$ is a cut-off function which satisfies, say, $\chi(\mathbf{w}) = 1$ if $|\mathbf{w}| \leq 1$, and $\chi(\mathbf{w}) = 0$ if $|\mathbf{w}| \geq 2$. Integrating by parts and applying the dominated convergence theorem, we find that

$$(\mathscr{F} \star \mathscr{G})(\mathbf{w})$$
$$= \frac{1}{\pi^n} \int_{\mathbb{R}^{4n}} e^{2i\boldsymbol{\sigma}(\mathbf{w}',\mathbf{w}'')} (1+4|\mathbf{w}'|^{2k})^{-1}(1+(-\Delta_{\mathbf{w}'})^{\ell})\mathscr{F}(\mathbf{w}'+\mathbf{w})$$
$$\times (1+(-\Delta_{\mathbf{w}''})^{k})((1+4|\mathbf{w}''|^{2\ell})^{-1}(\mathscr{G}(\mathbf{w}''+\mathbf{w}))d\mathbf{w}'d\mathbf{w}'',$$

where the integral is absolutely convergent and independent of $k, \ell \in \mathbb{Z}_+$, provided that $2k > n+\gamma_1$ and $2\ell > n+\gamma_2$. Moreover, the analogue of (2.2.8) with $\hbar = 1$ holds true . In particular, we have

$$\mathscr{F} \star \mathscr{G} - \mathscr{F}\mathscr{G} \in \Gamma_\rho^{\gamma_1+\gamma_2-2\rho}(\mathbb{R}^{2n}).$$

which implies

$$\mathscr{F} \star \mathscr{G} \in \Gamma_\rho^{\gamma_1+\gamma_2}(\mathbb{R}^{2n}).$$

(see [185, Subsection 23.6] for the case $\rho \in (0,1]$).

Proposition 2.3.5 *Let $\mathscr{F} \in \Gamma_\rho^\gamma(\mathbb{R}^{2n})$ with $\gamma \leq 0$, $\rho \in [0,1]$. Then* $\mathrm{Op}^w(\mathscr{F})$ *extends to an operator bounded in* $L^2(\mathbb{R}^n)$.

Proof. Evidently, $\gamma \leq 0$ implies $\mathscr{F} \in \Gamma_\rho^\gamma(\mathbb{R}^{2n}) \subset \Gamma(\mathbb{R}^{2n})$. Therefore the claim follows immediately from Proposition 2.3.4. ☐

Proposition 2.3.6 [185, Theorems 24.4] *Let $\mathscr{F} \in \Gamma_\rho^\gamma(\mathbb{R}^{2n})$ with $\gamma < 0$, $\rho \in (0,1]$. Then the operator* $\mathrm{Op}^w(\mathscr{F})$ *is compact in* $L^2(\mathbb{R}^n)$.

We omit the details of the proof but we note that it could be based on approximation of the Weyl ΨDO $\mathrm{Op}^w(\mathscr{F})$ by an anti-Wick ΨDO introduced in 3.4, and follows easily from Proposition 3.4.2.

Now, let

$$\mathbf{H} := -\Delta + |\mathbf{x}|^2, \tag{2.3.18}$$

be the standard harmonic oscillator, self-adjoint in $L^2(\mathbb{R}^n)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. As is well known, $\mathbf{H}$ is strictly positive; more precisely, $\mathbf{H} \geq nI$ (see the spectrum of $\mathbf{H}$ for $n = 1$ in (2.7.13)). Of course, the Weyl symbol of $\mathbf{H}$ is equal to

$$|\boldsymbol{\xi}|^2 + |\mathbf{x}|^2, \quad (\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^{2n},$$

and the calculus of the powers of ΨDOs (see [185]) implies that for each $s \in \mathbb{R}$ the symbol of the operator $\mathbf{H}^s$ coincides with $(|\boldsymbol{\xi}|^2 + |\mathbf{x}|^2)^s$ modulo a symbol in $\Gamma_1^{2s-1}(\mathbb{R}^{2n})$; in particular, the symbol of $\mathbf{H}^{s/2}$ is in $\Gamma_1^s(\mathbb{R}^{2n})$. For $s \geq 0$ define the Hilbert space $\mathfrak{L}^s(\mathbb{R}^n) := \mathfrak{D}(\mathbf{H}^{\frac{s}{2}})$ with norm

$$\|u\|_{\mathfrak{L}^s(\mathbb{R}^n)} := \|\mathbf{H}^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}, \quad u \in \mathfrak{D}(\mathbf{H}^{\frac{s}{2}}).$$

In particular, $\mathfrak{L}^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

Proposition 2.3.7 [185, Theorems 25.2] *Let* $\mathscr{F} \in \Gamma_\rho^\gamma(\mathbb{R}^{2n})$ *with* $\gamma > 0$ *and* $\rho \in (0,1]$. *Then for any* $s \geq \gamma$*, the operator* $\mathrm{Op^w}(\mathscr{F})$ *extends to a continuous mapping* $\mathrm{Op^w}(\mathscr{F}) : \mathfrak{L}^s(\mathbb{R}^n) \to \mathfrak{L}^{s-\gamma}(\mathbb{R}^n)$.

We again omit the details of the proof, but note that for every $s > 0$ the operator $\mathbf{H}^{s/2}$ is a unitary mapping from $\mathfrak{L}^s(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$. Hence, it suffices to check the the boundedness in $L^2(\mathbb{R}^n)$ of the operator $\mathbf{H}^{(s-\gamma)/2}\,\mathrm{Op^w}(\mathscr{F})\,\mathbf{H}^{-s/2}$ whose symbol is in Γ_ρ^0 and hence is bounded in $L^2(\mathbb{R}^n)$ by Proposition 2.3.5.

The Weyl pseudo-differential calculus for $\Gamma_\rho^\gamma(\mathbb{R}^{2n})$ admits extensions for considerably more general classes of symbols (see [95] or [96, Section[18.5]). Following [185], we consider $\Gamma_\rho^\gamma(\mathbb{R}^{2n})$ as reasonable model classes which are sufficient for the applications considered in the book; in particular, Γ_1^γ with $\gamma \in \mathbb{Z}_+$ contains all the polynomials of order less or or equal of degree γ.

Finally, we describe *the metaplectic unitary equivalence* of Weyl ΨDOs whose symbols are mapped into each other by a linear symplectic change of the variables.

Proposition 2.3.8 [63, Chapter 7, Theorem A.2] *Let* $\kappa : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $n \geq 1$, *be a linear symplectic transformation, i.e.*

$$\boldsymbol{\sigma}(\kappa\mathbf{w}, \kappa\mathbf{w}') = \boldsymbol{\sigma}(\mathbf{w}, \mathbf{w}'), \quad \mathbf{w}, \mathbf{w}' \in \mathbb{R}^{2n},$$

$\boldsymbol{\sigma}$ *being the canonic bi-linear form defined in* (2.1.17). *Let* $\mathscr{F} \in \Gamma(\mathbb{R}^{2n})$, *or* $\mathscr{F} \in \Gamma_\rho^\gamma(\mathbb{R}^{2n})$ *with* $\gamma \in \mathbb{R}$ *and* $\rho \in [0,1]$, *and* $\mathscr{G} := \mathscr{F} \circ \kappa$. *Then there exists a unitary operator* $U_\kappa : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *such that*

$$\mathrm{Op^w}(\mathscr{G}) = U_\kappa^* \mathrm{Op^w}(\mathscr{F}) U_\kappa. \tag{2.3.19}$$

The operator U_κ is called *the metaplectic operator* corresponding to the linear symplectic transformation κ. There exists a one-to-one correspondence between metaplectic operators and linear symplectic transformations, up to a constant factor of modulus 1 (see [96, Theorem 18.5.9]). Moreover, every linear symplectic transformation κ is a composition of a finite number of elementary linear symplectic maps (see [96, Lemma 18.5.8]), and for each elementary linear symplectic map there exists an explicit simple metaplectic operator (see the proof of [96, Theorem 18.5.9]).
The conditions imposed on $\mathscr{F}$ in Proposition 2.3.8 are chosen bearing in mind their further applications of this proposition in the book, and hence they might seem somewhat too restrictive and artificial. As a matter of fact, as explained above, the metaplectic operator U_κ is determined by the symplectomorphism κ, and not by the symbol $\mathscr{F}$. Therefore, loosely speaking, we may say that for a given symplectomorphism κ, relation (2.3.19) holds true for *any reasonable* Weyl ΨDO $\mathrm{Op^w}(\mathscr{F})$, provided that $\mathscr{G} = \mathscr{F} \circ \kappa$.

2.4 Magnetic momentum. Magnetic Weyl quantization

Assume $A \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and introduce the operators

$$\Pi_j(A) = \mathfrak{Q}(\xi_j - A_j) := -i\hbar \frac{\partial}{\partial x_j} - A_j, \quad j = 1, \dots, n,$$

which are self-adjoint in $L^2(\mathbb{R}^n)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. We will call the operator

$$\Pi(A) = (\Pi_1(A), \dots, \Pi_n(A)) := -i\hbar\nabla - A$$

the magnetic quantum momentum. Its components satisfy the commutation relations

$$[\Pi_j(A), \Pi_k(A)] = i\hbar B_{jk}, \quad j, k = 1, \dots, n, \tag{2.4.1}$$

where B_{jk} are the components of the magnetic field B, defined in (2.1.6). Then, $\Pi(A)$ can be interpreted as *a connection* in $\mathbb{R}^n$, and the magnetic field as *the curvature* of this connection. Thus the connection is not trivial if and only if B does not vanish identically.

Assume now that the magnetic potentials $A^{(1)}$ and $A^{(2)}$ generate the same magnetic field, i.e. that

$$d(A^{(1)} - A^{(2)}) = 0. \tag{2.4.2}$$

Since $\mathbb{R}^n$ is simply connected and all closed 1-forms are exact, there exists a function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that

$$A^{(1)} - A^{(2)} = \nabla\phi. \tag{2.4.3}$$

In this case we will say that the potentials $A^{(1)}$ and $A^{(2)}$ are *gauge equivalent*. Then the operators $\Pi(A^{(1)})$ and $\Pi(A^{(2)})$ are *gauge covariant*, i.e

$$e^{i\phi/\hbar}\Pi(A^{(2)})e^{-i\phi/\hbar} = \Pi(A^{(1)}). \tag{2.4.4}$$

For any given magnetic potential A the operator $\Pi(A)$ is gauge unitarily equivalent to $\Pi(A')$ with $\operatorname{div} A' = 0$ (Coulomb gauge). To see this it suffices to pick up a solution ϕ of the Poisson equation

$$-\Delta\phi = \operatorname{div} A,$$

and set $A' = A + \nabla\phi$.

Let us mention another important symmetry of $\Pi(A)$. Denote by $\mathscr{C}$ the anti-linear operator of the complex conjugation, i.e.

$$(\mathscr{C}u)(\mathbf{x}) = \overline{u(\mathbf{x})}, \quad \mathbf{x} \in \mathbb{R}^n, \quad u \in L^2(\mathbb{R}^n).$$

Then we have

$$\mathscr{C}\Pi(A)\mathscr{C} = -\Pi(-A). \tag{2.4.5}$$

As will see in the several following sections, the operator $\Pi(A)$ plays a central role in the constructions of the magnetic quantum Hamiltonians such as the Schrödinger, Pauli, and Dirac operators. Within the framework of the so called *magnetic quantization* $\mathfrak{Q}_B$ we can choose

$$\mathfrak{Q}_B(\xi_j) := \Pi_j(A) = -i\hbar\frac{\partial}{\partial x_j} - A_j, \quad j = 1,\dots,n. \tag{2.4.6}$$

Let us summarize briefly the main properties of $\mathfrak{Q}_B$. First, the underlying symplectic manifold on which the classical observables are defined, is $T^*\mathbb{R}^n$, equipped with the magnetic symplectic form (see (2.1.24)). Next, $\mathfrak{Q}_B$ satisfies the minimal list of axioms stated in Section 5.4.2 except that Axiom 2 is replaced by

Axiom 2_B: If $\mathscr{F}$ and $\mathscr{G}$ coincide with one of the components of the vectors $\mathbf{x} = (x_1,\dots,x_n)$ or $\boldsymbol{\xi} = (\xi_1,\dots,\xi_n)$, then $Q_B(\{\mathscr{F},\mathscr{G}\}_B) = \frac{1}{i\hbar}[\mathfrak{Q}_B(\mathscr{F}),\mathfrak{Q}_B(\mathscr{G})]$, the magnetic Poisson bracket $\{\cdot,\cdot\}_B$ being defined in (2.1.26).

In other words, in agreement with (2.1.27), we have

$$[\mathfrak{Q}_B(x_j),\mathfrak{Q}_B(x_k)] = 0, \quad [\mathfrak{Q}_B(\xi_j),\mathfrak{Q}_B(\xi_k)] = i\hbar\mathfrak{Q}_B(B_{jk}),$$
$$[\mathfrak{Q}_B(x_j),\mathfrak{Q}_B(\xi_k)] = i\hbar\,\delta_{jk}I, \quad j,k = 1,\dots,n. \tag{2.4.7}$$

Note that the only difference between (2.2.3) and (2.4.7) is the second relation, i.e. the commutation relation between $\mathfrak{Q}_B(\xi_j)$ and $\mathfrak{Q}_B(\xi_k)$, $j,k = 1,\dots,n$. We can define again $\mathfrak{Q}_B(x_j)$, $j = 1,\dots,n$, as the multiplier by x_j, and consequently $\mathfrak{Q}_B(\mathscr{F})$ with $\mathscr{F} = \mathscr{F}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, as the multiplier by $\mathscr{F}$; in particular, $\mathfrak{Q}_B(B_{jk}) = B_{jk}$, $j,k = 1,\dots,n$. At the same time, we can define $\mathfrak{Q}_B(\xi_j)$, $j = 1,\dots,n$, as in (2.4.6). Then (2.4.1) coincides with second commutation relation in (2.4.7).
M. Măntoiu and R. Purice constructed in [125] (see also [99]) a quantization $\mathfrak{Q}_B$ which satisfies the minimal list of assumptions and possesses some extra nice properties. This quantization denoted here by $\mathrm{Op}^w_{\hbar,A}$ where A is a magnetic potential which generates B, could be considered as an appropriate "magnetic" analogue of the Weyl quantization. We will introduce the operator $\mathrm{Op}^w_{\hbar,A}(\mathscr{F})$ in a slightly different manner than in [125]. Our staring point will be a "magnetic" version of (2.3.3) with arbitrary $\hbar > 0$, namely

$$\mathrm{Op}^w_{\hbar,A}(\mathscr{F}) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} \widehat{\mathscr{F}}(\mathbf{q},\mathbf{p})\, e^{i(\mathbf{q}\cdot X+\mathbf{p}\cdot\Pi(A))}\, d\mathbf{q}\, d\mathbf{p}, \tag{2.4.8}$$

with, say, $\mathscr{F} \in \mathscr{S}(\mathbb{R}^n)$ and $A \in C^1(\mathbb{R}^n;\mathbb{R}^n)$, the integral being considered again the strong sense. Let us obtain a suitable representation of the operator-valued function

$$e^{i(\mathbf{q}\cdot X+\mathbf{p}\cdot\Pi(A))}, \quad \mathbf{q},\mathbf{p} \in \mathbb{R}^n, \tag{2.4.9}$$

analogous to (2.3.6). To this end, we write the analogue of the Cauchy problem (2.3.5), that is

$$\begin{cases} -i\frac{\partial w}{\partial t}(t,\mathbf{x}) = -i\hbar\mathbf{p}\cdot\nabla_{\mathbf{x}} w(t,\mathbf{x}) + (\mathbf{q}\cdot\mathbf{x} - \mathbf{p}\cdot A(\mathbf{x}))\, w(t,\mathbf{x}), \\ w(0,\mathbf{x}) = u(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

and resolving it again by the method of characteristics, we get this time

$$\left(e^{i(\mathbf{q}\cdot X+\mathbf{p}\cdot\Pi(A))}u\right)(\mathbf{x})$$

$$= \exp\left(i \left(\frac{\mathbf{p}\cdot\mathbf{q}}{2} + \mathbf{q}\cdot\mathbf{x} - \mathbf{p}\cdot\int_0^1 A(\mathbf{x}+s\hbar\mathbf{p})ds \right)\right) u(\mathbf{x}+\mathbf{p}), \quad \mathbf{x}\in\mathbb{R}^n. \qquad (2.4.10)$$

Substituting (2.4.10) and (2.4.8), we arrive at

$$(\mathrm{Op}^w_{\hbar,A}(\mathscr{F})u)(\mathbf{x})$$

$$:= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \mathscr{F}\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\boldsymbol{\xi}\right) e^{i(\mathbf{x}-\mathbf{x}')\cdot\boldsymbol{\xi}/\hbar} e^{i\mathscr{M}(\mathbf{x},\mathbf{x}')/\hbar} u(\mathbf{x}')d\mathbf{x}'d\boldsymbol{\xi}, \quad u\in L^2(\mathbb{R}^n), \qquad (2.4.11)$$

which is the magnetic counterpart of (2.3.1). Here

$$\mathscr{M}(\mathbf{x},\mathbf{x}') := (\mathbf{x}-\mathbf{x}')\cdot\int_0^1 A(\mathbf{x}'+s(\mathbf{x}-\mathbf{x}'))ds, \quad \mathbf{x},\mathbf{x}'\in\mathbb{R}^n,$$

is *the circulation* of the magnetic potential A along the rectilinear segment in $\mathbb{R}^n$ connecting $\mathbf{x}$ and $\mathbf{x}'$. Note that $\mathrm{Op}^w_{\hbar,A}(\mathscr{F})$ is gauge covariant, i.e. if $A' = A + \nabla\phi$ with $\phi : \mathbb{R}^n \to \mathbb{R}$, then, similarly to (2.4.4), we have

$$e^{i\phi/\hbar}\mathrm{Op}^w_{\hbar,A}(\mathscr{F})e^{-i\phi/\hbar} = \mathrm{Op}^w_{\hbar,A'}(\mathscr{F}).$$

Moreover, the quantization $\mathfrak{Q}_B$ satisfies not only Axiom 2_B but, more generally, for a fairly large class of classical observables $\mathscr{F}$ and $\mathscr{G}$, the operator $\frac{1}{i\hbar}[\mathfrak{Q}_B(\mathscr{F}),\mathfrak{Q}_B(\mathscr{G})]$ coincides with the quantization $\mathfrak{Q}_B$ of an appropriate "magnetic" Moyal bracket of $\mathscr{F}$ and $\mathscr{G}$ (see the details in [125]).

2.5 Magnetic Schrödinger operators

In what follows, we assume $\hbar = 1$.
Our next goal is to construct the magnetic Schrödinger operator $H_S = H_S(A,V)$ which is the Hamiltonian of a spinless non-relativistic quantum particle subject to an electromagnetic potential (A,V). The operator $H_S(A,V)$ is the quantization of the classical Hamiltonian function $\mathscr{H}$ defined in (2.1.14). If we assume that the potential (A,V) is smooth, and use the Weyl quantization described in Section 2.3, we obtain

$$H_S(A,V) := \mathrm{Op}^w(\mathscr{H}) = \frac{1}{2m}\sum_{j=1}^n \Pi_j(A)^2 + V \qquad (2.5.1)$$

where $\Pi_j(A) = -i\frac{\partial}{\partial x_j} - A_j$, $j = 1,\dots,n$, are the components of the magnetic momentum operator introduced in the previous section. Then H_S can be written as

$$H_S(A,V) = \frac{1}{2m}(-i\nabla - A)^2 + V = \frac{1}{2m}\left(-\Delta + 2i A\cdot\nabla + i\,\mathrm{div}\,A + |A|^2\right) + V.$$

We have as well

$$H_S(A,V) = \mathrm{Op}^w_{1,A}(\widetilde{\mathscr{H}})$$

where $\widetilde{\mathscr{H}}$ is the classical Hamiltonian defined in (2.1.22), and $\mathrm{Op}^{\mathrm{w}}_{1,\mathrm{A}}$ is the magnetic Weyl quantization introduced in (2.4.11). We would like to define the operator $\mathrm{H_S(A,V)}$ self-adjoint operator in $\mathrm{L}^2(\mathbb{R}^n)$, under minimal regularity assumptions on (A,V). Suppose that

$$\mathrm{A} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n), \quad \mathrm{V}_+ \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n), \quad \mathrm{V}_+ \geq 0. \tag{2.5.2}$$

Then the operators $\Pi_j(\mathrm{A})$, $j = 1,\ldots,n$, and the multiplier by $\mathrm{V}_+^{1/2}$, are well defined as operators from $\mathrm{C}_0^\infty(\mathbb{R}^n)$ to $\mathrm{L}^2(\mathbb{R}^n)$. Introduce the quadratic form

$$\mathrm{h}[u;\mathrm{A},\mathrm{V}_+] := \int_{\mathbb{R}^n} \left(\frac{1}{2m} \sum_{j=1}^{n} |\Pi_j(\mathrm{A})u|^2 + \mathrm{V}_+|u|^2 \right) d\mathbf{x}, \quad u \in \mathrm{C}_0^\infty(\mathbb{R}^n), \tag{2.5.3}$$

and then close in $\mathrm{L}^2(\mathbb{R}^n)$. By [117, Theorem 1], under the assumptions (2.5.2), the quadratic form $\mathrm{h}(\mathrm{A},\mathrm{V}_+)$ is closed on the domain

$$\mathfrak{D}\,(\mathrm{h}(\mathrm{A},\mathrm{V}_+)) :=$$
$$\left\{ u \in \mathrm{L}^2(\mathbb{R}^n) \,|\, \Pi_j(\mathrm{A})u \in \mathrm{L}^2(\mathbb{R}^n),\ j = 1,\ldots,n,\ \mathrm{V}_+^{1/2}u \in \mathrm{L}^2(\mathbb{R}^n) \right\} \tag{2.5.4}$$

where, as usual, the derivatives of u are understood in the distributional sense. Then the operator $\mathrm{H_S}(\mathrm{A},\mathrm{V}_+)$ is defined as the self-adjoint operator generated in $\mathrm{L}^2(\mathbb{R}^n)$ by the closed quadratic form h_+. By [117, Theorem 2], if

$$\mathrm{A} \in \mathrm{L}^4_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n), \quad \mathrm{div}\,\mathrm{A} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^n), \quad \mathrm{V}_+ \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^n), \quad \mathrm{V}_+ \geq 0,$$

then the operator $\mathrm{H_S}(\mathrm{A},\mathrm{V}_+)$ is essentially self-adjoint on $\mathrm{C}_0^\infty(\mathbb{R}^n)$.

The operator $\mathrm{H_S}(\mathrm{A};\mathrm{V}_+)$ is still not general enough since we assume $\mathrm{V}_+ \geq 0$. In order to allow general real electric potentials V, we will need *the diamagnetic inequality* satisfied by the magnetic Schrödinger operator. In order to state this inequality, we introduce some auxiliary concepts and notations.

Let $\mathrm{T},\mathrm{S} \in \mathfrak{B}(\mathrm{L}^2(\mathrm{M},d\mu))$ where M is a space with measure μ. We will write

$$\mathrm{T} \overset{\leq}{\cdot} \mathrm{S}$$

if

$$|(\mathrm{T}u)(x)| \leq (\mathrm{S}|u|)(x), \quad u \in \mathrm{L}^2(\mathrm{M},d\mu),$$

for μ-almost every $x \in \mathrm{M}$. Note that if $\mathrm{T} \overset{\leq}{\cdot} \mathrm{S}$, then S should necessarily preserve positivity.

Theorem 2.5.1 *Let* $\mathrm{T} \overset{\leq}{\cdot} \mathrm{S}$.
(i) [64, 147] *If* $\mathrm{S} \in \mathfrak{S}_\infty(\mathrm{L}^2(\mathrm{M},d\mu))$*, then* $\mathrm{T} \in \mathfrak{S}_\infty(\mathrm{L}^2(\mathrm{M},d\mu))$.
(ii) [188, Theorem 2.13] *If* $\mathrm{S} \in \mathfrak{S}_{2\ell}(\mathrm{L}^2(\mathrm{M},d\mu))$ *with* $\ell \in \mathbb{N}$*, then* $\mathrm{T} \in \mathfrak{S}_{2\ell}(\mathrm{L}^2(\mathrm{M},d\mu))$.

Note that second part of the theorem is false if we replace $\mathfrak{S}_{2\ell}$ with $\ell \in \mathbb{N}$, by $\mathfrak{S}_p$ with $p \in [1,\infty) \setminus 2\mathbb{N}$ (see [139], [140, Chapter 6], and [186]).

Theorem 2.5.2 *Assume that* (2.5.2) *holds true. Then* the diamagnetic inequality

$$\exp(-tH_S(A,V_+)) \overset{\leq}{\cdot} \exp(-tH_S(0,V_+)) \overset{\leq}{\cdot} \exp(-tH_S(0,0)) \tag{2.5.5}$$

holds true for each $t \geq 0$.

Idea of the proof: Due to the gauge covariance, we can assume that div $A = 0$ without loss of generality. Then, if A and V_+ are sufficiently regular, the operator $\exp(-tH_S(A,V_+))$ with $t > 0$ has an integral kernel

$$\mathscr{K}_{A,V_+}(\mathbf{x},\mathbf{y};t)$$

$$= \int \exp\left(-i\int_0^t A(\omega(s))\cdot d\omega(s)\right) \exp\left(-\int_0^t V_+(\omega(s))ds\right) dE_{0,\mathbf{x};t,\mathbf{y}}(\omega)$$

where $E_{0,\mathbf{x};t,\mathbf{y}}(\omega(s))$ is the conditional Wiener measure on set of paths

$$\{\omega \in C([0,t];\mathbb{R}^n) \mid \omega(0) = \mathbf{x},\ \omega(t) = \mathbf{y}\}$$

(see [187, Theorem 15.5] and [32]). In particular, we have

$$|\mathscr{K}_{A,V_+}(\mathbf{x},\mathbf{y};t)| \leq \mathscr{K}_{0,V_+}(\mathbf{x},\mathbf{y};t) \leq \mathscr{K}_{0,0}(\mathbf{x},\mathbf{y};t)$$

for $t > 0$, $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2d}$ which implies (2.5.5) in the regular case. For general (A,V_+), (2.5.5) follows from an appropriate approximation argument. □
Note that $H_S(0,0)$ is just the operator $-\frac{1}{2m}\Delta$.
A proof of (2.5.5) which does not use the Wiener integral, can be found, for example, in the proof of [56, Theorem 1.13].

Corollary 2.5.1 *Assume* (2.5.2). *Let* $E > 0$ *and* $\gamma > 0$. *Then we have*

$$(H_S(A,V_+)+E)^{-\gamma} \overset{\leq}{\cdot} (H_S(0,V_+)+E)^{-\gamma} \overset{\leq}{\cdot} (H_S(0,0)+E)^{-\gamma}. \tag{2.5.6}$$

Proof. If $S = S^* \geq 0$, and $E > 0$, $\gamma > 0$, then

$$(S+E)^{-\gamma} = \frac{1}{\Gamma(\gamma)}\int_0^\infty t^{\gamma-1}e^{-tE}e^{-tS}\,dt, \tag{2.5.7}$$

the integral being considered as a Riemann one in the uniform topology. Now, (2.5.6) follows easily from (2.5.5) and (2.5.7). □

Proposition 2.5.1 *Let* $V_- \geq 0$ *be a measurable function over* $\mathbb{R}^n$. *Assume* (2.5.2). *If the multiplier by* V_- *is* Δ*-bounded* (*resp.,* $-\Delta$*-form-bounded*) *with relative bound* α, *then* V_- *is* $H_S(A,V_+)$*-bounded* (*resp.,* $H_S(A,V_+)$*-form-bounded*) *with relative bound* (*resp., relative form-bound*) *at most* α.

Proof. Using (2.5.6) with $\gamma = 1$ (resp., with $\gamma = 1/2$), we get

$$\|V_-(H_S(A,V_+)+E)^{-1}\| \le \|V_-(H_S(0,0)+E)^{-1}\|,$$

or, respectively,

$$\|V_-^{1/2}(H_S(A,V_+)+E)^{-1}\| \le \|V_-^{1/2}(H_S(0,0)+E)^{-1/2}\|.$$

In order to complete the proof, we have to recall that if T and S are nonnegative self-adjoint operators and T is S-bound (resp., S-form-bound), then the S-relative bound (resp., S-relative form-bound) of T can be written as $\lim_{E\to\infty}\|T(S+E)^{-1}\|$ or, respectively, as $\lim_{E\to\infty}\|T^{1/2}(S+E)^{-1/2}\|^2$. □

The following theorem concerns the self-adjoint realizations of the operator $H_S(A,V)$ with $V = V_+ - V_-$, $V_+ \ge 0$ and $V_- \ge 0$. In particular, V_+ (resp., V_-) can be, as usual, the positive (resp., the negative) part of V so that $V_+V_- = 0$.

Theorem 2.5.3 *Let the potential* (A,V_+) *satisfy* (2.5.2). *Assume that the multiplier by the measurable function* $V_- : \mathbb{R}^n \to [0,\infty)$ *is* Δ*-bounded* (*resp.,* $-\Delta$*-form-bounded*) *with relative bound* (*resp., relative form bound*) *smaller than one. Set* $V = V_+ - V_-$. *Then the operator sum* (*resp., form sum*)

$$H_S(A,V) := H_S(A,V_+) - V_-$$

is self-adjoint in $L^2(\mathbb{R}^n)$. *Moreover we have*

$$\exp(-tH_S(A,V)) \overset{<}{\cdot} \exp(-tH_S(0,V)), \quad t \ge 0. \tag{2.5.8}$$

Proof. The self-adjointness of $H(A,V)$ follows immediately from Proposition 2.5.1 and the Kato-Rellich theorem [163, Theorem X.12] (resp., the KLMN theorem [163, Theorem X.17]). For the proof of (2.5.8), see [7, Theorems 2.4–2.5]. □

Inequality (2.5.8) is *the semigroup version* of the diamagnetic inequality. There exists also its *quadratic-form* version. In order to describe it, let us assume that (A,V) satisfies the hypotheses of the form version of Theorem 2.5.3, and set

$$h[u;A,V] = h[u;A,V_+] - \int_{\mathbb{R}^n} V_-|u|^2 d\mathbf{x}, \quad u \in \mathfrak{D}(h(A,V_+)),$$

the quadratic form $h(A,V_+)$ being defined in (2.5.3), and its domain $\mathfrak{D}(h(A,V_+))$ in (2.5.4). Note that $h(A,V)$ with domain $\mathrm{Dom}(h(A,V_+))$, is closed in $L^2(\mathbb{R}^n)$.

Proposition 2.5.2 *Let* (A,V) *satisfy the hypotheses of the form version of Theorem 2.5.3, and* $u \in \mathrm{Dom}(h(A,V_+))$. *Then* $|u| \in \mathrm{Dom}(h(0,V_+))$ *and we have*

$$h[u;A,V] \ge h[|u|;0,V]. \tag{2.5.9}$$

Proof. Let $u \in C_0^\infty(\mathbb{R}^n)$. Then $|u|$ is in the Sobolev space $W^{1,\infty}(\mathbb{R}^n)$ and has a compact support; hence $|u| \in W^{1,2}(\mathbb{R}^n)$ and $\nabla|u|$ coincides almost everywhere with the vector-valued function equal to $\mathrm{Re}(i\bar{u}|u|^{-1}\Pi(A)u)(\mathbf{x})$ if $u(\mathbf{x}) \neq 0$, and to zero if $u(\mathbf{x}) = 0$ (see e.g. [120, Section 7.21]). Therefore,

$$|\nabla|u||(\mathbf{x}) \le |(\Pi(A)u)(\mathbf{x})|$$

for almost each $\mathbf{x} \in \mathbb{R}^n$ which easily implies the desired results. □

Note that (2.5.9) is equivalent to the first inequality with $\gamma = 1/2$ and $V_- = 0$ in (2.5.6).

Corollary 2.5.2 *Assume that* (A, V) *satisfies the form version of Theorem 2.5.3. Then we have*

$$\inf \sigma(H_S(A,V)) \geq \inf \sigma(H_S(0,V)), \tag{2.5.10}$$

$$\inf \sigma_{ess}(H_S(A,V)) \geq \inf \sigma_{ess}(H_S(0,V)). \tag{2.5.11}$$

Proof. Let Ω be a domain in $\mathbb{R}^n$. Set

$$e_{A,V}(\Omega) := \inf_{\substack{0 \neq u \in C_0^\infty(\mathbb{R}^n): \\ \operatorname{supp} u \subset \Omega}} \frac{h[u; A, V]}{\int_{\mathbb{R}^n} |u|^2 \, d\mathbf{x}}.$$

By the mini-max principle,

$$\inf \sigma(H_S(A,V)) = e_{A,V}(\mathbb{R}^n). \tag{2.5.12}$$

For $R > 0$ set $\Omega_R := \{\mathbf{x} \in \mathbb{R}^n \,|\, |\mathbf{x}| > R\}$. Then the results of [142] easily imply

$$\inf \sigma_{ess}(H_S(A,V)) = \lim_{R \to \infty} e_{A,V}(\Omega_R). \tag{2.5.13}$$

Now (2.5.10) (resp., (2.5.11)) follows from (2.5.9) and (2.5.12) (resp., and (2.5.13)). □

Corollary 2.5.3 *Let* $A \in L^2_{loc}(\mathbb{R}^n, \mathbb{R}^n)$*, and* $V : \mathbb{R}^n \to \mathbb{R}$ *be a measurable function.*
(i) *If the multiplier by* V *is* Δ*-compact, then it is also* $H_S(A,0)$*-compact, and*

$$\sigma_{ess}(H_S(A,V)) = \sigma_{ess}(H_S(A,0)). \tag{2.5.14}$$

(ii) *If the multiplier by* V *is* $-\Delta$*-form-compact, then it is also* $H_S(A,0)$*-form-compact, and (2.5.14) remains valid.*

Let us recall some simple examples of potentials V which are Δ-compact and $-\Delta$-form-compact. We shall say that the function V is in the class $\mathscr{L}_p = \mathscr{L}_p(\mathbb{R}^n)$, $p \geq 1$, $n \geq 1$, if for every $\varepsilon > 0$ we have $V = V_1 + V_2$ with $V_1 \in L^p(\mathbb{R}^n)$ and $\sup_{\mathbf{x} \in \mathbb{R}^n} |V_2(\mathbf{x})| \leq \varepsilon$. In contrast to the usual Lebesgue spaces $L^p(\mathbb{R}^n)$, the classes $\mathscr{L}_p$ are embedded, i.e. $\mathscr{L}_p \subset \mathscr{L}_r$ if $p > r$.

Lemma 2.5.1 (i) *Let* $V \in \mathscr{L}_p(\mathbb{R}^n)$ *with* $p = 2$ *if* $n = 1, 2, 3$, $p > 2$ *if* $n = 4$*, and* $p = n/2$ *if* $n \geq 5$*. Then* V *is* Δ*-compact.*
(ii) *Let* $V \in \mathscr{L}_p(\mathbb{R}^n)$ *with* $p = 1$ *if* $n = 1$, $p > 1$ *if* $n = 2$*, and* $p = n/2$ *if* $n \geq 3$*. Then* V *is* $-\Delta$*-form-compact.*

At the end of this section we discuss briefly the gauge covariance and the $\mathscr{C}$-symmetry of the Schrödinger operator $H_S(A,V)$. If the magnetic potentials $A^{(j)}$, $j = 1, 2$, generate the same magnetic field (see (2.4.2)) so that (2.4.3) holds true with a function $\phi \in C^1(\mathbb{R}^n;\mathbb{R})$, then (2.4.4) implies

$$e^{i\phi} H_S(A^{(2)}, V) e^{-i\phi} = H_S(A^{(1)}, V). \tag{2.5.15}$$

Here we do not specify the properties of V since for any reasonable V we have $V = e^{i\phi} V e^{-i\phi}$. In fact, the gauge covariance (2.5.15) remains valid also for more general magnetic potentials:

Proposition 2.5.3 [116, Theorem 1.2] *Assume that the electromagnetic potentials* $(A^{(j)}, V)$*,* $j = 1, 2$*, satisfy the operator or the form version of Theorem* 2.5.3*, and* (2.4.2) *holds true in the distribution sense. Then there exists a function* $\phi \in W^{1,2}(\mathbb{R}^n;\mathbb{R})$ *such that* (2.4.3) *is fulfilled, and* (2.5.15) *remains valid.*

Note by $\phi \in W^{1,2}(\mathbb{R}^n;\mathbb{R})$, the function ϕ is measurable and real-valued. Hence the multiplier by $e^{i\phi}$ is a unitary operator.
The gauge covariance (2.5.15) implies that the spectra of the operators $H(A^{(j)}, V)$, $j = 1, 2$, and all their components are the same. Here we would like to warn the reader that the gauge covariance (2.5.15) and the consequent invariance of the spectrum and its components under gauge transformations strongly depends on the fact that $\mathbb{R}^n$ is simply connected. It is possible to define the operator $H_S(A, V)$ also on more general manifolds, but if the manifold is not simply connected, then, generally, the gauge covariance does not hold true any more. Let us give a simple counter-example borrowed from [91]. Let $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ be the 1D torus, and $a \in \mathbb{R}$. Consider the operator

$$H(a) := \left(-i\frac{d}{d\theta} - a\right)^2$$

with domain $W^{2,2}(\mathbb{T})$, self-adjoint in $L^2(\mathbb{T})$. The 1-form $a d\theta$ is of course closed, but is not exact which is related to the fact that $\mathbb{T}$ is not simply connected. The spectrum of $H(a)$ consists of eigenvalues $(k - a)^2$, corresponding to eigenfunctions $e^{ik\theta}$, $\theta \in \mathbb{T}$, $k \in \mathbb{Z}$. Therefore, $\sigma(H(a)) = \sigma(H(0))$ if and only if $a \in \mathbb{Z}$. The existence of these "exceptional" values of a is explained by the fact that the function $e^{ia\theta}$, $\theta \in \mathbb{T}$, which is the natural candidate for $e^{i\phi}$, is well defined as a smooth function on $\mathbb{T}$ (even if $a\theta$ is not), if and only if a is an integer; in this case, by analogy with (2.5.15), we have

$$H(a) = e^{ia\theta} H(0) e^{-ia\theta}.$$

Finally, we note that, similarly to (2.4.5), we have

$$\mathscr{C} H_S(A, V) \mathscr{C} = H_S(-A, V), \tag{2.5.16}$$

say, under the hypotheses of Theorem 2.5.3. We recall that $\mathscr{C}$ is the anti-unitary operator of complex conjugation. Hence, the operators $H_S(A, V)$ and $H_S(-A, V)$ are anti-unitarily equivalent. In particular, the spectra of these operators, and all their spectral components coincide.

2.6 Magnetic Pauli and Dirac operators

The Schrödinger operator H_S does not take account of the spin effects. The appropriate Hamiltonian operator of a quantum non-relativistic particle of $\frac{1}{2}$-spin is the Pauli operator (see e.g. [131, Chapter XII, Section 18]). Although it is possible to define this operator in arbitrary dimension (see e.g. [183]), here we will consider it only for $n = 2, 3$ which are the most interesting cases from physics point of view. Introduce the Pauli matrices $\hat{\sigma}_j$, $j = 1, 2, 3$, which are constant 2×2 Hermitian matrices satisfying

$$\hat{\sigma}_j \hat{\sigma}_k + \hat{\sigma}_k \hat{\sigma}_j = 2\delta_{jk} I_2, \quad j, k = 1, 2, 3. \tag{2.6.1}$$

In the standard representation which we will use in the sequel,

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.6.2}$$

In this case the cyclic permutations

$$\hat{\sigma}_1 = -i\hat{\sigma}_2 \hat{\sigma}_3, \quad \hat{\sigma}_2 = -i\hat{\sigma}_3 \hat{\sigma}_1, \quad \hat{\sigma}_3 = -i\hat{\sigma}_1 \hat{\sigma}_2, \tag{2.6.3}$$

hold true. Assume $A \in L^2_{loc}(\mathbb{R}^n; \mathbb{R}^n)$, and introduce the operator

$$\Sigma(A) := \sum_{j=1}^{n} \hat{\sigma}_j \Pi_j(A) \tag{2.6.4}$$

as the closure in $L^2(\mathbb{R}^n; \mathbb{C}^2)$ of the operator defined originally on $C_0^\infty(\mathbb{R}^n; \mathbb{C}^2)$. If A is sufficiently regular, say, $A \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, then $\Sigma(A)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n; \mathbb{C}^2)$ (see e.g [206, Theorem 4.3]). If $n = 2$, then we have

$$\Sigma(A) = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} \tag{2.6.5}$$

where

$$a := \Pi_1 + i\Pi_2 = -i\frac{\partial}{\partial x_1} - A_1 + \frac{\partial}{\partial x_2} - iA_2 \tag{2.6.6}$$

is *the magnetic annihilation operator*, and

$$a^* := \Pi_1 - i\Pi_2 = -i\frac{\partial}{\partial x_1} - A_1 - \frac{\partial}{\partial x_2} + iA_2 \tag{2.6.7}$$

is *the magnetic creation operator*. Under generic assumptions on A and B, the operators a and a^* are mutually adjoint in $L^2(\mathbb{R}^2)$, and

$$[a, a^*] = 2B_{12}. \tag{2.6.8}$$

For $A \in L^2_{loc}(\mathbb{R}^n; \mathbb{R}^n)$, set

$$S(A) := \Sigma(A)^* \Sigma(A).$$

Of course, if A is regular enough, and $\Sigma(A)$ is self-adjoint, then $S(A) = \Sigma(A)^2$. If $n = 2$, then (2.6.5) implies

$$S(A) = \begin{pmatrix} a^*a & 0 \\ 0 & aa^* \end{pmatrix}. \tag{2.6.9}$$

The purely magnetic Pauli operator is defined just as

$$H_P(A,0) := \frac{1}{2m} S(A), \tag{2.6.10}$$

where, as above, $m > 0$ is the mass of the particle. By (2.4.1) and (2.6.1) we have

$$H_P(A,0) = H_S(A,0)\, I_2 + \frac{i}{2m} \sum_{1 \le j < k \le n} B_{jk} \hat{\sigma}_j \hat{\sigma}_k. \tag{2.6.11}$$

In particular, if the magnetic field B is bounded, then (2.6.11) implies $\mathfrak{D}(H_P(A,0)) = \mathfrak{D}(H_S(A,0)) \otimes \mathbb{C}^2$. In the standard representation (2.6.2) of the Pauli matrices, we can use (2.6.3), and find that (2.6.11) can be re-written as

$$H_P(A,0) = H_S(A,0) I_2 - \frac{1}{2m} \hat{\sigma}_3 B_{12} \tag{2.6.12}$$

if $n = 2$, or as

$$H_P(A,0) = H_S(A,0) I_2 - \frac{1}{2m} \sum_{j=1}^{3} \hat{\sigma}_j B_j \tag{2.6.13}$$

if $n = 3$; in the latter case we have used the identification (2.1.7) of the entries of the matrix B with the components of the vector $\mathbf{B}$.

Let us now introduce the Pauli operator $H_P(A, \mathbf{V})$ with a *matrix valued* Hermitian electric potential $\mathbf{V}$. Denote by $\mathscr{M}_\ell$, $\ell \ge 2$, the set of complex $\ell \times \ell$ matrices.

Proposition 2.6.1 *Assume that* $A \in L^2_{loc}(\mathbb{R}^n; \mathbb{R}^n)$ *and* B *is bounded and measurable. Let* $\mathbf{V} : \mathbb{R}^n \to \mathscr{M}_2$ *be a Hermitian Lebesgue measurable function such that the multiplier by* $\|\mathbf{V}\|$ *is* $H_S(0,0)$*-bounded (resp.,* $H_S(0,0)$*-form bounded) with relative (resp., form relative) bound less than one. Then the operator (resp., form) sum*

$$H_P(A, \mathbf{V}) := H_P(A,0) + \mathbf{V}$$

is self-adjoint on the domain $\mathfrak{D}(H_P(A,0))$. *Moreover, if the multiplier by* $\|\mathbf{V}\|$ *is* Δ*-compact (resp.,* $-\Delta$*-form compact), then* $\mathbf{V}$ *is* $H_P(A,0)$*-compact (resp.,* $H_P(A,0)$*-form compact), and*

$$\sigma_{ess}(H_P(A, \mathbf{V})) = \sigma_{ess}(H_P(A,0)).$$

Results on the self-adjointness of $H_P(A, \mathbf{V})$ with unbounded magnetic fields in dimensions $n = 2$ and $n = 3$ can be found in [196] and [197] respectively.

In contrast to its Schrödinger counterpart, the Pauli operator does not satisfy the diamagnetic inequality. In fact, the Pauli operator with constant magnetic field satisfies various versions of the so-called *paramagnetic inequality* (see e.g. [71, Section II]), which is related to an effect opposite to the diamagnetic one: the operator $H_P(A, V)$ "decreases" as the magnetic field "increases" in a suitable sense. The paramagnetic inequality however is much less general than the diamagnetic one, and generically does not hold true in variable magnetic fields (see e.g. [8]).

Further, if the magnetic potentials $A^{(1)}$ and $A^{(2)}$ are gauge equivalent, i. e. they generate the same magnetic field so that we have $A^{(1)} = A^{(2)} + \nabla\phi$ with $\phi : \mathbb{R}^n \to \mathbb{R}$ (see (2.4.2) amd (2.4.3)), then (2.4.4) implies that the Pauli operators $H_P(A^{(1)}, \mathbf{V})$ and $H_P(A^{(2)}, \mathbf{V})$ are gauge covariant, that is

$$e^{i\phi} H_P(A^{(2)}, \mathbf{V}) e^{-i\phi} = H_P(A^{(1)}, \mathbf{V}).$$

Finally, let us discuss briefly how the Pauli operator transforms under the complex conjugation. Define the operator

$$U_{Pauli} := i\hat{\sigma}_2,$$

unitary in $\mathbb{C}^2$ and hence in $L^2(\mathbb{R}^n; \mathbb{C}^2)$. We have

$$U^*_{Pauli} \bar{\hat{\sigma}}_j U_{Pauli} = -\hat{\sigma}_j, \quad j = 1, 2, 3.$$

Moreover, U_{Pauli} commutes with the complex conjugation $\mathscr{C}$. Therefore, by (2.4.5),

$$U^*_{Pauli} \mathscr{C} H_P(A, \mathbf{V}) \mathscr{C} U_{Pauli} = H_P(-A, \widetilde{\mathbf{V}}) \tag{2.6.14}$$

where

$$\widetilde{\mathbf{V}} = U^*_{Pauli} \mathscr{C} \mathbf{V} \mathscr{C} U_{Pauli} = \begin{pmatrix} V_{22} & -V_{12} \\ -V_{21} & V_{11} \end{pmatrix}.$$

Hence, the operators $H_P(A, \mathbf{V}$ and $H_P(-A, \widetilde{\mathbf{V}})$ are anti-unitarily equivalent.

The Pauli operator does not take into account the relativistic effects. The appropriate Hamiltonian operator of a quantum relativistic particle of $\frac{1}{2}$-spin is the Dirac operator (see e.g. [206]). As in the case of the Pauli operator, we will consider it only for $n = 2, 3$. First, we introduce the Dirac matrices $\hat{\alpha}_j$, $j = 1, \ldots, n$, and $\hat{\beta}$. For $\ell_2 := 2$, $\ell_3 := 4$, they are constant Hermitian $\ell_n \times \ell_n$ matrices satisfying

$$\hat{\alpha}_j \hat{\alpha}_k + \hat{\alpha}_k \hat{\alpha}_j = 2\delta_{jk} I_{\ell_n}, \quad j, k = 1, \ldots, n,$$

$$\hat{\alpha}_j \hat{\beta} + \hat{\beta} \hat{\alpha}_j = 0, \quad j = 1, \ldots, n, \quad \hat{\beta}^2 = I_{\ell_n}. \tag{2.6.15}$$

In what follows we will use the standard representation of the Dirac matrices: if $n = 2$, then $\hat{\alpha}_j = \hat{\sigma}_j$, $j = 1, 2$, $\hat{\beta} = \hat{\sigma}_3$, and if $n = 3$, then

$$\hat{\alpha}_j = \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix}, \quad j = 1, 2, 3, \quad \hat{\beta} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

Introduce the purely magnetic Dirac operator

$$H_D(A, 0) := \sum_{j=1}^{n} \hat{\alpha}_j \Pi_j(A) + m\hat{\beta}$$

acting in $L^2(\mathbb{R}^n;\mathbb{C}^{\ell_n})$. As above we denote by $m > 0$ the mass of the particle, and assume $\hbar = 1$, $c = 1$, $e = 1$. For simplicity, let $A \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$. Then $H_D(A,0)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n;\mathbb{C}^{\ell_n})$ (see e.g. [206, Theorem 4.3]). The Dirac operator $H_D(A,0)$ can be written as

$$H_D(A,0) = \begin{pmatrix} m & Q^* \\ Q & -m \end{pmatrix} \tag{2.6.16}$$

where $Q = a$ (see (2.6.6)) if $n = 2$, and $Q = \Sigma(A)$ (see (2.6.4)) if $n = 3$. Moreover, $H_D(A,0)$ satisfies

$$H_D(A,0)^2 = \begin{cases} S(A) + m^2 I & \text{if} \quad n = 2, \\ \begin{pmatrix} S(A)+m^2 I & 0 \\ 0 & S(A)+m^2 I \end{pmatrix} & \text{if} \quad n = 3, \end{cases} \tag{2.6.17}$$

which, in particular, implies that $H_D(A,0)$ is invertible and

$$\sigma(H_D(A,0)) \cap (-m,m) = \emptyset.$$

There is a deep connection between (2.6.17) and (2.6.16) through *the Foldy–Wouthuysen transformation* defined by

$$\mathscr{U}_{FW} := a_+ + \hat{\beta}\left(\operatorname{sgn}\left(H_D(A,0) - m\hat{\beta}\right)\right) a_-$$

with

$$a_\pm := \frac{1}{\sqrt{2}}\sqrt{I \pm m|H_D(A,0)|^{-1}}.$$

Proposition 2.6.2 [206, Theorem 5.13] *Let* $A \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ *and* $m > 0$. *Then the operator* $\mathscr{U}_{FW}$ *is unitary in* $L^2(\mathbb{R}^n;\mathbb{C}^{\ell_n})$ *and we have*

$$\mathscr{U}_{FW}\, H_D(A,0)\, \mathscr{U}_{FW}^* = \begin{pmatrix} \sqrt{Q^* Q + m^2 I} & 0 \\ 0 & -\sqrt{Q Q^* + m^2 I} \end{pmatrix}. \tag{2.6.18}$$

In particular, (2.6.17) and (2.6.18) imply that the operators $\mathscr{U}_{FW}$ and $H_D(A,0)^2$ commute. Note moreover that both operators $Q^* Q$ and $Q Q^*$ coincide with $S(A)$ if $n = 3$. However, if $n = 2$, we have $Q^* Q = a^* a$ and $Q Q^* = a a^*$, so that (2.6.8) implies

$$Q^* Q \neq Q Q^* \tag{2.6.19}$$

in the case of non-vanishing magnetic field.

Further, let $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$ with $\mathbf{V}_1 = \mathbf{V}_1^* \in C^\infty(\mathbb{R}^n;\mathscr{M}_{\ell_n})$ and $\mathbf{V}_2 = \mathbf{V}_2^* \in L^\infty(\mathbb{R}^n;\mathscr{M}_{\ell_n})$. Set

$$H_D(A,\mathbf{V}) := H_D(A,0) + \mathbf{V}. \tag{2.6.20}$$

The operator $H_D(A,\mathbf{V})$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n;\mathbb{C}^{\ell_n})$ (see e.g. [206, Theorem 4.3]). Since the magnetic potential A enters linearly the expression for $H_D(A,0)$ we have evidently

$$H_D(A,\mathbf{V}) = H_D\left(0, \mathbf{V} - \sum_{j=1}^n \hat{\alpha}_j A_j\right).$$

An alternative assumption on $\mathbf{V}$ which guarantees that the operator $H_D(A, \mathbf{V})$ is self-adjoint on $\mathfrak{D}(H_D(A,0))$, is that the $H_D(A,0)$-relative bound of the multiplier by $\mathbf{V}$ is smaller than one. This assumption is somewhat implicit but we can estimate the $H_D(A,0)$-relative bound of $\mathbf{V}$ using the following

Lemma 2.6.1 *Let* $n = 2,3$, $A \in C^\infty_{loc}(\mathbb{R}^n;\mathbb{R}^n)$. *Then the relative* $H_D(A,0)$*-bound of the multiplier by* $\mathbf{V} = \mathbf{V}^*$ *does not exceed the relative* $H_P(A,0)$*-form-bound of the multiplier by* $\mathbf{V}^2$. *Moreover, if the multiplier by* $\mathbf{V}^2$ *is* $H_P(A,0)$*-form-compact, then the operator* $\mathbf{V}$ *is* $H_D(A,0)$*-compact, and we have*

$$\sigma_{ess}(H_D(A;\mathbf{V})) = \sigma_{ess}(H_D(A;0)).$$

Further, as in the case of the Schrödinger and Pauli operators, (2.4.4) implies that two Dirac operators corresponding to gauge equivalent magnetic potentials, are gauge covariant.

Finally, we discuss the transformation of $H_D(A,\mathbf{V})$ under the complex conjugation. Again, we assume that $\mathbf{V} := VI_{\ell_n}$ with appropriate $V : \mathbb{R}^n \to \mathbb{R}$. Set

$$U_{Dirac} := \hat{\sigma}_1$$

if $n = 2$, and

$$U_{Dirac} := \begin{pmatrix} 0 & i\hat{\sigma}_2 \\ -i\hat{\sigma}_2 & 0 \end{pmatrix},$$

if $n = 3$. We have

$$U^*_{Dirac}\bar{\hat{\alpha}}_j U_{Dirac} = \hat{\alpha}_j, \quad j = 1,2,3, \quad n = 2,3,$$

$$U^*_{Dirac}\bar{\hat{\beta}} U_{Dirac} = -\hat{\beta},$$

and therefore, by (2.4.5),

$$U^*_{Dirac}\mathscr{C} H_D(A, VI_{\ell_n})\mathscr{C} U_{Dirac} = -H_D(-A, -VI_{\ell_n}). \tag{2.6.21}$$

Hence, the operators $H_D(A, VI_{\ell_n})$ and $-H_D(-A, -VI_{\ell_n})$ are anti-unitarily equivalent. Taking into account (2.1.2), we conclude that the mapping $H_D(A, VI_{\ell_n}) \mapsto H_D(-A, -VI_{\ell_n})$ could be interpreted as the change of sign of the charge of the particle. Similarly, recalling the unitary group e^{itH_D} with $t \in \mathbb{R}$, we find that the mapping $H_D(A, VI_{\ell_n}) \mapsto -H_D(A, VI_{\ell_n})$ could be related to the time reversal $t \mapsto -t$.

In what follows, we will set $m = 1$ in our considerations of the Schrödinger and Pauli operators but we prefer not to fix the parameter $m > 0$ when dealing with the Dirac operator.

2.7 Hamiltonians with constant magnetic fields

In this section we will discuss the case where the magnetic field B is constant, i.e. its entries B_{ij}, $i,j = 1,\dots,n$, are independent of $\mathbf{x} \in \mathbb{R}^n$. In this case B can be regarded as a antisymmetric linear mapping $\mathbb{R}^n \to \mathbb{R}^n$. Set $2d := \dim \operatorname{Ran} B$, and $k := n - 2d =$

dim Ker B. Throughout the section we assume that $B \neq 0$, and hence $d \geq 1$. The spectral theory of the operators $H_S(A,0)$, $H_P(A,0)$, and $H_D(A,0)$ is quite different in the case $k = 0$ and $k \geq 1$; that is why we will consider them separately. The two leading examples illustrating these two cases, are respectively $n = 2$, i.e. $d = 1$ and $k = 0$, and $n = 3$, i.e. $d = 1$ and $k = 1$.
Let $b_1 \geq \ldots \geq b_d > 0$ be such numbers that the non-zero eigenvalues of B coincide with $-ib_j$ and ib_j, $j = 1,\ldots,d$. Then there exist Cartesian coordinates $(x_1,y_1,\ldots,x_d,y_d) \in \mathbb{R}^{2d} = \mathbb{R}^n = \mathrm{Ran}\, B$ if $k = 0$ (resp., $(x_1,y_1,\ldots,x_d,y_d,w_1,\ldots,w_k)$ with $(x_1,y_1,\ldots,x_d,y_d) \in \mathbb{R}^{2d} = \mathrm{Ran}\, B$, and $(w_1,\ldots,w_k) \in \mathbb{R}^k = \mathrm{Ker}\, B$ if $k \geq 1$), in which the operator $H_S(A,0)$ can be written as

$$H_S(A,0) = \sum_{j=1}^{d} \left\{ \left(-i\frac{\partial}{\partial x_j} + \frac{b_j y_j}{2}\right)^2 + \left(-i\frac{\partial}{\partial y_j} - \frac{b_j x_j}{2}\right)^2 \right\} \tag{2.7.1}$$

if $k = 0$, or, respectively, as

$$H_S(A,0) = \sum_{j=1}^{d} \left\{ \left(-i\frac{\partial}{\partial x_j} + \frac{b_j y_j}{2}\right)^2 + \left(-i\frac{\partial}{\partial y_j} - \frac{b_j x_j}{2}\right)^2 \right\} - \sum_{\ell=1}^{k} \frac{\partial^2}{\partial w_\ell^2} \tag{2.7.2}$$

if $k \geq 1$. In both cases $k = 0$ and $k \geq 1$ we have $B = \sum_{j=1}^d b_j dx_j \wedge dy_j$.

First, we consider the case $k = 0$; then (2.7.1) is valid. Let us start with the leading example $n = 2$, i.e. $d = 1$. In this case we set $b := b_1 = B_{12}$, and assume without loss of generality that $b > 0$. Hence, (2.7.1) reduces to

$$H_S(A,0) = \Pi_1(A)^2 + \Pi_2(A)^2 \tag{2.7.3}$$

with $A = (-by/2, bx/2)$ and $(x,y) = (x_1,y_1)$, which is coherent with (2.5.1). Moreover, in this case (2.6.6) - (2.6.7) imply

$$a^* = a^*(b) = -2ie^{\varphi}\frac{\partial}{\partial z}e^{-\varphi}, \quad z = x + iy, \tag{2.7.4}$$

$$a = a(b) = -2ie^{-\varphi}\frac{\partial}{\partial \bar{z}}e^{\varphi}, \quad \bar{z} = x - iy, \tag{2.7.5}$$

where

$$\varphi(x,y) = \varphi_b(x,y) := \frac{b(x^2+y^2)}{4}, \quad (x,y) \in \mathbb{R}^2,$$

so that $\Delta\varphi = b$. By (2.6.8), we have

$$[a, a^*] = 2b\, I, \tag{2.7.6}$$

and, hence,

$$H_S(A,0) = a^*a + bI = aa^* - bI. \tag{2.7.7}$$

For $\mathbf{y} \in \mathbb{R}^2$, introduce the *magnetic translation*

$$(\mathscr{T}_{\mathbf{y}}u)(\mathbf{x}) := e^{-i\frac{b}{2}(\mathbf{x}\wedge\mathbf{y})}u(\mathbf{x}-\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{2.7.8}$$

where

$$\mathbf{x} \wedge \mathbf{y} := x_1 y_2 - x_2 y_1. \tag{2.7.9}$$

Evidently,the operator $\mathscr{T}_{\mathbf{y}}$ is unitary in $L^2(\mathbb{R}^2)$. Moreover, a direct calculation yields

$$\mathscr{T}_{\mathbf{y}}^* \, a(b) \, \mathscr{T}_{\mathbf{y}} = a(b), \quad \mathscr{T}_{\mathbf{y}}^* \, a(b)^* \, \mathscr{T}_{\mathbf{y}} = a(b)^*, \tag{2.7.10}$$

and, hence, by (2.7.7), we have

$$\mathscr{T}_{\mathbf{y}}^* \, H_S(A, 0) \, \mathscr{T}_{\mathbf{y}} = H_S(A, 0) \tag{2.7.11}$$

Next, we will show that $H_S(A,0)$ is unitarily equivalent, under an appropriate metaplectic mapping, to the operator $(b\mathfrak{h}) \otimes I_y$ where

$$\mathfrak{h} := -\frac{d^2}{dx^2} + x^2, \tag{2.7.12}$$

is the harmonic oscillator, self-adjoint in $L^2(\mathbb{R}_x)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R})$, while I_y is the identity in $L^2(\mathbb{R}_y)$. Let us recall the well known spectral properties of $\mathfrak{h}$. We have

$$\mathfrak{h} = \alpha^* \alpha + I = \alpha \alpha^* - I,$$

where

$$\alpha := -i \frac{d}{dx} - ix, \quad \alpha^* := -i \frac{d}{dx} + ix,$$

are the standard annihilation and creation operators which are closed on

$$\mathfrak{D}(\alpha) = \mathfrak{D}(\alpha^*) = \mathfrak{D}(\mathfrak{h}^{1/2}),$$

and are mutually adjoint in $L^2(\mathbb{R})$. Moreover, they satisfy the commutation relation

$$[\alpha, \alpha^*] = 2I.$$

Therefore,

$$\sigma(\mathfrak{h}) = \bigcup_{q \in \mathbb{Z}_+} \{2q + 1\}, \tag{2.7.13}$$

$$\mathrm{Ker}\,(\mathfrak{h} - (2q+1)I) = (\alpha^*)^q \mathrm{Ker}\,\alpha, \quad q \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}.$$

Since

$$\mathrm{Ker}\,\alpha = \left\{ u \in L^2(\mathbb{R}) \,|\, u(x) = ce^{-x^2/2}, x \in \mathbb{R}, \quad c \in \mathbb{C} \right\},$$

we get

$$\dim \mathrm{Ker}\,(\mathfrak{h} - (2q+1)I) = 1, \quad q \in \mathbb{Z}_+.$$

Moreover, the functions

$$\widetilde{\psi}_q(x) := \left(-\frac{d}{dx} + x \right)^q e^{-x^2/2}, \quad x \in \mathbb{R},$$

satisfy $\mathfrak{h}\widetilde{\psi}_q = (2q+1)\widetilde{\psi}_q$, $q \in \mathbb{Z}_+$, and form an orthogonal eigenbasis in $L^2(\mathbb{R})$. A simple calculation shows that

$$\psi_q := \widetilde{\psi}_q/\|\widetilde{\psi}_q\| = \frac{H_q(x)e^{-x^2/2}}{(\sqrt{\pi}2^q q!)^{1/2}}, \quad x \in \mathbb{R}, \tag{2.7.14}$$

where

$$H_q(x) := e^{x^2/2}\left(-\frac{d}{dx}+x\right)^q e^{-x^2/2} = (-1)^q e^{x^2}\frac{d^q}{dx^q}e^{-x^2}, \quad x \in \mathbb{R}, \tag{2.7.15}$$

is the Hermite polynomial of degree q (see e.g. [130, vol.1, Appendix B, Section III] for the details). Thus the functions ψ_q, $q \in \mathbb{Z}_+$, form an orthonormal basis in $L^2(\mathbb{R})$. For $\mathbf{x} = (x,y) \in \mathbb{R}^2$, $\boldsymbol{\xi} = (\xi,\eta) \in \mathbb{R}^2$, set

$$\kappa_b(\mathbf{x},\boldsymbol{\xi}) := \left(\frac{1}{\sqrt{b}}(x-\eta), \frac{1}{\sqrt{b}}(\xi-y), \frac{\sqrt{b}}{2}(\xi+y), -\frac{\sqrt{b}}{2}(\eta+x)\right). \tag{2.7.16}$$

Evidently, the mapping κ_b is linear and symplectic. Introduce the Weyl symbol

$$\mathscr{H}_b(\mathbf{x},\boldsymbol{\xi}) = (\xi+\frac{1}{2}by)^2 + (\eta-\frac{1}{2}bx)^2, \quad \mathbf{x}=(x,y)\in\mathbb{R}^2, \quad \boldsymbol{\xi}=(\xi,\eta)\in\mathbb{R}^2, \tag{2.7.17}$$

of the operator defined in (2.7.3). Then we have

$$(\mathscr{H}_b \circ \kappa_b)(\boldsymbol{\xi},\mathbf{x}) = b(x^2+\xi^2), \quad (\boldsymbol{\xi},\mathbf{x}) \in T^*\mathbb{R}^d. \tag{2.7.18}$$

Note that the function on the r.h.s. of (2.7.18) coincides with the Weyl symbol of the operator $(b\mathfrak{h}) \otimes I_y$ where I_y is the identity in $L^2(\mathbb{R}_y)$.
Next, define the unitary operator $\mathscr{W}_b : L^2(\mathbb{R}^2_{x,y}) \to L^2(\mathbb{R}^2_{x,y})$ by

$$(\mathscr{W}_b u)(x,y) := \frac{\sqrt{b}}{2\pi}\int_{\mathbb{R}^2} e^{i\phi_b(x,y;x',y')}u(x',y')dx'dy' \tag{2.7.19}$$

where

$$\phi_b(x,y;x',y') := b\frac{xy}{2} + b^{1/2}(xy'-yx') - x'y'.$$

Writing κ_b as a product of elementary linear symplectic transformations (see e.g. [96, Lemma 18.5.8]), and composing the corresponding elementary metaplectic operators, we easily check that $\mathscr{W}_b$ is a metaplectic operator corresponding to the symplectic mapping κ_b in (2.7.16). Then (2.7.18) and (2.3.19) imply

$$\mathscr{W}_b^* a \mathscr{W}_b = (\sqrt{b}\alpha) \otimes I_y, \quad \mathscr{W}_b^* a^* \mathscr{W}_b = (\sqrt{b}\alpha^*) \otimes I_y, \tag{2.7.20}$$

and

$$\mathscr{W}_b^* H_S(A,0)\mathscr{W}_b = (b\mathfrak{h}) \otimes I_y. \tag{2.7.21}$$

The unitary equivalence between the operators $H_S(A,0)$ and $(b\mathfrak{h}) \otimes I_y$, established in (2.7.21), and the explicit properties of the harmonic oscillator $\mathfrak{h}$ allow us to describe the spectrum of $H_S(A,0)$. Namely, we have

$$\sigma(H_S(A,0)) = \bigcup_{q\in\mathbb{Z}_+}\{\Lambda_q\}, \tag{2.7.22}$$

where

$$\Lambda_q := b(2q+1), \quad q \in \mathbb{Z}_+,$$

are the so called *Landau levels*, and

$$\mathrm{Ker}\,(H_S(A,0) - \Lambda_q I) = (a^*)^q\, \mathrm{Ker}\, a, \quad q \in \mathbb{Z}_+. \tag{2.7.23}$$

Note that by (2.7.5) we have

$$\mathrm{Ker}\ a = \left\{ u \in L^2(\mathbb{R}^2) \,|\, u = e^{-\varphi}\, g, \quad \frac{\partial g}{\partial \bar{z}} = 0 \right\}, \tag{2.7.24}$$

so that Ker a coincides, up to the unitary mapping $u = e^{-\varphi}\, g \mapsto g$ with *Fock-Segal-Bargmann space*, i.e. the space of entire functions $g \in L^2(\mathbb{R}^2; e^{-2\varphi}\, d\mathbf{x})$.
By the spectral theorem, (2.7.21) implies that for each $q \in \mathbb{Z}_+$ we have

$$\mathscr{W}_b^* p_q \mathscr{W}_b = \pi_q \otimes I_y \tag{2.7.25}$$

where $p_q = p_q(b)$ is the orthogonal projection onto $\mathrm{Ker}\,(H_S(A,0) - \Lambda_q I)$, and

$$\pi_q := \langle \cdot, \psi_q \rangle_{L^2(\mathbb{R})} \psi_q \tag{2.7.26}$$

is the orthogonal projection onto $\mathrm{Ker}\,(\mathfrak{h} - (2q+1)I)$.
Moreover, (2.7.11) and the spectral theorem imply

$$\mathscr{T}_{\mathbf{y}}^*\, p_q\, \mathscr{T}_{\mathbf{y}} = p_q, \quad \mathbf{y} \in \mathbb{R}^2, \quad q \in \mathbb{Z}_+. \tag{2.7.27}$$

where $\mathscr{T}_{\mathbf{y}}$ are the magnetic translations introduced in (2.7.8).
Next, we will use (2.7.23) in order to obtain an explicit representation of the integral kernel K_q of the orthogonal projection p_q. Denote by π_q the spectral projection onto $\mathrm{Ker}\,(\mathfrak{h} - (2q+1)I)$. Evidently, π_q admits an integral kernel

$$\psi_q(x)\, \psi_q(x'), \quad x, x' \in \mathbb{R}.$$

Then (2.7.21) implies

$$K_q(x,y;x',y') = K_{q,b}(x,y;x',y')$$

$$= \frac{b}{(2\pi)^2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} e^{i(\phi_b(x,y;\xi_2,\eta) - \phi_b(x',y';\xi_1,\eta))} \psi_q(\xi_1) \psi_q(\xi_2) d\xi_1 d\xi_2 \right) d\eta. \tag{2.7.28}$$

Let us recall now a well-known integral formula relating Hermite and Laguerre polynomials, namely

$$\int_{\mathbb{R}} e^{-t^2} H_q(t-z) H_q(t+\bar{z}) dt = \sqrt{\pi}\, 2^q q!\, L_q(2|z|^2), \quad z \in \mathbb{C}, \quad q \in \mathbb{Z}_+, \tag{2.7.29}$$

(see [86, Eq. (7.377)]), where

$$L_q(t) := \frac{e^t}{q!} \frac{d^q}{dt^q} \left(t^q e^{-t} \right) = \sum_{\ell=0}^{q} \binom{q}{q-\ell} \frac{(-t)^\ell}{\ell!}, \quad t \in \mathbb{R}, \tag{2.7.30}$$

is the Laguerre polynomial of degree $q \in \mathbb{Z}_+$. For further references, we recall that

$$\int_0^\infty L_q(t) L_\ell(t) e^{-t} dt = \delta_{q\ell}, \quad q, \ell \in \mathbb{Z}_+. \tag{2.7.31}$$

Taking into account that ψ_q is an eigenfunction of the Fourier transform with eigenvalue i^{-q}, we easily find that (2.7.28) and (2.7.29) imply

$$K_q(\mathbf{x}, \mathbf{x}') = K_{q,b}(\mathbf{x}, \mathbf{x}')$$

$$= \frac{b}{2\pi} \exp\{-\frac{b}{4}(|\mathbf{x}-\mathbf{x}'|^2 + 2i\mathbf{x} \wedge \mathbf{x}')\} L_q\left(\frac{b}{2}|\mathbf{x}-\mathbf{x}'|^2\right), \quad q \in \mathbb{Z}_+, \quad \mathbf{x}, \mathbf{x}' \in \mathbb{R}^2, \tag{2.7.32}$$

the notation $\mathbf{x} \wedge \mathbf{x}'$ being defined in (2.7.9). In particular, we have

$$K_q(\mathbf{x}, \mathbf{x}) = \frac{b}{2\pi}, \quad q \in \mathbb{Z}_+, \quad \mathbf{x} \in \mathbb{R}^2. \tag{2.7.33}$$

We believe that here is the appropriate place to introduce as well the so called *canonical basis* of

$$\operatorname{Ran} p_q = \operatorname{Ker}(H_S(A, 0) - \Lambda_q I) = (a^*)^q \operatorname{Ker} a, \quad q \in \mathbb{Z}_+.$$

Let at first $q = 0$. Then the functions

$$\widetilde{\varphi}_{k,0}(\mathbf{x}) = z^k e^{-b|\mathbf{x}|^2/4}, \quad \mathbf{x} = (x, y) \in \mathbb{R}^2, \quad z = x + iy, \quad k \in \mathbb{Z}_+, \tag{2.7.34}$$

form an orthogonal basis of $\operatorname{Ker} a = \operatorname{Ran} p_0$ (see e.g. [87, Sections 3.1-3.2]). Normalizing, we obtain the following orthonormal basis of $\operatorname{Ran} p_0$:

$$\varphi_{k,0}(\mathbf{x}) := \frac{\widetilde{\varphi}_{k,0}(\mathbf{x})}{\|\widetilde{\varphi}_{k,0}\|_{L^2(\mathbb{R}^2)}} = \sqrt{\frac{b}{2\pi}} \sqrt{\frac{1}{k!}} \left(\sqrt{\frac{b}{2}}\, z\right)^k e^{-b|\mathbf{x}|^2/4}, \; \mathbf{x} \in \mathbb{R}^2, \; k \in \mathbb{Z}_+. \tag{2.7.35}$$

Note that

$$K_0(\mathbf{x}, \mathbf{x}') = \sum_{k \in \mathbb{Z}_+} \varphi_{k,0}(\mathbf{x}) \overline{\varphi_{k,0}(\mathbf{x}')} = \frac{b}{2\pi} \exp\{-\frac{b}{4}(|\mathbf{x}-\mathbf{x}'|^2 + 2i\mathbf{x} \wedge \mathbf{x}')\}, \quad \mathbf{x}, \mathbf{x}' \in \mathbb{R}^2,$$

which is equivalent to (2.7.32) with $q = 0$. Let now $q \geq 1$. Set

$$\widetilde{\varphi}_{k,q} = (a^*)^q \varphi_{k,0}, \quad k \in \mathbb{Z}_+. \tag{2.7.36}$$

The commutation relation (2.7.6) easily implies

$$\langle \widetilde{\varphi}_{k,q}, \widetilde{\varphi}_{\ell,q} \rangle_{L^2(\mathbb{R}^2)} = (2b)^q q! \delta_{k\ell}, \quad k, \ell \in \mathbb{Z}_+.$$

Therefore, the functions

$$\varphi_{k,q} := \frac{\widetilde{\varphi}_{k,q}}{\|\widetilde{\varphi}_{k,q}\|_{L^2(\mathbb{R}^2)}} = \frac{\widetilde{\varphi}_{k,q}}{\sqrt{(2b)^q q!}}, \quad k \in \mathbb{Z}_*, \tag{2.7.37}$$

form an orthonormal basis of $\mathrm{Ran}\, p_q$, $q \in \mathbb{N}$. The functions $\varphi_{k,q}$ admit an explicit expression in terms of the generalized Laguerre polynomials

$$L_q^{(\alpha)}(t) := \frac{t^{-\alpha} e^t}{q!} \frac{d^q}{dt^q} \left(t^{q+\alpha} e^{-t}\right), \quad t > 0, \quad \alpha \in \mathbb{R}, \quad q \in \mathbb{Z}_+, \tag{2.7.38}$$

which is coherent with (2.7.30) for $\alpha = 0$. We will need these polynomials for $\alpha = k - q$ with $k \in \mathbb{Z}_+$. Then we have

$$L_q^{(k-q)}(t) = \sum_{\ell=0}^{q} \binom{k}{q-\ell} \frac{(-t)^\ell}{\ell!}, \quad t \in \mathbb{R},$$

where as usual $\binom{k}{q-\ell} = \frac{k(k-1)\ldots(k-q+\ell+1)}{(q-\ell)!}$ if $\ell < q$ and $\binom{k}{0} = 1$. Note that $\binom{k}{q-\ell} = 0$ if $k < q - \ell$. Applying the identity

$$(\alpha - t) L_q^{(\alpha)}(t) + t \frac{d}{dt} L_q^{(\alpha)}(t) = (q+1) L_{q+1}^{(\alpha-1)}(t), \quad t \in \mathbb{R},$$

which follows from formulae (8.971.3) and (8.971.5) of [86], we easily find that

$$\varphi_{k,q}(\mathbf{x})$$

$$= \frac{1}{i^q} \sqrt{\frac{b}{2\pi}} \sqrt{\frac{q!}{k!}} \left(\sqrt{\frac{b}{2}}\, z\right)^{k-q} L_q^{(k-q)}\left(\frac{b|\mathbf{x}|^2}{2}\right) e^{-b|\mathbf{x}|^2/4}, \ \mathbf{x} \in \mathbb{R}^2, \ k, q \in \mathbb{Z}_+. \tag{2.7.39}$$

The following lemma contains an important supplement to (2.3.8).

Lemma 2.7.1 [45, Lemma 3.5] *We have*

$$\mathscr{W}_b^* \varphi_{k,q} = i^{q-k} \psi_q \otimes \psi_k, \quad k, q \in \mathbb{Z}_+, \tag{2.7.40}$$

where $\mathscr{W}_b$ *is the unitary operator defined in* (2.7.19) ψ_q, $q \in \mathbb{Z}_+$, *are the Hermite functions defined in* (2.7.14).

Proof. By (2.7.35) – (2.7.37), and (2.7.20), we get

$$\mathscr{W}_b^* \varphi_{k,q} = \sqrt{\frac{b^{k+1}}{\pi 2^{k+q+1} k! q!}} ((\alpha^*)^q \otimes I_y) \mathscr{U}_b^* \widetilde{\varphi}_{k,0}. \tag{2.7.41}$$

Using (2.7.19), we easily find that

$$\left(\mathscr{W}_b^* \widetilde{\varphi}_{k,0}\right)(x, y) = \frac{1}{2\pi\sqrt{b}} \left(\frac{2}{\sqrt{b}}\right)^k e^{ixy} \left(\frac{\partial}{\partial \bar{z}}\right)^k J(x, y) \tag{2.7.42}$$

where

$$J(x, y) := \int_{\mathbb{R}^2} e^{-i(ty - sx)} e^{-its/2} e^{-(t^2+s^2)/4} \, dt\, ds, \quad (x, y) \in \mathbb{R}^2.$$

An elementary calculation yields

$$J(x, y) = \sqrt{2}(2\pi) e^{-ixy} e^{-(x^2+y^2)/2}. \tag{2.7.43}$$

Inserting (2.7.43) into (2.7.42), we get

$$\left(\mathscr{W}_b^* u_k\right)(x,y) = \sqrt{\frac{2}{b^{k+1}}}\, e^{-x^2/2}(-1)^k(\alpha^*)^k e^{-y^2/2}, \tag{2.7.44}$$

and inserting (2.7.44) into (2.7.41), we obtain (2.7.40). □

Let us go back now to the general case $\dim \operatorname{Ker} B = 0$ and $\dim \operatorname{Ran} B = 2d$ with $d \geq 1$ where $H_S(A,0)$ can be written as in (2.7.1). Hence, in this case (2.7.22) should be replaced by

$$\sigma(H_S(A,0)) = \bigcup_{q_1 \in \mathbb{Z}_+} \ldots \bigcup_{q_d \in \mathbb{Z}_+} \{(2q_1+1)b_1 + \ldots + (2q_d+1)b_d\}. \tag{2.7.45}$$

Let us re-write (2.7.45) using the increasing sequence of the *Landau levels* $\{\Lambda_q\}_{q \in \mathbb{Z}_+}$:

$$\begin{cases} \Lambda_0 := b_1 + \ldots + b_d = \frac{1}{2}\operatorname{Tr}\sqrt{B^*B}, \\ \Lambda_q := \inf\left\{\lambda \in \mathbb{R} | \lambda > \Lambda_{q-1}, \lambda = \sum_{j=1}^d (2s_j+1)b_j,\ (s_1,\ldots,s_d) \in \mathbb{Z}_+^d\right\},\ q \in \mathbb{N}. \end{cases} \tag{2.7.46}$$

Thus we find that if B is constant and has a full rank, i.e. $k = \dim \operatorname{Ker} B = 0$, then

$$\sigma(H_S(A,0)) = \sigma_{ess}(H_S(A,0)) = \sigma_{pp}(H_S(A,0)) = \cup_{q \in \mathbb{Z}_+} \{\Lambda_q\}. \tag{2.7.47}$$

Taking into account equality (2.7.23) concerning the case $d = 1$, we find that in the general case $d \geq 1$ we have

$$\operatorname{Ker}(H_S(A,0) - \Lambda_q I) = \bigoplus_{\substack{(q_1,\ldots,q_d) \in \mathbb{Z}_+^d: \\ \sum_{j=1}^d b_j(2q_j+1) = \Lambda_q}} \bigotimes_{j=1}^d \left(\left(a(b_j)^*\right)^{q_j} \operatorname{Ker} a(b_j)\right), \quad q \in \mathbb{Z}_+, \tag{2.7.48}$$

where $a(b_j)^*$ and $a(b_j)$ are the creation and annihilation operators defined in (2.7.4) and (2.7.5) respectively. In particular,

$$\dim \operatorname{Ker}(H_S(A,0) - \Lambda_q I) = \infty, \quad q \in \mathbb{Z}_+.$$

Moreover, bearing in mind (2.7.32), we conclude that the integral kernel of the orthogonal projection onto $\operatorname{Ker}(H_S(A,0) - \Lambda_q I)$ can be written as

$$K_q(\mathbf{x},\mathbf{x}') = \sum_{\substack{(q_1,\ldots,q_d) \in \mathbb{Z}_+^d: \\ \sum_{j=1}^d b_j(2q_j+1) = \Lambda_q}} \prod_{j=1}^d K_{q_j,b_j}(\mathbf{x}_j, \mathbf{x}'_j) \tag{2.7.49}$$

with $\mathbf{x} = (\mathbf{x}_1,\ldots,\mathbf{x}_d) \in \mathbb{R}^{2d}$, $\mathbf{x}' = (\mathbf{x}'_1,\ldots,\mathbf{x}'_d) \in \mathbb{R}^{2d}$. In particular, we have

$$K_q(\mathbf{x},\mathbf{x}) = \frac{b_1 \ldots b_d}{(2\pi)^d}\kappa_q, \quad \mathbf{x} \in \mathbb{R}^d,$$

where

$$\kappa_q := \#\left\{(q_1,\dots,q_d)\in\mathbb{Z}_+ \mid \sum_{j=1}^{d} b_j(2q_j+1)=\Lambda_q\right\} \tag{2.7.50}$$

could be called the *multiplicity of the Landau level* Λ_q, $q\in\mathbb{Z}_+$. Hence, the integral kernel $\mathscr{E}_E(\mathbf{x},\mathbf{x}')$ of $\mathbb{1}_{(-\infty,E)}(H_S(A,0))$, can be written explicitly

$$\mathscr{E}_E(\mathbf{x},\mathbf{x}') = \sum_{q\in\mathbb{Z}_+} \mathbb{1}_{(0,\infty)}(E-\Lambda_q)\,K_q(\mathbf{x},\mathbf{x}'), \quad \mathbf{x},\mathbf{x}'\in\mathbb{R}^{2d}, \quad E\in\mathbb{R}. \tag{2.7.51}$$

In particular, $\mathscr{E}_E$ is constant on the diagonal, i.e.

$$\mathscr{E}_E(\mathbf{x},\mathbf{x}) = \frac{b_1\dots b_d}{(2\pi)^d}\sum_{q\in\mathbb{Z}_+} \mathbb{1}_{(0,\infty)}(E-\Lambda_q)\,\kappa_q, \quad \mathbf{x}\in\mathbb{R}^{2d}, \quad E\in\mathbb{R}. \tag{2.7.52}$$

Next, we describe the spectra of the 2D Pauli and Dirac operators in constant magnetic fields. At first we consider Pauli operator $H_P(A,0)$. Again, we assume that $b = B_{12} > 0$. Hence, (2.6.12) reads

$$H_P(A,0) = \begin{pmatrix} H_S(A,0)-bI & 0 \\ 0 & H_S(A,0)+bI \end{pmatrix}. \tag{2.7.53}$$

Therefore, (2.7.22) and (2.7.47) imply

$$\sigma(H_P(A,0)) = \sigma(H_S(A,0)-bI)\bigcup\sigma(H_S(A,0)+bI) = \bigcup_{q=0}^{\infty}\{2bq\},$$

and

$$\sigma(H_P(A,0)) = \sigma_{ess}(H_P(A,0)) = \sigma_{pp}(H_P(A,0)) = \bigcup_{q=0}^{\infty}\{2bq\}. \tag{2.7.54}$$

Let us now describe the spectrum of the 2D Dirac operator with constant magnetic field. Assume at first as above that $b = B_{12} > 0$. Introduce the Dirac-Landau levels

$$\Lambda_q^- := -\sqrt{2bq+m^2}, \quad q\in\mathbb{N}, \quad \Lambda_q^+ := \sqrt{2bq+m^2}, \quad q\in\mathbb{Z}_+.$$

By Proposition 2.6.2, the operator $H_D(A,0)$ is unitarily equivalent to

$$\begin{pmatrix} \sqrt{a^*a+m^2} & 0 \\ 0 & -\sqrt{aa^*+m^2} \end{pmatrix}.$$

Hence, (2.7.7) and (2.7.47) yield

$$\sigma(H_D(A,0)) = \sigma_{ess}(H_D(A,0)) = \sigma_{pp}(H_D(A,0)) = \left(\cup_q\left\{\Lambda_q^-\right\}\right)\bigcup\left(\cup_q\left\{\Lambda_q^+\right\}\right). \tag{2.7.55}$$

Note that due to (2.6.19), the spectrum of $H_D(A,0)$ is not symmetric with respect to the origin. If $b < 0$, we apply (2.6.21) and obtain

$$\sigma(H_D(A,0)) = \sigma(-H_D(-A,0)), \tag{2.7.56}$$

and the spectrum of $-H_D(-A,0)$ can be recovered from (2.7.55). Since $\sigma(H_D(-A,0))$ is not symmetric with respect to the origin, i.e. $\sigma(-H_D(-A,0)) \neq \sigma(H_D(-A,0))$, relation (2.7.56) implies that $\sigma(H_D(A,0)) \neq \sigma(H_D(-A,0))$.
Further, we discuss $\sigma(H_S(A,0))$ in the case where the constant magnetic field B has a non-trivial kernel, i.e. $k = \dim \mathrm{Ker}\, B \geq 1$. We use the representation $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^{2d}) \otimes L^2(\mathbb{R}^k)$, and for $\mathbf{x} = (x_1, y_1, \dots, x_d, y_d, w_1, \dots, w_k)$ we write $\mathbf{x} = (x_\perp, x_\parallel)$ with $x_\perp := (x_1, y_1, \dots, x_d, y_d) \in \mathbb{R}^{2d}$ and $x_\parallel = (w_1, \dots, w_k) \in \mathbb{R}^k$. In particular, if $n = 3$, i.e. $d = 1$ and $k = 1$, and $\mathbf{B} = (0,0,b)$, then $x_\perp = (x,y)$ are the variables perpendicular to $\mathbf{B}$, and $x_\parallel = w$ is the variable along the magnetic field. In the general case, we have

$$H_S(A,0) = H_\perp \otimes I_\parallel + I_\perp \otimes H_\parallel \tag{2.7.57}$$

where

$$H_\perp := \sum_{j=1}^{d} \left\{ \left(-i\frac{\partial}{\partial x_j} + \frac{b_j y_j}{2}\right)^2 + \left(-i\frac{\partial}{\partial y_j} - \frac{b_j x_j}{2}\right)^2 \right\}, \quad H_\parallel := -\sum_{\ell=1}^{k} \frac{\partial^2}{\partial w_\ell^2},$$

and $I_\parallel$, $I_\perp$, are the identities in $L^2(\mathbb{R}^{2d})$ and $L^2(\mathbb{R}^k)$ respectively. In what follows we will use systematically the spectral properties of operators in the form (2.7.57). That is why, here we recall briefly these properties. Let $\mathfrak{H}_j$, $j = 1,2$, be two Hilbert spaces, H_j be operators self-adjoint and lower bounded in $\mathfrak{H}_j$, and I_j be the identities in $\mathfrak{H}_j$. Introduce the closure

$$H := H_1 \otimes I_2 + I_1 \otimes H_2 \tag{2.7.58}$$

of the operator defined originally on $\mathfrak{D}(H_1) \otimes \mathfrak{D}(H_2)$ (here and in the sequel we use the terminology concerning tensor products, established in [162, Section VIII.10]).

Lemma 2.7.2 [162, Theorem VIII.33], [4, Section 8.2.3]
(i) *The operator* H *defined in* (2.7.58) *is self-adjoint in the Hilbert space* $\mathfrak{H}_1 \otimes \mathfrak{H}_2$.
(ii) *We have*

$$\sigma(H) = \overline{\{\lambda \in \mathbb{R} \,|\, \lambda = \lambda_1 + \lambda_2, ; \lambda_j \in \sigma(H_j),\ j = 1,2\}}. \tag{2.7.59}$$

(iii) *We have* $\lambda \in \sigma_{pp}(H)$ *if and only if* $\lambda = \lambda_1 + \lambda_2$ *with* $\lambda_j \in \sigma_{pp}(H_j)$, $j = 1,2$, *and*

$$\mathbb{1}_{\{\lambda\}}(H) = \sum_{\lambda_1+\lambda_2=\lambda} \mathbb{1}_{\{\lambda_1\}}(H_1) \otimes \mathbb{1}_{\{\lambda_2\}}(H_2).$$

(iv) *If one of the operators* H_j, $j = 1,2$, *has a purely a.c. spectrum, then the spectrum of* H *is purely a.c.*

The operator $H_\parallel$ in (2.7.57) is unitarily equivalent under the Fourier transform to the multiplier by $|\upsilon|^2$, $\upsilon \in \mathbb{R}^k$, acting in $L^2(\mathbb{R}^k)$. Therefore, the spectrum of $H_\parallel$ is purely a.c., and

$$\sigma(H_\parallel) = [0, \infty). \tag{2.7.60}$$

Applying Lemma 2.7.2, we find that, by (2.7.57), the spectrum of $\sigma(H_S(A,0))$ is purely a.c., and by the combination of (2.7.57), (2.7.47), and (2.7.60), we have

$$\sigma(H_S(A,0)) = \bigcup_{q \in \mathbb{Z}_+} [\Lambda_q, \infty) = [\Lambda_0, \infty). \tag{2.7.61}$$

Representation (2.7.58) allows us to write explicitly the integral kernel $\mathscr{E}_E(\mathbf{x},\mathbf{w};\mathbf{x}',\mathbf{w}')$ of $\mathbb{1}_{(-\infty,E)}(H_S(A,0))$. Namely, we have

$$\mathscr{E}_E(\mathbf{x},\mathbf{w};\mathbf{x}',\mathbf{w}') = \sum_{q\in\mathbb{Z}_+} \mathscr{E}^0_{E-\Lambda_q}(\mathbf{w},\mathbf{w}')\, K_q(\mathbf{x},\mathbf{x}'), \quad (\mathbf{x},\mathbf{w}),(\mathbf{x}',\mathbf{w}') \in \mathbb{R}^n, \quad E\in\mathbb{R},$$

where K_q is the integral kernel defined in (2.7.49), and $\mathscr{E}^0_\lambda$, $\lambda\in\mathbb{R}$, is the integral kernel of the spectral projection $\mathbb{1}_{(-\infty,E)}(-\Delta_{\mathbf{w}})$. Evidently,

$$\mathscr{E}^0_\lambda(\mathbf{w},\mathbf{w}') = (2\pi)^{-k}\int_{\mathbb{R}^k} \mathbb{1}_{(0,\infty)}(\lambda-|\xi|^2)e^{i\boldsymbol{\xi}\cdot(\mathbf{w}-\mathbf{w}')}\,d\boldsymbol{\xi}, \quad \lambda\in\mathbb{R}, \quad \mathbf{w},\mathbf{w}'\in\mathbb{R}^k,$$

and, in particular,

$$\mathscr{E}^0_\lambda(\mathbf{w},\mathbf{w}) = \frac{\omega_k}{(2\pi)^k}\lambda_+^{k/2}, \quad \lambda\in\mathbb{R}, \quad \mathbf{w}\in\mathbb{R}^k.$$

Thus, we find again that $\mathscr{E}_E$ is constant on the diagonal, namely

$$\mathscr{E}_E(\mathbf{x},\mathbf{w};\mathbf{x},\mathbf{w}) = \frac{\omega_k\, b_1\ldots b_d}{(2\pi)^{d+k}}\sum_{q\in\mathbb{Z}_+}(E-\Lambda_q)_+^{k/2}\,\kappa_q, \quad (\mathbf{x},\mathbf{w})\in\mathbb{R}^n, \quad E\in\mathbb{R}. \qquad (2.7.62)$$

Note that the higher Landau levels Λ_q, $q\ge 1$, play the role of *spectral thresholds* embedded in $\sigma_{ac}(H_S(A,0))$. Moreover, the structure of $H_S(A,0)$ in (2.7.57) resembles, in particular in the case $k=1$, the structure of *a quantum waveguide*, i.e., say, the Dirichlet Laplacian $-\Delta^D_\Omega$ in a cylindrical domain

$$\Omega = \left\{\mathbf{x} = (x_\perp, x_\parallel)\in\mathbb{R}^n \,|\, x_\perp\in\omega,\, x_\parallel\in\mathbb{R}\right\}$$

where the cross-section ω is a bounded domain in $\mathbb{R}^{n-1}$. Let $\{\mu_j\}_{j\in\mathbb{N}}$ be the non-decreasing sequence of the eigenvalues of the transversal Dirichlet Laplacian $-\Delta^D_\omega$, self-adjoint in $L^2(\omega)$. Then the spectrum of $-\Delta^D_\Omega$ is purely a.c., and, similarly to (2.7.61) we have

$$\sigma\left(-\Delta^D_\Omega\right) = \bigcup_{j=1}^\infty [\mu_j,\infty) = [\mu_1,\infty),$$

while the embedded spectral threshold are μ_j, $j\ge 2$. The essential difference between $H_S(A,0)$ and $-\Delta^D_\Omega$ is that the Landau levels Λ_q, $q\in\mathbb{Z}_+$, are eigenvalues of *infinite* multiplicity of $H_\perp$, while the multiplicities of the eigenvalues μ_j, $j\in\mathbb{N}$, of $-\Delta^D_\omega$ are *finite*.

Finally, we describe the spectra of the three-dimensional Pauli and Dirac operators in constant magnetic fields. If $n=3$, and $\mathbf{B} = (0,0,b)$, $b>0$, then taking into account (2.6.13), we find that (2.7.53) holds true again. Therefore, the spectrum $\sigma(H_P(A,0))$ of the Pauli operator is purely a.c., and

$$\sigma(H_P(A,0)) = \sigma(H_S(A,0)-bI)\bigcup\sigma(H_S(A,0)+bI) = [0,\infty). \qquad (2.7.63)$$

In this case, the embedded spectral thresholds are 2bq, $q \in \mathbb{N}$.
Further, if n = 3, the Proposition 2.6.2 implies that the Dirac operator $H_D(A,0)$ is unitarily equivalent to

$$\begin{pmatrix} \sqrt{S(A)+m^2 I} & 0 \\ 0 & -\sqrt{S(A)+m^2 I} \end{pmatrix}.$$

Since according to our conventions $S(A) = H_P(A,0)$, we find that $\sigma(H_D(A,0))$ is purely a.c., and by (2.7.63) we have

$$\sigma(H_D(A,0)) = \sigma\left(\sqrt{S(A)+m^2 I}\right) \bigcup \sigma\left(-\sqrt{S(A)+m^2 I}\right) = (-\infty,-m] \cup [m,\infty), \tag{2.7.64}$$

the embedded spectral thresholds being this time equal to $\pm\sqrt{2bq+m^2}$, $q \in \mathbb{N}$. Note also that in the 3D case $\sigma(H_D(A,0))$ is symmetric with respect to the origin.

2.8 Pauli Hamiltonians with admissible non-constant magnetic fields

Let n = 2. As above, we will use the short-hand notation

$$b(\mathbf{x}) = B_{12}(\mathbf{x}) = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}, \quad \mathbf{x} = (x,y) \in \mathbb{R}^2.$$

For simplicity, we will assume that $b \in C(\mathbb{R}^2;\mathbb{R}) \cap L^\infty(\mathbb{R}^2)$. Let $\varphi \in C^2(\mathbb{R}^2;\mathbb{R})$ be a solution of the Poisson equation

$$\Delta\varphi = b. \tag{2.8.1}$$

Then the magnetic potential

$$A = (A_1, A_2) = \left(-\frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial x}\right) \tag{2.8.2}$$

generates the magnetic field b. If we add a harmonic function to the solution φ of (2.8.1), we will obtain another solution of this equation, and hence another magnetic potential which generates b, gauge equivalent to the original A defined in (2.8.2).
Then, similarly to (2.7.4) - (2.7.5), the magnetic creation and annihilation operators can be written as

$$a^* = a^*(b) = -2i\,e^{\varphi}\frac{\partial}{\partial z}e^{-\varphi}, \quad a = a(b) = -2i\,e^{-\varphi}\frac{\partial}{\partial \bar{z}}e^{\varphi}. \tag{2.8.3}$$

Since by (2.6.10), (2.6.9), and our convention m = 1/2, we have

$$H_P(A,0) = \begin{pmatrix} a^* a & 0 \\ 0 & a a^* \end{pmatrix}, \tag{2.8.4}$$

we find that

$$\mathrm{Ker}\,(H_P(A,0)) = \{\mathbf{u} = (u_1,u_2) \in L^2(\mathbb{R}^2;\mathbb{C}^2) \,|\, u_1 \in \mathrm{Ker}\,a(b),\ u_2 \in \mathrm{Ker}\,a^*(b)\}. \tag{2.8.5}$$

Here we have used the elementary fact that

$$\mathrm{Ker}\, a^* a = \mathrm{Ker}\, a, \quad \mathrm{Ker}\, a a^* = \mathrm{Ker}\, a^*. \tag{2.8.6}$$

In particular, it follows from (2.8.5) that

$$\dim \mathrm{Ker}\, (H_P(A,0)) = \dim \mathrm{Ker}\, a(b) + \dim \mathrm{Ker}\, a^*(b). \tag{2.8.7}$$

In accordance with (2.4.5) and (2.6.6) - (2.6.7), we have

$$\mathscr{C} a(-b) \mathscr{C} = -a^*(b), \quad \mathscr{C} a^*(-b) \mathscr{C} = -a(b), \tag{2.8.8}$$

where, as above, $\mathscr{C}$ denotes the complex conjugation. Therefore,

$$\dim \mathrm{Ker}\, a(-b) = \dim \mathrm{Ker}\, a^*(b), \quad \dim \mathrm{Ker}\, a^*(-b) = \dim \mathrm{Ker}\, a(b). \tag{2.8.9}$$

Further, (2.8.3) implies

$$\mathrm{Ker}\, a^*(b) = \left\{ f \in L^2(\mathbb{R}^2) | f = g e^{\varphi}, \frac{\partial g}{\partial z} = 0 \right\}, \tag{2.8.10}$$

$$\mathrm{Ker}\, a(b) = \left\{ f \in L^2(\mathbb{R}^2) | f = g e^{-\varphi}, \frac{\partial g}{\partial \bar{z}} = 0 \right\}. \tag{2.8.11}$$

Our next goal is to discuss classes of magnetic fields which admit explicit description of Ker a(b) and Ker $a^*(b)$. If any of the kernels of a or a^* is not trivial, then $\mathrm{Ker}\, H_P(A,0) \neq \{0\}$ as well, and in this case we are also interested in estimating the distance from the origin to the rest of the spectrum of $H_P(A,0)$. We start with the classical *Aharonov-Casher theorem*. Let

$$F := \frac{1}{2\pi} \int_{\mathbb{R}^2} b(\mathbf{x}) d\mathbf{x}$$

be *the flux* of the magnetic field b. For $t \in (0,\infty)$ we denote by $\lfloor t \rfloor$ the greatest integer less that t, and $\lfloor 0 \rfloor := 0$.

Proposition 2.8.1 [2] *Assume that* $b \in C(\mathbb{R}^2;\mathbb{R})$ *satisfies*

$$|b(\mathbf{x})| \leq C(1+|\mathbf{x}|)^{-2-\varepsilon}, \quad \mathbf{x} \in \mathbb{R}^2, \quad C \in [0,\infty), \quad \varepsilon \in (0,\infty). \tag{2.8.12}$$

Then

$$\dim \mathrm{Ker}\, a = \lfloor F_+ \rfloor, \quad \dim \mathrm{Ker}\, a^* = \lfloor F_- \rfloor, \tag{2.8.13}$$

and hence

$$\dim \mathrm{Ker}\, H_P(A,0) = \lfloor |F| \rfloor. \tag{2.8.14}$$

Proof. We can choose

$$\varphi(\mathbf{x}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |\mathbf{x} - \mathbf{x}'|\, b(\mathbf{x}')\, d\mathbf{x}', \quad \mathbf{x} \in \mathbb{R}^2.$$

Then we have

$$\varphi(\mathbf{x}) = F \ln |\mathbf{x}| + o(1), \quad |\mathbf{x}| \to \infty,$$

and, therefore,

$$e^{-\varphi(\mathbf{x})} = |x|^{-F}(1+o(1)), \quad e^{\varphi(\mathbf{x})} = |x|^{F}(1+o(1)), \quad |\mathbf{x}| \to \infty. \tag{2.8.15}$$

Assume $F \geq 0$. Let $u \in \mathrm{Ker}\, a$. Then, according to (2.8.11), we have $L^2(\mathbb{R}^2) \ni u = ge^{-\varphi}$ with entire g. Since g is harmonic, [74, Chapter 2, Theorem 7] implies that for very $k \in \mathbb{Z}_+$ there exists a constant C_k such that

$$|g^{(k)}(z))| \leq \frac{C_k}{r^{2+k}} \int_{\mathscr{B}_r(\mathbf{x})} |g(z')| d\mathbf{x}'$$

for any $r > 0$ and $\mathbf{x} \in \mathbb{R}^2$. Here, $\mathbf{x} = (x, y)$, $z = x + iy$, and $\mathscr{B}_r(\mathbf{x}) := \left\{\mathbf{x}' \in \mathbb{R}^2 \,|\, |\mathbf{x} - \mathbf{x}'| < r\right\}$. Applying the Cauchy-Schwarz inequality, we find that (2.8.15) implies

$$|g^{(k)}(z))| \leq C' r^{F-k-1} \underset{r\to\infty}{\longrightarrow} 0, \quad z \in \mathbb{C},$$

if $k > F-1$. Therefore, $g^{(k)}(z) = 0$ for any $z \in \mathbb{C}$ if $k > F-1$, and hence g is a polynomial of z. Since $u \in L^2(\mathbb{R}^2)$, we find that (2.8.15) implies that the degree of this polynomial is at most $\lfloor F \rfloor - 1$ if $F > 1$, and $g = 0$ if $F \leq 1$. Thus, we arrive at the first identity in (2.8.13) with $F \geq 0$.
Let now $u \in \mathrm{Ker}\, a^*$. Then it follows form (2.8.10) that $L^2(\mathbb{R}^2) \ni u = ge^{\varphi}$ with $\frac{\partial g}{\partial z} = 0$. By (2.8.15), we conclude that $g \in L^2(\mathbb{R}^2)$ which implies $g = 0$, i.e. the second inequality in (2.8.13) for $F \geq 0$ holds true. If $F < 0$, we apply the result for $F \geq 0$, and (2.8.9). Finally, (2.8.14) follows from (2.8.13) and (2.8.7). ☐

Next, note that if $b(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, and b is sufficiently regular, say $b \in C^\infty(\mathbb{R}^2; \mathbb{R})$, then

$$\sigma(H_P(A, 0)) = \sigma_{ess}(H_P(A, 0)) = [0, \infty)$$

(see e.g. [56, Theorem 6.1]). Therefore, if $b \in C^\infty(\mathbb{R}^2; \mathbb{R})$ satisfies (2.8.12) and $|F| > 1$, then the zero eigenvalue of $H_P(A, 0)$ is *not isolated* in the spectrum of $H_P(A, 0)$.
The following proposition contains a sufficient condition which guarantees that the dimension of Ker a and hence of Ker $H_P(A, 0)$ is infinite.

Proposition 2.8.2 [183, Lemma 3.4] *Assume that* $b \in C^1(\mathbb{R}^2)$*, and*

$$|\mathbf{x}|^2 b(\mathbf{x}) \to \infty \quad \textit{as} \quad |\mathbf{x}| \to \infty. \tag{2.8.16}$$

Then dim Ker a = ∞.

Various extensions of Propositions 2.8.1 and 2.8.2 can be found in [73, 81, 170, 70]. The dimension of Ker a can be infinite even if (2.8.16) is quite far from being fulfilled. Propositions 2.8.3 and 2.8.4 below contain examples of such situations. For their formulations we need several definitions and notations. Let

$$b(\mathbf{x}) = b_0 + \widetilde{b}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{2.8.17}$$

with $b_0 \in \mathbb{R}$, and $\widetilde{b} \in C(\mathbb{R}^2; \mathbb{R})$. Assume that there exists a solution $\widetilde{\varphi} \in C^2(\mathbb{R}^2; \mathbb{R})$ of the Poisson equation

$$\Delta \widetilde{\varphi}(\mathbf{x}) = \widetilde{b}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{2.8.18}$$

such that

$$\widetilde{\varphi}(\mathbf{x}) = o(|\mathbf{x}|^2), \quad |\mathbf{x}| \to \infty. \tag{2.8.19}$$

Then will say that the magnetic field b is *pre-admissible*. Set $\varphi_0(\mathbf{x}) = b_0|\mathbf{x}|^2/4$ so that $\Delta\varphi_0 = b_0$. Then $\varphi := \varphi_0 + \widetilde{\varphi}$ is a solution of (2.8.1). If b is pre-admissible, then its representation in the form (2.8.17) is unique. We will call b_0 *the mean value* of the pre-admissible b, and $\widetilde{b}$ *the background* of b.

Note that by a generalization of the Liouville theorem (see e.g. [74, Chapter 2, Theorems 7, 8]), the solution $\widetilde{\varphi} \in C^2(\mathbb{R}^2;\mathbb{R})$ of (2.8.18) which satisfies (2.8.19), is defined uniquely up to an affine function.

Our leading example of a pre-admissible background $\widetilde{b}$ has the form

$$\widetilde{b}(\mathbf{x}) = \int_{\mathbb{R}^2} e^{i\lambda\cdot\mathbf{x}} d\nu(\lambda), \quad \mathbf{x} \in \mathbb{R}^2, \tag{2.8.20}$$

where ν is a complex charge (i.e. a complex-valued measure) which satisfies

$$|\nu|(\mathbb{R}^2) < \infty, \tag{2.8.21}$$

$$\nu(\delta) = \overline{\nu(-\delta)}, \tag{2.8.22}$$

for every Borel set $\delta \subset \mathbb{R}^2$, and

$$\nu(\{0\}) = 0. \tag{2.8.23}$$

In this case, the solution $\widetilde{\varphi} \in C^2(\mathbb{R}^2;\mathbb{R})$ of (2.8.18) which satisfies (2.8.19) can be chosen as

$$\widetilde{\varphi}(\mathbf{x}) = \int_{\mathbb{R}^2} \frac{(\lambda\cdot\mathbf{x})^2}{|\lambda|^2} \int_0^1 (1-s)e^{is\lambda\cdot\mathbf{x}} ds\, d\nu(\lambda), \quad \mathbf{x} \in \mathbb{R}^2. \tag{2.8.24}$$

In particular, we are interested in the case $b \in WAP(\mathbb{R}^2)$, the Wiener class of almost periodic functions which is an important subclass of pre-admissible b satisfying (2.8.20) - (2.8.23), corresponding to a discrete charge ν. Let us recall briefly the definition and the main features of the class $WAP(\mathbb{R}^n)$, $n \geq 1$, (for more details see, for instance, [184] or [25]). Let $C_b(\mathbb{R}^n)$ be the Banach space of bounded functions $f \in C(\mathbb{R}^n)$ with norm

$$\|f\|_{C_b(\mathbb{R}^n)} = \sup_{\mathbf{x}\in\mathbb{R}^n} |f(\mathbf{x})|.$$

Set

$$e_\lambda(\mathbf{x}) := e^{i\lambda\cdot\mathbf{x}}, \quad \lambda \in \mathbb{R}^n, \quad \mathbf{x} \in \mathbb{R}^n,$$

and

$$\mathrm{Trig}(\mathbb{R}^n) := \left\{ u = \sum_{j=1}^N c_j e_{\lambda_j}, \quad c_j \in \mathbb{C}, \quad \lambda_j \in \mathbb{R}^n, \quad j = 1,\dots,N < \infty \right\}.$$

Then the Banach space of continuous almost periodic function $CAP(\mathbb{R}^n)$ is the closure of $\mathrm{Trig}(\mathbb{R}^n)$ in $C_b(\mathbb{R}^n)$. For $f \in CAP(\mathbb{R}^n)$, let

$$\mathscr{M}(f) := \lim_{T\to\infty} T^{-n} \int_{(-T/2,T/2)^n} f(\mathbf{x})\, d\mathbf{x} \in \mathbb{C}$$

stands for the mean value of f. For $\lambda \in \mathbb{R}^n$ denote by

$$f_\lambda := \mathscr{M}(fe_{-\lambda})$$

the Fourier coefficient of f, so that $f_0 := \mathscr{M}(f)$. Put

$$J(f) := \{\lambda \in \mathbb{R}^n \,|\, f_\lambda \neq 0\}, \quad J_0(f) := J(f) \setminus \{0\}.$$

It is well known that for any given $f \in CAP(\mathbb{R}^n)$, the set J(f) is countable, and f is uniquely determined by the set $\{f_\lambda\}_{\lambda \in \mathbb{R}^n}$. Define

$$WAP(\mathbb{R}^n) := \left\{ f \in CAP(\mathbb{R}^n) \,|\, \sum_{\lambda \in J(f)} |f_\lambda| < \infty \right\}.$$

If $f \in WAP(\mathbb{R}^n)$, then $f(\mathbf{x}) = \sum_{\lambda \in J(f)} f_\lambda\, e_\lambda(\mathbf{x})$ i.e. f coincides with the sum of its Fourier series which is absolutely convergent, uniformly with respect to $\mathbf{x} \in \mathbb{R}^n$. In particular, if $b \in WAP(\mathbb{R}^2;\mathbb{R})$, we have

$$b(\mathbf{x}) = \sum_{\lambda \in J(b)} b_\lambda\, e_\lambda(\mathbf{x}) = b_0 + \widetilde{b}(\mathbf{x}), \quad \widetilde{b}(\mathbf{x}) = \sum_{\lambda \in J_0(b)} b_\lambda\, e_\lambda(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2,$$

and b is a pre-admissible magnetic field. In this case the solution $\widetilde{\varphi} \in C^2(\mathbb{R}^2;\mathbb{R})$ of (2.8.18) introduced in (2.8.24) is

$$\widetilde{\varphi}(\mathbf{x}) = \sum_{\lambda \in J_0(b)} b_\lambda \frac{(\lambda \cdot \mathbf{x})^2}{|\lambda|^2} \int_0^1 (1-s) e_{s\lambda}(\mathbf{x})\, ds, \quad \mathbf{x} \in \mathbb{R}^2. \tag{2.8.25}$$

Next, we will say that the magnetic field $b \in C(\mathbb{R}^2;\mathbb{R})$ is *admissible* if it has the form (2.8.17), and there exists a solution $\widetilde{\varphi} \in C^2(\mathbb{R}^2;\mathbb{R})$ of (2.8.18) which is bounded together with its derivatives of order up to two. If b is of the form described in (2.8.20) - (2.8.23), then b is admissible if the charge ν satisfies in addition

$$\int_{\mathbb{R}^2} |\lambda|^{-2} d|\nu|(\lambda) < \infty. \tag{2.8.26}$$

In this case we may choose

$$\widetilde{\varphi}(\mathbf{x}) = -\int_{\mathbb{R}^2} |\lambda|^{-2} e^{i\lambda \cdot \mathbf{x}} d\nu(\lambda), \quad \mathbf{x} \in \mathbb{R}^2. \tag{2.8.27}$$

In particular, $b \in WAP(\mathbb{R}^2;\mathbb{R})$ is admissible if

$$\sum_{\lambda \in J_0(b)} |\lambda|^{-2} |b_\lambda| < \infty. \tag{2.8.28}$$

In this case the solution $\widetilde{\varphi} \in C^2(\mathbb{R}^2;\mathbb{R})$ of (2.8.18) defined in (2.8.27) is

$$\widetilde{\varphi}(\mathbf{x}) = -\sum_{\lambda \in J_0(b)} b_\lambda |\lambda|^{-2} e_\lambda(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2.$$

Consider a $(2\pi\mathbb{Z})^2$-*periodic* magnetic field

$$b(\mathbf{x}) = \sum_{k\in\mathbb{Z}^2} b_k e_k(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2,$$

with $\sum_{k\in\mathbb{Z}^2} |b_k| < \infty$, and $\overline{b_k} = b_{-k}$, $k \in \mathbb{Z}^2$. Such b is a special case of an admissible magnetic field $b \in WAP(\mathbb{R}^2)$. Then we have

$$\widetilde{b}(\mathbf{x}) = \sum_{0\neq k\in\mathbb{Z}^2} b_k e_k(\mathbf{x}), \quad \widetilde{\varphi}(\mathbf{x}) = -\sum_{0\neq k\in\mathbb{Z}^2} b_k |k|^{-2} e_k(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2,$$

Examples of pre-admissible and admissible magnetic fields satisfying (2.8.20) – (2.8.24), which correspond to charges ν which are absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^2$, can be found in [157].

Proposition 2.8.3 [154, Proposition 1.2] *Let* $b = b_0 + \widetilde{b}$ *be admissible.*
(i) *We have*

$$\dim \operatorname{Ker} a(b) = \begin{cases} \infty & \text{if } b_0 > 0, \\ 0 & \text{if } b_0 \leq 0, \end{cases} \quad \dim \operatorname{Ker} a^*(b) = \begin{cases} \infty & \text{if } b_0 < 0, \\ 0 & \text{if } b_0 \geq 0, \end{cases} \tag{2.8.29}$$

and, hence,

$$\dim \operatorname{Ker} H_P(A,0) = \begin{cases} \infty & \text{if } b_0 \neq 0, \\ 0 & \text{if } b_0 = 0. \end{cases} \tag{2.8.30}$$

(ii) *Assume* $b_0 \neq 0$. *Then*

$$\operatorname{dist}(0, \sigma(H_P(A,0)) \setminus \{0\}) \geq 2|b_0| e^{-2\operatorname{osc}\widetilde{\varphi}} \tag{2.8.31}$$

where

$$\operatorname{osc}\widetilde{\varphi} := \sup_{\mathbf{x}\in\mathbb{R}^2} \widetilde{\varphi}(\mathbf{x}) - \inf_{\mathbf{x}\in\mathbb{R}^2} \widetilde{\varphi}(\mathbf{x}),$$

i.e. the zero eigenvalue of $H_P(A,0)$ *is isolated in the spectrum of* $H_P(A,0)$.

Proof. Assume $b_0 > 0$. We have

$$\varphi(\mathbf{x}) = \frac{b_0|\mathbf{x}|^2}{4} + \mathscr{O}(1), \quad \mathbf{x} \in \mathbb{R}^2. \tag{2.8.32}$$

Then all the functions $z^k e^{-\varphi}$ with $k \in \mathbb{Z}_+$ are in $\operatorname{Ker} a(b)$, and hence $\dim \operatorname{Ker} a(b) = \infty$. On the other hand, if $u = ge^{\varphi} \in \operatorname{Ker} a^*(b)$ with $\frac{\partial g}{\partial z} = 0$, then it follows from (2.8.32) that $g \in L^2(\mathbb{R}^2)$, and hence $g = u = 0$. Thus we obtain (2.8.29) with $b_0 > 0$. The case $b_0 < 0$ follows from the result for $b_0 > 0$, and (2.8.9).
Let now $b_0 = 0$. Then (2.8.32) implies

$$e^{\pm\varphi(\mathbf{x})} = \mathscr{O}(1), \quad \mathbf{x} \in \mathbb{R}^2.$$

Then $u = ge^{-\varphi} \in \operatorname{Ker} a(b)$ with $\frac{\partial g}{\partial \bar{z}} = 0$ implies $g \in L^2(\mathbb{R}^2)$ and thus $g = u = 0$. Similarly, it follows from $u = ge^{\varphi} \in \operatorname{Ker} a^*(b)$ with $\frac{\partial g}{\partial z} = 0$ that $u = 0$. This we obtain (2.8.29) with

b_0. Finally, (2.8.30) follows from (2.8.29) and (2.8.7).
(ii) By (2.8.4), we have

$$\sigma(H_P(A,0)) = \sigma(a^* a) \cup \sigma(a a^*). \tag{2.8.33}$$

Assume $b_0 > 0$, and denote by $p : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ the orthogonal projection onto $\mathrm{Ker}\, a$. Then the restriction of $a^* a$ onto $(I-p)\mathfrak{D}(a^* a)$ is unitarily equivalent to $a a^*$. Hence, (2.8.33) and Part (i) imply that

$$\sigma(H_P(A,0)) = \{0\} \cup \sigma(a a^*).$$

Therefore, in order to prove (2.8.31) with $b_0 > 0$, it suffices to show that

$$\inf \sigma(a(b)\, a^*(b)) \geq 2b_0 e^{-2\,\mathrm{osc}\,\widetilde{\varphi}}. \tag{2.8.34}$$

We have

$$\begin{aligned} \inf \sigma(a(b)\, a^*(b)) &= \inf_{0 \neq u \in \mathfrak{D}(a^*(b))} \frac{\int_{\mathbb{R}^2} |a^*(b) u|^2 d\mathbf{x}}{\int_{\mathbb{R}^2} |u|^2 d\mathbf{x}} \\ &= \inf_{0 \neq u \in \mathfrak{D}(a^*(b))} \frac{4 \int_{\mathbb{R}^2} e^{2\varphi_0 + 2\widetilde{\varphi}} |\frac{\partial}{\partial z}\left(e^{-\varphi_0 - \widetilde{\varphi}} u\right)|^2 d\mathbf{x}}{\int_{\mathbb{R}^2} |u|^2 d\mathbf{x}}, \end{aligned} \tag{2.8.35}$$

applying (2.8.3) at the second step. Further, the mapping $u \mapsto e^{\widetilde{\varphi}} u =: w$ is a bijection in $\mathfrak{D}(a^*(b))$. Therefore, bearing in mind (2.8.35), we get

$$\begin{aligned} \inf \sigma(a(b)\, a^*(b)) &= \inf_{0 \neq w \in \mathfrak{D}(a^*(b))} \frac{4 \int_{\mathbb{R}^2} e^{2\varphi_0 + 2\widetilde{\varphi}} |\frac{\partial}{\partial z}\left(e^{-\varphi_0} w\right)|^2 d\mathbf{x}}{\int_{\mathbb{R}^2} e^{2\widetilde{\varphi}} |w|^2 d\mathbf{x}} \\ &\geq e^{-2\,\mathrm{osc}\,\widetilde{\varphi}} \frac{4 \int_{\mathbb{R}^2} e^{2\varphi_0} |\frac{\partial}{\partial z}\left(e^{-\varphi_0} w\right)|^2 d\mathbf{x}}{\int_{\mathbb{R}^2} |w|^2 d\mathbf{x}} \\ &= e^{-2\,\mathrm{osc}\,\widetilde{\varphi}} \inf \sigma(a(b_0)\, a^*(b_0)) \\ &= 2b_0\, e^{-2\,\mathrm{osc}\,\widetilde{\varphi}}, \end{aligned}$$

applying (2.7.7) and (2.7.22) at the last step. Thus we arrive at (2.8.34), and obtain (2.8.31) with $b_0 > 0$. The result for $b_0 < 0$ follows from the case $b_0 > 0$ and (2.6.14). □

If b is a continuous periodic magnetic field with non-vanishing mean value b_0, the fact that the zero eigenvalue is isolated in the spectrum of $H_P(A,0)$ was noticed in [66] without proof, and was proved later in [14, Example 6]. If b is periodic and $b_0 = 0$, is shown in [21] that $\sigma(H_P(A,0))$ is a.c.

If b with $b_0 \neq 0$, is only pre-admissible, we have

$$b(\mathbf{x}) = \frac{b_0 |\mathbf{x}|^2}{4}(1 + o(1)), \quad |\mathbf{x}| \to \infty,$$

instead of (2.8.27). This, however, is sufficient to obtain the following result which is analogous to (2.8.29) - (2.8.30) with $b_0 \neq 0$:

Proposition 2.8.4 *Assume that* $b = b_0 + \widetilde{b}$ *with* $b_0 \neq 0$ *is a pre-admissible magnetic field. Then we have*

$$\dim \operatorname{Ker} a(b) = \begin{cases} \infty & \textit{if } b_0 > 0, \\ 0 & \textit{if } b_0 < 0, \end{cases} \quad \dim \operatorname{Ker} a^*(b) = \begin{cases} \infty & \textit{if } b_0 < 0, \\ 0 & \textit{if } b_0 > 0, \end{cases}$$

and, hence,

$$\dim \operatorname{Ker} H_P(A, 0) = \infty.$$

The result of Proposition 2.8.4 follows also from [170, Theorem 3.11]. If, under the hypotheses of this proposition, there exists no bounded solution of (2.8.18), then estimate (2.8.34) is not applicable. However, [170, Theorem 3.11] implies that in this case still there is a gap in $\sigma(H_P(A,0))$ adjoining the origin.
If b is a pre-admissible magnetic field with $b_0 = 0$, which is not admissible, then the situation is drastically different from (2.8.29) - (2.8.30) with $b_0 = 0$. Namely, Propositions 2.8.5 and 2.8.6 below contain examples of $b \in WAP(\mathbb{R}^2;\mathbb{R})$ with $b_0 = 0$ which do not satisfy (2.8.28) and are not admissible but for which $\dim \operatorname{Ker} H_P(A,0)$ can be equal to infinity, or to any given natural number. Let

$$C > 0, \quad K \in \mathbb{N}, \quad \gamma_k \in \mathbb{S}^1, \quad k = 1, \dots, K,$$

with $\gamma_k \neq \gamma_\ell$ if $k \neq \ell$, and

$$s > 1, \quad t > 0, \quad s - 2t \leq 1.$$

We will consider magnetic fields of the form

$$b(x) = C \sum_{k=1}^{K} \sum_{n=1}^{\infty} n^{-s} \cos(n^{-t} \gamma_k \cdot x), \quad x \in \mathbb{R}^2. \tag{2.8.36}$$

Then, $b \in WAP(\mathbb{R}^2;\mathbb{R})$ but it does not satisfy (2.8.28). Moreover, (2.8.25) yields

$$\varphi(\mathbf{x}) = \widetilde{\varphi}(\mathbf{x}) = 2C \sum_{k=1}^{K} g_{s,t}(|\gamma_k \cdot \mathbf{x}|/2), \quad \mathbf{x} \in \mathbb{R}^2,$$

where

$$g_{s,t}(r) := \sum_{n=1}^{\infty} n^{-s+2t} \sin^2(n^{-t} r), \quad r \geq 0.$$

Evidently,

$$0 \leq g_{s,t}(r) \leq \zeta(s) r^2, \quad r \geq 0,$$

where ζ is the Riemann zeta function, and $g_{s,t}(r) = 0$ if and only if $r = 0$. Note that $g_{s,t}$ is represented by *a Dirichlet series*, and extends to an entire function on the complex plane. More precisely, we have

$$g_{s,t}(z) = 2z^2 \sum_{n=0}^{\infty} \frac{(-4)^n \zeta(s + 2nt)}{(2(n+1))!} z^{2n}, \quad z \in \mathbb{C}.$$

Proposition 2.8.5 [25, Theorem 4.1] *Suppose that* b *has the form* (2.8.36) *with*

$$s > 1, \quad s-2t < 1, \quad C = 1, \quad K = 2, \quad \gamma_1 = (1,0), \quad \gamma_2 = (0,1).$$

Then,

$$\dim \mathrm{Ker}\, a(b) = \infty, \quad \dim \mathrm{Ker}\, a^*(b) = 0,$$

and, hence,

$$\dim \mathrm{Ker}\, H_P(A,0) = \infty.$$

We do not know yet whether the zero eigenvalue of $H_P(A,0)$ is isolated in $\sigma(H_P(A,0))$ under the general hypotheses of Proposition 2.8.5. However, [25, Proposition 4.7] implies that there is no gap if $s-t > 1$.
The proof of Proposition 2.8.5 is based on arguments similar to those used in the proof of Proposition 2.8.4, and the following lemma:

Lemma 2.8.1 [25, Proposition 4.3] *If* $s > 1$ *and* $s-2t < 1$, *then*

$$g_{s,t}(r) = C_{s,t} r^{\frac{-s+2t+1}{t}}(1+o(1)), \quad r \to \infty,$$

where $C_{s,t} := \frac{1}{t}\int_0^\infty u^{\frac{s-3t-1}{t}} \sin^2(u)\, du$.

Further, we consider the border-line case $s = 1+2t$ where $g_{1+2t,t}(r)$ has a logarithmic growth as $r \to \infty$, and Ker a(b) is finite-dimensional but generically non-trivial.

Proposition 2.8.6 [25, Theorem 4.2] *Assume that* b *has the form* (2.8.36) *with* $t > 0$, $s = 1+2t$,

$$C = \frac{1}{K}, \quad \gamma_k = (\cos\theta_k, \sin\theta_k), \quad \theta_k = \frac{2\pi k}{K}, \quad k = 1, \ldots, K. \tag{2.8.37}$$

Moreover, suppose that $t^{-1} \notin \mathbb{N}$, $K \geq 3$ *is odd, and*

$$\lfloor t^{-1} \rfloor < \frac{K-1}{Kt} < \frac{K+1}{Kt} < \lfloor t^{-1} \rfloor + 1. \tag{2.8.38}$$

Then $\dim \mathrm{Ker}\, a(b) = \lfloor t^{-1} \rfloor$, $\dim \mathrm{Ker}\, a^*(b) = 0$, *and hence,*

$$\dim \mathrm{Ker}\, H_P(A,0) = \lfloor t^{-1} \rfloor.$$

The ergodic properties of $H_P(A,0)$ (see [25, Corollary 3.1]) imply that the zero eigenvalue of $H_P(A,0)$ is not isolated in $\sigma(H_P(A,0))$.
Proposition 2.8.6 is valid under much more general hypotheses on the family $\{\gamma_k\}_{k=1}^K \subset \mathbb{S}^1$, In particular, if γ_k are defined as in (2.8.37), we can assume that $K \geq 4$ is even, replacing the numbers $K \pm 1$ in (2.8.38) by $K \pm 2$.
The proof of Proposition 2.8.6 is based on arguments similar to those applied in the proof of Proposition 2.8.1, and the following highly non-trivial lemma:

Lemma 2.8.2 [25, Proposition 4.5] *Let* $t > 0$. *Then we have*

$$g_{1+2t,t}(r) = \frac{1}{2t} \ln r(1+o(1)), \quad r \to \infty.$$

Let now $n = 3$. Again, we write $\mathbf{x} \in \mathbb{R}^3$ as $\mathbf{x} = (x_\perp, x_\|)$ with $x_\perp = (x,y) \in \mathbb{R}^2$ and $x_\| = w \in \mathbb{R}$. Let us consider a magnetic field $\mathbf{B}$ of a constant direction. i.e.

$$\mathbf{B} = (0,0,b). \tag{2.8.39}$$

In this case div $\mathbf{B} = 0$ is equivalent to $\frac{\partial b}{\partial w} = 0$. Therefore, we should have $b = b(x_\perp)$. Then there exists a magnetic potential

$$A = A(x_\perp) = (A_1, A_2, 0),$$

such that $\operatorname{curl} A = \mathbf{B}$. For example, A can be chosen as in (2.1.8). Alternatively, the components A_j, $j = 1,2$, can be defined by (2.8.2). The resulting magnetic potentials will not be necessarily identical, but they will be gauge equivalent. Hence,

$$H_P(A,0) = \begin{pmatrix} H_P^- & 0 \\ 0 & H_P^+ \end{pmatrix} \tag{2.8.40}$$

where, similarly to (2.7.57),

$$H_P^\pm := H_\perp^\pm \otimes I_\| + I_\perp \otimes H_\|, \tag{2.8.41}$$

with

$$H_\perp^- := a^* a, \quad H_\perp^+ := a^* a,$$

the operators a and a^* being defined in (2.6.6) and (2.6.7) respectively,

$$H_\| := -\frac{\partial^2}{\partial w^2},$$

and $I_\perp$, $I_\|$, being the identities in $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R})$ respectively. Set

$$M^\pm := \inf \sigma(H_P^\pm), \quad M := \min\{M^-, M^+\}.$$

Proposition 2.8.7 *Let* $n = 3$. *Assume that* $\mathbf{B}$ *has the form (2.8.39). Then the spectra of the operators* $H_P^\pm$ *are a.c. and*

$$\sigma(H_P^\pm(A,0)) = [M^\pm, \infty).$$

Hence, the spectrum of $H_P(A,0)$ *is a.c. and*

$$\sigma(H_P(A,0)) = [M, \infty).$$

Proof. The claims concerning the operators $H_P^\pm$ follow from (2.8.41), the fact that $\sigma(H_\|) = [0,\infty)$ is a.c., and Lemma 2.7.2. The claims about $H_P(A,0)$ now follow from (2.8.40) which implies

$$\sigma(H_P(A,0)) = \sigma(H_P^-) \cup \sigma(H_P^+),$$

and analogous identities for the spectral components of $H_P(A,0)$. □

The spectrum of the 3D Pauli operator $H_P(A,0)$ can be purely a.c. also for magnetic fields the direction of which is not constant. For instance, it follows from the results of [199] that this is the case for periodic and sufficiently regular A. On the other hand, [122] and [72] contain examples of magnetic potentials generating magnetic fields **B** of variable direction such that Ker $\Sigma(A) \neq \{0\}$, and hence Ker $H_P(A,0) \neq \{0\}$ which implies in particular that $\sigma(H_P(A,0))$ is not purely a.c. The problem of describing of Ker $H_P(A,0)$ for general A in the 3D case remains open and is considerably more difficult than in the 2D case. Various partial results on the zero modes of the 3D Pauli operator $H_P(A,0)$ can be found in [9] and [69].

Most of the results of this section admit straightforward extensions concerning the 2D and 3D Dirac operators $H_D(A,0)$ with variable magnetic fields. We do not state explicitly the corresponding claims because they follow quite easily from the relations between $H_P(A,0)$ and $H_D(A,0)$ established in Section 2.7.

Chapter 3

Berezin-Toeplitz operators

Abstract: As we will see in the next chapter, the spectral analysis of the magnetic operators leads naturally to the study of Berezin-Toeplitz operators. They are compact operators who will play the role of an effective Hamiltonian in the study of the spectral distribution for magnetic hamiltonians near thresholds (the Landau levels in general). We will also see that the anti-Wick quantization is sometimes appropriate to these spectral analysis. In Section 3.1 we introduce a general class of operators with contravariant symbols. Then we discuss the special case of Berezin-Toeplitz operators in general holomorphic spaces in Section of 3.2 and in Fock-Segal-Bargmann spaces in Section 3.3. This last class of Berezin-Toeplitz operators will play an important role for quantum hamiltonians with constant magnetic fields. The other particular class of operators with contravariant symbols, introduced in Section 3.4, is a class of generalized anti-Wick pseudodifferential operators. We end this chapter with Section 3.5 where we conduct a sharp study of Berezin-Toeplitz operators useful for the high energy asymptotic study of eigenvalue clusters. This chapter contains fundamental results for the sequel, but, for a first reading, some technical and theorical parts of this chapter can be skipped.

3.1 Operators with contravariant symbols

The aim of this chapter is to introduce the Berezin-Toeplitz operators which play an essential role in the spectral analysis of the magnetic quantum Hamiltonians considered in this book. The Berezin-Toeplitz operators are an important special case of a more general class of *operators with contravariant symbols*, which we describe in this section, adhering mainly to [12], [13, Chapter V, Section 2], and [185, Section 24].
Let $\mathfrak{H}$ be a separable Hilbert space with scalar product $\langle \cdot,\cdot \rangle_{\mathfrak{H}}$ and norm $\|\cdot\|_{\mathfrak{H}}$, and let M be a space with σ-finite measure. Assume that the family $\{\mathrm{f_m}\}_{\mathrm{m}\in \mathrm{M}} \subset \mathfrak{H}$ satisfies the following two conditions:

- for every $\mathrm{f} \in \mathfrak{H}$ the function $\mathrm{M} \ni \mathrm{m} \mapsto \langle \mathrm{f},\mathrm{f_m}\rangle_{\mathfrak{H}} \in \mathbb{C}$ is μ-measurable, and

$$\|\mathrm{f}\|_{\mathfrak{H}}^2 = \int_{\mathrm{M}} |\langle \mathrm{f},\mathrm{f_m}\rangle|^2 d\mu(\mathrm{m}). \tag{3.1.1}$$

- for each $m \in M$ we have

$$\|f_m\|_{\mathfrak{H}} = 1; \tag{3.1.2}$$

In accordance with [13, Chapter V, Section 2], we will call $\{f_m\}_{m\in M}$ a *system, overcomplete with respect to the measure* μ.

Suppose that $\mathscr{F} \in L^\infty(M;d\mu)+L^1(M;d\mu)$ and introduce the sesquilinear form

$$\mathscr{C}(f,g) := \int_M \mathscr{F}(m)\,\langle f,f_m\rangle_{\mathfrak{H}}\,\overline{\langle g,f_m\rangle_{\mathfrak{H}}}\,d\mu(m), \quad f,g \in \mathfrak{H}. \tag{3.1.3}$$

It is easy to check that $\mathscr{C}$ is bounded in $\mathfrak{H}$. Indeed, if $\mathscr{F} \in L^\infty(M;d\mu)$, then by the Cauchy–Schwarz inequality and (3.1.1) we get

$$\begin{aligned} |\mathscr{C}(f,g)| &\le \|\mathscr{F}\|_{L^\infty(M;d\mu)} \int_M |\langle f,f_m\rangle_{\mathfrak{H}}|\,|\langle g,f_m\rangle_{\mathfrak{H}}|\,d\mu(m) \\ &\le \|\mathscr{F}\|_{L^\infty(M;d\mu)}\,\|f\|_{\mathfrak{H}}\,\|g\|_{\mathfrak{H}}. \end{aligned} \tag{3.1.4}$$

If $\mathscr{F} \in L^1(M;d\mu)$, then (3.1.2) yields

$$\begin{aligned} |\mathscr{C}(f,g)| &\le \|\mathscr{F}\|_{L^1(M;d\mu)} \sup_{m\in M} |\langle f,f_m\rangle_{\mathfrak{H}}\,\langle f_m,g\rangle_{\mathfrak{H}}| \\ &\le \|\mathscr{F}\|_{L^1(M;d\mu)}\,\|f\|_{\mathfrak{H}}\,\|g\|_{\mathfrak{H}}. \end{aligned} \tag{3.1.5}$$

We will call the linear bounded operator generated in $\mathfrak{H}$ by the sesquilinear form (3.1.3), an *operator with contravariant symbol* $\mathscr{F}$, and will denote it by $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})$. In other words, if

$$\Pi_m := \langle\cdot,f_m\rangle_{\mathfrak{H}}\,f_m, \quad m \in M,$$

is the rank-one orthogonal projection onto the subspace generated by f_m, then

$$\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}) = \int_M \mathscr{F}(m)\,\Pi_m\,d\mu(m), \tag{3.1.6}$$

the integral being understood in the weak sense.
By (3.1.1) and (3.1.6), we have $\mathrm{Op}^{\mathrm{cnv}}(1) = I$. Note also that (3.1.6) implies

$$\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})^* = \mathrm{Op}^{\mathrm{cnv}}(\overline{\mathscr{F}}).$$

In particular, $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})$ is self-adjoint if $\mathscr{F}$ is real-valued. Moreover, it follows from (3.1.6) that $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})$ is monotone with respect to $\mathscr{F}$, i.e. $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}) \ge 0$ if $\mathscr{F}(m) \ge 0$ for μ-almost every $m \in M$.
Further, if $\mathscr{F} \in L^\infty(M;d\mu)$, then by (3.1.4),

$$\|\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})\| \le \|\mathscr{F}\|_{L^\infty(M;d\mu)}. \tag{3.1.7}$$

Of course, if $\mathscr{F} \in L^1(M;d\mu)$, then (3.1.5) implies an analogous estimate where the L^∞-norm of $\mathscr{F}$ is replaced by its L^1-norm. However, we will show that in this case $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})$ is not only bounded but also a trace-class operator, and

$$\|\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})\|_1 \le \|\mathscr{F}\|_{L^1(M;d\mu)}. \tag{3.1.8}$$

To this end, define the operator $\mathscr{I}_{\mathscr{F}} : \mathfrak{H} \to L^2(M; d\mu)$ by

$$(\mathscr{I}_{\mathscr{F}} f)(m) := |\mathscr{F}(m)|^{1/2} \langle f, f_m \rangle_{\mathfrak{H}}, \quad m \in M.$$

If $\{\varphi_j\}$ is an orthonormal basis of $\mathfrak{H}$, it is easy to see that

$$\|\mathscr{I}_{\mathscr{F}}\|_2^2 = \sum_j \int_M |\langle \varphi_j, f_m \rangle_{\mathfrak{H}}|^2 |\mathscr{F}(m)| \, d\mu(m) = \|\mathscr{F}\|_{L^1(M;d\mu)}. \qquad (3.1.9)$$

Moreover, we have

$$\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}) = \mathscr{I}_{\mathscr{F}}^* \, e^{i \arg \mathscr{F}} \, \mathscr{I}_{\mathscr{F}}. \qquad (3.1.10)$$

Now, (3.1.9) and (3.1.10) imply (3.1.8). It is worth noting that if $\mathscr{F} \geq 0$, then we have an equality in (3.1.8), i.e.

$$\|\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})\|_1 = \mathrm{Tr}\, \mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}) = \int_M \mathscr{F}(m) \, d\mu(m) = \|\mathscr{F}\|_{L^1(M;d\mu)}. \qquad (3.1.11)$$

Summarizing (3.1.7) and (3.1.8) and interpolating between these two estimates with the aid of [19, Theorem 3.1], we obtain the following

Proposition 3.1.1 (i) *Let* $\mathscr{F} \in L^\infty(M; d\mu)$. *Then* $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}) \in \mathfrak{B}(\mathfrak{H})$*, and* (3.1.7) *holds true.*
(ii) *Let* $\mathscr{F} \in L^p(M; d\mu)$*,* $p \in [1, \infty)$. *Then* $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}) \in \mathfrak{S}_p(\mathfrak{H})$*, and*

$$\|\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})\|_p \leq \|\mathscr{F}\|_{L^p(M;d\mu)}. \qquad (3.1.12)$$

(iii) *Let* $\mathscr{F} \in L^p_w(M; d\mu)$*,* $p \in (1, \infty)$. *Then* $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}) \in \mathfrak{S}_{p,w}(\mathfrak{H})$*, and*

$$\|\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})\|_{p,w} \leq \|\mathscr{F}\|_{L^p_w(M;d\mu)}. \qquad (3.1.13)$$

A simple *sufficient* condition for the compactness of $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})$ follows from Proposition 3.1.1. Here and in the sequel, we write $\mathscr{F} \in L^1_\varepsilon(M; d\mu)$ if for any $\varepsilon > 0$ we can represent $\mathscr{F}$ as the sum

$$\mathscr{F} = \mathscr{F}_1 + \mathscr{F}_2 \qquad (3.1.14)$$

with $\mathscr{F} \in L^1(M; d\mu)$ and $\|\mathscr{F}\|_{L^\infty(M;d\mu)} < \varepsilon$.

Corollary 3.1.1 *Let* $\mathscr{F} \in L^1_\varepsilon(M; d\mu)$. *Then* $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}) \in \mathfrak{S}_\infty(\mathfrak{H})$.

Proof. Pick any $\varepsilon > 0$ and write $\mathscr{F}$ as the sum (3.1.14). Then $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}_1) \in \mathfrak{S}_1(\mathfrak{H}) \subset \mathfrak{S}_\infty(\mathfrak{H})$, and

$$\|\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}) - \mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}_1)\| = \|\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}_2)\| \leq \|\mathscr{F}\|_{L^\infty(M;d\mu)} < \varepsilon.$$

Thus, $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})$ can be approximated arbitrarily well in norm by a compact operator. Therefore, $\mathrm{Op}^{\mathrm{cnv}}(\mathscr{F}) \in \mathfrak{S}_\infty(\mathfrak{H})$. □

Next, let $T \in \mathfrak{B}(\mathfrak{H})$. We define the *covariant symbol* $\mathscr{F}^{\mathrm{cov}}(T)$ of T by

$$\left(\mathscr{F}^{\mathrm{cov}}(T)\right)(m) := \langle T f_m, f_m \rangle_{\mathfrak{H}}, \quad m \in M. \qquad (3.1.15)$$

Evidently, if $T = T^*$ (resp., $T \geq 0$), then $\mathscr{F}^{\rm cov}(T)$ is real-valued (resp., $(\mathscr{F}^{\rm cov}(T))(m) \geq 0$ for every $m \in M$).
If $T = \mathrm{Op}^{\rm cnv}(\mathscr{F})$, then the contravariant symbol $\mathscr{F}$ of the operator T and its covariant symbol $\mathscr{F}^{\rm cov} = \mathscr{F}^{\rm cov}(\mathrm{Op}^{\rm cnv}(\mathscr{F}))$ are related by

$$\mathscr{F}^{\rm cov}(m) = \int_M |\langle f_m, f_{m'}\rangle_{\mathfrak{H}}|^2 \mathscr{F}(m')\, d\mu(m'), \quad m \in M. \tag{3.1.16}$$

If $\mathscr{F} \in L^1(M; d\mu)$, then (3.1.16) implies $\mathscr{F}^{\rm cov} \in L^1(M; d\mu)$, and

$$\int_M \mathscr{F}^{\rm cov}(m) d\mu(m) = \int_M \mathscr{F}(m) d\mu(m). \tag{3.1.17}$$

Further, obviously, we have

$$\sup_{m\in M} \left|\left(\mathscr{F}^{\rm cov}(T)\right)(m)\right| \leq \sup_{f\in\mathfrak{H}: \|f\|_{\mathscr{H}}=1} |\langle Tf, f\rangle_{\mathfrak{H}}| \leq \|T\|. \tag{3.1.18}$$

We will show now that if $T \in \mathfrak{S}_1(\mathfrak{H})$, then $\mathscr{F}^{\rm cov}(T) \in L^1(M; d\mu)$ and

$$\|\mathscr{F}^{\rm cov}(T)\|_{L^1(M;d\mu)} \leq \|T\|_1. \tag{3.1.19}$$

Assume $T \in \mathfrak{S}_1(\mathfrak{H})$, and $T \neq 0$. Let, as above, $\{s_j\}_{j=1}^{\mathrm{rank}\, T}$ be the non-increasing set of the non-zero singular numbers of T, and let

$$T = \sum_j s_j \langle \cdot, \varphi_j\rangle_{\mathfrak{H}} \psi_j$$

be its canonic Schmidt representation where $\{\varphi_j\}_j$ and $\{\psi_j\}_j$ are two orthonormal systems in $\mathfrak{H}$ (see e.g. [**?**, Theorem VI.17]). Then we have

$$\langle Tf_m, f_m\rangle_{\mathfrak{H}} = \sum_j s_j \langle f_m, \varphi_j\rangle_{\mathfrak{H}} \langle \psi_j, f_m\rangle_{\mathfrak{H}},$$

so that

$$|\langle Tf_m, f_m\rangle_{\mathfrak{H}}| \leq \sum_j s_j \frac{|\langle f_m, \varphi_j\rangle_{\mathfrak{H}}|^2 + |\langle f_m, \psi_j\rangle_{\mathfrak{H}}|^2}{2}, \quad m \in M.$$

Integrating with respect to $m \in M$, and taking into account (3.1.1), we obtain

$$\int_M |\langle Tf_m, f_m\rangle_{\mathfrak{H}}|\, d\mu(m) \leq \sum_j s_j,$$

which is identical with (3.1.19). Now, similarly to Proposition 3.1.1, we find that (3.1.18) and (3.1.19) entail the following

Proposition 3.1.2 (i) *Let* $T \in \mathfrak{B}(\mathfrak{H})$. *Then* $\mathscr{F}^{\rm cov} \in L^\infty(M; d\mu)$*, and*

$$\|\mathscr{F}^{\rm cov}(T)\|_{L^\infty(M;d\mu)} \leq \|T\|. \tag{3.1.20}$$

(ii) *Let* $T \in \mathfrak{S}_p(\mathfrak{H})$, $p \in [1,\infty)$. *Then* $\mathscr{F}^{cov}(T) \in L^p(M;d\mu)$, *and*

$$\|\mathscr{F}^{cov}(T)\|_{L^p(M;d\mu)} \le \|T\|_p. \tag{3.1.21}$$

(iii) *Let* $T \in \mathfrak{S}_{p,w}(\mathfrak{H})$, $p \in (1,\infty)$. *Then* $\mathscr{F}^{cov}(T) \in L^p_w(M;d\mu)$, *and*

$$\|\mathscr{F}^{cov}(T)\|_{L^p_w(M;d\mu)} \le \|T\|_{p,w}. \tag{3.1.22}$$

In addition to Corollary 3.1.1, we have the following two *necessary* conditions for the compactness of the operator T:

Corollary 3.1.2 *Let* $T \in \mathfrak{S}_\infty(\mathfrak{H})$. *Then*

$$\mu\left(\left\{m \in M \,|\, \left|\left(\mathscr{F}^{cov}(T)\right)(m)\right| > \varepsilon\right\}\right) < \infty \tag{3.1.23}$$

for every $\varepsilon > 0$.

Proof. Fix $\varepsilon > 0$ and write $T = T_1 + T_2$ where $\operatorname{rank} T_1 < \infty$ and $\|T_2\| < \varepsilon/2$. Then

$$\mathscr{F}^{cov}(T) = \mathscr{F}^{cov}(T_1) + \mathscr{F}^{cov}(T_2),$$

and

$$\begin{aligned} &\mu\left(\left\{m \in M \,|\, \left|\left(\mathscr{F}^{cov}(T)\right)(m)\right| > \varepsilon\right\}\right) \\ &\le \mu\left(\left\{m \in M \,|\, \left|\left(\mathscr{F}^{cov}(T_1)\right)(m)\right| > \varepsilon/2\right\}\right) \\ &+\mu\left(\left\{m \in M \,|\, \left|\left(\mathscr{F}^{cov}(T_2)\right)(m)\right| > \varepsilon/2\right\}\right). \end{aligned} \tag{3.1.24}$$

Since $T_1 \in \mathfrak{B}(\mathfrak{H})$ and $\operatorname{rank} T_1 < \infty$, we have $T_1 \in \mathfrak{S}_1(\mathfrak{H})$. Then, by (3.1.22) with $p = 1$, we have $\mathscr{F}^{cov}(T_1) \in L^1(M;d\mu)$, and, hence,

$$\mu\left(\left\{m \in M \,|\, \left|\left(\mathscr{F}^{cov}(T_1)\right)(m)\right| > \varepsilon/2\right\}\right) < \infty. \tag{3.1.25}$$

On the other hand, (3.1.20) implies

$$\|\mathscr{F}^{cov}(T_2)\|_{L^\infty(M;d\mu)} \le \|T_2\| < \varepsilon/2.$$

so that

$$\mu\left(\left\{m \in M \,|\, \left|\left(\mathscr{F}^{cov}(T_2)\right)(m)\right| > \varepsilon/2\right\}\right) = 0. \tag{3.1.26}$$

Putting together (3.1.24) – (3.1.26), we obtain (3.1.23). □

Corollary 3.1.3 *Assume that there exists a sequence* $\{m_k\}_{k\in\mathbb{N}} \subset M$ *such that*

$$w-\lim_{k\to\infty} f_{m_k} = 0. \tag{3.1.27}$$

Suppose that $T \in \mathfrak{S}_\infty(\mathfrak{H})$. *Then*

$$\lim_{k\to\infty} (\mathscr{F}^{cov}(T))(m_k) = 0. \tag{3.1.28}$$

Proof. Since $\|f_{m_k}\|_{\mathfrak{H}} = 1$, we have

$$|(\mathscr{F}^{cov}(T))(m_k)| \leq \|Tf_{m_k}\|_{\mathfrak{H}}. \tag{3.1.29}$$

Since the operator T is compact, (3.1.27) implies

$$\lim_{k\to\infty} \|Tf_{m_k}\|_{\mathfrak{H}} = 0. \tag{3.1.30}$$

Now, (3.1.28) follows from (3.1.29) and (3.1.30). □

Proposition 3.1.1 on the one hand, and Proposition 3.1.2 on the other, suggest the general wisdom that the operator norm and the Schatten-von Neumann norms of $\mathrm{Op}^{cnv}(\mathscr{F})$ are *upper bounded* by the norms of the *contravariant symbol* $\mathscr{F}$ in appropriate Lebesgue spaces, while the norms of the operator T are *lower bounded* by the norms of the *covariant symbol* $\mathscr{F}^{cov}(T)$.
A natural question arises whether the lower or the upper bounds are sharper. We will give a partial answer to this question in the following few sections where we will consider the Berezin-Toeplitz operators and the operators with anti-Wick symbols. We will see that in many cases the lower bound (3.1.20) of the operator norm involving the covariant symbol is sharper than the upper bound (3.1.7) involving the contravariant one. On the other hand, if $\mathscr{F} \geq 0$, we have an equality in (3.1.8) so that this estimate is sharp. From this point of view, upper bounds (3.1.12) with $p > 1$, and (3.1.13), are obtained by interpolation between a sharp estimate which corresponds to $p = 1$, and an estimate corresponding to $p = \infty$, which may turn not to be sharp.
The definition of the operator $\mathrm{Op}^{cnv}(\mathscr{F})$ is possible for much more general symbols than $\mathscr{F} \in L^1(M; d\mu) + L^\infty(M; d\mu)$ but the resulting operators may turn to be unbounded and, hence, not defined on the entire space $\mathfrak{H}$. Similarly, the covariant symbol $\mathscr{F}^{cov}(T)$ could be introduced not only for bounded T operators but in the case of unbounded T one should ensure that $f_m \in \mathfrak{D}(T)$. More comments on these issues can be found in the following few sections where we consider special examples of operators with contravariant symbols.

3.2 Berezin–Toeplitz operators in general holomorphic spaces

In this and the next section we introduce the Berezin-Toeplitz operators in holomorphic spaces and their generalizations, and describe those of their properties which are needed in the study of their role as effective Hamiltonians in the asymptotic spectral analysis of the magnetic Schrödinger, Pauli, and Dirac operators. Our brief introduction to the Berezin-Toeplitz operators will be subordinated to these applications, so will omit various interesting aspects of their general theory. We refer the reader to monographs like [220], [29], [27], and [115], containing a wealth of information from this theory which, for absence of space, has not found place in our book.
Let Ω be a domain, i.e. an open, connected, non empty set in $\mathbb{C}^d \cong \mathbb{R}^{2d}$, $d \geq 1$, and let $\alpha(\mathbf{z}) > 0$, $\mathbf{z} \in \Omega$, be a continuous function. Denote by $d\lambda$ the 2d-dimensional Lebesgue

measure on Ω. Note that the measures $d\lambda$ and $\alpha d\lambda$ are equivalent. Put

$$.\mathscr{A}(\Omega;\alpha) := \left\{ u \in \mathrm{Hol}(\Omega) \mid \int_\Omega |u(\mathbf{z})|^2 \alpha(\mathbf{z})\, d\lambda(\mathbf{z}) < \infty \right\} \tag{3.2.1}$$

where $\mathrm{Hol}(\Omega)$ stands for the set of holomorphic functions on Ω. Thus, $\mathscr{A}(\Omega;\alpha)$ is a subspace of $L^2(\Omega;\alpha d\lambda)$ which we will call a *holomorphic space* over Ω. We introduce also the *anti-holomorphic* space $\mathscr{A}^*(\Omega;\alpha)$, replacing in (3.2.1) $\mathrm{Hol}(\Omega)$ by the set of anti-holomorphic functions on Ω. If $\alpha = 1$, we will write $\mathscr{A}(\Omega)$ and $\mathscr{A}^*(\Omega)$ instead of $\mathscr{A}(\Omega;1)$ and $\mathscr{A}^*(\Omega;1)$ respectively

Using the mean-value formula for holomorphic functions, we find that for any fixed $\mathbf{z} \in \Omega$ there exists its neighborhood $U \subset \Omega$ and a constant $c_{\mathbf{z}} \geq 0$ such that

$$|u(\boldsymbol{\zeta})| \leq c_{\mathbf{z}} \|u\|_{L^2(\Omega;\alpha d\lambda)}, \quad \boldsymbol{\zeta} \in U, \tag{3.2.2}$$

for every $u \in \mathscr{A}(\Omega;\alpha)$, which easily implies that $\mathscr{A}(\Omega;\alpha)$ is a *closed* subspace of $L^2(\Omega;\alpha d\lambda)$ (see e.g. [87, Theorem 2.2]). Let P be the orthogonal projection onto $\mathscr{A}(\Omega;\alpha)$. Then P admits an integral kernel $\mathscr{R}(\mathbf{z},\boldsymbol{\zeta}) = \overline{\mathscr{R}(\boldsymbol{\zeta},\mathbf{z})}$, $\mathbf{z},\boldsymbol{\zeta} \in \Omega$, which is holomorphic with respect to $\mathbf{z}$, and anti-holomorphic with respect to $\boldsymbol{\zeta}$ (see e.g. [87, Theorem 2.3]). Then

$$u(\mathbf{z}) = \int_\Omega \mathscr{R}(\mathbf{z},\boldsymbol{\zeta}) u(\boldsymbol{\zeta}) \alpha(\boldsymbol{\zeta}) d\lambda(\boldsymbol{\zeta}), \quad \mathbf{z} \in \Omega,$$

for every $u \in \mathscr{A}(\Omega;\alpha)$. In particular,

$$\mathscr{R}(\mathbf{z},\boldsymbol{\zeta}) = \int_\Omega \mathscr{R}(\mathbf{z},\mathbf{z}')\mathscr{R}(\mathbf{z}',\boldsymbol{\zeta})\alpha(\mathbf{z}')d\lambda(\mathbf{z}'), \quad \mathbf{z},\boldsymbol{\zeta} \in \Omega.$$

The function $\mathscr{R}$ is called the *reproducing kernel* of the holomorphic space $\mathscr{A}(\Omega;\alpha)$. Then, of course, $\overline{\mathscr{R}}$ is the reproducing kernel of the anti-holomorphic space $\mathscr{A}^*(\Omega;\alpha)$. Generally speaking, we can have $\mathscr{A}(\Omega;\alpha) = \{0\}$; this is the case, for example, if $\Omega = \mathbb{C}^d$ and $\alpha = 1$. Assume that $\mathscr{A}(\Omega;\alpha)$ is not a null space. Let $\{\varphi_j\}$ be an orthonormal basis in $\mathscr{A}(\Omega;\alpha)$. Then

$$\mathscr{R}(\mathbf{z},\boldsymbol{\zeta}) = \sum_j \varphi_j(\mathbf{z})\overline{\varphi_j(\boldsymbol{\zeta})}, \quad \mathbf{z},\boldsymbol{\zeta} \in \Omega, \tag{3.2.3}$$

the series being locally uniformly absolutely convergent in $\Omega \times \Omega$. For $\mathbf{z} \in \Omega$ put

$$\rho(\mathbf{z}) := \mathscr{R}(\mathbf{z},\mathbf{z}) = \int_\Omega |\mathscr{R}(\mathbf{z},\boldsymbol{\zeta})|^2 \alpha(\boldsymbol{\zeta}) d\lambda(\boldsymbol{\zeta}). \tag{3.2.4}$$

Then, (3.2.3) implies that

$$|\mathscr{R}(\mathbf{z},\boldsymbol{\zeta})| \leq \rho(\mathbf{z})^{1/2}\rho(\boldsymbol{\zeta})^{1/2}, \quad \mathbf{z},\boldsymbol{\zeta} \in \Omega. \tag{3.2.5}$$

Next, evidently the multiplier by $\alpha^{1/2}$ is a unitary operator from $L^2(\Omega;\alpha d\lambda)$ onto $L^2(\Omega) = L^2(\Omega;d\lambda)$. Set

$$\widetilde{\mathscr{A}}(\Omega;\alpha) = \alpha^{1/2}\mathscr{A}(\Omega;\alpha).$$

Then $\widetilde{\mathscr{A}}(\Omega;\alpha)$ is a closed subspace of $L^2(\Omega)$, which we will call a *weighted holomorphic space*. Denote by $\widetilde{P} : L^2(\Omega) \to L^2(\Omega)$ the orthogonal projection onto $\widetilde{\mathscr{A}}(\Omega;\alpha)$. Obviously,

$$\widetilde{P} = \alpha^{1/2} P \alpha^{-1/2}$$

where, as above, $\mathrm{P} : \mathrm{L}^2(\Omega;\alpha d\lambda) \to \mathrm{L}^2(\Omega;\alpha d\lambda)$ is the orthogonal projection onto $\mathscr{A}(\Omega;\alpha)$. Therefore, $\widetilde{\mathrm{P}}$ has an integral kernel

$$\widetilde{\mathscr{R}}(\mathbf{z},\boldsymbol{\zeta}) := \alpha(\mathbf{z})^{1/2}\,\mathscr{R}(\mathbf{z},\boldsymbol{\zeta})\,\alpha(\boldsymbol{\zeta})^{1/2}, \quad \mathbf{z},\boldsymbol{\zeta}\in\Omega, \tag{3.2.6}$$

$\mathscr{R}$ being the reproducing kernel of $\mathscr{A}(\Omega;\alpha)$.

Let us now set

$$\mathfrak{H} = \mathscr{A}(\Omega;\alpha), \quad \mathrm{M} = \Omega,$$
$$d\mu = \rho\alpha d\lambda, \tag{3.2.7}$$

and

$$\mathrm{f}_{\mathbf{z}}(\boldsymbol{\zeta}) = \rho(\mathbf{z})^{-1/2}\,\mathscr{R}(\boldsymbol{\zeta},\mathbf{z}), \quad \mathbf{z},\boldsymbol{\zeta}\in\Omega, \tag{3.2.8}$$

the function ρ being defined in (3.2.4). Note that the measures $d\mu$ and $\alpha d\lambda$ and, hence, $d\mu$ and $d\lambda$ are equivalent. In particular,

$$\mathrm{L}^\infty(\Omega;d\mu) = \mathrm{L}^\infty(\Omega;\alpha d\lambda) = \mathrm{L}^\infty(\Omega).$$

Let us show that the system $\{\mathrm{f}_{\mathbf{z}}\}_{\mathbf{z}\in\Omega}$ is overcomplete with respect to the measure $d\mu$. First, we have

$$\|\mathrm{f}_{\mathbf{z}}\|^2_{\mathfrak{H}} = \rho(\mathbf{z})^{-1}\int_\Omega |\mathscr{R}(\boldsymbol{\zeta},\mathbf{z})|^2\,\alpha(\boldsymbol{\zeta})\,d\lambda(\boldsymbol{\zeta}) = 1, \quad \mathbf{z}\in\Omega.$$

Further,

$$\langle \mathrm{f},\mathrm{f}_{\mathbf{z}}\rangle_{\mathfrak{H}} = \rho(\mathbf{z})^{-1/2}\int_\Omega \mathscr{R}(\mathbf{z},\boldsymbol{\zeta})\,\mathrm{f}(\boldsymbol{\zeta})\,\alpha(\boldsymbol{\zeta})\,d\lambda(\boldsymbol{\zeta}) = \rho(\mathbf{z})^{-1/2}\mathrm{f}(\mathbf{z}), \quad \mathbf{z}\in\Omega,$$

so that

$$\int_\Omega |\langle \mathrm{f},\mathrm{f}_{\mathbf{z}}\rangle_{\mathfrak{H}}|^2\,d\mu(\mathbf{z}) = \int_\Omega \rho(\mathbf{z})^{-1}|\mathrm{f}(\mathbf{z})|^2\,d\mu(\mathbf{z}) = \int_\Omega |\mathrm{f}(\mathbf{z})|^2\,\alpha(\boldsymbol{\zeta})\,d\lambda(\boldsymbol{\zeta}) = \|\mathrm{f}\|^2_{\mathfrak{H}},$$

and therefore the system is overcomplete with respect to $d\mu$.
Assume $\mathscr{F}\in\mathrm{L}^\infty(\Omega;\alpha d\lambda)+\mathrm{L}^1(\Omega;\alpha d\lambda)$ and define the *Berezin-Toeplitz operator* $\mathrm{T}_{\mathscr{F}} : \mathscr{A}(\Omega;\alpha)\to\mathscr{A}(\Omega;\alpha)$ by

$$\mathrm{T}_{\mathscr{F}} := \mathrm{P}\mathscr{F} = \mathrm{P}\mathscr{F}\mathrm{P}.$$

We will call $\mathscr{F}$ the *symbol* of $\mathrm{T}_{\mathscr{F}}$. Thus, $\mathrm{T}_{\mathscr{F}}$ is the operator with contravariant symbol $\mathscr{F}$ in the particular setting where $\mathfrak{H} = \mathscr{A}(\Omega;\alpha)$, and the overcomplete system is defined as in (3.2.8). Evidently, $\mathrm{T}^*_{\mathscr{F}} = \mathrm{T}_{\overline{\mathscr{F}}}$; in particular, $\mathrm{T}_{\mathscr{F}}$ is self-adjoint if $\mathscr{F}$ is real-valued. In particular, $\mathrm{T}_1 = \mathrm{I}$ where I is the identity in $\mathscr{A}(\Omega;\alpha)$.
Next, define the *Berezin transform* $\mathscr{B}(\mathscr{F})$ of $\mathscr{F}\in\mathrm{L}^\infty(\Omega)$ by

$$(\mathscr{B}(\mathscr{F}))(\mathbf{z}) = \rho(\mathrm{z})^{-1}\int_\Omega |\mathscr{R}(\mathbf{z},\boldsymbol{\zeta})|^2\,\mathscr{F}(\boldsymbol{\zeta})\,\alpha(\boldsymbol{\zeta})\,d\lambda(\boldsymbol{\zeta}), \quad \mathbf{z}\in\Omega. \tag{3.2.9}$$

The function $\mathscr{B}(\mathscr{F})$ is well defined and bounded on Ω. Moreover,

$$\sup_{\mathbf{z}\in\Omega} |(\mathscr{B}(\mathscr{F}))(\mathbf{z})| \le \|\mathscr{F}\|_{\mathrm{L}^\infty(\Omega)}. \tag{3.2.10}$$

Putting together (3.2.4) and (3.1.15), we find that in our particular setting $\mathscr{B}(\mathscr{F})$ is nothing else than the covariant symbol of the operator $T_{\mathscr{F}}$. For the reader's convenience we translate below Propositions 3.1.1 and 3.1.2 into the language of Berezin-Toeplitz operators.

Proposition 3.2.1 (i) *If* $\mathscr{F} \in L^\infty(\Omega; d\mu)$*, then* $T_{\mathscr{F}}$ *is bounded in* $L^2(\Omega; \alpha d\lambda)$.
If $T_{\mathscr{F}}$ *is bounded in* $L^2(\Omega; \alpha d\lambda)$*, then* $\mathscr{B}(\mathscr{F}) \in L^\infty(\Omega; d\mu)$*. Moreover,*

$$\|\mathscr{B}(\mathscr{F})\|_{L^\infty(\Omega;d\mu)} \leq \|T_{\mathscr{F}}\| \leq \|\mathscr{F}\|_{L^\infty(\Omega;d\mu)}. \tag{3.2.11}$$

(ii) *Let* $p \in [1, \infty)$*. If* $\mathscr{F} \in L^p(\Omega; d\mu)$*, then* $T_{\mathscr{F}} \in \mathfrak{S}_p(L^2(\Omega; \alpha d\lambda))$.
If $T_{\mathscr{F}} \in \mathfrak{S}_p(L^2(\Omega; \alpha d\lambda))$*, then* $\mathscr{B}(\mathscr{F}) \in L^p(\Omega; d\mu)$*. Moreover,*

$$\|\mathscr{B}(\mathscr{F})\|_{L^p(\Omega;d\mu)} \leq \|T_{\mathscr{F}}\|_p \leq \|\mathscr{F}\|_{L^p(\Omega;d\mu)}. \tag{3.2.12}$$

(iii) *Let* $p \in (1, \infty)$*. If* $\mathscr{F} \in L^p_w(\Omega; d\mu)$*, then* $T_{\mathscr{F}} \in \mathfrak{S}_{p,w}(L^2(\Omega; \alpha d\lambda))$.
If $T_{\mathscr{F}} \in \mathfrak{S}_{p,w}(L^2(\Omega; \alpha d\lambda))$*, then* $\mathscr{B}(\mathscr{F}) \in L^p_w(\Omega; d\mu)$*. Moreover,*

$$\|\mathscr{B}(\mathscr{F})\|_{L^p_w(\Omega;d\mu)} \leq \|T_{\mathscr{F}}\|_{p,w} \leq \|\mathscr{F}\|_{L^p_w(\Omega;d\mu)}.$$

Define now the *weighted Berezin-Toeplitz operator* $\widetilde{T}_{\mathscr{F}} : \widetilde{\mathscr{A}}(\Omega; \alpha) \to \widetilde{\mathscr{A}}(\Omega; \alpha)$ by

$$\widetilde{T}_{\mathscr{F}} := \alpha^{1/2} T_{\mathscr{F}} \alpha^{-1/2}.$$

Evidently, the operators $T_{\mathscr{F}}$ and $\widetilde{T}_{\mathscr{F}}$ are unitarily equivalent under the unitary operator $\alpha^{1/2} : \mathscr{A}(\Omega; \alpha) \to \widetilde{\mathscr{A}}(\Omega; \alpha)$. Then Proposition 3.2.1 remains valid if we replace $T_{\mathscr{F}}$ by $\widetilde{T}_{\mathscr{F}}$, and the spaces $L^p(\Omega; \alpha d\lambda)$ with $p \in [1, \infty]$ (resp., $L^p_w(\Omega; \alpha d\lambda)$ with $p \in (1, \infty)$) by $L^p(\Omega)$ (resp., $L^p_w(\Omega)$). Note, however, that the definition of the measure $d\mu$ and of the Berezin transform $\mathscr{B}(\mathscr{F})$ of the symbol $\mathscr{F}$ remain invariant.

Our leading example of a holomorphic space is the *Fock-Segal-Bargmann space* discussed together with some of its extensions in the following section. Here we consider two other examples of independent interest.
Let us consider first the *Bergman space*. In this case Ω is the unit disk in $\mathbb{C}$, i.e.

$$\Omega = \mathbb{D} := \{z \in \mathbb{C} \,|\, |z| < 1\}, \tag{3.2.13}$$

and $\alpha = 1$. Then $\{z^j\}_{j \in \mathbb{Z}_+}$ is an orthogonal basis, and $\left\{\left(\frac{j+1}{\pi}\right)^{1/2} z^j\right\}_{j \in \mathbb{Z}_+}$ is an orthonormal basis in $\mathscr{A}(\mathbb{D})$, so that

$$\mathscr{R}(z, \zeta) = \frac{1}{\pi} \sum_{j=0}^{\infty} (j+1)(z\overline{\zeta})^j = \frac{1}{\pi}(1 - z\overline{\zeta})^{-2}, \quad z, \zeta \in \mathbb{D}, \tag{3.2.14}$$

(see [87, Subsection 3.1] for the details). A generalization of this space is the case where still $\Omega = \mathbb{D}$ but $\alpha(z) = (1 - |z|^2)^a$ with $a > -1$. Then

$$\mathscr{R}(z, \zeta) = \frac{a+1}{\pi}(1 - z\overline{\zeta})^{-a-2}, \quad z, \zeta \in \mathbb{D}.$$

Another important generalization of the Bergman space is the case where Ω is a *bounded* domain in $\mathbb{C}^d$, $d \geq 1$, and $\alpha = 1$. Using this example, we will explain now how we can define more general Berezin-Toeplitz operators whose symbols can be measures or distributions on Ω.
Let ν be a finite finite complex Borel measure on Ω. Then we define the Berezin-Toeplitz operator T_ν by

$$(T_\nu u)(\mathbf{z}) := \int_\Omega \mathscr{R}(\mathbf{z}, \boldsymbol{\zeta}) u(\boldsymbol{\zeta})\, d\nu(\boldsymbol{\zeta}), \quad \mathbf{z} \in \Omega, \quad u \in \mathscr{A}(\Omega),$$

where $\mathscr{R}$ is the reproducing kernel of $\mathscr{A}(\Omega)$. Of course, if $d\nu(\mathbf{z}) = \mathscr{F}(\mathbf{z}) d\lambda(\mathbf{z})$ with $\mathscr{F} \in L^1(\Omega)$, then $T_\nu = T_{\mathscr{F}}$. By analogy with (3.2.9), define the Berezin transform of the measure ν by

$$(\mathscr{B}(\nu))(\mathbf{z}) := \rho(\mathbf{z})^{-1} \int_\Omega |\mathscr{R}(\mathbf{z}, \boldsymbol{\zeta})|^2\, d\nu(\boldsymbol{\zeta}), \quad \mathbf{z} \in \Omega.$$

Then we have the following

Theorem 3.2.1 [219] *Let* $\Omega \subset \mathbb{C}^d$, $d \geq 1$, *be a bounded symmetric domain, and* ν *be a finite positive Borel measure on* Ω. *Then:*
(i) T_ν *is bounded in* $\mathscr{A}(\Omega)$ *if and only if* $\mathscr{B}(\nu) \in L^\infty(\Omega)$*;*
(ii) T_ν *is compact in* $\mathscr{A}(\Omega)$ *if and only if* $(\mathscr{B}(\nu))(\mathbf{z}) \to 0$ *as* $\mathbf{z} \to \partial\Omega$*;*
(iii) $T_\nu \in \mathfrak{S}_p(\mathscr{A}(\Omega))$ *with* $p \in [1, \infty)$, *if and only if* $\mathscr{B}(\nu) \in L^p(\Omega)$.

Thus we find that in the case of $\mathscr{A}(\Omega)$, the sharp bounds in (3.2.11) and (3.2.12) are the lower ones.

Let now $\phi \in \mathscr{E}'(\Omega)$. Define the Berezin-Toeplitz operator T_ϕ as the operator with integral kernel

$$\mathscr{K}_\phi(\mathbf{z}, \boldsymbol{\zeta}) := (\phi, \mathscr{R}(\mathbf{z}, \cdot)\mathscr{R}(\cdot, \boldsymbol{\zeta}))_{\mathscr{E}'(\Omega)}, \quad \mathbf{z}, \boldsymbol{\zeta} \in \Omega, \tag{3.2.15}$$

where $(\phi, u)_{\mathscr{E}'(\Omega)}$ is the standard pairing between a distribution $\phi \in \mathscr{E}'(\Omega)$ and a test function $u \in C^\infty(\Omega)$. Since $\operatorname{supp}\phi$ is compact in Ω and $\mathscr{R} \in C^\infty(\Omega \times \Omega)$, the kernel $\mathscr{K}_\phi(\mathbf{z}, \boldsymbol{\zeta})$ is well defined for every $\mathbf{z}, \boldsymbol{\zeta} \in \Omega$ but a priori it is not clear whether the operator with such an integral kernel is bounded in $\mathscr{A}(\Omega)$. However, we can show in many cases that in fact $K_\phi \in C^\infty(\overline{\Omega} \times \overline{\Omega})$ (see e.g. the example where $\Omega = \mathbb{D}$, and we have the explicit reproducing kernel (3.2.14), or the general pseudo-convex domains $\Omega \subset \mathbb{C}^2$ of finite type, considered in [136]). In this case, T_ϕ is not only bounded in $\mathscr{A}(\Omega)$ but also is in $\mathfrak{S}_p(\mathscr{A}(\Omega)$ for any $p \in (0, \infty)$ (see [19]). If $\phi \in \mathscr{E}'(\Omega)$ satisfies

$$(\phi, u)_{\mathscr{E}'(\Omega)} = \int_\Omega u(\mathbf{z}) d\nu(\mathbf{z}), \quad u \in C^\infty(\Omega),$$

where ν is a complex Borel measure, compactly supported in Ω, then $T_\phi = T_\nu$.
Finally, if $\phi \in \mathscr{D}'(\Omega) \setminus \mathscr{E}'(\Omega)$ possesses some additional properties, the integral kernel $\mathscr{K}_\phi$ in (3.2.15) may again turn out to be well defined, and even determine an operator bounded in $\mathscr{A}(\Omega)$. In the case $\Omega = \mathbb{D}$, examples of bounded and compact operators T_ϕ

distributions $\phi \in \mathscr{D}'(\mathbb{D}) \setminus \mathscr{E}'(\mathbb{D})$ can be found in [146].
The Berezin-Toeplitz operators on compact Kähler manifold which could be regarded as generalizations of the operators $T_{\mathscr{F}}$ in $\mathscr{A}(\Omega)$ have been recently examined in the monograph in [115].

Our second example diverges in a way from our general definition of a holomorphic space since now the measure $\alpha d\lambda$, absolutely continuous with respect to the Lebesgue measure $d\lambda$, is substituted by a measure singular with respect to $d\lambda$. We will restrict ourselves to the simplest case where $\Omega = \mathbb{D}$ (see (3.2.13)) and the support of the measure is the unit circle $\partial\mathbb{D}$. We define the analogue $\mathscr{A}(\mathbb{D}; \delta_{\partial\mathbb{D}})$ of the holomorphic space $\mathscr{A}(\Omega; \alpha)$ by

$$\mathscr{A}(\mathbb{D}; \delta_{\partial\mathbb{D}}) := \left\{ u \in \mathrm{Hol}(\mathbb{D}) \,|\, \lim_{r\uparrow 1} \int_0^{2\pi} |u(re^{i\theta})|^2 \, d\theta < \infty \right\}.$$

Then $\mathscr{A}(\mathbb{D}; \delta_{\partial\mathbb{D}})$ is a Hilbert space with scalar product

$$\langle u, v \rangle := \lim_{r\uparrow 1} \int_0^{2\pi} u(re^{i\theta}) \overline{v(re^{i\theta})} \, d\theta,$$

called the *Hardy space*. The elements u of $\mathscr{A}(\mathbb{D}; \delta_{\partial\mathbb{D}})$ admit absolutely convergent Taylor series

$$u(z) := \sum_{j\in\mathbb{Z}_+} u_j z^j, \quad z \in \mathbb{D}, \tag{3.2.16}$$

with $\{u_j\}_{j\in\mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+)$. Set

$$\mathscr{L}_+ := \left\{ u \in L^2(0, 2\pi) \,|\, \int_0^{2\pi} u(\theta) e^{-ij\theta} \, d\theta = 0, \quad j \in -\mathbb{N} \right\}.$$

Then $\mathscr{L}_+$ is a closed subspace of $L^2(0, 2\pi)$, and $\mathscr{A}(\mathbb{D}; \delta_{\partial\mathbb{D}})$ is unitarily equivalent to $\mathscr{L}_+$ under the mapping $\mathscr{W}$ defined by

$$(\mathscr{W}u)(\theta) = \sum_{j\in\mathbb{Z}_+} u_j \, e^{ij\theta}, \quad \theta \in (0, 2\pi), \quad u \in \mathscr{A}(\mathbb{D}; \delta_{\partial\mathbb{D}}),$$

where u_j are the coefficients appearing in (3.2.16). Thus, $\mathscr{W}u \in \mathscr{L}_+$ is the boundary trace of $u \in \mathscr{A}(\mathbb{D}; \delta_{\partial\mathbb{D}})$, while u is the holomorphic extension of $\mathscr{W}u$ to $\mathbb{D}$. Denote by $P_+ : L^2(0, 2\pi) \to L^2(0, 2\pi)$ the orthogonal projection onto $\mathscr{L}_+$.
Let $\mathscr{F} \in L^1(0, 2\pi)$. Then the Toeplitz operator $T_{\mathscr{F}} : \mathscr{A}(\mathbb{D}; \delta_{\partial\mathbb{D}}) \to \mathscr{A}(\mathbb{D}; \delta_{\partial\mathbb{D}})$ can be defined as

$$T_{\mathscr{F}} := \mathscr{W}^* P_+ \mathscr{F} \mathscr{W}.$$

Some authors identify $T_{\mathscr{F}}$ with the operator $P_+\mathscr{F} : \mathscr{L}_+ \to \mathscr{L}_+$, unitarily equivalent to $T_{\mathscr{F}}$ (see [29]). In the orthonormal basis $\left\{(2\pi)^{-1/2} e^{ij\theta}\right\}_{j\in\mathbb{Z}_+}$, $\theta \in (0, 2\pi)$, of $\mathscr{L}_+$, the infinite matrix $\left\{a_{j,k}\right\}_{j\in\mathbb{Z}_+}$ of this operator has elements

$$a_{j,k} := \frac{1}{2\pi} \int_0^{2\pi} \mathscr{F}(\theta) \, e^{-i(j-k)\theta} \, d\theta, \quad j, k \in \mathbb{Z}_+,$$

which depend only on the difference $j-k$ of the indices j and k. Traditionally, such matrices are known as *Toeplitz matrices*.
An important generalization of the Toeplitz operators in Hardy spaces are the operators considered by L. Boutet de Monvel and V. Guillemin in [27]. There the unit disk $\mathbb{D}$ is replaced by a bounded pseudo-convex domain Ω with smooth boundary, $\mathscr{L}_+$ is replaced by the closed subspace of $L^2(\partial\Omega; d\nu)$ with suitable measure $d\nu$, consisting of functions which admit a holomorphic extension into Ω, and $\mathscr{F}$ is an appropriate ΨDO on the boundary $\partial\Omega$.

3.3 Berezin-Toeplitz operators in Fock-Segal-Bargmann spaces

Let us now consider *Fock-Segal-Bargmann space* which is a holomorphic space playing an important role in the spectral analysis of quantum Hamiltonians with constant magnetic fields. For $b \in (0,\infty)$ introduce the Gaussian function

$$G_b(z) := e^{-b|z|^2/2}, \quad z \in \mathbb{C},$$

and for $d \geq 1$ set

$$G_{\mathbf{b}}(\mathbf{z}) = \prod_{j=1}^{d} G_{b_j}(z_j), \quad \mathbf{z} = (z_1, \dots, z_d) \in \mathbb{C}^d, \tag{3.3.1}$$

where $\mathbf{b} = (b_1, \dots, b_d) \in (0,\infty)^d$. We define the *Fock-Segal-Bargmann space* as the holomorphic space $\mathscr{A}(\mathbb{C}^d; G_{\mathbf{b}})$ with $d \geq 1$. Note that, traditionally, the weight in this space is multiplied by the normalizing factor $\frac{b_1 \dots b_d}{(2\pi)^d}$ so that the measure of $\mathbb{C}^d$ is equal to one (see e. g. [87, Subsection 3.2] and [220]). We use another convention which is more suitable for our purposes; in any case, we will work mostly in the weighted space $\widetilde{\mathscr{A}}(\mathbb{C}^d; G_{\mathbf{b}})$ which, of course, is invariant to the numeric normalization of the measure $G_{\mathbf{b}} d\lambda$.
In order to calculate the reproducing kernel, let us consider at first the case $d = 1$. Then $\left\{z^j\right\}_{j\in\mathbb{Z}_+}$ is an orthogonal basis, while $\left\{\sqrt{\frac{b^{j+1}}{\pi j! 2^{j+1}}}\, z^j\right\}_{j\in\mathbb{Z}_+}$ is an orthonormal basis in $\mathscr{A}(\mathbb{C}; G_b)$. Therefore,

$$\mathscr{R}(z,\zeta) = \frac{b}{2\pi} \sum_{j=0}^{\infty} \frac{1}{j!} (bz\overline{\zeta}/2)^j = \frac{b}{2\pi} e^{bz\overline{\zeta}/2}, \quad z, \zeta \in \mathbb{C}. \tag{3.3.2}$$

For $d \geq 1$, we take into account (3.3.1) and (3.3.2), and obtain

$$\mathscr{R}(\mathbf{z}, \boldsymbol{\zeta}) = \prod_{j=1}^{d} \frac{b_j}{2\pi} e^{b_j z_j \overline{\zeta_j}/2}, \quad \mathbf{z} = (z_1, \dots, z_d) \in \mathbb{C}^d, \quad \boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_d) \in \mathbb{C}^d.$$

Hence, the reproducing kernel $\widetilde{\mathscr{R}}$ of the weighted space

$$\widetilde{\mathscr{A}}(\mathbb{C}^d; G_{\mathbf{b}}) = G_{\mathbf{b}}^{1/2} \mathscr{A}(\mathbb{C}^d; G_{\mathbf{b}})$$

is equal to

$$\widetilde{\mathscr{R}}(\mathbf{z},\boldsymbol{\zeta}) = G_{\mathbf{b}}(\mathbf{z})^{1/2}\mathscr{R}(\mathbf{z},\boldsymbol{\zeta})G_{\mathbf{b}}(\boldsymbol{\zeta})^{1/2}$$
$$= \prod_{j=1}^{d} \frac{b_j}{2\pi} e^{-b_j(|z_j|^2+|\zeta_j|^2-2z_j\overline{\zeta_j})/4}, \quad \mathbf{z},\boldsymbol{\zeta} \in \mathbb{C}^d. \tag{3.3.3}$$

Comparing (3.3.3) with (2.7.49) and (2.7.32), we establish the crucial fact that

$$\widetilde{\mathscr{A}}(\mathbb{C}^d; G_{\mathbf{b}}) = \mathrm{Ker}(H_S(A,0) - \Lambda_0 I) \tag{3.3.4}$$

with $A(\mathbf{x}) := \frac{1}{2}B\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^{2d}$, and B is a magnetic field represented by a $(2d)\times(2d)$-matrix B whose eigenvalues are $-ib_j/2, ib_j/2$, $j = 1\ldots,d$. In other words, the eigenspace of the Schrödinger operator $H_S(A,0)$ with constant full-rank magnetic field B associated with the ground-state energy $\Lambda_0 = b_1 + \ldots b_d$, coincides with the weighted holomorphic space $\widetilde{\mathscr{A}}(\mathbb{C}^d; G_{\mathbf{b}})$. Using (2.7.49), we can give an similar interpretation of the eigenspace of $H_S(A,0)$ associated with the higher Landau levels Λ_q, $q \in \mathbb{N}$. In order to avoid tedious technical complications, we will restrict our attention to the case $d = 1$ and will consider (2.7.23) rather than (2.7.49). So, let $d = 1$ and $b > 0$. Fix $q \in \mathbb{Z}_+$ and set

$$\widetilde{\mathscr{A}_q} := p_q L^2(\mathbb{R}^2), \quad \mathscr{A}_q = G_b^{-1/2}\,\widetilde{\mathscr{A}_q},$$

$p_q = p_q(b) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ being the orthogonal projection onto the kernel of $H_S(A,0) - \Lambda_q I$ introduced in Section 2.7. In particular, we have seen that $\mathscr{A}_0 = \mathscr{A}(\mathbb{C}; G_b)$. By (2.7.27), we find that the spaces $\widetilde{\mathscr{A}_q}$, $q \in \mathbb{Z}_+$, are invariant with respect to the magnetic translations defined in (2.7.8).
Further, according to (2.7.32), the space $\widetilde{\mathscr{A}_q}$ admits a reproducing kernel

$$\widetilde{\mathscr{R}}_q(z,\zeta) := \frac{b}{2\pi}\exp\left(-\frac{b}{4}(|z|^2+|\zeta|^2-2z\overline{\zeta})\right)L_q\left(\frac{b}{2}|z-\zeta|^2\right), \quad z,\zeta \in \mathbb{C},$$

L_q being as above the Laguerre polynomials of degree q. Therefore, (3.2.6) implies that $\mathscr{A}_q$ has a reproducing kernel

$$\mathscr{R}_q(z,\zeta) := G_b(z)^{-1/2}\widetilde{\mathscr{R}}_q(z,\zeta)G_b(\zeta)^{-1/2} = \frac{b}{2\pi}\exp\left(\frac{b}{2}z\overline{\zeta}\right)L_q\left(\frac{b}{2}|z-\zeta|^2\right), \quad z,\zeta \in \mathbb{C}. \tag{3.3.5}$$

By analogy with (3.2.4) and (3.2.7), set

$$\rho_q(z) := \mathscr{R}_q(z,z) = \frac{b}{2\pi}e^{b|z|^2/2}, \quad d\mu_q := \rho_q(z)G_b(z)d\lambda(z) = \frac{b}{2\pi}d\lambda(z).$$

Note that, in fact, ρ_q and $d\mu_q$ are independent of q since we have $L_q(0) = 1$ for all $q \in \mathbb{Z}_+$. Thus, for any fixed $q \in \mathbb{Z}_+$, the system

$$f_{z,q}(\zeta) := \rho_q(z)^{-1/2}\mathscr{R}_q(\zeta,z) = \sqrt{\frac{b}{2\pi}}\,e^{-\frac{b}{4}(|z|^2-2z\overline{\zeta})}L_q\left(\frac{b}{2}|z-\zeta|^2\right), \quad z,\zeta \in \mathbb{C},$$

is overcomplete in $L^2(\mathbb{C}; G_b d\lambda)$ while the system

$$\tilde{f}_{z,q}(\zeta)$$

$$:= G_b(\zeta)^{1/2} f_{z,q}(\zeta) = \sqrt{\frac{b}{2\pi}}\, e^{-\frac{b}{4}(|z|^2 - 2z\overline{\zeta} + |\zeta|^2)} L_q\left(\frac{b}{2}|z-\zeta|^2\right), \; z, \zeta \in \mathbb{C}, \tag{3.3.6}$$

is overcomplete in $L^2(\mathbb{C})$. Let us give an interpretation of $\mathscr{A}_q$, $q \in \mathbb{Z}_+$, as a subspace of the space of *polyanalytic functions*. It is straightforward to check that

$$\mathscr{A}_q = \left\{ u \in L^2(\mathbb{C}; G_b d\lambda) \,|\, u = \left(\frac{\partial}{\partial z} - \frac{b}{2}\overline{z}\right)^q v, \; v \in \mathrm{Hol}(\mathbb{C}) \right\}, \quad q \in \mathbb{Z}_+.$$

Evidently, if $u \in \mathscr{A}_q$, then u is $(q+1)$-polyanalitc, i.e. it is a solution of the equation

$$\frac{\partial^{q+1} u}{\partial \overline{z}^{q+1}} = 0.$$

Note that the orthogonal sum $\bigoplus_{j=0}^{\ell-1} \mathscr{A}_j$, $\ell \in \mathbb{N}$, coincides with the Fock space of ℓ-polyanalytic functions

$$\left\{ u \in L^2(\mathbb{C}; G_b d\lambda) \,|\, \frac{\partial^\ell u}{\partial \overline{z}^\ell} = 0 \right\}, \tag{3.3.7}$$

called sometimes the ℓth poly-Fock space; accordingly, $\mathscr{A}_{\ell-1}$ is called the *true* ℓth poly-Fock space (see e. g. [212], [1], [173]).

Further, if, say, $\mathscr{F} \in L^\infty(\mathbb{R}^2)$, then the operator $p_q \mathscr{F} = p_q \mathscr{F} p_q$ is equal to the weighted Berezin-Toeplitz operator $\widetilde{T}_{\mathscr{F}}$ acting in $\widetilde{\mathscr{A}_q} = \mathrm{Ran}\, p_q$. Since sometimes we consider the operators $p_q \mathscr{F} p_q$ on the domain $L^2(\mathbb{R}^2)$, we prefer to write $p_q \mathscr{F} p_q$ instead of $p_q \mathscr{F}$, indicating explicitly whether the operator is considered on $\mathrm{Ran}\, p_q$ or on $L^2(\mathbb{R}^2)$.
By analogy with (3.2.9), set

$$\begin{aligned}(\mathscr{B}_q(\mathscr{F}))(z) :&= \langle \mathscr{F} f_{z,q}, f_{z,q} \rangle_{L^2(\mathbb{R}^2; G_b d\lambda)} \\ &= \langle \mathscr{F} \tilde{f}_{z,q}, \tilde{f}_{z,q} \rangle_{L^2(\mathbb{R}^2)} \\ &= \frac{b}{2\pi} \int_{\mathbb{C}} e^{-b|z-\zeta|^2/2} L_q(b|z-\zeta|^2/2)^2 \mathscr{F}(\zeta)\, d\lambda(\zeta).\end{aligned}$$

Then by analogy with Proposition 3.2.1, we obtain

Proposition 3.3.1 *Let* $b > 0$. *Fix* $q \in \mathbb{Z}_+$.
(i) *If* $\mathscr{F} \in L^\infty(\mathbb{R}^2)$, *then* $p_q \mathscr{F} p_q$ *is bounded in* $\widetilde{\mathscr{A}_q}$.
If $p_q \mathscr{F} p_q$ *is bounded in* $\widetilde{\mathscr{A}_q}$, *then* $\mathscr{B}_q(\mathscr{F}) \in L^\infty(\mathbb{R}^2)$. *Moreover,*

$$\|\mathscr{B}_q(\mathscr{F})\|_{L^\infty(\mathbb{R}^2)} \le \|p_q \mathscr{F} p_q\| \le \|\mathscr{F}\|_{L^\infty(\mathbb{R}^2)}. \tag{3.3.8}$$

(ii) *Let* $p \in [1, \infty)$. *If* $\mathscr{F} \in L^p(\mathbb{R}^2)$, *then* $p_q \mathscr{F} p_q \in \mathfrak{S}_p(\widetilde{\mathscr{A}_q})$.

If $p_q \mathscr{F} p_q \in \mathfrak{S}_p(\widetilde{\mathscr{A}_q})$*, then* $\mathscr{B}_q(\mathscr{F}) \in L^p(\mathbb{R}^2)$*. Moreover,*

$$\frac{b}{2\pi}\|\mathscr{B}_q(\mathscr{F})\|^p_{L^p(\mathbb{R}^2)} \leq \|p_q \mathscr{F} p_q\|^p_p \leq \frac{b}{2\pi}\|\mathscr{F}\|^p_{L^p(\mathbb{R}^2)}. \tag{3.3.9}$$

(iii) *Let* $p \in (1,\infty)$*. If* $\mathscr{F} \in L^p_w(\mathbb{R}^2)$*, then* $p_q \mathscr{F} p_q \in \mathfrak{S}_{p,w}(\widetilde{\mathscr{A}_q})$*.*

If $p_q \mathscr{F} p_q \in \mathfrak{S}_{p,w}(\widetilde{\mathscr{A}_q})$*, then* $\mathscr{B}_q(\mathscr{F}) \in L^p_w(\mathbb{R}^2)$*. Moreover,*

$$\frac{b}{2\pi}\|\mathscr{B}_q(\mathscr{F})\|^p_{L^p_w(\mathbb{R}^2)} \leq \|p_q \mathscr{F} p_q\|^p_{p,w} \leq \frac{b}{2\pi}\|\mathscr{F}\|^p_{L^p_w(\mathbb{R}^2)}. \tag{3.3.10}$$

Corollary 3.3.1 *Fix* $q \in \mathbb{Z}_+$*.*
(i) *Let* $\mathscr{F} \in L^1_{loc}(\mathbb{R}^2)$ *and* $\lim_{|z|\to\infty} \mathscr{F}(z) = 0$*. Then the operator* $p_q\mathscr{F}p_q$ *is compact in* $\widetilde{\mathscr{A}_q}$*.*
(ii) *Let* $p_q\mathscr{F}p_q \in \mathfrak{S}_\infty(\widetilde{\mathscr{A}_q})$*. Then*

$$\lim_{|z|\to\infty} (\mathscr{B}_q(\mathscr{F})(z)) = 0. \tag{3.3.11}$$

Proof. (i) The claim follows from Corollary 3.1.1 since our assumptions imply $\mathscr{F} \in L^1_\varepsilon(\mathbb{R}^2)$.
(ii) It is easy to check that

$$w-\lim_{|z|\to\infty} \tilde{f}_{z,q} = 0,$$

where the system $\{\tilde{f}_{z,q}\}_{z\in\mathbb{C}}$ is defined in (3.3.6). Thus, we find that (3.3.11) follows from Corollary 3.1.3. □

The next theorem shows that if $q = 0$ and $\mathscr{F} \geq 0$ satisfies a reasonable integrability assumption, then the necessary conditions concerning the membership of $p_0\mathscr{F}p_0$ to $\mathfrak{B}(\widetilde{\mathscr{A}_0})$ or $\mathfrak{S}_p(\widetilde{\mathscr{A}_0})$ with $p \in [1,\infty]$, in Proposition 3.3.1 and Corollary 3.3.1 are also sufficient.

Theorem 3.3.1 *Assume that* $\mathscr{F} : \mathbb{R}^2 \to [0,\infty)$ *is a Lebesgue measurable function which satisfies*

$$\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} \mathscr{F}(\zeta) e^{-\frac{b}{2}(|\zeta|^2 - 2\mathrm{Re}\, z\bar{\zeta})}\, d\lambda(\zeta) < \infty.$$

Then the following assertions hold true.
(i) [220, Theorem 6.19] *The operator* $p_0\mathscr{F}p_0$ *is bounded in* $\widetilde{\mathscr{A}_0}$ *if and only if* $\mathscr{B}_0(\mathscr{F}) \in L^\infty(\mathbb{R}^2)$.
(ii) [220, Theorem 6.23] *The operator* $p_0\mathscr{F}p_0$ *is compact in* $\widetilde{\mathscr{A}_0}$ *if and only if*

$$\lim_{|z|\to\infty} (\mathscr{B}_0(\mathscr{F}))(z) = 0.$$

(iii) [220, Corollary 6.33] *Let* $p \in [1,\infty)$*. Then* $p_0\mathscr{F}p_0 \in \mathfrak{S}_p(\widetilde{\mathscr{A}_0})$ *if and only if* $\mathscr{B}_0(\mathscr{F}) \in L^p(\mathbb{R}^2)$.

The class of bounded operators $p_q \mathscr{F} p_q$, $q \in \mathbb{Z}_+$, with symbols $\mathscr{F} \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ is too restrictive for our purposes. If $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^2)$, we could define, similarly to (3.2.15), the operator $p_q \mathscr{F} p_q$ as the operator with integral kernel

$$\mathscr{K}_{\mathscr{F},q}(\mathbf{x},\mathbf{x}') := \left(\mathscr{F}, K_q(\mathbf{x},\cdot)K_q(\cdot,\mathbf{x}')\right)_{\mathscr{S}'(\mathbb{R}^2)}, \quad \mathbf{x},\mathbf{x}' \in \mathbb{R}^2, \tag{3.3.12}$$

where K_q are the kernels defined in (2.7.32), and $(\phi, u)_{\mathscr{S}'(\mathbb{R}^N)}$ is the pairing between the distribution $\phi \in \mathscr{S}'(\mathbb{R}^N)$ and the test function $u \in \mathscr{S}(\mathbb{R}^N)$, $N \geq 1$. Since for every fixed $\mathbf{x} \in \mathbb{R}^2$ we have $K_q(\mathbf{x},\cdot) \in \mathscr{S}(\mathbb{R}^2)$, the kernel $\mathscr{K}_{\mathscr{F},q}(\mathbf{x},\mathbf{x}')$ is well defined for every $\mathbf{x},\mathbf{x}' \in \mathbb{R}^2$. However, generally speaking, the operator with integral kernel $\mathscr{K}_{\mathscr{F},q}(\mathbf{x},\mathbf{x}')$ may turn out not to be bounded. Next, we describe a class of distributions $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^2)$ such that the operator $p_q \mathscr{F} p_q$ is bounded. Let $\ell \in \mathbb{Z}_+$, $N \in \mathbb{N}$. Set

$$\mathscr{S}'_\ell(\mathbb{R}^N)$$

$$:= \left\{ \mathscr{F} \in \mathscr{S}'(\mathbb{R}^N) \,|\, \mathscr{F} = \sum_{\alpha \in \mathbb{Z}_+^N : |\alpha| \leq \ell} D^\alpha \mathscr{G}_\alpha,\ \mathscr{G}_\alpha \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \right\}. \tag{3.3.13}$$

Note that by the representation theorem for $\mathscr{S}'(\mathbb{R}^N)$ (see e.g. [162, Theorem V.10]), each $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^N)$ can be represented as $\mathscr{F} = D^\beta \mathscr{G}$ where $\beta \in \mathbb{Z}_+^N$, and $\mathscr{G} \in C(\mathbb{R}^N)$ admits the estimate

$$|\mathscr{G}(\mathbf{x})| \leq C\langle \mathbf{x} \rangle^k, \quad \mathbf{x} \in \mathbb{R}^N,$$

with some $k \in \mathbb{Z}_+$. Thus, roughly speaking, the restriction which we impose on the elements of $\mathscr{S}'_\ell(\mathbb{R}^N)$ is that $\mathscr{G}_\alpha$ are not allowed to have a polynomial growth at infinity.

Proposition 3.3.2 *Let $\mathscr{F} \in \mathscr{S}'_\ell(\mathbb{R}^2)$ with some $\ell \in \mathbb{Z}_+$. Then*

$$p_q \mathscr{F} p_q \in \mathfrak{B}(\widetilde{\mathscr{A}_q}), \quad q \in \mathbb{Z}_+. \tag{3.3.14}$$

Proof. By (3.3.12) and (2.7.32), the operator $p_q \mathscr{F} p_q$ has an integral kernel

$$\mathscr{K}_{\mathscr{F},q}(\mathbf{x},\mathbf{x}') := \left(\frac{b}{2\pi}\right)^2 \sum_{\alpha \in \mathbb{Z}_+^2 : |\alpha| \leq \ell} (-1)^{|\alpha|} \int_{\mathbb{R}^2} \mathscr{G}_\alpha(\mathbf{y})$$

$$\times D_{\mathbf{y}}^\alpha \left(e^{-\frac{b}{4}(|\mathbf{y}-\mathbf{x}|^2 + |\mathbf{x}'-\mathbf{y}|^2 - 2i(\mathbf{x}-\mathbf{x}')\wedge \mathbf{y})} L_q\left(\frac{b}{2}|\mathbf{x}-\mathbf{y}|^2\right) L_q\left(\frac{b}{2}|\mathbf{x}'-\mathbf{y}|^2\right)\right) d\mathbf{y}.$$

Differentiating, we find that

$$|\mathscr{K}_{\mathscr{F},q}(\mathbf{x},\mathbf{x}')| \leq C\langle \mathbf{x}-\mathbf{x}' \rangle^\ell e^{-\frac{b'}{8}|\mathbf{x}-\mathbf{x}'|^2}, \tag{3.3.15}$$

with $b' \in (0,b)$, and C which may depend on b, b', ℓ, q. and $\sup_\alpha \|\mathscr{G}_\alpha\|_{L^1(\mathbb{R}^2)+L^\infty(\mathbb{R}^2)}$ but is independent of $\mathbf{x},\mathbf{x}' \in \mathbb{R}^2$. Applying the Schur test, we find that (3.3.14) follows from (3.3.15). □

Remark: It is easy to see that if $\mathscr{F} \in \mathscr{E}'(\mathbb{R}^2)$, then $\mathscr{K}_{\mathscr{F},q} \in \mathscr{S}(\mathbb{R}^{2n})$ so that $p_q \mathscr{F} p_q \in \mathfrak{S}_p(\widetilde{\mathscr{A}_q})$ for any $p \in (0, \infty)$. If we suppose in addition that $\operatorname{supp} \mathscr{F}$ is finite, then $\operatorname{rank}(p_q \mathscr{F} p_q) < \infty$.

In our next proposition, we will describe a class of bounded operators $p_q \mathscr{F} p_q$ with radially symmetric symbols $\mathscr{F}$, whose spectrum is pure point and admits an explicit description. We recall that if $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^2)$ and $\mathscr{O} : \mathbb{R}^2 \to \mathbb{R}^2$ is an orthogonal mapping, then the element $\mathscr{F} \circ \mathscr{O}$ is defined by

$$(\mathscr{F} \circ \mathscr{O}, \phi)_{\mathscr{S}'(\mathbb{R}^2)} = \left(\mathscr{F}, \phi \circ \mathscr{O}^{-1}\right)_{\mathscr{S}'(\mathbb{R}^2)}, \quad \phi \in \mathscr{S}(\mathbb{R}^2).$$

Then $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^2)$ is called radially symmetric if $\mathscr{F} \circ \mathscr{O} = \mathscr{F}$ for any orthogonal $\mathscr{O}$. Of course, if $\mathscr{F} \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$, then there exists a function $\mathscr{R}_{\mathscr{F}} : [0, \infty] \to \mathbb{C}$ such that

$$\mathscr{F}(x, \xi) = \mathscr{R}_{\mathscr{F}}(x^2 + \xi^2), \quad (x, \xi) \in \mathbb{R}^2.$$

Proposition 3.3.3 *Let $\mathscr{F} \in \mathscr{S}'_\ell(\mathbb{R}^2)$, $\ell \in \mathbb{Z}_+$, be radially symmetric. Fix $q \in \mathbb{Z}_+$. Then the functions $\{\varphi_{q,k}\}_{k \in \mathbb{Z}_+}$ from the canonic orthogonal basis of $\widetilde{\mathscr{A}_q}$ defined in* (2.7.39)*, are eigenfunctions of $p_q \mathscr{F} p_q$ with eigenvalues*

$$\lambda_{k,q}(\mathscr{F}) := \left(\mathscr{F}, |\varphi_{k,q}|^2\right)_{\mathscr{S}'(\mathbb{R}^2)}, \quad k \in \mathbb{Z}_+.$$

In particular, if $\mathscr{F} \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$, then

$$\begin{aligned} \lambda_{k,q}(\mathscr{F}) &= \int_{\mathbb{R}^2} \mathscr{F}(\mathbf{x}) |\varphi_{k,q}(\mathbf{x})|^2 \, d\mathbf{x} \\ &= \frac{q!}{k!} \int_0^\infty \mathscr{R}_{\mathscr{F}}(2t/b) \, t^{k-q} e^{-t} L_q^{(k-q)}(t)^2 \, dt, \quad k \in \mathbb{Z}_+. \end{aligned} \tag{3.3.16}$$

Proof. We have

$$(p_q \mathscr{F} p_q)\varphi_{k,q} = \sum_{\ell \in \mathbb{Z}_+} \langle (p_q \mathscr{F} p_q)\varphi_{k,q}, \varphi_{\ell,q} \rangle_{L^2(\mathbb{R}^2)} \varphi_{\ell,q},$$

where the series is convergent in $\widetilde{\mathscr{A}_q}$, and

$$\langle (p_q \mathscr{F} p_q)\varphi_{k,q}, \varphi_{\ell,q} \rangle_{L^2(\mathbb{R}^2)} = \left(\mathscr{F}, \varphi_{k,q} \overline{\varphi_{\ell,q}}\right)_{\mathscr{S}'(\mathbb{R}^2)}, \quad k, \ell \in \mathbb{Z}_+.$$

Due to the radial symmetry of $\mathscr{F}$,

$$(\mathscr{F}, u)_{\mathscr{S}'(\mathbb{R}^2)} = 0$$

for any $u \in \mathscr{S}(\mathbb{R}^2)$ such that

$$\int_0^{2\pi} u(re^{i\theta}) d\theta = 0, \quad r \in [0, \infty).$$

Since the functions

$$e^{-i(k-\ell)\theta}\varphi_{k,q}(re^{i\theta})\overline{\varphi_{\ell,q}(re^{i\theta})}, \quad k,\ell \in \mathbb{Z}_+,$$

depend only on r but not on θ, we find that

$$\left(\mathscr{F}, \varphi_{k,q}\overline{\varphi_{\ell,q}}\right)_{\mathscr{S}'(\mathbb{R}^2)} = 0$$

if $k \neq \ell$. Therefore,

$$(p_q \mathscr{F} p_q)\varphi_{k,q} = \lambda_k\, \varphi_{k,q}, \quad k \in \mathbb{Z}_+$$

□

An important extension of the Fock-Segal-Bargmann space is the holomorphic space $\mathscr{A}(\Omega;\alpha)$ with $\Omega = \mathbb{C}^d$, $d \geq 1$, and $\alpha = e^{-2\varphi}$ where $\varphi \in C^2(\mathbb{C}^d)$ is a general uniformly strictly plurisubharmonic function, i.e. its Levi matrix

$$\left\{4\frac{\partial^2\varphi}{\partial z_j \partial \overline{z_k}}(\mathbf{z})\right\}_{j,k=1}^{d}$$

is positive definite, uniformly with respect to $\mathbf{z} \in \mathbb{C}^d$. In the case of the Fock-Segal-Bargmann space we have

$$\varphi(\mathbf{z}) = \frac{1}{4}\sum_{j=1}^{d} b_j |z_j|^2, \quad \mathbf{z} = (z_1, \dots, z_d) \in \mathbb{C}^d,$$

so that the Levi matrix of φ is diagonal and is equal to $\operatorname{diag}\{b_1, \dots, b_d\}$.
If $d = 1$, then φ should simply satisfy

$$\Delta\varphi(z) \geq C, \quad z \in \mathbb{C}, \tag{3.3.17}$$

with a constant $C > 0$.
Assume now that

$$b \in C(\mathbb{R}^2;\mathbb{R}) \cap L^\infty(\mathbb{R}^2). \tag{3.3.18}$$

Let $\varphi \in C^2(\mathbb{R}^2;\mathbb{R})$ be a solution of the poisson equation $\Delta\varphi = b$. Then the kernels of the annihilation and creation operators $a(b)$ and $a(b)^*$ which determine according to (2.8.5) the kernel of the 2D Pauli operator $H_P(A,0)$ with magnetic field b, satisfy

$$\operatorname{Ker} a(b) = \widetilde{\mathscr{A}}(\mathbb{C}; e^{-2\varphi}) = e^{-\varphi}\mathscr{A}(\mathbb{C}; e^{-2\varphi})$$

(see (2.8.11)), and

$$\operatorname{Ker} a(b)^* = \widetilde{\mathscr{A}^*}(\mathbb{C}; e^{2\varphi}) = e^{\varphi}\mathscr{A}^*(\mathbb{C}; e^{2\varphi})$$

(see (2.8.10)). Denote by $\mathfrak{p}_{ann} = \mathfrak{p}_{ann}(b)$ (resp., by $\mathfrak{p}_{cre} = \mathfrak{p}_{cre}(b)$) the orthogonal projection onto $\operatorname{Ker} a(b)$ (resp., onto $\operatorname{Ker} a(b)^*$).

Proposition 3.3.4 *Assume that* (3.3.18) *holds true. Let* $\mathscr{F} \in \mathrm{L}^1(\mathbb{R}^2)$*. Then* $\mathfrak{p}_\natural \mathscr{F} \mathfrak{p}_\natural \in \mathfrak{S}_1(\widetilde{\mathscr{A}}(\mathbb{C};\mathrm{e}^{-2\varphi}))$*,* $\natural = \mathrm{an}, \mathrm{cr}$*, and*

$$\|\mathfrak{p}_\natural \mathscr{F} \mathfrak{p}_\natural\|_1 \leq \mathrm{C}_\natural \|\mathscr{F}\|_{\mathrm{L}^1(\mathbb{R}^2)} \tag{3.3.19}$$

with $\mathrm{C}_{\mathrm{ann}} := \pi \|(\mathrm{a}^*\mathrm{a}+\mathrm{I})^{-1}(\mathrm{a}^*\mathrm{a}+\mathrm{b}+\mathrm{I})\|^2$ *and* $\mathrm{C}_{\mathrm{cre}} := \pi \|(\mathrm{a}^*\mathrm{a}+\mathrm{I})^{-1}(\mathrm{a}\,\mathrm{a}^*-\mathrm{b}+\mathrm{I})\|^2$.

Proof. Let at first $\natural = \mathrm{ann}$. For brevity set $\mathfrak{p} := \mathfrak{p}_{\mathrm{ann}}(\mathrm{b})$. Evidently,

$$\begin{aligned}\|\mathfrak{p}\mathscr{F}\mathfrak{p}\|_1 &\leq \|\mathfrak{p}\,|\mathscr{F}|^{1/2}\|_2^2 = \|\mathfrak{p}\,(\mathrm{a}^*\mathrm{a}+\mathrm{I})^{-1}\,|\mathscr{F}|^{1/2}\|_2^2 \leq \|(\mathrm{a}^*\mathrm{a}+\mathrm{I})^{-1}\,|\mathscr{F}|^{1/2}\|_2^2 \\ &\leq \|(\mathrm{a}^*\mathrm{a}+\mathrm{I})^{-1}(\mathrm{a}^*\mathrm{a}+\mathrm{b}+\mathrm{I})\|^2\,\|(\mathrm{a}^*\mathrm{a}+\mathrm{b}+\mathrm{I})^{-1}\,|\mathscr{F}|^{1/2}\|_2^2. \end{aligned} \tag{3.3.20}$$

The diamagnetic inequality (2.5.6) and Theorem 2.5.1 (ii) imply

$$\|(\mathrm{a}^*\mathrm{a}+\mathrm{b}+\mathrm{I})^{-1}\,|\mathscr{F}|^{1/2}\|_2^2 \leq \|(-\Delta+\mathrm{I})^{-1}\,|\mathscr{F}|^{1/2}\|_2^2 = \pi\|\mathscr{F}\|_{\mathrm{L}^1(\mathbb{R}^2)}. \tag{3.3.21}$$

Now, (3.3.19) with $\natural = \mathrm{ann}$ follows immediately from (3.3.20) and (3.3.21). The proof for $\natural = \mathrm{cre}$ is quite similar. □

Interpolating between (3.3.19) and the trivial estimates

$$\|\mathfrak{p}_\natural \mathscr{F} \mathfrak{p}_\natural\| \leq \|\mathscr{F}\|_{\mathrm{L}^\infty(\mathbb{R}^2)}, \quad \natural = \mathrm{ann}, \mathrm{cre}. \tag{3.3.22}$$

we can obtain the analogues of the upper estimates in (3.3.9) and (3.3.10) but we omit the obvious details. Moreover, similarly to Corollary 3.3.1 (i), we can prove the following corollary, using (3.3.19) and (3.3.22):

Corollary 3.3.2 *Let* b *satisfy the hypotheses of Proposition 3.3.4. Assume that* $\mathscr{F} \in \mathrm{L}^1_\varepsilon(\mathbb{R}^2)$*. Then the operators* $\mathfrak{p}_\natural(\mathrm{b})\mathscr{F}\mathfrak{p}_\natural(\mathrm{b})$ *with* $\natural = \mathrm{ann}, \mathrm{cre}$ *are compact in* $\mathrm{Ran}\,\mathfrak{p}_\natural(\mathrm{b})$.

Remark: If $\mathscr{F} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^2)$, and $\lim_{|\mathbf{x}|\to\infty} \mathscr{F}(\mathbf{x}) = 0$, then, evidently, $\mathscr{F} \in \mathrm{L}^1_\varepsilon(\mathbb{R}^2)$.

Note that in contrast to the case of a constant magnetic field $\mathrm{b} \neq 0$, in the general case where b satisfies just (3.3.18), we do not dispose of an explicit expression of $\widetilde{\rho}(\mathrm{z}) = \widetilde{\mathscr{R}}(\mathrm{z},\mathrm{z})$, $\mathrm{z} \in \mathbb{C}$, where $\widetilde{\mathscr{R}}$ is the reproducing kernel of $\widetilde{\mathscr{A}}(\mathbb{C};\mathrm{e}^{-2\varphi})$. Thus, in this case, we cannot claim a priori, that the Lebesgue measure $d\lambda$ is equivalent to the measure $d\mu = \widetilde{\rho}\,d\lambda$. However, if $\mathrm{b} = \mathrm{b}_0 + \tilde{\mathrm{b}}$ with $\mathrm{b}_0 > 0$, is an admissible magnetic field, and $\varphi = \varphi_0 + \widetilde{\varphi}$ satisfies $\Delta\varphi = \mathrm{b}$, then, arguing as in the proof of [154, Eq. (3.32)], we can show that

$$\frac{\mathrm{b}_0}{2\pi}\mathrm{e}^{-2\operatorname{osc}\widetilde{\varphi}} \leq \widetilde{\rho}(\mathrm{z}) \leq \frac{\mathrm{b}_0}{2\pi}\mathrm{e}^{2\operatorname{osc}\widetilde{\varphi}}, \quad \mathrm{z} \in \mathbb{C},$$

i.e. the measures $d\lambda$ and $d\mu$ are equivalent.

Assume now that b is a pre-admissible magnetic field which satisfies (2.8.17). Let $\varphi = \varphi_0 + \widetilde{\varphi}$ where $\varphi_0 = \frac{\mathrm{b}_0}{4}|\mathrm{z}|^2$ and $\widetilde{\varphi}$ satisfies the Poisson equation (2.8.18) so that

$$\Delta\varphi = \mathrm{b} = \mathrm{b}_0 + \tilde{\mathrm{b}}. \tag{3.3.23}$$

Note that if b is a pre-admissible magnetic field with mean value $b_0 > 0$ and background $\tilde{b}$ of the form (2.8.20), then φ may not satisfy (3.3.17) point-wise but still, by (3.3.23) an averaged version of (3.3.17) holds true, namely

$$\lim_{T\to\infty} T^{-2} \int_{\mathbf{x}+(-T/2,T/2)^2} \Delta\varphi(\mathbf{y})\, d\mathbf{y} = b_0 > 0,$$

for any $\mathbf{x} \in \mathbb{R}^2$. From this point of view, $\widetilde{\mathscr{A}}(\mathbb{C}; e^{-2\varphi}) = \operatorname{Ker} a(b)$ is a direct generalization of the weighted Fock-Segal-Bargmann space $\widetilde{\mathscr{A}}(\mathbb{C}; G_{b_0})$.
Similarly, if b is a pre-admissible magnetic field with mean value $b_0 < 0$, then the weighted anti-holomorphic space $\widetilde{\mathscr{A}^*}(\mathbb{C}; e^{2\varphi}) = \operatorname{Ker} a(b)^*$ is a generalization of $\widetilde{\mathscr{A}^*}(\mathbb{C}; G_{-b_0})$.

3.4 Operators with anti-Wick symbols

In this subsection we introduce generalized anti-Wick ΨDOs which again are a special case of the operators with contravariant symbols considered in Section 3.1. Let $\psi \in \mathscr{S}(\mathbb{R}^n)$, $n \geq 1$, with $\|\psi\|_{L^2(\mathbb{R}^n)} = 1$. Introduce the so called *coherent states*

$$\psi_{\mathbf{x},\boldsymbol{\xi}}(y) := e^{i\boldsymbol{\xi}\cdot\mathbf{y}}\psi(\mathbf{y}-\mathbf{x}), \quad \mathbf{y} \in \mathbb{R}^n, \quad \mathbf{x},\boldsymbol{\xi} \in \mathbb{R}^{2n}, \tag{3.4.1}$$

built on ψ. Evidently, $\|\psi_{\mathbf{x},\boldsymbol{\xi}}\|_{L^2(\mathbb{R}^n)} = 1$ for all $\mathbf{x},\boldsymbol{\xi} \in \mathbb{R}^{2n}$. If $f \in \mathfrak{H} := L^2(\mathbb{R}^n)$, then it is easy to check that

$$\|f\|^2_{\mathfrak{H}} = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} |\langle f, \psi_{\mathbf{x},\boldsymbol{\xi}}\rangle_{\mathfrak{H}}|^2\, d\mathbf{x}\, d\boldsymbol{\xi}. \tag{3.4.2}$$

Thus the system $\{\psi_{\mathbf{x},\boldsymbol{\xi}}\}_{(\mathbf{x},\boldsymbol{\xi})\in\mathbb{R}^{2n}}$ is overcomplete with respect to the measure

$$d\mu(\mathbf{x},\boldsymbol{\xi}) := (2\pi)^{-n}\, d\mathbf{x}\, d\boldsymbol{\xi},$$

proportional to the Lebesgue measure $d\mathbf{x}\, d\boldsymbol{\xi}$ in $\mathbb{R}^{2n}$. Then the operator $\operatorname{Op}^{cnv}(\mathscr{F})$ with contravariant symbol $\mathscr{F} \in L^\infty(\mathbb{R}^{2n}) + L^1(\mathbb{R}^{2n})$ is well defined. In this particular setting, we will call $\operatorname{Op}^{cnv}(\mathscr{F})$ a ΨDO with *anti-Wick symbol* $\mathscr{F}$ built on ψ, and will denote it by $\operatorname{Op}^{aw}_\psi(\mathscr{F})$. Accordingly, if $T \in \mathfrak{B}(L^2(\mathbb{R}^{2n}))$, we will call the covariant symbol of T the *Wick symbol* of T built on ψ, and will denote it by $\mathscr{F}^{wick}_\psi(T)$. In other words,

$$(\mathscr{F}^{wick}_\psi(T))(\mathbf{x},\boldsymbol{\xi}) := \langle T\psi_{\mathbf{x},\boldsymbol{\xi}}, \psi_{\mathbf{x},\boldsymbol{\xi}}\rangle_{L^2(\mathbb{R}^n)}, \quad (\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^{2n}. \tag{3.4.3}$$

For $f, g \in L^2(\mathbb{R}^2)$ define the *Husimi transform* $Q_{f,g}$ of the pair (f, g) by

$$Q_{f,g}(\mathbf{x},\boldsymbol{\xi}) := (2\pi)^{-n}\langle f, \psi_{\mathbf{x},\boldsymbol{\xi}}\rangle_{L^2(\mathbb{R}^n)}\overline{\langle g, \psi_{\mathbf{x},\boldsymbol{\xi}}\rangle_{L^2(\mathbb{R}^n)}}, \quad (\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^{2n}.$$

Note that $Q_{f,g} = \overline{Q_{g,f}}$. Moreover, $Q_{f,g}$ depends on ψ as a functional parameter. If $f \in \mathscr{S}(\mathbb{R}^n)$, then it is easy to check that $\langle f, \psi_{\mathbf{x},\boldsymbol{\xi}}\rangle_{L^2(\mathbb{R}^2)} \in \mathscr{S}(\mathbb{R}^{2n})$; hence, $Q_{f,g} \in \mathscr{S}(\mathbb{R}^{2n})$ if $f, g \in \mathscr{S}(\mathbb{R}^n)$. Moreover, we have

$$\langle \operatorname{Op}^{aw}_\psi(\mathscr{F})f, g\rangle_{L^2(\mathbb{R}^n)} = \langle \mathscr{F}, Q_{g,f}\rangle_{L^2(\mathbb{R}^{2n})}, \quad f, g \in \mathscr{S}(\mathbb{R}^2).$$

Thus, if $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^{2n})$, then, by analogy with (2.3.13), we can define the continuous mapping $\mathrm{Op}^{\mathrm{aw}}_{\psi}(\mathscr{F}) : \mathscr{S}(\mathbb{R}^{2n}) \to \mathscr{S}'(\mathbb{R}^{2n})$ by

$$\left(\mathrm{Op}^{\mathrm{aw}}_{\psi}(\mathscr{F})\mathrm{f}, \mathrm{g}\right)_{\mathscr{S}'(\mathbb{R}^n)} = \left(\mathscr{F}, \mathrm{Q}_{\mathrm{f},\bar{\mathrm{g}}}\right)_{\mathscr{S}'(\mathbb{R}^{2n})}, \quad \mathrm{f}, \mathrm{g} \in \mathscr{S}(\mathbb{R}^n). \tag{3.4.4}$$

In this case, we set

$$\mathscr{F}^{\mathrm{wick}}_{\psi}(\mathrm{Op}^{\mathrm{w}}(\mathscr{F}))(\mathbf{x}, \boldsymbol{\xi}) := \left(\mathscr{F}, \mathrm{W}_{\psi_{\mathbf{x},\boldsymbol{\xi}}, \psi_{\mathbf{x},\boldsymbol{\xi}}}\right)_{\mathscr{S}'(\mathbb{R}^{2n})}, \quad (\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^{2n}, \tag{3.4.5}$$

where $\mathrm{Op}^{\mathrm{w}}(\mathscr{F})$ is the ΨDO with Weyl symbol $\mathscr{F}$ defined in (2.3.13), and $\mathrm{W}_{\psi_{\mathbf{x},\boldsymbol{\xi}}, \psi_{\mathbf{x},\boldsymbol{\xi}}}$ is the Wigner function for the pair $(\psi_{\mathbf{x},\boldsymbol{\xi}}, \psi_{\mathbf{x},\boldsymbol{\xi}})$ defined in (2.3.10). Thus, (3.4.5) is compatible with (3.4.3) if, say, $\mathscr{F} \in \Gamma(\mathbb{R}^{2n})$ so that $\mathrm{Op}^{\mathrm{w}}(\mathscr{F})$ is bounded in $\mathrm{L}^2(\mathbb{R}^n)$.
In Proposition 3.4.1 below we establish the relation between the anti-Wick, Weyl, and Wick symbols of a given ΨDO . For its formulation we need some additional notations and facts.
Further, if $\phi \in \mathscr{S}'(\mathbb{R}^N)$ and $\mathrm{u} \in \mathscr{S}(\mathbb{R}^N)$, we define, as usual, the convolution of ϕ with u as the function

$$(\phi * \mathrm{u})(\mathbf{x}) := (\phi, \mathrm{u}(\mathbf{x} - \cdot))_{\mathscr{S}'(\mathbb{R}^N)}, \quad \mathbf{x} \in \mathbb{R}^N.$$

Thus, if $\mathrm{f}, \mathrm{g} \in \mathscr{S}(\mathbb{R}^N)$, then, of course, $\mathrm{f} * \mathrm{g} \in \mathscr{S}(\mathbb{R}^N)$, and

$$(\mathrm{f} * \mathrm{g})(\mathbf{x}) = \int_{\mathbb{R}^N} \mathrm{f}(\mathbf{x} - \mathbf{y})\mathrm{g}(\mathbf{y})d\mathbf{y} = (\mathrm{g} * \mathrm{f})(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^N.$$

In our next lemma summarize the necessary properties of $\phi * \mathrm{u}$.

Lemma 3.4.1 *Let* $\phi \in \mathscr{S}'(\mathbb{R}^N)$ *and* $\mathrm{u} \in \mathscr{S}(\mathbb{R}^N)$*,* $\mathrm{N} \in \mathbb{N}$*. Then:*
(i) *We have* $\phi * \mathrm{u} \in \mathrm{C}^\infty(\mathbb{R}^N)$ *and the exists* $\mathrm{k} \in \mathbb{Z}_+$ *such that the estimates*

$$|\mathrm{D}^\alpha(\phi * \mathrm{u})(\mathbf{x})| \le \mathrm{C}_\alpha \langle \mathbf{x} \rangle^{\mathrm{k}}, \quad \alpha \in \mathbb{Z}^N_+, \quad \mathbf{x} \in \mathbb{R}^N, \tag{3.4.6}$$

hold true with some constants $\mathrm{C}_\alpha \ge 0$*. In particular,* $\phi * \mathrm{u} \in \mathscr{S}'(\mathbb{R}^N)$.
(ii) *If* $\phi \in \mathscr{S}'_\ell(\mathbb{R}^N)$*, then* $\phi * \mathrm{u} \in \mathrm{C}^\infty_{\mathrm{b}}(\mathbb{R}^N)$.
(iii) *If* $\phi \in \mathscr{E}'(\mathbb{R}^N)$*, then* $\phi * \mathrm{u} \in \mathscr{S}(\mathbb{R}^N)$.

Proof. The claims of the lemma are well known to the experts. For the reader's convenience we include here a short outline of their proofs.
(i) Applying the representation formula for ϕ (see e.g. [**?**, Theorem V.10]), we write

$$\phi = \mathrm{D}^\beta \mathrm{g} \tag{3.4.7}$$

with some $\beta \in \mathbb{Z}^N_+$ and $\mathrm{g} \in \mathrm{C}(\mathbb{R}^N)$ such that

$$|\mathrm{g}(\mathbf{x})| \le \mathrm{C}\langle \mathbf{x} \rangle^{\mathrm{k}}, \quad \mathbf{x} \in \mathbb{R}^N,$$

with $\mathrm{k} \in \mathbb{Z}_+$ and a constant $\mathrm{C} \ge 0$. Then we have

$$(\phi * \mathrm{u})(\mathbf{x}) = \int_{\mathbb{R}^N} \mathrm{g}(\mathbf{y})\mathrm{D}^\beta \mathrm{u}(\mathbf{x} - \mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^N.$$

Differentiating with respect to $\mathbf{x}$, we find that $\phi * u \in C^\infty(\mathbb{R}^N)$ and

$$(D^\alpha(\phi * u))(\mathbf{x}) = \int_{\mathbb{R}^N} g(\mathbf{y}) D^{\alpha+\beta} u(\mathbf{x}-\mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^N. \tag{3.4.8}$$

Then we have

$$\begin{aligned} &|D^\alpha(\phi * u)(\mathbf{x})| \\ &\le \int_{\mathbb{R}^N} |g(\mathbf{y})| \langle \mathbf{y}\rangle^{-k} \frac{\langle \mathbf{y}\rangle^k}{\langle \mathbf{x}-\mathbf{y}\rangle^k} \langle \mathbf{x}-\mathbf{y}\rangle^k |D^{\alpha+\beta} u(\mathbf{x}-\mathbf{y})|\, d\mathbf{y} \\ &\le 2^k C \langle \mathbf{x}\rangle^k \int_{\mathbb{R}^N} \langle \mathbf{y}\rangle^k |D^{\alpha+\beta} u(\mathbf{y})|\, d\mathbf{y} \quad \mathbf{x} \in \mathbb{R}^N, \end{aligned}$$

which implies (3.4.6).
(ii) Let now ϕ has the form (3.4.7) with $g = g_1 + g_2$, $g_1 \in L^1(\mathbb{R}^N)$ and $g_2 \in L^\infty(\mathbb{R}^N)$. Then (3.4.8) implies

$$\sup_{\mathbf{x}\in\mathbb{R}^N} |D^\alpha(\phi * u)(\mathbf{x})| \le \|g_1\|_{L^1(\mathbb{R}^N)} \|D^{\alpha+\beta} u\|_{L^\infty(\mathbb{R}^N)} + \|g_2\|_{L^\infty(\mathbb{R}^N)} \|D^{\alpha+\beta} u\|_{L^1(\mathbb{R}^N)},$$

i.e. $\phi * u \in C_b^\infty(\mathbb{R}^N)$. Now note that $\phi \in \mathscr{S}'_\ell(\mathbb{R}^N)$ is a finite sum of terms of the form (3.4.7) with $g \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$.
(iii) Assume now that ϕ in (3.4.7) is compactly supported. By (3.4.8), we get

$$D^\alpha(\phi * u)(\mathbf{x}) = \int_{\operatorname{supp}\eta} g(\mathbf{y}) D^{\alpha+\beta}(\eta u))(\mathbf{x}-\mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^N,$$

where $\eta \in C_0^\infty(\mathbb{R}^N; [0,1])$ is a cut-off function such that $\eta = 1$ on $\operatorname{supp}\phi$. Therefore,

$$\sup_{\mathbf{x}\in\mathbb{R}^N} \langle \mathbf{x}\rangle^m |D^\alpha(\phi * u)(\mathbf{x})| \le 2^m \sup_{\mathbf{w}\in\mathbb{R}^N} \langle \mathbf{w}\rangle^m |D^{\alpha+\beta}(\eta u))(\mathbf{w})| \int_{\operatorname{supp}\eta} |g(\mathbf{y})| \langle \mathbf{y}\rangle^m\, d\mathbf{y},$$

for any $m \in \mathbb{Z}_+$, i.e. $\phi * u \in \mathscr{S}(\mathbb{R}^N)$. □

Let, as above, $\psi \in \mathscr{S}(\mathbb{R}^n)$, $n \ge 1$, with $\|\psi\|_{L^2(\mathbb{R}^n)} = 1$. Set

$$\Psi := W_{\psi,\psi},$$

i.e. Ψ is the Wigner transform of the pair (ψ, ψ). By (2.3.10), we have

$$\Psi(\mathbf{x}, \boldsymbol{\xi}) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\mathbf{x}'\cdot\boldsymbol{\xi}} \psi(\mathbf{x}+\mathbf{x}'/2) \overline{\psi(\mathbf{x}-\mathbf{x}'/2)}\, d\mathbf{x}', \quad (\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^{2n}. \tag{3.4.9}$$

Note that $\Psi \in \mathscr{S}(\mathbb{R}^{2n})$, Ψ is real-valued, and

$$\int_{\mathbb{R}^{2n}} \Psi(\mathbf{x}, \boldsymbol{\xi})\, d\mathbf{x}\, d\boldsymbol{\xi} = 1. \tag{3.4.10}$$

If f is a function with domain $\Omega \subset \mathbb{R}^N$, $N \ge 1$, invariant with respect to the reflection $\mathbf{x} \mapsto -\mathbf{x}$, we set $f^\#(\mathbf{x}) := f(-\mathbf{x})$, $\mathbf{x} \in \Omega$.
In our next lemma we establish a fundamental relation between the Husimi and the Wigner transforms

Lemma 3.4.2 *Let* $\psi \in \mathscr{S}(\mathbb{R}^n)$, $n \geq 1$, *with* $\|\psi\|_{L^2(\mathbb{R}^n)} = 1$. *Then*

$$Q_{f,g} = \Psi^{\#} * W_{f,g}, \quad f, g \in \mathscr{S}(\mathbb{R}^n). \tag{3.4.11}$$

Proof. We have

$$\begin{aligned}
&\left(\Psi^{\#} * W_{f,g}\right)(\mathbf{x}, \boldsymbol{\xi}) \\
&= \int_{\mathbb{R}^{2n}} \Psi(\mathbf{x}' - \mathbf{x}, \boldsymbol{\xi}' - \boldsymbol{\xi}) W_{f,g}(\mathbf{x}', \boldsymbol{\xi}') \, d\mathbf{x}' \, d\boldsymbol{\xi}' \\
&= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}^n} e^{i(\boldsymbol{\xi}' - \boldsymbol{\xi}) \cdot \mathbf{t}} \, \psi\left(\mathbf{x}' - \mathbf{x} - \frac{\mathbf{t}}{2}\right) \overline{\psi\left(\mathbf{x}' - \mathbf{x} + \frac{\mathbf{t}}{2}\right)} \, d\mathbf{t} \right) \\
&\times \left(\int_{\mathbb{R}^n} e^{i\boldsymbol{\xi}' \cdot \mathbf{s}} \, f\left(\mathbf{x}' - \frac{\mathbf{s}}{2}\right) \overline{g\left(\mathbf{x}' + \frac{\mathbf{s}}{2}\right)} \, d\mathbf{s} \right) d\mathbf{x}' \, d\boldsymbol{\xi}' \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{-i\boldsymbol{\xi} \cdot \mathbf{t}} \, \psi\left(\mathbf{x}' - \mathbf{x} - \frac{\mathbf{t}}{2}\right) \overline{\psi\left(\mathbf{x}' - \mathbf{x} + \frac{\mathbf{t}}{2}\right)} \, f\left(\mathbf{x}' + \frac{\mathbf{t}}{2}\right) \overline{g\left(\mathbf{x}' - \frac{\mathbf{t}}{2}\right)} \, d\mathbf{t} \, d\mathbf{x}' \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi} \cdot \mathbf{y}} e^{i\boldsymbol{\xi} \cdot \mathbf{y}'} \, \psi\left(\mathbf{y}' - \mathbf{x}\right) \overline{\psi(\mathbf{y} - \mathbf{x})} \, f(\mathbf{y}) \, \overline{g(\mathbf{y}')} \, d\mathbf{y} \, d\mathbf{y}' \\
&= \frac{1}{(2\pi)^n} \langle f, \psi_{\mathbf{x},\boldsymbol{\xi}} \rangle_{L^2(\mathbb{R}^n)} \overline{\langle g, \psi_{\mathbf{x},\boldsymbol{\xi}} \rangle}_{L^2(\mathbb{R}^n)} \\
&= Q_{f,g}(\mathbf{x}, \boldsymbol{\xi}), \quad (\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^{2n},
\end{aligned}$$

i.e. we obtain (3.4.11). □

Remark: An appropriate density argument shows that (3.4.11) remains valid for $f, g \in L^2(\mathbb{R}^n)$.

Proposition 3.4.1 *Let* $\psi \in \mathscr{S}(\mathbb{R}^n)$, $n \geq 1$, *with* $\|\psi\|_{L^2(\mathbb{R}^n)} = 1$. *Assume* $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^{2n})$.
(i) *We have*

$$\mathrm{Op}^{\mathrm{aw}}_{\psi}(\mathscr{F}) = \mathrm{Op}^{\mathrm{w}}(\mathscr{F} * \Psi), \tag{3.4.12}$$

where Ψ *is the Wigner transform of* (ψ, ψ) *defined in* (3.4.9), *and* $\mathrm{Op}^{\mathrm{w}}(\mathscr{F} * \Psi)$ *is the* ΨDO *with Weyl symbol* $\mathscr{F} * \Psi$.
(ii) *The Wick symbol of the Weyl* ΨDO $\mathrm{Op}^{\mathrm{w}}(\mathscr{F})$ *satisfies*

$$\mathscr{F}^{\mathrm{wick}}_{\psi}\left(\mathrm{Op}^{\mathrm{w}}(\mathscr{F})\right) = \mathscr{F} * \Psi^{\#}. \tag{3.4.13}$$

(iii) *The Wick symbol of the anti-Wick* ΨDO $\mathrm{Op}^{\mathrm{aw}}_{\psi}(\mathscr{F})$ *satisfies*

$$\mathscr{F}^{\mathrm{wick}}_{\psi}\left(\mathrm{Op}^{\mathrm{aw}}_{\psi}(\mathscr{F})\right) = \mathscr{F} * \Psi * \Psi^{\#}. \tag{3.4.14}$$

Proof. (i) By (2.3.13) we have

$$\left(\mathrm{Op}^{\mathrm{w}}(\mathscr{F} * \Psi) f, g\right)_{\mathscr{S}'(\mathbb{R}^n)} = \left(\mathscr{F} * \Psi, W_{f,\bar{g}}\right)_{\mathscr{S}'(\mathbb{R}^{2n})} = \left(\mathscr{F}, \Psi^{\#} * \Psi_{f,\bar{g}}\right)_{\mathscr{S}'(\mathbb{R}^{2n})} \tag{3.4.15}$$

Putting together (3.4.4), (3.4.15) and (3.4.11), we obtain (3.4.12).
(ii) A simple calculation yields

$$W_{\psi_{\mathbf{x},\boldsymbol{\xi}}, \psi_{\mathbf{x},\boldsymbol{\xi}}}(\mathbf{x}', \boldsymbol{\xi}') = W_{\psi,\psi}(\mathbf{x}' - \mathbf{x}, \boldsymbol{\xi}' - \boldsymbol{\xi}), \quad (\mathbf{x}, \boldsymbol{\xi}), (\mathbf{x}', \boldsymbol{\xi}') \in \mathbb{R}^{2n}. \tag{3.4.16}$$

Now, (3.4.13) follows directly from (3.4.5) and (3.4.16).
(iii) Putting together (3.4.12) and (3.4.13), we immediately obtain (3.4.14). □

By Lemma 3.4.1 (i), we find that for a general $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^{2n})$, there exists $k \in \mathbb{Z}_+$ such that $\mathscr{F} * \Psi \in \Gamma_0^k(\mathbb{R}^{2n})$.
By Lemma 3.4.1 (ii), if $\mathscr{F} \in \mathscr{S}_\ell'(\mathbb{R}^{2n})$, then $\mathscr{F} * \Psi \in C_b(\mathbb{R}^{2n}) \subset \Gamma(\mathbb{R}^{2n})$. Therefore, the Calderón-Vaillancourt theorem (see Proposition 2.3.4) and Proposition 3.4.1 (i) imply that in this case the operator $\mathrm{Op}_\psi^{aw}(\mathscr{F}) = \mathrm{Op}^w(\mathscr{F} * \Psi)$ is bounded in $L^2(\mathbb{R}^{2n})$.
Similarly, by Lemma 3.4.1 (iii), if $\mathscr{F} \in \mathscr{E}'(\mathbb{R}^{2n})$, then $\mathscr{F} * \Psi \in \mathscr{S}(\mathbb{R}^{2n})$ so that the integral kernel of the $\mathrm{Op}_\psi^{aw}(\mathscr{F})$ is in $\mathscr{S}(\mathbb{R}^{2n})$. Therefore, $\mathrm{Op}_\psi^{aw}(\mathscr{F}) \in \mathfrak{S}_p(L^2(\mathbb{R}^{2n}))$ for any $p \in (0,\infty)$ (see [19]).
In our next proposition, we prepare the discussion of the properties of the operators $\mathrm{Op}_\psi^{aw}(\mathscr{F})$ with $\mathscr{F} \in \Gamma_\rho^\gamma(\mathbb{R}^n)$.

Proposition 3.4.2 *Assume that $\psi \in \mathscr{S}(\mathbb{R}^n)$, $n \geq 1$, with $\|\psi\|_{L^2(\mathbb{R}^n)} = 1$. Let $\mathscr{F} \in \Gamma_\rho^\gamma(\mathbb{R}^{2n})$ with $\gamma \in \mathbb{R}$ and $\rho \in (0,1]$. Then for each $N \in \mathbb{Z}_+$ we have*

$$\mathscr{F} * \Psi - \sum_{\alpha \in \mathbb{Z}_+^{2n}: |\alpha| \leq N} c_\alpha D^\alpha \mathscr{F} \in \Gamma_\rho^{\gamma - \rho(N+1)}(\mathbb{R}^{2n}), \tag{3.4.17}$$

where, as above, $\Psi = W_{\psi,\psi}$, and

$$c_\alpha := \frac{1}{\alpha!} \int_{\mathbb{R}^{2n}} \mathbf{w}^\alpha \Psi(-\mathbf{w})\, d\mathbf{w}, \quad \alpha \in \mathbb{Z}_+^{2n}. \tag{3.4.18}$$

Proof. We have

$$(\mathscr{F} * \Psi)(\mathbf{w}') = \int_{\mathbb{R}^{2n}} \mathscr{F}(\mathbf{w}' + \mathbf{w}) \Psi(-\mathbf{w})\, d\mathbf{w}, \quad \mathbf{w}' \in \mathbb{R}^{2n}, \tag{3.4.19}$$

and

$$\mathscr{F}(\mathbf{w}' + \mathbf{w}) = \sum_{\alpha \in \mathbb{Z}_+^{2n}: |\alpha| \leq N} \frac{\mathbf{w}^\alpha}{\alpha!} (D^\alpha \mathscr{F})(\mathbf{w}') + r_N(\mathbf{w}, \mathbf{w}'), \quad \mathbf{w}, \mathbf{w}' \in \mathbb{R}^{2n}, \tag{3.4.20}$$

where

$$r_N(\mathbf{w}, \mathbf{w}') = \sum_{\alpha \in \mathbb{Z}_+^{2n}: |\alpha| = N+1} c'_\alpha \int_0^1 (D^\alpha \mathscr{F})(\mathbf{w} + t\mathbf{w}')(1-t)^N\, dt$$

with some constant coefficients c'_α. Inserting (3.4.20) into (3.4.19), we obtain

$$(\mathscr{F} * \Psi)(\mathbf{w}') = \sum_{\alpha \in \mathbb{Z}_+^{2n}: |\alpha| \leq N} c_\alpha (D^\alpha \mathscr{F})(\mathbf{w}') + R_N(\mathbf{w}'), \quad \mathbf{w}' \in \mathbb{R}^{2n},$$

where

$$R_N(\mathbf{w}') = \int_{\mathbb{R}^{2n}} r_N(\mathbf{w}, \mathbf{w}') \Psi(-\mathbf{w})\, d\mathbf{w}.$$

Hence, in order to prove (3.4.17), it remains to check for any $\alpha \in \mathbb{Z}_+^{2n}$ with $|\alpha| = N+1$ we have

$$R_{N,\alpha} \in \Gamma_\rho^{\gamma - \rho(N+1)}(\mathbb{R}^{2n}) \tag{3.4.21}$$

where

$$R_{N,\alpha}(\mathbf{w}') := \int_{\mathbb{R}^{2n}} \int_0^1 \mathbf{w}^\alpha D^\alpha \mathscr{F}(\mathbf{w}' + t\mathbf{w})(1-t)^N \Psi(-\mathbf{w})\, dt\, d\mathbf{w}.$$

Since $D^\alpha \mathscr{F} \in \Gamma_\rho^{\gamma-\rho|\alpha|}(\mathbb{R}^{2n})$ for every $\alpha \in \mathbb{Z}_+^{2n}$, and $\rho > 0$, we can assume without loss of generality that $\gamma - \rho(N+1) < 0$. For $\beta \in \mathbb{Z}_+^{2n}$, we have

$$(D^\beta R_{N,\alpha})(\mathbf{w}') = \int_{\mathbb{R}^{2n}} \int_0^1 \mathbf{w}^\alpha D^{\alpha+\beta} \mathscr{F}(\mathbf{w}' + t\mathbf{w})(1-t)^N \Psi(-\mathbf{w})\, dt\, d\mathbf{w},$$

and therefore,

$$|(D^\beta R_{N,\alpha})(\mathbf{w}')| \\ \leq C_{\alpha,\beta} \int_{\mathbb{R}^{2n}} \left(\int_0^1 \langle \mathbf{w}' + t\mathbf{w} \rangle^{\gamma-\rho(N+1+|\beta|)}\, dt \right) \langle \mathbf{w} \rangle^\alpha |\Psi(-\mathbf{w})|\, d\mathbf{w}, \quad \mathbf{w}' \in \mathbb{R}^{2n}. \tag{3.4.22}$$

Now that if $|\mathbf{w}| \leq |\mathbf{w}'|/2$ and $\delta > 0$, then

$$\int_0^1 \langle \mathbf{w}' + t\mathbf{w} \rangle^{-\delta}\, dt \leq \int_0^1 (1-t/2)^{-\delta}\, dt\, \langle \mathbf{w}' \rangle^{-\delta},$$

and if $|\mathbf{w}| > |\mathbf{w}'|/2$ and $\delta > 0$, then

$$\int_0^1 \langle \mathbf{w}' + t\mathbf{w} \rangle^{-\delta}\, dt \leq 1 \leq 2^{\delta'} \langle \mathbf{w} \rangle^{\delta'} \langle \mathbf{w}' \rangle^{-\delta'},$$

for any $\delta' > 0$. Since $\Psi \in \mathscr{S}(\mathbb{R}^{2n})$, we find that (3.4.22) implies (3.4.21). □

Remark: Since $\int_{\mathbb{R}^{2n}} \Psi(\mathbf{w})\, d\mathbf{w} = 1$, we have $c_0 = 1$ in (3.4.18). Moreover, if Ψ is invariant under the reflection $\mathbf{w} \mapsto -\mathbf{w}$, then $c_\alpha = 0$ for $|\alpha|$ odd.

Combining Proposition 3.4.2 and 2.3.7, we obtain the following

Corollary 3.4.1 *Let ψ satisfy the assumptions of Proposition 3.4.2. Suppose that $\mathscr{F} \in \Gamma_\rho^\gamma(\mathbb{R}^{2n})$ with $\gamma > 0$ and $\rho \in (0,1]$. Then for each $s \geq \gamma$ the operator $\mathrm{Op}_\psi^{aw}(\mathscr{F})$ extends to a continuous mapping from $\mathfrak{L}^s(\mathbb{R}^n)$ into $\mathfrak{L}^{s-\gamma}(\mathbb{R}^n)$.*

Next, we state a variant of Proposition 3.4.2 for the case of polynomial $\mathscr{F}$.

Proposition 3.4.3 *Let ψ satisfy the assumptions of Proposition 3.4.2. Suppose that $\mathscr{F}$ is a polynomial of degree $\gamma \in \mathbb{Z}_+$. Then we have*

$$\mathscr{F} * \Psi = \sum_{\alpha \in \mathbb{Z}_+^{2n} : |\alpha| \leq \gamma} c_\alpha D^\alpha \mathscr{F},$$

the coefficients c_α defined in (3.4.18).

Proposition 3.4.2 is a straightforward extension of [185, Theorem 24.1] where

$$\psi(\mathbf{x}) = \pi^{-n/4} e^{-|\mathbf{x}|^2}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{3.4.23}$$

and consequently

$$\Psi(\mathbf{x},\boldsymbol{\xi}) = \pi^{-n} e^{-|\mathbf{x}|^2-|\boldsymbol{\xi}|^2}, \quad \mathbf{x},\boldsymbol{\xi} \in \mathbb{R}^{2n}.$$

The operators $\mathrm{Op}^{\mathrm{aw}}_{\psi}(\mathscr{F})$ with such ψ are the standard anti-Wick ΨDOs . Let us assume that ψ is of form (3.4.23), and discuss in more detail the properties of $\mathrm{Op}^{\mathrm{aw}}(\mathscr{F}) = \mathrm{Op}^{\mathrm{aw}}_{\psi}(\mathscr{F})$. First, note the elementary fact that if $\mathscr{F}$ is a polynomial of degree $\gamma \le 1$, then the Weyl symbol of $\mathrm{Op}^{\mathrm{aw}}(\mathscr{F})$ coincides with its anti-Wick symbol $\mathscr{F}$. Therefore, the mapping $\mathscr{F} \mapsto \mathrm{Op}^{\mathrm{aw}}(\mathscr{F})$ is a quantization which satisfies the Axioms 1, 2, and 3, in Section 5.4.2. As a special case of the mapping $\mathscr{F} \mapsto \mathrm{Op}^{\mathrm{cnv}}(\mathscr{F})$, this quantization possesses also the important positivity property, i.e $\mathscr{F} \ge 0$ implies $\mathrm{Op}^{\mathrm{aw}}(\mathscr{F}) \ge 0$, which is not valid for the Weyl quantization. Note, however, that not any given Weyl ΨDO $\mathrm{Op}^{\mathrm{w}}(\mathscr{F}^{\mathrm{w}})$ with Weyl symbol $\mathscr{F}^{\mathrm{w}} \in \mathscr{S}'(\mathbb{R}^{2n})$ has an anti-Wick symbol $\mathscr{F}^{\mathrm{aw}} \in \mathscr{S}'(\mathbb{R}^{2n})$. In fact, by (3.4.12), the symbol $\mathscr{F}^{\mathrm{aw}}$ should satisfy the equation

$$\mathscr{F}^{\mathrm{w}} = \mathscr{F}^{\mathrm{aw}} * \Psi, \tag{3.4.24}$$

i.e. in order to find $\mathscr{F}^{\mathrm{aw}}$ for a given $\mathscr{F}^{\mathrm{w}}$, we should invert the Weierstrass transform, or, equivalently, to solve the inverse heat equation (see [185, Remark 24.2]). Note that the equation (3.4.24) is equivalent to

$$\widehat{\mathscr{F}^{\mathrm{w}}} = (2\pi)^{n}\widehat{\Psi}\widehat{\mathscr{F}^{\mathrm{aw}}}, \tag{3.4.25}$$

the distribution $\widehat{\Psi}\widehat{\mathscr{F}^{\mathrm{aw}}} \in \mathscr{S}'(\mathbb{R}^{2n})$ being well defined since $\widehat{\Psi} \in \mathscr{S}(\mathbb{R}^{2n})$ and $\widehat{\mathscr{F}^{\mathrm{aw}}} \in \mathscr{S}'(\mathbb{R}^{2n})$. Thus, for example, if $0 \ne \mathscr{F}^{\mathrm{w}} \in C_0^\infty(\mathbb{R}^{2n})$, then there are no solutions $\mathscr{F}^{\mathrm{aw}} \in \mathscr{S}'(\mathbb{R}^{2n})$ of (3.4.24). On the other hand, if $\widehat{\mathscr{F}^{\mathrm{w}}} \in C_0^\infty(\mathbb{R}^{2n})$, this equation admits even a solution $\mathscr{F}^{\mathrm{aw}} \in \mathscr{S}(\mathbb{R}^{2n})$. Since any anti-Wick ΨDO admits a Weyl symbol, the products of two anti-Wick ΨDOs is a Weyl ΨDO , but this product may not have an anti-Wick symbol which is a serious drawback of the anti-Wick quantization.
Further, if (3.4.24) has a solution $\mathscr{F}^{\mathrm{aw}} \in \mathscr{S}'(\mathbb{R}^{2n})$, then this solution is unique. This follows from the fact that

$$\widehat{\Psi}(\mathbf{x},\boldsymbol{\xi}) = (2\pi)^{-n} e^{-(|\mathbf{x}|^2+\boldsymbol{\xi}|^2)/4}, \quad (\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^{2n},$$

so that $\widehat{\Psi}$ in (3.4.25) vanishes nowhere in $\mathbb{R}^{2n}$. Note that if Ψ in (3.4.24) is the Wigner transform of (ψ,ψ) for an arbitrary $\psi \in \mathscr{S}(\mathbb{R}^n)$ with $\|\psi\|_{L^2(\mathbb{R}^n)} = 1$, then the last claim may be false since in the general case the set

$$\mathscr{Z} := \left\{(\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^{2n} \,|\, \widehat{\Psi}(\mathbf{x},\boldsymbol{\xi}) = 0\right\}$$

may be non-empty, so that the linear mapping $\mathscr{F}^{\mathrm{aw}} \mapsto \mathscr{F}^{\mathrm{aw}} * \Psi$ may have a non-trivial kernel

$$\left\{\mathscr{F}^{\mathrm{aw}} \in \mathscr{S}'(\mathbb{R}^{2n}) \,|\, \mathrm{supp}\, \widehat{\mathscr{F}^{\mathrm{aw}}} \subset \mathscr{Z}\right\}.$$

Finally, if the symbol $\mathscr{F}^{\mathrm{w}}$ in (3.4.24) is a polynomial of degree $\gamma \in \mathbb{Z}_+$, and Ψ in (3.4.24) is the Wigner transform of (ψ,ψ) for an arbitrary $\psi \in \mathscr{S}(\mathbb{R}^n) = 1$ with $\|\psi\|_{L^2(\mathbb{R}^n)} = 1$, then Proposition 3.4.3 implies that there is a solution $\mathscr{F}^{\mathrm{aw}} \in \mathscr{S}'(\mathbb{R}^{2n})$ which is a polynomial of degree γ. A circumstance which may help to understand the last claim, is the fact that if $\mathscr{F}$ is a polynomial of degree $\gamma \in \mathbb{Z}_+$, then $\mathrm{supp}\,\widehat{\mathscr{F}} = \{0\}$ while $\widehat{\Psi}(0) \neq 0$ since $\int_{\mathbb{R}^{2n}} \Psi(\mathbf{w})d\mathbf{w} = 1$.

Our next goal is to consider the anti-Wick quantifucation $\mathrm{Op}^{\mathrm{aw}}_{\psi}$ in the special cases where $n = 1$ and $\psi = \psi_q$, $q \in \mathbb{Z}_+$, are the normalized eigenfunctions of the harmonic oscillator $\mathfrak{h}$, introduced in (2.7.14). Since $\psi_q \in \mathscr{S}(\mathbb{R})$ and $\|\psi_q\|_{L^2(\mathbb{R})} = 1$, all the general theory of the operators $\mathrm{Op}^{\mathrm{aw}}_{\psi}(\mathscr{F})$ remains valid. If no confusion is likely to occur, we write $\mathrm{Op}^{\mathrm{aw}}_{q}(\mathscr{F})$ instead of $\mathrm{Op}^{\mathrm{aw}}_{\psi_q}(\mathscr{F})$. As mentioned above, the operator $\mathrm{Op}^{\mathrm{aw}}_{0}(\mathscr{F})$ coincides with the standard anti-Wick operators if $n = 1$. That is why, we will call the operator $\mathrm{Op}^{\mathrm{aw}}_{q}(\mathscr{F})$ the anti-Wick operator of order q. In accordance with our general notations, we set

$$\psi_{q;x,\xi}(y) := e^{iy\xi}\,\psi_q(y-x), \quad y \in \mathbb{R}, \quad (x,\xi) \in \mathbb{R}^2, \quad q \in \mathbb{Z}_+.$$

Moreover, put

$$\Psi_{q,\ell} := W_{\psi_q,\psi_\ell}, \quad q,\ell \in \mathbb{Z}_+. \tag{3.4.26}$$

i.e. $\Psi_{q,\ell}$ is the Wigner transform of the pair (ψ_q,ψ_ℓ). If $\ell = q$ we write

$$\Psi_q := \Psi_{q,q} = W_{\psi_q,\psi_q}, \quad q \in \mathbb{Z}_+. \tag{3.4.27}$$

Proposition 3.4.1 (i) immediately entails the following

Corollary 3.4.2 *Let $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^2)$. Then we have*

$$\mathrm{Op}^{\mathrm{aw}}_{q}(\mathscr{F}) = \mathrm{Op}^{\mathrm{w}}(\mathscr{F} * \Psi_q), \quad q \in \mathbb{Z}_+. \tag{3.4.28}$$

In the sequel,we will need explicit expressions for $\Psi_{q,r}$ and the Fourier transform $\widehat{\Psi_q}$. The following two lemmas contains the corresponding calculations.

Lemma 3.4.3 *Let $q,\ell \in \mathbb{Z}_+$. Then for $(x,\xi) \in \mathbb{R}^2$ we have*

$$\Psi_{q,\ell}(x,\xi) =$$

$$\begin{cases} \frac{1}{\pi}(-1)^{\ell}2^{\frac{q-\ell}{2}}\left(\frac{\ell!}{q!}\right)^{1/2}(x+i\xi)^{q-\ell}L^{(q-\ell)}_{\ell}(2(x^2+\xi^2))e^{-(x^2+\xi^2)}, & q \geq \ell, \\ \frac{1}{\pi}(-1)^{q}2^{\frac{\ell-q}{2}}\left(\frac{q!}{\ell!}\right)^{1/2}(x-i\xi)^{\ell-q}L^{(\ell-q)}_{q}(2(x^2+\xi^2))e^{-(x^2+\xi^2)}, & q \leq \ell, \end{cases} \tag{3.4.29}$$

where $L^{(\alpha)}_q$ are the generalized Laguerre polynomials defined in (2.7.38).
In particular,

$$\Psi_{q,\ell}(r\cos\theta, r\sin\theta) = e^{i(k-\ell)\theta}\Phi_{k,\ell}(r), \quad k,\ell \in \mathbb{Z}_+, \quad \theta \in [0,2\pi), \quad r \in [0,\infty), \tag{3.4.30}$$

where $\left\{\Phi_{q,\ell}(r)\right\}_{q,\ell\in\mathbb{Z}_+}$ is a symmetric real valued matrix. Moreover,

$$\Psi_q(x,\xi) = \frac{1}{\pi}(-1)^k L_q(2(x^2+\xi^2))e^{-(x^2+\xi^2)}, \quad q \in \mathbb{Z}_+, \quad (x,\xi) \in \mathbb{R}^2, \tag{3.4.31}$$

where L_q are the Laguerre polynomials defined in (2.7.30).

Proof. An elementary calculation taking into account the parity of the Hermite polynomials easily yields

$$\Psi_{q,\ell}(x,\xi) = \frac{(-1)^k}{(2\pi)\sqrt{\pi}(k!\ell!)^{1/2}2^{\frac{k+\ell}{2}}} e^{-(x^2+\xi^2)} \int_{\mathbb{R}} e^{-(\frac{y}{2}+i\xi)^2} H_q\left(\frac{y}{2}-x\right) H_\ell\left(\frac{y}{2}+x\right) dy. \tag{3.4.32}$$

Changing the variable $\frac{y}{2}+i\xi = t$, and applying a standard complex-analysis argument in order to replace the interval of integration $\mathbb{R}+i\xi$ by $\mathbb{R}$, we get

$$\int_{\mathbb{R}} e^{-(\frac{y}{2}+i\xi)^2} H_q\left(\frac{y}{2}+x\right) H_\ell\left(\frac{y}{2}-x\right) dy = 2\int_{\mathbb{R}} e^{-t^2} H_q\left(t-x-i\xi\right) H_\ell\left(t+x-i\xi\right) dt. \tag{3.4.33}$$

By [86, Eq. (7.377)],

$$\int_{\mathbb{R}} e^{-t^2} H_q\left(t-x-i\xi\right) H_\ell\left(t+x-i\xi\right) dt = \begin{cases} 2^q\sqrt{\pi}\ell!(-x-i\xi)^{q-\ell}L_\ell^{(q-\ell)}(2(x^2+\xi^2)), & q \geq \ell, \\ 2^\ell\sqrt{\pi}q!(x-i\xi)^{\ell-q}L_q^{(\ell-q)}(2(x^2+\xi^2)), & q \leq \ell. \end{cases} \tag{3.4.34}$$

Putting together (3.4.32), (3.4.33), and (3.4.34), we obtain (3.4.29). □

Lemma 3.4.4 *Let* $q \in \mathbb{Z}_+$. *Then*

$$\widehat{\Psi_q}(x,\xi) = \frac{1}{2\pi} L_q((x^2+\xi^2)/2)e^{-(x^2+\xi^2)/4}, \quad (x,\xi) \in \mathbb{R}^2. \tag{3.4.35}$$

Proof. By

$$\Psi_q(x,\xi) = (2\pi)^{-1} \int_{\mathbb{R}} e^{itx}\,\psi_q(x-t/2)\,\psi_q(x+t/2)\,dt, \quad (x,\xi) \in \mathbb{R}^2,$$

we easily find that

$$\widehat{\Psi_q}(x,\xi) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-itx}\,\psi_q(t-\xi/2)\,\psi_q(t+\xi/2)\,dt, \quad (x,\xi) \in \mathbb{R}^2.$$

Changing the variables $t = y/2$ and taking into account the parity of ψ_q, we find that

$$\widehat{\Psi_q}(x,\xi) = \frac{(-1)^q}{2}\Psi_q\left(\xi/2, x/2\right), \quad (x,\xi) \in \mathbb{R}^2. \tag{3.4.36}$$

Now (3.4.36) and (3.4.31) entail (3.4.35). □

Corollary 3.4.3 (i) *Let* $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^2)$. *Then* $\mathscr{F} * \Psi_0 = 0$ *or, equivalently,* $\mathrm{Op}_0^{aw}(\mathscr{F}) = 0$, *if and only if* $\mathscr{F} = 0$.
(ii) *Let* $q \in \mathbb{Z}_+$. *Then* $\mathscr{F} * \Psi_q = 0$ *or, equivalently,* $\mathrm{Op}_q^{aw}(\mathscr{F}) = 0$, *if and only if*

$$\mathrm{supp}\,\mathscr{F} \subset \bigcup_{m=1}^{q} \left\{(x,\xi) \in \mathbb{R}^2 \,|\, x^2+\xi^2 = 2r_{m,q}\right\}$$

where $r_{m,q} > 0$ *is the* m*th root of the Laguerre polynomial* L_q.

Proof. Similarly to (3.4.25), we have

$$\widehat{\mathscr{F}*\Psi_q} = 2\pi\widehat{\Psi_q}\widehat{\mathscr{F}}.$$

Therefore, the kernel of the linear mapping $\mathscr{F}\mapsto \mathrm{Op}_q^{aw}(\mathscr{F})$ coincides with

$$\left\{\mathscr{F}\in\mathscr{S}'(\mathbb{R}^2)\,|\,\mathrm{supp}\,\widehat{\mathscr{F}}\subset\mathscr{Z}_q\right\}$$

where

$$\mathscr{Z}_q = \left\{(x,\xi)\in\mathbb{R}^2\,|\,\widehat{\Psi}_q(x,\xi)=0\right\}.$$

By (3.4.35), we get

$$\mathscr{Z}_0=\emptyset,\quad \mathscr{Z}_q=\bigcup_{m=1}^{q}\left\{(x,\xi)\in\mathbb{R}^2\,|\,x^2+\xi^2=2r_{m,q}\right\},\quad q\in\mathbb{N},$$

which implies the claim. □

In our next proposition close in spirit to Proposition 3.3.3, we introduce a class of normal Weyl ΨDOs with radial symbols whose spectrum is pure point and the eigenvalues admit an explicit description.

Proposition 3.4.4 *Let $\mathscr{F}\in\mathscr{S}'(\mathbb{R}^2)$ be radially symmetric. Then $\mathrm{Op}^w(\mathscr{F}):\mathscr{S}(\mathbb{R})\to\mathscr{S}'(\mathbb{R})$ extends to an operator normal in $L^2(\mathbb{R})$ for which the Hermite functions $\{\psi_q\}_{q\in\mathbb{Z}_+}$ defined in* (2.7.14) *are eigenfunctions with eigenvalues*

$$\widetilde{\lambda}_q(\mathscr{F}) := \left(\mathscr{F},\Psi_q\right)_{\mathscr{S}'(\mathbb{R}^2)},\quad q\in\mathbb{Z}_+, \tag{3.4.37}$$

where Ψ_q is the Wigner function defined in (3.4.27)*. In particular, if $\mathscr{F}\in L^1(\mathbb{R}^2)+L^\infty(\mathbb{R}^2)$, then*

$$\begin{aligned}\widetilde{\lambda}_q(\mathscr{F}) &= \int_{\mathbb{R}^2}\mathscr{F}(x,\xi)\Psi_q(x,\xi)\,dx d\xi\\ &= \frac{(-1)^q}{2}\int_0^\infty \mathscr{R}_{\mathscr{F}}(t/2)\,L_q(t)\,e^{-t/2}\,dt,\quad q\in\mathbb{Z}_+,\end{aligned} \tag{3.4.38}$$

where

$$\mathscr{R}_{\mathscr{F}}(x^2+\xi^2)=\mathscr{F}(x,\xi),\quad (x,\xi)\in\mathbb{R}^2.$$

Moreover,

$$\mathfrak{D}(\mathrm{Op}^w(\mathscr{F}))=\left\{u=\sum_{q\in\mathbb{Z}_+}u_q\psi_q\,|\,\sum_{q\in\mathbb{Z}_+}(1+|\widetilde{\lambda}_q(\mathscr{F})|^2)|u_q|^2<\infty\right\}. \tag{3.4.39}$$

Remark: A result closely related to Proposition 3.4.4 is [214, Theorem 24.5].
Proof of Proposition 3.4.4: By (3.4.30), for any $\ell, q\in\mathbb{Z}_+$ we have

$$\left(\mathrm{Op}^w(\mathscr{F})\psi_\ell,\psi_q\right)_{\mathscr{S}'(\mathbb{R})}=\left(\mathscr{F},\Psi_{q,\ell}\right)_{\mathscr{S}'(\mathbb{R}^2)}=\delta_{q\ell}\left(\mathscr{F},\Psi_q\right)_{\mathscr{S}'(\mathbb{R}^2)}=\delta_{q\ell}\widetilde{\lambda}_q(\mathscr{F}).$$

Therefore, if

$$\mathscr{S}(\mathbb{R}) \ni u = \sum_{q \in \mathbb{Z}_+} u_q \psi_q$$

where $u_q := \langle u, \psi_q \rangle_{L^2(\mathbb{R})}$, then $\mathrm{Op}^w(\mathscr{F})u \in \mathscr{S}'(\mathbb{R})$ has Fourier coefficients

$$\left(\mathrm{Op}^w(\mathscr{F})u, \psi_q\right)_{\mathscr{S}'(\mathbb{R})} = \widetilde{\lambda}_q, \quad q \in \mathbb{Z}_+.$$

Let us now prove that for any $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^2)$ there exists $p \in \mathbb{Z}_+$ and $C \in [0, \infty)$ such that

$$|\widetilde{\lambda}_q(\mathscr{F})| \leq C(1+q)^p, \quad q \in \mathbb{Z}_+. \tag{3.4.40}$$

By the representation theorem for $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^2)$ (see e.g. [162, Theorem V.10]),there exist $\mathscr{G} \in C(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, $\beta \in \mathbb{Z}_+^2$, and $\ell \in \mathbb{Z}_+$ such that

$$\mathscr{F} = D^\beta \left(\langle \cdot \rangle^\ell \mathscr{G} \right).$$

Therefore,

$$\widetilde{\lambda}_q(\mathscr{F}) = (-1)^{|\beta|} \int_{\mathbb{R}^2} \mathscr{G}(\mathbf{w}) \langle \mathbf{w} \rangle^\ell D^\beta \Psi_q(\mathbf{w}) d\mathbf{w}. \tag{3.4.41}$$

Let $\widetilde{\mathbf{H}} := -\frac{1}{2}\Delta + 2|\mathbf{w}|^2$ be the distorted harmonic oscillator, self-adjoint in $L^2(\mathbb{R}^2)$. By (3.4.31), [150, Section A1], and (2.7.14), we have

$$\begin{aligned}
\Psi_q(x,\xi) &= \frac{(-1)^q}{\pi} L_q(2(x^2+\xi^2)) e^{-(x^2+\xi^2)} \\
&= \frac{1}{2^{2q}\pi} \sum_{m=0}^{q} \frac{H_{2m}(\sqrt{2}x) H_{2q-2m}(\sqrt{2}\xi)}{m!(q-m)!} e^{-(x^2+\xi^2)} \\
&= \frac{1}{2^q} \sum_{m=0}^{q} \frac{\psi_{2m}(\sqrt{2}x)\psi_{2q-2m}(\sqrt{2}\xi)}{\sqrt{m!(q-m)!}}, \quad (x,\xi) \in \mathbb{R}^2.
\end{aligned}$$

Therefore, Ψ_q is an eigenfunction of $\widetilde{\mathbf{H}}$ with eigenvalue $2(q+1)$, and, by (3.4.41), we have

$$\widetilde{\lambda}_q(\mathscr{F}) = (-1)^{|\beta|} 2^p (1+q)^p \int_{\mathbb{R}^2} \mathscr{G}(\mathbf{w}) \langle \mathbf{w} \rangle^\ell (D^\beta \widetilde{\mathbf{H}}^{-p} \Psi_q)(\mathbf{w}) d\mathbf{w}. \tag{3.4.42}$$

for any $p \in \mathbb{Z}_+$.On the other hand, evidently, there exists $p \in \mathbb{Z}_+$ such that the operator $\langle \cdot \rangle^\ell D^\beta \widetilde{\mathbf{H}}^{-p}$is bounded in $L^2(\mathbb{R}^2)$. Then, (3.4.42) implies

$$|\widetilde{\lambda}_q(\mathscr{F})| \leq 2^p (1+q)^p \| \langle \cdot \rangle^\ell D^\beta \widetilde{\mathbf{H}}^{-p} \| \, \|\mathscr{G}\|_{L^2(\mathbb{R}^2)} \|\Psi_q\|_{L^2(\mathbb{R}^2)}. \tag{3.4.43}$$

Moreover,

$$\|\Psi_q\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{\pi} \int_0^\infty e^{-2t} L_q(2t)^2 \, dt = \frac{1}{2\pi}. \tag{3.4.44}$$

Now, (3.4.40) follows from (3.4.43) and (3.4.44). Further, by [162, Theorem V.13], $u \in \mathscr{S}(\mathbb{R})$ implies

$$\sup_{q \in \mathbb{Z}_+} (1+q)^j |u_q| < \infty$$

for each $j \in \mathbb{Z}_+$. Then, by (3.4.40), we have

$$\sup_{q\in\mathbb{Z}_+}(1+q)^j|\widetilde{\lambda}_q(\mathscr{F})||u_q| < \infty$$

for each $j \in \mathbb{Z}_+$, and again by [162, Theorem V.13] we have $\mathrm{Op}^w(\mathscr{F})u \in \mathscr{S}(\mathbb{R})$. Thus $\mathrm{Op}^w(\mathscr{F}) : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$,

$$\mathrm{Op}^w(\mathscr{F})\psi_q = \widetilde{\lambda}_q(\mathscr{F})\psi_q, \quad q \in \mathbb{Z}_+,$$

and

$$\|\mathrm{Op}^w(\mathscr{F})u\|^2_{L^2(\mathbb{R})} = \sum_{q\in\mathbb{Z}_+}|\widetilde{\lambda}_q(\mathscr{F})|^2|u_q|^2, \quad u \in \mathscr{S}(\mathbb{R}),$$

which allows us to extend $\mathrm{Op}^w(\mathscr{F})$ to an operator with domain $\mathfrak{D}(\mathrm{Op}^w(\mathscr{F}))$ defined in (3.4.39), normal in $L^2(\mathbb{R})$. □

In our next theorem we establish the relation between the operator $p_q\mathscr{F}p_q$ and an appropriate anti-Wick ΨDO. For its formulation we need the following notations

$$(O_b u)(x,y) = u(-b^{-1/2}y, -b^{-1/2}x), \quad (x,y) \in \mathbb{R}^2. \tag{3.4.45}$$

By duality, O_b extends to $\mathscr{S}'(\mathbb{R}^2)$. In particular, $b^{-1/2}O_b : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a unitary operator. If no misunderstanding is likely to occur, we write

$$\phi_b := O_b\,\phi, \quad \phi \in \mathscr{S}'(\mathbb{R}^2). \tag{3.4.46}$$

In the statement and the proof of the following theorem, we systematically use the representation

$$L^2(\mathbb{R}^2_{x,y}) = L^2(\mathbb{R}_x) \otimes L^2(\mathbb{R}_y).$$

Theorem 3.4.1 *Let* $q \in \mathbb{Z}_+$ *and* $\mathscr{F} \in \mathscr{S}'_\ell(\mathbb{R}^2)$, $\ell \in \mathbb{Z}_+$. *Then we have*

$$\mathscr{W}_b^* p_q\mathscr{F}p_q\mathscr{W}_b = \pi_q \otimes \mathrm{Op}_q^{aw}(\mathscr{F}_b), \tag{3.4.47}$$

where $\mathscr{W}_b$ *is the unitary operator defined in* (2.7.19)*,* $p_q\mathscr{F}p_q$ *is considered as an operator acting in* $L^2(\mathbb{R}^2)$*, and* π_q *is the orthogonal projection onto*

$$\mathrm{Ker}(\mathfrak{h} - (2q+1)I),$$

introduced in (2.7.26)*.*

Proof. Assume at first $\mathscr{F} \in \mathscr{S}(\mathbb{R}^2)$. Then, by (2.3.19) and (2.7.18),

$$\mathscr{W}_b^* p_q\mathscr{F}p_q\mathscr{W}_b = (\pi_q \otimes I_y)\mathrm{Op}^w(\mathscr{F}\circ\kappa_b)(\pi_q \otimes I_y), \tag{3.4.48}$$

where κ_b is the symplectic mapping defined in (2.7.16). Writing $\mathscr{F}\circ\kappa_b$, we consider $\mathscr{F}(x,y)$ as a function of the variables (x,y,ξ,η), constant with respect to (ξ,η), so that for $(x,y;\xi,\eta) \in \mathbb{R}^4$ we have

$$(\mathscr{F}\circ\kappa_b)(x,y;\xi,\eta) = \mathscr{F}(b^{-1/2}(x-\eta), b^{-1/2}(\xi-y)) = \mathscr{F}_b(\eta-x, y-\xi).$$

Let $u \in \mathscr{S}(\mathbb{R}^2)$. Set

$$u_q(y) := \int_{\mathbb{R}} u(x,y)\psi_q(x)dx. \tag{3.4.49}$$

Then by (3.4.48), we have

$$\begin{aligned}
&\langle \mathscr{W}_b^* p_q \mathscr{F} p_q \mathscr{W}_b u, u\rangle_{L^2(\mathbb{R}^2)} \\
&= \langle \mathrm{Op}^w(\mathscr{F}\circ\kappa_b)(\psi_q\otimes u_q), (\psi_q\otimes u_q)\rangle_{L^2(\mathbb{R}^2)} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^6} \mathscr{F}_b((y_1+y_2)/2-\xi, \eta-(x_1+x_2)/2) \\
&\times e^{-i((x_1-x_2)\xi+(y_1-y_2)\eta)}\, \psi_q(x_1)u_q(y_1)\, \psi_q(x_2)\overline{u_q(y_2)}\, dx_1 dx_2\, dy_1 dy_2\, d\xi\, d\eta \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^5} \mathscr{F}_b((y_1+y_2)/2-y', \eta-\eta') \\
&\times e^{-i(y_1-y_2)\eta}\left(\int_{\mathbb{R}} \psi_q(\eta'+v/2)\psi_q(\eta'-v/2)e^{ivy'}dv\right) u_q(y_1)\overline{u_q(y_2)} d\eta' d\eta dy' dy_1 dy_2 \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^5} \mathscr{F}_b((y_1+y_2)/2-y', \eta-\eta')\Psi_q(y',\eta') \\
&\times e^{-i(y_1-y_2)\eta} u_q(y_1)\overline{u_q(y_2)} dy_1 dy_2 dy' d\eta d\eta' \\
&= \langle \mathrm{Op}^w(\mathscr{F}_b * \Psi_q)u_q, u_q\rangle_{L^2(\mathbb{R})} = \langle \mathrm{Op}_q^{aw}(\mathscr{F}_b)u_q, u_q\rangle_{L^2(\mathbb{R})} \\
&= \langle (\pi_q\otimes \mathrm{Op}_q^{aw}(\mathscr{F}_b))u, u\rangle_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{3.4.50}$$

Thus, (3.4.50) entails (3.4.47) in the case $\mathscr{F} \in \mathscr{S}(\mathbb{R}^2)$.
Let us consider now the general case $\mathscr{F} \in \mathscr{S}'_\ell(\mathbb{R}^2)$. Since the operators at both hand sides of (3.4.47) are bounded, it suffices to show that

$$\langle \mathscr{W}_b^*(p_q \mathscr{F} p_q)\mathscr{W}_b u, u\rangle_{L^2(\mathbb{R}^2)} = \langle ((\pi_q\otimes \mathrm{Op}_q^{aw}(\mathscr{F}_b))u, u\rangle_{L^2(\mathbb{R}^2)} \tag{3.4.51}$$

for any $u \in \mathscr{S}(\mathbb{R}^2)$. We have

$$\langle \mathscr{W}_b^*(p_q \mathscr{F} p_q)\mathscr{W}_b u, u\rangle_{L^2(\mathbb{R}^2)} = \left(\mathscr{F}_b, |w_q|^2\right)_{\mathscr{S}'(\mathbb{R}^2)}$$

where $w_q := p_q \mathscr{W}_b u \in \mathscr{S}(\mathbb{R}^2)$. On the other hand,

$$\langle (\pi_q\otimes \mathrm{Op}_q^{aw}(\mathscr{F}_b))u, u\rangle_{L^2(\mathbb{R}^2)} = \left(\mathscr{F}_b, Q_{u_q,u_q}\right)_{\mathscr{S}'(\mathbb{R}^2)} = \left(\mathscr{F}, O_b^*(Q_{u_q,u_q})\right)_{\mathscr{S}'(\mathbb{R}^2)}$$

where $u_q \in \mathscr{S}(\mathbb{R})$ is the function defined in (3.4.49).
Pick a sequence $\left\{\mathscr{F}^{(m)}\right\}_{m\in N} \subset \mathscr{S}(\mathbb{R}^2)$ such that

$$\lim_{m\to\infty}\left(\mathscr{F}^{(m)}, v\right)_{\mathscr{S}'(\mathbb{R}^2)} = (\mathscr{F}, v)_{\mathscr{S}'(\mathbb{R}^2)}, \quad v \in \mathscr{S}(\mathbb{R}^2).$$

By the first part of the proof concerning Schwartz-class symbols $\mathscr{F}$, we get

$$\left(\mathscr{F}, |w_q|^2\right) = \lim_{m\to\infty}\left(\mathscr{F}^{(m)}, |w_q|^2\right) = \lim_{m\to\infty}\left(\mathscr{F}^{(m)}, O_b^*(Q_{u_q,u_q})\right) = \left(\mathscr{F}, O_b^*(Q_{u_q,u_q})\right)$$

i.e. (3.4.51) holds true. □

Theorem 3.4.1 is a generalization of [150, Theorem 2.11] which concerned the case $\mathscr{F} \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$.

Combining Theorem 3.4.1 and Corollary 3.4.2, we obtain the following

Corollary 3.4.4 *Let* $q \in \mathbb{Z}_+$, *and* $\mathscr{F} \in \mathscr{S}'_\ell(\mathbb{R}^2)$, $\ell \in \mathbb{Z}_+$. *Then*

$$\mathscr{W}_b^* p_q \mathscr{F} p_q \mathscr{W}_b = \pi_q \otimes \mathrm{Op}^w(\mathscr{F}_b * \Psi_q). \tag{3.4.52}$$

The unitary equivalence between the weighted Berezin-Toeplitz operator $p_0 \mathscr{F} p_0$ and the standard 1D anti-Wick operator $\mathrm{Op}_0^{aw}(\mathscr{F})$ is closely related to the Segal-Bargmann transform which, in one form or another, plays an important role in the semiclassical analysis of quantum Hamiltonians (see e.g. [208, 207, 211]). The Segal-Bargmann transform $S_0 : L^2(\mathbb{R}) \to p_0 L^2(\mathbb{R}^2)$ is a unitary operator with integral kernel

$$\frac{1}{\sqrt{2}} \left(\frac{b}{\pi}\right)^{3/4} e^{-b((x+iy+2t)^2-2t^2+|\mathbf{x}|^2)/4}, \quad \mathbf{x} = (x,y) \in \mathbb{R}^2, \quad t \in \mathbb{R},$$

(see [138, Lemma 3.1]). Fix $q \in \mathbb{Z}_+$. Denote by $M_q : L^2(\mathbb{R}) \to (\pi_q \otimes I_y) L^2(\mathbb{R}^2)$ the unitary operator which maps $u \in L^2(\mathbb{R})$ into $b^{-1/4} \psi_q(x) u(b^{-1/2} y)$, $(x,y) \in \mathbb{R}^2$, and by $\iota : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ the unitary operator generated by the rotation by angle $\pi/2$, i.e. $(\iota u)(x,y) = u(y,-x)$, $(x,y) \in \mathbb{R}^2$; note that ι commutes with the Landau Hamiltonian, and hence $[\iota, p_q] = 0$. Then we have

$$S_0 = \iota \mathscr{W}_b M_0.$$

From this point of view the operators $S_q := \iota \mathscr{W}_b M_q$, $q \in \mathbb{N}$, could be considered as generalized Segal-Bargmann transforms.

In our next theorem we establish the unitary equivalence between the operators $p_q \mathscr{F} p_q$ with $q \in \mathbb{Z}_+$ and $\mathscr{F} \in \mathscr{S}'_\ell(\mathbb{R}^2)$, $\ell \in \mathbb{Z}_+$, and $p_0(\mathscr{D}_{q,b}\mathscr{F})p_0$ where

$$\mathscr{D}_{q,b} := L_q\left(-\frac{\Delta}{2b}\right). \tag{3.4.53}$$

Thus, $\mathscr{D}_{0,b} = I$, and $\mathscr{D}_{q,b}$ with $q \in \mathbb{N}$ is a partial differential operator of order $2q$ with constant coefficients. For the proof of this theorem we need the following

Lemma 3.4.5 *Let* $q \in \mathbb{Z}_+$ *and* $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^2)$. *Then*

$$(O_b \mathscr{F}) * \Psi_q = (O_b(\mathscr{D}_{q,b}\mathscr{F})) * \Psi_0, \tag{3.4.54}$$

where O_b *is the transform described in* (3.4.45), Ψ_q, $q \in \mathbb{Z}_+$, *are the Wigner functions introduced in* (3.4.27), *and* $\mathscr{D}_{q,b}$ *is the operator defined in* (3.4.53).

Proof. Note that Ψ_q is real-valued, and $\Psi_q = \Psi_q^{\#}$, i.e. Ψ_q is invariant with respect to the reflection $\mathbf{w} \mapsto -\mathbf{w}$, $\mathbf{w} \in \mathbb{R}^{2n}$. Then, by (3.4.35), for any $\mathscr{G} \in \mathscr{S}'(\mathbb{R}^2)$ and $u \in \mathscr{S}(\mathbb{R}^2)$, we have

$$\left(\mathscr{G} * \Psi_q, u\right) = \left(\mathscr{G}, \Psi_q * u\right) = \left(\widehat{\mathscr{G}}, 2\pi \widehat{\Psi_q}\, \check{u}\right) = \left(\widehat{\mathscr{G}}, 2\pi L_q(|\cdot|^2/2)\, \widehat{\Psi_0}\, \check{u}\right)$$

$$= \left(\mathscr{G}, \mathscr{D}_{q,1}(\Psi_0 * u)\right) = \left(\mathscr{D}_{q,1}\mathscr{G}, \Psi_0 * u\right) = \left((\mathscr{D}_{q,1}\mathscr{G}) * \Psi_0, u\right).$$

Therefore,

$$\mathscr{G} * \Psi_q = (\mathscr{D}_{q,1}\mathscr{G}) * \Psi_0, \quad \mathscr{G} \in \mathscr{S}'(\mathbb{R}^2). \tag{3.4.55}$$

Moreover, evidently

$$O_b^{-1}\, \mathscr{D}_{q,1}\, O_b = \mathscr{D}_{q,b}. \tag{3.4.56}$$

Combining (3.4.55) and (3.4.56), we obtain (3.4.54). □

Theorem 3.4.2 *Let* $q \in \mathbb{N}$ *and* $\mathscr{F} \in \mathscr{S}'_\ell(\mathbb{R}^2)$, $\ell \in \mathbb{Z}_+$. *Then the operator* $p_q \mathscr{F} p_q$ *with domain* $\widetilde{\mathscr{A}_q}$ *is unitarily equivalent to* $p_0\,(\mathscr{D}_{q,b}\mathscr{F})\,p_0$ *with domain* $\widetilde{\mathscr{A}_0}$.

Proof. Let us consider at first the operators $p_q \mathscr{F} p_q$ and $p_0\,(\mathscr{D}_{q,b}\mathscr{F})\,p_0$ on $L^2(\mathbb{R}^2)$. By Corollary 3.4.4 and Lemma 3.4.5, we have

$$\mathscr{W}_b^* p_q \mathscr{F} p_q \mathscr{W}_b = \pi_q \otimes \mathrm{Op}^w\left(O_b\left(\mathscr{D}_{q,b}\mathscr{F}\right) * \Psi_0\right). \tag{3.4.57}$$

Introduce the operator $\mathscr{J}_q : L^2(\mathbb{R}^2_{x,y}) \to L^2(\mathbb{R}^2_{x,y})$ which interchanges the qth Fourier coefficient of $u_q(y)$ of the function $u(x,y)$ (see (3.4.49)) with its zeroth one. In other words, if

$$u(x,y) = \sum_{k \in \mathbb{Z}_+} \psi_k(x) u_k(y), \quad (x,y) \in \mathbb{R}^2,$$

the series being convergent in $L^2(\mathbb{R}^2)$, then

$$(\mathscr{J}_q u)(x,y) = \sum_{\substack{k \in \mathbb{Z}_+: \\ k \neq 0,q}} \psi_k(x) u_k(y) + \psi_0(x) u_q(y) + \psi_q(x) u_0(y), \quad (x,y) \in \mathbb{R}^2.$$

Evidently, $\mathscr{J}_q$ is a unitary operator from $L^2(\mathbb{R}^2)$ onto $L^2(\mathbb{R}^2)$, and the restriction of $\mathscr{J}_q$ onto $\widetilde{\mathscr{A}_q}$ is a unitary mapping from $\widetilde{\mathscr{A}_q}$ onto $\widetilde{\mathscr{A}_0}$. Moreover, if $T \in \mathfrak{B}(L^2(\mathbb{R}_y))$, then

$$\pi_q \otimes T = \mathscr{J}_q^*\,(\pi_0 \otimes T)\,\mathscr{J}_q. \tag{3.4.58}$$

Bearing in mind (3.4.27) and (3.4.52), we get

$$\begin{aligned}\pi_q \otimes \mathrm{Op}^w\left(O_b\left(\mathscr{D}_{q,b}\mathscr{F}\right) * \Psi_0\right) &= \mathscr{J}_q^*\left(\pi_0 \otimes \mathrm{Op}^w\left(O_b\left(\mathscr{D}_{q,b}\mathscr{F}\right) * \Psi_0\right)\right)\mathscr{J}_q \\ &= \mathscr{J}_q^*\,\mathscr{W}_b^*\left(p_0\left(\mathscr{D}_{q,b}\mathscr{F}\right)p_0\right)\mathscr{W}_b\,\mathscr{J}_q.\end{aligned} \tag{3.4.59}$$

Combining (3.4.57) and (3.4.59), we obtain

$$p_q \mathscr{F} p_q = \mathscr{W}_b\, \mathscr{J}_q^*\, \mathscr{W}_b^*\left(p_0\left(\mathscr{D}_{q,b}\mathscr{F}\right)p_0\right)\mathscr{W}_b\, \mathscr{J}_q\, \mathscr{W}_b^*,$$

i.e. the operators $p_q \mathscr{F} p_q$ and $p_0 \left(\mathscr{D}_{q,b}\mathscr{F}\right) p_0$ with domain $L^2(\mathbb{R}^2)$ are unitarily equivalent under the unitary operator $\mathscr{W}_b \mathscr{J}_q \mathscr{W}_b^* : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. In order to prove the unitary equivalence of $p_q \mathscr{F} p_q$ with domain $\widetilde{\mathscr{A}_q}$ and $p_0 \left(\mathscr{D}_{q,b}\mathscr{F}\right) p_0$ with domain $\widetilde{\mathscr{A}_0}$, we just have to note that the restriction of $\mathscr{W}_b \mathscr{J}_q \mathscr{W}_b^*$ onto $\widetilde{\mathscr{A}_q}$ is a unitary mapping from $\widetilde{\mathscr{A}_q}$ onto $\widetilde{\mathscr{A}_0}$. □

Theorem 3.4.2 was first proved as [39, Corollary 9.3] in the case where $\mathscr{F} \in C^{2q}(\mathbb{R}^2)$ and $\Delta^s \mathscr{F} \in L^\infty(\mathbb{R}^2)$, $s = 0, \ldots, q$. The proof in [39] was based on the fact that if $\mathscr{F} \in \mathscr{S}'(\mathbb{R}^2)$, then

$$\left(\mathscr{F}, \varphi_{k,q}\overline{\varphi_{\ell,q}}\right)_{\mathscr{S}'(\mathbb{R}^2)} = \left(\mathscr{D}_{q,b}\mathscr{F}, \varphi_{k,0}\overline{\varphi_{\ell,0}}\right)_{\mathscr{S}'(\mathbb{R}^2)}, \quad k, \ell \in \mathbb{Z}_+, \tag{3.4.60}$$

(see [39, Lemma 9.2]). The proof of (3.4.60) contained in [39] which is of combinatorial nature and exploits essentally the commutation relation (2.7.6), might be of independent interest (see the considerations in [173]).
A more abstract and less explicit point of view concerning the unitary equivalence between $p_q \mathscr{F} p_q$ and $p_0 \left(\mathscr{D}_{q,b}\mathscr{F}\right) p_0$, could be found in [82].

3.5 q-dependence of the norms of the operators $p_q \mathscr{F} p_q$

3.5.1 Motivation and main estimates

As mentioned in Section 3.1, the upper bounds in (3.3.8), (3.3.9) with $p > 1$, and (3.3.10) are not sharp. In particular, the estimating quantities are independent of $q \in \mathbb{Z}_+$. In this section we obtain some sharp in q estimates of $\|p_q \mathscr{F} p_q\|$ and deduce from them bounds on $\|p_q \mathscr{F} p_q\|_\ell$ with appropriate ℓ. In particular, we see that if $\mathscr{F}$ decays at infinity, then the norms of $p_q \mathscr{F} p_q$ tend to zero as $q \to \infty$. These estimates will play a crucial role in Chapter 7 where we investigate the high-energy asymptotic density of the eigenvalue clusters for the 2D Landau Hamiltonian.
Assume that $\mathscr{F} \in C(\mathbb{R}^2)$ satisfies the estimate

$$|\mathscr{F}(\mathbf{x})| \leq C\langle \mathbf{x}\rangle^{-\gamma}, \quad \mathbf{x} \in \mathbb{R}^2, \tag{3.5.1}$$

with some constants $C \geq 0$ and $\gamma > 0$.

Theorem 3.5.1 *Assume that* $\mathscr{F}$ *satisfies* (3.5.1) *with* $\gamma \in (0,\infty)$. *Then, there exists a constant* c_∞ *independent of q such that*

$$\|p_q \mathscr{F} p_q\| \leq c_\infty \begin{cases} \Lambda_q^{-\gamma/2} & \text{if } \gamma \in (0,1), \\ \Lambda_q^{-1/2}\left(1 + |\ln \Lambda_q|\right) & \text{if } \gamma = 1, \\ \Lambda_q^{-1/2} & \text{if } \gamma > 1, \end{cases} \quad q \in \mathbb{Z}_+. \tag{3.5.2}$$

Estimates (3.5.2) are sharp for $\gamma \neq 1$. In fact, let $\{\varphi_{k,q}\}_{k \in \mathbb{Z}_+}$ be the canonic orthonormal basis of $p_q L^2(\mathbb{R}^2)$, $q \in \mathbb{Z}_+$ defined in (2.7.39), and let χ_R be the characteristic function of a disk of radius $R > 0$ centered at the origin. Then

$$\liminf_{q\to\infty} \Lambda_q^{1/2} \langle \chi_R \varphi_{0,q}, \varphi_{0,q}\rangle_{L^2(\mathbb{R}^2)} > 0, \tag{3.5.3}$$

which implies the sharpness of estimate (3.5.2) for $\gamma > 1$. Similarly, if $\gamma \in (0,1)$ we can show that

$$\liminf_{q\to\infty} \Lambda_q^{1/2} \langle\langle\cdot\rangle^{-\gamma}\varphi_{0,q}, \varphi_{0,q}\rangle_{L^2(\mathbb{R}^2)} > 0, \tag{3.5.4}$$

which entails the sharpness for $\gamma \in (0,1)$ (see the details in [112] for $\gamma \in (0,1)$, and in [123] for $\gamma > 1$). We do not know whether the estimate for $\gamma = 1$ is sharp.

In our second theorem we estimate the Schatten-von Neumann estimates of the operators $p_q \mathscr{F} p_q$.

Theorem 3.5.2 (i) *Assume that* $\mathscr{F}$ *satisfies* (3.5.1) *with* $\gamma > 1$. *Then, for each* $\ell > 1/(\gamma-1)$ *we have* $p_q \mathscr{F} p_q \in \mathfrak{S}_\ell$, *and there exists a constant* c_ℓ *such that*

$$\|p_q \mathscr{F} p_q\|_\ell \le c_\ell \Lambda_q^{\frac{1}{2\ell}-\frac{1}{2}}, \quad q \in \mathbb{Z}_+. \tag{3.5.5}$$

(ii) *Assume that* $\mathscr{F}$ *satisfies* (3.5.1) *with* $\gamma \in (0,1)$. *Then, for each* $\ell > 2/\gamma$ *we have* $p_q \mathscr{F} p_q \in \mathfrak{S}_\ell$, *there exists a constant* c_ℓ *such that*

$$\|p_q \mathscr{F} p_q\|_\ell \le c_\ell \Lambda_q^{\frac{1}{\ell}-\frac{\gamma}{2}} (1 + |\ln \Lambda_q|)^{1/\ell}, \quad q \in \mathbb{Z}_+. \tag{3.5.6}$$

Estimate (3.5.6) should be considered an *a priori* estimate, sufficient for the purposes of this exposition, but not necessarily sharp.

3.5.2 Proof of the operator norm estimates

In this subsection we prove Theorem 3.5.1. We will divide the proof into several propositions. The main steps of the proof are as follows:

- we apply Theorem 3.4.1 where we established the unitary equivalence of the operators $p_q \mathscr{F} p_q$ and $\pi_q \otimes \mathrm{Op}_q^{aw}(\mathscr{F}_b)$;
- assuming that $\mathscr{F} \in \Gamma_1^{-\gamma}$ with $\gamma > 0$, we approximate in the Hilbert-Schmidt norm $\mathrm{Op}_q^{aw}(\mathscr{F}_b) = \mathrm{Op}^w(\mathscr{F}_b * \Psi_q)$ by $\mathrm{Op}^w(\mathscr{F}_b * \delta_{\sqrt{2q+1}})$;
- we estimate $\|\mathrm{Op}^w(\mathscr{F}_b) * \delta_{\sqrt{2q+1}}\|$ using the Calderón-Vaillancourt theorem (see Proposition 2.3.4).

For $k > 0$ define $\delta_k \in \mathscr{E}'(\mathbb{R}^2)$ by

$$\left(\delta_k, u\right)_{\mathscr{E}'(\mathbb{R}^2)} := \frac{1}{2\pi}\int_0^{2\pi} u(k\cos\theta, k\sin\theta)d\theta, \quad u \in C^\infty(\mathbb{R}^2). \tag{3.5.7}$$

As usual, we define the function $u * \delta_k$ by

$$(u * \delta_k)(\mathbf{x}) := \left(\delta_k, u(\mathbf{x}-\cdot)\right)_{\mathscr{E}'(\mathbb{R}^2)}, \quad u \in C_\infty(\mathbb{R}^2), \quad \mathbf{x} \in \mathbb{R}^2.$$

By analogy with Proposition 3.4.2, we can show that if $\mathscr{F} \in \Gamma_\rho^\gamma(\mathbb{R}^2)$ with $\gamma \in \mathbb{R}$ and $\rho \in [0,1]$, then $\mathscr{F} * \delta_k \in \Gamma_\rho^\gamma(\mathbb{R}^2)$, and $\mathscr{F} - \mathscr{F} * \delta_k \in \Gamma_\rho^{\gamma-\rho}(\mathbb{R}^2)$.

Proposition 3.5.1 *Assume that* $\mathscr{F} \in \Gamma_1^{-\gamma}(\mathbb{R}^2)$ *with* $\gamma \in (0,\infty)$*. Then*

$$\mathrm{Op}^{\mathrm{w}}(\mathscr{F}_{\mathrm{b}} * \Psi_{\mathrm{q}}) - \mathrm{Op}^{\mathrm{w}}(\mathscr{F}_{\mathrm{B}} * \delta_{\sqrt{2q+1}}) \in \mathfrak{S}_2,$$

and there exists a constant c_2 *independent of* q*, such that*

$$\|\mathrm{Op}^{\mathrm{w}}(\mathscr{F}_{\mathrm{b}} * \Psi_{\mathrm{q}}) - \mathrm{Op}^{\mathrm{w}}(\mathscr{F}_{\mathrm{b}} * \delta_{\sqrt{2q+1}})\|_2 \le c_2 \Lambda_q^{-3/4}, \quad q \in \mathbb{Z}_+. \tag{3.5.8}$$

Proof. By (2.3.15) we have

$$\|\mathrm{Op}^{\mathrm{w}}(\mathscr{F}_{\mathrm{b}} * \Psi_{\mathrm{q}}) - \mathrm{Op}^{\mathrm{w}}(\mathscr{F}_{\mathrm{b}} * \delta_{\sqrt{2q+1}})\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}^2} |(\mathscr{F}_{\mathrm{b}} * \Psi_{\mathrm{q}})(\mathbf{w}) - (\mathscr{F}_{\mathrm{b}} * \delta_{\sqrt{2q+1}})(\mathbf{w})|^2 d\mathbf{w}$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}^2} |(\widehat{\mathscr{F}_{\mathrm{b}} * \Psi_{\mathrm{q}}})(\mathbf{w}) - (\widehat{\mathscr{F}_{\mathrm{b}} * \delta_{\sqrt{2q+1}}})(\mathbf{w})|^2 d\mathbf{w}. \tag{3.5.9}$$

An explicit calculation (see (3.4.35) and [150, Eq. (3.9)]) yields

$$(\widehat{\mathscr{F}_{\mathrm{b}} * \Psi_{\mathrm{q}}})(\mathbf{w}) - (\widehat{\mathscr{F}_{\mathrm{b}} * \delta_{\sqrt{2q+1}}})(\mathbf{w})$$

$$= \left(L_q(|\mathbf{w}|^2/2)e^{-|\mathbf{w}|^2/4} - J_0(\sqrt{2q+1}|\mathbf{w}|)\right)\widehat{\mathscr{F}_{\mathrm{b}}}(\mathbf{w}), \quad \mathbf{w} \in \mathbb{R}^2, \tag{3.5.10}$$

where L_q is the Laguerre polynomial defined in (2.7.30), and J_0 is the Bessel function of zeroth order. Moreover, there exists a constant $\tilde{c}_2$ such that

$$\left|L_q(r)e^{-r/2} - J_0(\sqrt{(4q+2)r})\right|$$

$$\le \tilde{c}_2\left((q+1)^{-3/4}r^{5/4} + (q+1)^{-1}r^3\right), \quad q \in \mathbb{Z}_+, \quad r > 0, \tag{3.5.11}$$

(see [150, (Eq. (3.10)] for the generic case $q \in \mathbb{N}$; if $q = 0$, then (3.5.11) follows from $|e^{-r/2} - J_0(\sqrt{2r})| = O(r^2)|$, $r \in (0,1)$, and $|e^{-r/2} - J_0(\sqrt{2r})| = O(1)$, $r \ge 1$). Further,

$$|\widehat{\mathscr{F}_{\mathrm{b}}}(\mathbf{w})| = \begin{cases} \mathscr{O}(|\mathbf{w}|^{-2+\gamma}) & \text{if } \gamma \in (0,2), \\ \mathscr{O}(|\ln|\mathbf{w}||) & \text{if } \gamma = 2, \\ \mathscr{O}(1) & \text{if } \gamma > 2, \end{cases} \qquad |\mathbf{w}| \le 1/2,$$

and

$$|\widehat{\mathscr{F}_{\mathrm{b}}}(\mathbf{w})| = O(|\mathbf{w}|^{-N}), \quad |\mathbf{w}| > 1/2, \quad N > 0,$$

(see [204, Chapter XII, Lemma 3.1]). In particular, the functions $|\mathbf{w}|^m \widehat{\mathscr{F}_{\mathrm{b}}}(\mathbf{w})$, $\mathbf{w} \in \mathbb{R}^2$, with $m > 1-\gamma$ if $\gamma \in (0,2)$ or with $m > -1$ if $\gamma \ge 2$, are in $L^2(\mathbb{R}^2)$. Combining (3.5.9), (3.5.10), and (3.5.11), we get

$$\|\mathrm{Op}(\mathscr{F}_{\mathrm{b}} * \Psi_{\mathrm{q}}) - \mathrm{Op}(\mathscr{F}_{\mathrm{b}} * \delta_{\sqrt{2q+1}})\|_2^2$$

$$\le \frac{\tilde{c}_2^2}{\pi}\int_{\mathbb{R}^2}\left((q+1)^{-3/2}|\mathbf{w}|^5 + (q+1)^{-2}|\mathbf{w}|^{12}\right)|\widehat{\mathscr{F}_{\mathrm{b}}}(\mathbf{w})|^2 d\mathbf{w},$$

which yields (3.5.8). $\square$

Estimate (3.5.8) could be interpreted as a manifestation of the equipartition of the eigenfunctions of the harmonic oscillator $\mathfrak{h}$ (see (2.7.12)), i.e. the appropriate weak convergence as $q \to \infty$ of the Wigner function Ψ_q associated with the qth normalized eigenfunction of $\mathfrak{h}$, to the measure invariant with respect to the classical flow (see e.g. [49, 218, 28] for related results concerning various ergodic quantum systems).

Proposition 3.5.2 *Assume that* $\mathscr{F} \in \Gamma_1^{-\gamma}(\mathbb{R}^2)$ *with* $\gamma \in (0,\infty)$*. Then the operator* $\mathrm{Op}(\mathscr{F}_b * \delta_k)$*,* $k > 0$*, is bounded and there exists a constant* c_1 *such that*

$$\|\mathrm{Op}^w(\mathscr{F}_b * \delta_k)\| \le c_1 \begin{cases} k^{-\gamma} & \text{if} \quad \gamma \in (0,1), \\ k^{-1}\ln k & \text{if} \quad \gamma = 1, \\ k^{-1} & \text{if} \quad \gamma \in (1,\infty), \end{cases} \qquad k \in [2,\infty). \tag{3.5.12}$$

Proof. By Proposition 2.3.4,

$$\|\mathrm{Op}^w(\mathscr{F}_b * \delta_k)\| \le c_0 \max_{\alpha \in \mathbb{Z}_+^2 : 0 \le |\alpha| \le 2} \sup_{\mathbf{w} \in \mathbb{R}^2} |(D^\alpha \mathscr{F} * \delta_k)(\mathbf{w})|. \tag{3.5.13}$$

Since $\mathscr{F} \in \Gamma_1^{-\gamma}(\mathbb{R}^2)$, we have

$$|D^\alpha \mathscr{F}(\mathbf{x})| \le c_{1,\alpha} \langle \mathbf{x} \rangle^{-|\alpha|-\gamma} \le c_{1,\alpha} \langle \mathbf{x} \rangle^{-\gamma}, \quad \mathbf{x} \in \mathbb{R}^2, \quad \alpha \in \mathbb{Z}_+^2, \tag{3.5.14}$$

with constants $c_{1,\alpha}$ which may depend on b but are independent of $\mathbf{x}$. Now (3.5.13) and (3.5.14) imply

$$\|\mathrm{Op}^w(\mathscr{F}_b * \delta_k)\| \le c_1' \sup_{\mathbf{w} \in \mathbb{R}^2} \langle \cdot \rangle^{-\gamma} * \delta_k(\mathbf{w}). \tag{3.5.15}$$

Note that the function $\langle \cdot \rangle^{-\gamma} * \delta_k$ is radially symmetric. Then we have

$$\begin{aligned} (\langle \cdot \rangle^{-\gamma} * \delta_k)(\mathbf{w}) &= \frac{1}{2\pi} \int_0^{2\pi} ((k\cos\theta - |\mathbf{w}|)^2 + k^2 \sin^2\theta + 1)^{-\gamma/2} d\theta \\ &\le \frac{1}{2\pi} \int_0^{2\pi} (k^2 \sin^2\theta + 1)^{-\gamma/2} d\theta \\ &= \frac{2}{\pi} \int_0^{\pi/2} (k^2 \sin^2\theta + 1)^{-\gamma/2} d\theta \\ &\le \int_0^1 (k^2 t^2 + 1)^{-\gamma/2} dt =: I_\gamma(k). \end{aligned} \tag{3.5.16}$$

Elementary calculations yield

$$I_\gamma(k) = \begin{cases} \mathscr{O}(k^{-\gamma}) & \text{if} \quad \gamma \in (0,1), \\ \mathscr{O}(k^{-1}\ln k) & \text{if} \quad \gamma = 1, \\ \mathscr{O}(k^{-1}) & \text{if} \quad \gamma \in (1,\infty), \end{cases} \qquad k \in [2,\infty). \tag{3.5.17}$$

Putting together (3.5.15) – (3.5.17), we obtain (3.5.12). □

Now we are in position to prove Theorem 3.5.1. An elementary variational argument implies that we may assume without loss of generality that $\mathscr{F}(\mathbf{x}) = \langle \mathbf{x} \rangle^{-\gamma}$, $\mathbf{x} \in \mathbb{R}^2$; then, of course, $\mathscr{F} \in \Gamma_1^{-\gamma}(\mathbb{R}^2)$. By Theorem 3.4.1, and Propositions 3.5.1 and 3.5.2, we have

$$\begin{aligned}
\|p_q \mathscr{F} p_q\| &= \|\mathrm{Op}^w(\mathscr{F}_b * \Psi_q)\| \\
&\leq \|\mathrm{Op}^w(\mathscr{F}_b * \delta_{\sqrt{2q+1}})\| + \|\mathrm{Op}(\mathscr{F}_b * \Psi_q) - \mathrm{Op}(\mathscr{F}_b * \delta_{\sqrt{2q+1}})\| \\
&\leq \|\mathrm{Op}(\mathscr{F}_b * \delta_{\sqrt{2q+1}})\| + \|\mathrm{Op}^w(\mathscr{F}_b * \Psi_q) - \mathrm{Op}^w(\mathscr{F}_b * \delta_{\sqrt{2q+1}})\|_2 \\
&\leq \|\mathrm{Op}^w(\mathscr{F}_b * \delta_{\sqrt{2q+1}})\| + c_2 \Lambda_q^{-3/4}. \qquad (3.5.18)
\end{aligned}$$

Now, (3.5.18) and (3.5.12) yield (3.5.2).

3.5.3 Proof of the Schatten-von Neumann norm estimates

In this subsection we prove Theorem 3.5.2. As in the proof of Theorem 3.5.1, we may assume again $\mathscr{F}(\mathbf{x}) = \langle \mathbf{x} \rangle^{-\gamma}$, $\mathbf{x} \in \mathbb{R}^2$, without any loss of generality.
(i) First let us consider the case $\gamma > 2$, $\ell = 1$. By (3.3.9) with $p = 1$, we have

$$\|p_q \mathscr{F} p_q\|_1 = \frac{b}{2\pi} \int_{\mathbb{R}^2} \langle \mathbf{x} \rangle^{-\gamma} d\mathbf{x},$$

which proves (3.5.5) in this case.
Let us consider the case of a general ℓ. For a fixed $s > 1$ and any $\ell \in [1, \infty]$, let

$$M_q^{(\ell)} = \Lambda_q^{\frac{1}{2} - \frac{1}{2\ell}} p_q \langle \cdot \rangle^{-s(1+\frac{1}{\ell})} p_q;$$

for $\ell = \infty$, one should replace $1/\ell$ by 0. By the previous step of the proof and Theorem 3.5.1, we have

$$\sup_{q \geq 0} \|M_q^{(1)}\|_1 \leq C_1 < \infty, \quad \sup_{q \geq 0} \|M_q^{(\infty)}\| \leq C_\infty < \infty,$$

where the constants C_1, C_∞ depend only on b and s. Applying the Calderón-Lions interpolation theorem (see e.g. [163, Theorem IX.20]), we get

$$\sup_{q \geq 0} \|M_q^{(\ell)}\|_\ell \leq C_1^{1/\ell} C_\infty^{(\ell-1)/\ell} < \infty$$

for all $\ell \geq 1$. It is easy to see that the last statement is equivalent to (3.5.5).
(ii) Note that in the proof of estimate (3.5.6) we may assume that q is large enough since for any fixed q it follows from (3.3.8).
For brevity, set $T_q := p_q \mathscr{F} p_q$, $q \in \mathbb{Z}_+$; by [159], we have $\mathrm{rank}\, T_q = \infty$. Let $\{s_j(T_q)\}_{j \in N}$ the non-increasing sequence of the operator T_q. Under our assumptions, $\mathscr{F} \in L_w^{2/\gamma}(\mathbb{R}^2)$. Therefore, by (3.3.10) with $p = 2/\gamma$, we have

$$s_j(T_q) \leq \mathscr{C} j^{-\gamma/2}, \quad j \in \mathbb{N}, \quad q \in \mathbb{Z}_+. \qquad (3.5.19)$$

with a constant $\mathscr{C} \geq 0$. On the other hand, (3.5.5) implies

$$s_1(T_q) \leq c_\infty \Lambda_q^{-\gamma/2}, \quad q \in \mathbb{Z}_+. \tag{3.5.20}$$

Fix $\ell > 2/\gamma$. By (3.5.19) – (3.5.20), for any $N \in \mathbb{N}$, we have

$$\begin{aligned}
\|T_q\|_\ell^\ell &= \sum_{j=1}^{\infty} s_j(T_q)^\ell \\
&= \sum_{j=1}^{N} s_j(T_q)^\ell + \sum_{j=N+1}^{\infty} s_j(T_q)^\ell \\
&\leq s_1(T_q)^{\ell-\frac{2}{\gamma}} \sum_{j=1}^{N} s_j(T_q)^{\frac{2}{\gamma}} + \mathscr{C}^\ell \sum_{j=N+1}^{\infty} j^{-\frac{\ell\gamma}{2}} \\
&\leq c_\infty^{\ell-\frac{2}{\gamma}} \mathscr{C}^{2/\gamma} \Lambda_q^{1-\frac{\ell\gamma}{2}} \sum_{j=1}^{N} j^{-1} + \mathscr{C}^\ell \sum_{j=N+1}^{\infty} j^{-\frac{\ell\gamma}{2}} \\
&\leq C \left(\Lambda_q^{1-\frac{\ell\gamma}{2}} (1 + \ln N) + N^{1-\frac{\ell\gamma}{2}} \right)
\end{aligned}$$

with a constant C independent of N and q. Assuming that q is large enough, and choosing N equal to the integer part of Λ_q, we obtain (3.5.6).

Chapter 4

Eigenvalue asymptotics for magnetic quantum Hamiltonians

Abstract: This chapter has a central role in the book because it shows the interaction between chapters 2 and 3, and the eigenvalue asymptotics for Berezin-Toeplitz operators given in Section 4.2 will be fondamental for the following chapter. As discussed in Section 4.2.5, these asymptotics are semi-classical for symbols of power-like decay and not of semi-classical nature for compactly supported symbols. For symbols having intermediate behaviors (exponential or gaussian decaying) the asymptotic order can be semi-classical but not the coefficient. From these asymptotics for Berezin-Toeplitz operators, we deduce in Section 4.4, the first results of spectral asymptotics for perturbations (electric, magnetic, geometric or by an obstacle) of the 2D magnetic Schrödinger operator $H_S(A,0)$ with constant magnetic field $b \neq 0$, and of the 2D Pauli operator $H_P(A,0)$ with admissible b of non-zero mean value b_0. They will also be fundamental for the 3D magnetic operators. Moreover the study of Berezin-Toeplitz operators for compactly supported symbols gives a controllability result stated in Section 4.3. These "magnetic" results in comparison to analogous results for perturbations of the non-magnetic Laplacian are commented in Section 4.1. In this section we specify in particular the meaning that we give to the notion of asymptotics of the semi-classical type by pointing out the specificities of the magnetic frame. While the results of the spectral accumulation at each Landau level Λ_q, $q \in \mathbb{Z}_+$, given in sections 4.2 and 4.4 do not depend on q, in section 4.5 we describe the behavior of the eigenvalue distribution in the qth cluster, when q tends to infinity. These results exploit the norm estimates of Berezin-Toeplitz operators obtained in section 3.5.

4.1 Motivation and overview of the results

The present chapter has a central role in the book because here we start investigating the interplay between the spectral theory of the magnetic quantum Hamiltonians discussed in Chapter 2, and the properties of the Berezin-Toeplitz operators revealed in Chapter 3. We believe that it would be convenient for the reader to get acquainted with this interplay first within the framework of such a popular topic as the asymptotic distribution of the discrete eigenvalues for quantum Hamiltonians. We concentrate our attention on the 2D unperturbed operators $H_S(A,0)$ with constant magnetic field $b \neq 0$, and $H_P(A,0)$ with admissible b of non-zero mean value b_0. Thus, the Landau levels in the Schrödinger case and the origin in the Pauli case are eigenvalues of infinite multiplicity for the unperturbed operators. As perturbations, we consider mainly electric potentials but also survey results on perturbations of the magnetic field and the metrics, as well as spectral problems in the exterior Ω of a compact subset of $\mathbb{R}^2$ with Dirichlet, Neumann or Robin boundary conditions on $\partial\Omega$. In all these cases, our assumptions on the perturbations imply the conservation of the essential spectrum. However, generically, there appear discrete eigenvalues of the perturbed operators which accumulate at the energies corresponding to the eigenvalues of infinite multiplicity of the unperturbed operators.

The eigenvalue distribution has been traditionally considered as one of the central mathematical problems of quantum mechanics (see e.g. [162, Section VIII.11] or [13]). Next, we recall briefly some well known results in this area which may be useful for the better understanding of our motivation to consider the problems attacked in this chapter.

Let T be a self-adjoint operator in a separable Hilbert space, and (s,t) be an open interval with $-\infty \leq s < t \leq \infty$. Set

$$N_{(s,t)}(T) := \mathrm{Tr}\, \mathbb{1}_{(s,t)}(T). \tag{4.1.1}$$

Thus, if $(s,t) \cap \sigma_{ess}(T) = \emptyset$, then $N_{(s,t)}(T)$ is the number of the eigenvalues of T, lying on (s,t) and counted with the multiplicities.

Let us start our story with Hamiltonians $\mathrm{Op}^w(\mathscr{F})$ with purely discrete spectrum. Sufficient conditions for the discreteness of $\sigma(\mathrm{Op}^w(\mathscr{F}))$ are, for example, $\mathscr{F} \in \Gamma_\rho^\gamma(\mathbb{R}^{2n};\mathbb{R})$ with $\gamma > 0$, $\rho \in (0,1]$, and

$$\lim_{|\mathbf{w}|\to\infty} \mathscr{F}(\mathbf{w}) = \infty. \tag{4.1.2}$$

For $E \in \mathbb{R}$ introduce the *volume function*

$$\mathfrak{V}(E;\mathscr{F}) := (2\pi)^{-n} |\{(\mathbf{x},\boldsymbol{\xi}) \in T^*\mathbb{R}^n \,|\, \mathscr{F}(\mathbf{x},\boldsymbol{\xi}) < E\}| \tag{4.1.3}$$

where, as usual, $|\cdot|$ denotes the Lebesgue measure. Evidently, if (4.1.2) is valid, then $\mathfrak{V}(E;\mathscr{F}) < \infty$ for any $E \in \mathbb{R}$. Thus, $\mathfrak{V}(E;\mathscr{F})$ is the normalized measure of that part of the phase space $T^*\mathbb{R}^n$ where the *classical* Hamiltonian $\mathscr{F}$ is smaller than the energy E, i.e. $\mathfrak{V}(E;\mathscr{F})$ is the classical counterpart of $N_{(-\infty,E)}(\mathrm{Op}^w(\mathscr{F}))$, the number of bound states of the *quantum* Hamiltonian $\mathrm{Op}^w(\mathscr{F})$ smaller than E. Under suitable hypotheses on the regularity of the growth of $\mathscr{F}$ at infinity, the *Weyl asymptotic law*

$$N_{(-\infty,E)}(\mathrm{Op}^w(\mathscr{F})) = \mathfrak{V}(E;\mathscr{F})(1+o(1)) \tag{4.1.4}$$

holds true as $E \to \infty$. In this case we will say that the eigenvalue asymptotics for the operator $\mathrm{Op}^w(\mathscr{F})$ is *semi-classical*. This terminology is coherent with N. Bohr's correspondence principle according to which in the *limit of large quantum numbers*, the behaviour of a quantum system should be close to that of its classical counterpart (see [22]). An example of hypotheses on $\mathscr{F}$ which guarantee a refined version of (4.1.4), is contained in the following

Theorem 4.1.1 [185, Theorem 30.1] *Assume that* $\mathscr{F} \in C^\infty(\mathbb{R}^{2n};\mathbb{R})$ *satisfies*

$$\begin{aligned} c_1|\mathbf{w}|^{\gamma_1} \le \mathscr{F}(\mathbf{w}) \le c_2|\mathbf{w}|^{\gamma_2}, \\ |D^\alpha \mathscr{F}(\mathbf{w})| \le C_\alpha \mathscr{F}(\mathbf{w})^{1-\rho|\alpha|}, \\ |\mathbf{w}\cdot\nabla\mathscr{F}(\mathbf{w})| \ge c\mathscr{F}(\mathbf{w})^{1-\kappa}, \end{aligned} \tag{4.1.5}$$

for $\mathbf{w} \in \mathbb{R}^{2n}$, $|\mathbf{w}| \ge R_0$ *with some positive constants* c_j, γ_j, $j = 1,2$, C_α, $\alpha \in \mathbb{Z}_+^n$, c, R_0, *and* $\rho \in (0, \min\{1, 1/\gamma_2\}]$, $\kappa \in [0,\rho)$. *Then for any* $\varepsilon > 0$ *we have*

$$N_{(-\infty,E)}(\mathrm{Op}^w(\mathscr{F})) = \mathfrak{V}(E;\mathscr{F})(1 + o(E^{\kappa-\rho+\varepsilon})), \quad E \to \infty. \tag{4.1.6}$$

The proof of Theorem 4.1.1 is an application of the *method of the approximate spectral projection* (see e.g. [185, Theorem 28.1]), based on the following heuristic argument. We have

$$\mathfrak{V}(E;\mathscr{F}) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \mathbb{1}_{(-\infty,E)}(\mathscr{F}(\mathbf{x},\boldsymbol{\xi}))d\mathbf{x}\,d\boldsymbol{\xi}, \quad E \in \mathbb{R}.$$

Applying formally the Mercer theorem, we obtain

$$\mathfrak{V}(E;\mathscr{F}) = \mathrm{Tr}\ \mathrm{Op}^w(\mathbb{1}_{(-\infty,E)} \circ \mathscr{F}) \tag{4.1.7}$$

so that (4.1.4) becomes equivalent to

$$\mathrm{Tr}\,\mathbb{1}_{(-\infty,E)}(\mathrm{Op}^w(\mathscr{F})) = \mathrm{Tr}\ \mathrm{Op}^w(\mathbb{1}_{(-\infty,E)} \circ \mathscr{F})(1 + o(1)), \quad E \to \infty.$$

Since the function $\mathbb{1}_{(-\infty,E)} : \mathbb{R} \to \{0,1\}$ is not continuous, we must approximate it by a suitable function $\chi_E \in C^\infty(\mathbb{R};\mathbb{R})$ which admits appropriate estimates of its derivatives, and hence allows the application of the pseudo differential calculus. In this calculus which leads to the proof of (4.1.6), an important role is played by estimates (4.1.5) which imply that the derivatives $D^\alpha\mathscr{F}$ grow at infinity less rapidly than the symbol $\mathscr{F}$ itself.

Within the context of quantum mechanics, an important example of a Hamiltonian $\mathrm{Op}^w(\mathscr{F})$ with discrete spectrum is the Schrödinger operator $-\Delta + V$ with electric potential V which satisfies

$$V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}), \qquad \lim_{|\mathbf{x}|\to\infty} V(\mathbf{x}) = \infty. \tag{4.1.8}$$

Then $\mathscr{F}(\mathbf{x},\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2 + V(\mathbf{x})$,

$$\mathfrak{V}(E;\mathscr{F}) = (2\pi)^{-n}\omega_n \int_{\mathbb{R}^n} (E - V(\mathbf{x}))_+^{n/2}\,d\mathbf{x}, \quad E \in \mathbb{R},$$

and, by (4.1.8), we have

$$\int_{\mathbb{R}^n} (E - V(\mathbf{x}))_+^{n/2}\, d\mathbf{x} < \infty, \quad E \in \mathbb{R}, \tag{4.1.9}$$

In this case, (4.1.4) reduces to

$$N_{(-\infty,E)}(-\Delta + V) = (2\pi)^{-n}\omega_n \int_{\mathbb{R}^n} (E - V(\mathbf{x}))_+^{n/2}\, d\mathbf{x}(1 + o(1)), \quad E \to \infty, \tag{4.1.10}$$

under appropriate hypotheses on the growth of V at infinity (see e.g. [165, Theorem XIII.81]). An important example of V which satisfies (4.1.8) is $V(\mathbf{x}) = |\mathbf{x}|^2$ so that in this case $H_S(0,V)$ is just the isotropic harmonic oscillator whose eigenvalues according to (2.7.13) are $\sum_{j=1}^n (2q_j + 1)$ with $q_j \in \mathbb{Z}_+$, $j = 1, \dots, n$, so that in this case (4.1.10) can be directly verified.
Note that (4.1.8) and (4.1.9) are only sufficient conditions for the discreteness of $\sigma(-\Delta + V)$. By the corresponding criterion, first established in [135], and later refined in [128], the spectrum $\sigma(-\Delta + V)$ is discrete if and only if for any $a > 0$ we have

$$\lim_{|\mathbf{x}|\to\infty} \inf_F \int_{\mathscr{Q}_a(\mathbf{x})\setminus F} V(\mathbf{y})d\mathbf{y} = \infty$$

where $\mathscr{Q}_a(\mathbf{x}) := \mathbf{x} + (-a/2, a/2)^n$, $n \geq 2$, and the infimum is taken over the *negligible* sets F, i.e. the compact subsets F of $\overline{\mathscr{Q}_a(\mathbf{x})}$ which satisfy

$$\mathrm{cap}(F) \leq c\, \mathrm{cap}(\mathscr{Q}_a)(\mathbf{x})$$

for some constant $c \in (0,1)$. Here $\mathrm{cap}(\mathscr{E})$ stands for the *Wiener capacity* of the set $\mathscr{E} \subset \mathbb{R}^n$, $n \geq 2$ (see the details of the definition in [128, Section 3]). There exist potentials V which meet this criterion but however $\int_{\mathbb{R}^n} (E - V(\mathbf{x}))_+^{n/2}\, d\mathbf{x} = \infty$ for any $E \in \mathbb{R}$ large enough, so that (4.1.10) does not make sense any more. A simple example of such a potential V is

$$V(\mathbf{x}) = |\mathbf{w}|^2 y^2, \quad \mathbf{x} = (\mathbf{w}, y) \in \mathbb{R}^n,$$

where $\mathbf{w} \in \mathbb{R}^m$, $m \geq 2$, $y \in \mathbb{R}$, so that $n = m + 1$. In this case $\sigma(-\Delta + V)$ is discrete but $\int_{\mathbb{R}^n} (E - V(\mathbf{x}))_+^{n/2}\, d\mathbf{x} = \infty$ for any $E > 0$. Let us write

$$-\Delta + V = -\Delta_{\mathbf{w}} \otimes I_y + \int_{\mathbb{R}^m}^{\oplus} \mathfrak{h}(\mathbf{w})d\mathbf{w},$$

where $-\Delta_{\mathbf{w}}$ is the Laplacian, self-adjoint in $L^2(\mathbb{R}^m_{\mathbf{w}})$, I_y is the identity in $L^2(\mathbb{R}_y)$, and

$$\mathfrak{h}(\mathbf{w}) := -\frac{d^2}{dy^2} + |\mathbf{w}|^2 y^2$$

is an operator, self-adjoint in $L^2(\mathbb{R}_y)$, depending on the parameter $\mathbf{w} \in \mathbb{R}^m$. If $\mathbf{w} \neq 0$, then the scaling $|\mathbf{w}|^{1/2} y \mapsto y$ and relation (2.7.13) imply the eigenvalues of $\mathfrak{h}(\mathbf{w})$ are $(2q+1)|\mathbf{w}|$, $q \in \mathbb{Z}_+$. Following [189], we introduce the *sliced bread Hamiltonian*

$$H_{SB} := \bigoplus_{q \in \mathbb{Z}_+} (-\Delta_{\mathbf{w}} + (2q+1)|\mathbf{w}|),$$

self-adjoint in $\ell^2(\mathbb{Z}_+; L^2(\mathbb{R}^m))$. It turns out that H_{SB} is the effective Hamiltonian within the context of the high-energy asymptotic regime for $-\Delta + V$. Namely, arguing as in [189], we can show that

$$\begin{aligned} N_{(-\infty,E)}(-\Delta+V) &= N_{(-\infty,E)}(H_{SB})(1+o(1)) \\ &= (2\pi)^{-m}\omega_m \sum_{q\in\mathbb{Z}_+} \int_{\mathbb{R}^m} (E-(2q+1)|\mathbf{w}|)_+^{m/2}\, d\mathbf{w}(1+o(1)) \\ &= \frac{m\omega_m^2}{(2\pi)^m} B\left(\frac{m}{2}+1, m+1\right) \sum_{q\in\mathbb{Z}_+} (2q+1)^{-m} E^{3m/2}(1+o(1)), \end{aligned}$$

as $E \to \infty$, where B is the Euler beta function. Thus, the high-energy asymptotics of $N_{(-\infty,E)}(-\Delta+V)$ is *non-classical*, i.e. its quantum features are manifested already in the main asymptotic term of the eigenvalue counting function. Deeper and more sophisticated examples of "degenerate" potentials V for which the Schrödinger operator $-\Delta+V$ has purely discrete spectrum but non-classical high-energy eigenvalue asymptotics, can be found in [166] and [189].

Once our book is devoted to *magnetic* quantum Hamiltonians, let us mention a result on the magnetic Schrödinger operator $H_S(A,0)$ with discrete spectrum. In this case the high-energy asymptotics is obliged to be non-classical since the Weyl symbol $\mathscr{F}$ of $H_S(A,0)$ is

$$\mathscr{F}(\mathbf{x},\boldsymbol{\xi}) = |\boldsymbol{\xi} - A(\mathbf{x})|^2, \quad (\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^{2n},$$

and the change of the variables $\boldsymbol{\xi} - A(\mathbf{x}) \mapsto \boldsymbol{\xi}$, $\mathbf{x} \mapsto \mathbf{x}$, shows that the volume function $\mathfrak{V}(E;\mathscr{F})$ is independent of A, and hence, $\mathfrak{V}(E;\mathscr{F}) = \infty$ for any $E > 0$.

For the formulation of our result concerning the high-energy asymptotics for $H_S(A,0)$, we need the notion of the *integrated density of states* (IDS) for a locally elliptic differential operator T, self-adjoint in $L^2(\mathbb{R}^n)$. Let $T_{L,\mathbf{x}}$ be the Dirichlet realization of T on the cube

$$\mathscr{Q}_L = \mathscr{Q}_L(\mathbf{x}) := \mathbf{x} + (-L/2, L/2)^n, \quad \mathbf{x} \in \mathbb{R}^n, \quad L \in (0,\infty),$$

self-adjoint in $L^2(\mathscr{Q}_L)$, so that $T_{L,\mathbf{x}}$ has purely discrete spectrum. Then the non-decreasing left-continuous function $\mathfrak{N}(E;T)$, $E \in \mathbb{R}$ is said to be the IDS for the operator T if

$$\mathfrak{N}(E;T) = \lim_{L\to\infty} L^{-n} N_{(-\infty,E)}(T_{L,\mathbf{x}}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{4.1.11}$$

at its points $E \in \mathbb{R}$ of continuity. The almost sure existence of the IDS for the non-magnetic Schrödinger operator $-\Delta+V$ with *ergodic* V has been well known since long ago (see [109, 137]), and was proved in [210] and [98] for $H_S(A,V)$ with fairly general ergodic B and V (see also [154, 25] for the case of 2D Pauli operators $H_P(A,\mathbf{V})$ with almost periodic B and $\mathbf{V}$). In [65, 98], it was shown that if the IDS $\mathfrak{N}(\cdot;H_S(A,V))$ defined in (4.1.11) exists, then at its points of continuity we have

$$\mathfrak{N}(E;H_S(A,V)) = \lim_{L\to\infty} L^{-n} \mathrm{Tr}(\mathbb{1}_{\mathscr{Q}_L} \mathbb{1}_{(-\infty,E)}(H_S(A,V))\mathbb{1}_{\mathscr{Q}_L}), \tag{4.1.12}$$

where $\mathbb{1}_{\mathscr{Q}_L}$ denotes the multiplier by the characteristic function of the cube $\mathscr{Q}_L$. Moreover, we may replace in (4.1.11) the Dirichlet realization of $H_S(A,V)$ on $\mathscr{Q}_L$, by its

Neumann realization. Recall now that the integral kernel $\mathscr{E}_E$, $E \in \mathbb{R}$, of the spectral projection $\mathbb{1}_{(-\infty,E)}(H_S(A,0))$ with constant B, is constant on the diagonal (see (2.7.52) and (2.7.62)). Applying (4.1.12) and the Mercer theorem, we find that in this case

$$\mathfrak{N}(E; H_S(A,0)) = \mathscr{E}_E(0,0), \quad E \in \mathbb{R}. \tag{4.1.13}$$

In particular, if $B = 0$, then the IDS for the Laplacian $-\Delta$, self-adjoint in $L^2(\mathbb{R}^n)$, can be wrriten as

$$\mathfrak{N}(E; -\Delta) = \frac{\omega_n}{(2\pi)^n} E_+^{n/2}, \quad E \in \mathbb{R}.$$

Thus, (4.1.10) is equivalent to

$$N_{(-\infty,E)}(-\Delta + V) = \int_{\mathbb{R}^n} \mathfrak{N}(E - V(\mathbf{x}); -\Delta)\, d\mathbf{x}\,(1 + o(1)), \quad E \to \infty.$$

Moreover, if $n = 3$, $B \neq 0$ is constant, and $b := |B|$, then (2.7.62) implies

$$\mathfrak{N}(E; H_S(A,0)) = \mathfrak{N}_b(E) := \frac{b}{2\pi^2} \sum_{q \in \mathbb{Z}_+} (E - (2q+1)b)_+^{1/2}, \quad E \in \mathbb{R}. \tag{4.1.14}$$

Theorem 4.1.2 [201] *Let* $n = 3$ *and* $A \in C^2(\mathbb{R}^3; \mathbb{R}^3)$. *Set* $b(\mathbf{x}) := |\mathrm{curl}\, A(\mathbf{x})|$, $\mathbf{x} \in \mathbb{R}^3$, *and assume*

$$\lim_{|\mathbf{x}| \to \infty} b(\mathbf{x}) = \infty,$$

$$|D^\alpha A_j(\mathbf{x})| = o(b(\mathbf{x}))^{3/2}, \quad |\mathbf{x}| \to \infty,$$

for $j = 1,2,3$, *and* $\alpha \in \mathbb{Z}_+^3$ *with* $|\alpha| = 2$. *Put*

$$m(E) := |\{\mathbf{x} \in \mathbb{R}^3 \,|\, b(\mathbf{x}) < E\}|, \quad E \in \mathbb{R},$$

and suppose that $m(2E) \leq Cm(E)$ *for some* $C > 1$ *and* E *large enough. Then the spectrum of* $H_S(A,0)$ *is purely discrete and*

$$N_{(-\infty,E)}(H_S(A,0)) = \int_{\mathbb{R}^3} \mathfrak{N}_{b(\mathbf{x})}(E)\, d\mathbf{x}\,(1 + o(1)), \quad E \to \infty,$$

$\mathfrak{N}_b$ *being the function defined in* (4.1.14).

A generalization of Theorem 4.1.2 to any dimension $n \geq 2$ can be found in [50] where the result is again formulated in the terms of the IDS for the operator $H_S(A,0)$ with constant B (see (2.7.52) and (2.7.62)). Further investigation of semi-classical and non-classical high-energy eigenvalue asymptotics for the operator $H_S(A,V)$ can be found in [127].

Up to now we have discussed the eigenvalue asymptotics for quantum Hamiltonians with purely discrete spectrum. However, Hamiltonians with non-empty essential spectrum are of primary importance in quantum mechanics. The leading example here is the

Schrödinger operator $-\Delta + V$ with $V \in L^\infty(\mathbb{R}^n;\mathbb{R})$ (or, more generally, $V \in L^p(\mathbb{R}^n;\mathbb{R})$ with suitable $p \in [1,\infty)$) such that

$$\lim_{|\mathbf{x}|\to\infty} V(\mathbf{x}) = 0.$$

In this case

$$\sigma_{\mathrm{ess}}(-\Delta + V) = \sigma_{\mathrm{ess}}(-\Delta) = \sigma(-\Delta) = [0,\infty) \tag{4.1.15}$$

but if the negative part V_- of V is non-trivial, then $-\Delta + V$ may have non-empty negative discrete spectrum. If

$$V_-(\mathbf{x}) = O(|\mathbf{x}|^{-\gamma}), \quad \gamma > 2, \tag{4.1.16}$$

for large $|\mathbf{x}|$, then the discrete spectrum of $-\Delta + V$ is finite, i.e. we have

$$N_{(-\infty,0)}(-\Delta + V) < \infty. \tag{4.1.17}$$

If $n \geq 3$, then (4.1.17) follows from the celebrated *Cwikel-Lieb-Rozenblum estimate*

$$N_{(-\infty,0)}(-\Delta + V) \leq c_n \int_{\mathbb{R}^n} V_-(\mathbf{x})^{n/2}\, d\mathbf{x},$$

valid with a constant c_n which depends only on the dimension n, and $V : \mathbb{R}^n \to \mathbb{R}$ such that $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $V_- \in L^{n/2}(\mathbb{R}^n)$ (see e.g. [165, Theorem XIII.12]). If $n = 1,2$, then (4.1.17) is implied, say, by the results of [47]. Our following theorem shows that the condition $\gamma > 2$ in (4.1.16) is close to the optimal one.

Theorem 4.1.3 [165, Theorem XIII.82] *Let* $n \geq 1$, $\gamma \in (0,2)$, $V \in C^1(\mathbb{R}^n;\mathbb{R})$. *Assume that there exist constants* $C \in (0,\infty)$, $R \in (0,\infty)$, *such that*

$$|V(\mathbf{x})| \leq C\langle \mathbf{x}\rangle^{-\gamma}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{4.1.18}$$

$$|\nabla V(\mathbf{x})| \leq C\langle \mathbf{x}\rangle^{-\gamma-1}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{4.1.19}$$

$$V(\mathbf{x}) \leq -C\langle \mathbf{x}\rangle^{-\gamma}, \quad \mathbf{x} \in \mathbb{R}^n, \quad |\mathbf{x}| \geq R. \tag{4.1.20}$$

Then we have

$$N_{(-\infty,E)}(-\Delta + V) = \frac{\omega_n}{(2\pi)^n} \int_{\mathbb{R}^n} (E - V(\mathbf{x}))_+^{n/2}\, d\mathbf{x}\,(1 + o(1)) \asymp |E|^{-n(\frac{1}{\gamma}-\frac{1}{2})} \tag{4.1.21}$$

as $E \uparrow 0$.

Relation (4.1.21) describes the asymptotic distribution of the discrete spectrum of $-\Delta +$ V as the energy E approaches from below $\inf \sigma_{\mathrm{ess}}(-\Delta + V) = 0$, and shows that under the hypotheses of Theorem 4.1.3 the eigenvalue asymptotics of $-\Delta + V$ near the origin is semi-classical. An analogue of (4.1.21) under more general assumptions on V in the case $n \geq 3$, can be found in [168].
However, in the border-line case $\gamma = 2$, some corrections to the semi-classical Weyl law (4.1.21) are needed. Let $n \geq 1$. Assume that there exists a function $\phi \in L^\infty(\mathbb{S}^{n-1};\mathbb{R})$, such that

$$\lim_{r\to\infty} r^2 V(r\omega) = \phi(\omega) \tag{4.1.22}$$

uniformly with respect to $\omega \in \mathbb{S}^{n-1}$. If $n \geq 2$, let $\{\mu_j(\phi)\}_{j\in\mathbb{N}}$ be the non-decreasing sequence of the eigenvalues of $-\Delta_{\mathbb{S}^{n-1}} + \phi$ where $-\Delta_{\mathbb{S}^{n-1}}$ is the Beltrami-Laplace operator on the unit sphere $\mathbb{S}^{n-1}$. If $n = 1$, set

$$\mu_1(\phi) := \min\{\phi(-1), \phi(1)\}, \quad \mu_2(\phi) := \max\{\phi(-1), \phi(1)\}.$$

Theorem 4.1.4 [110, 90] *Let* $n \geq 1$, $V \in L^\infty(\mathbb{R}^n; \mathbb{R})$. *Assume that there exists* $\phi \in L^\infty(\mathbb{S}^{n-1}; \mathbb{R})$ *such that* (4.1.22) *holds true. Then we have*

$$\lim_{E\uparrow 0} |\ln|E||^{-1} N_{(-\infty,E)}(-\Delta+V) = \mathscr{C}_n(\phi) := \frac{1}{2\pi}\sum_j \left(\mu_j(\phi) + \frac{(n-2)^2}{4}\right)_-^{1/2}. \qquad (4.1.23)$$

If, moreover, $\mu_1(\phi) > -\frac{(n-2)^2}{4}$, *then*

$$N_{(-\infty,E)}(-\Delta+V) = O(1), \qquad (4.1.24)$$

as $E \uparrow 0$, *i.e. the discrete spectrum of* $-\Delta+V$ *is finite.*

Note that the asymptotic order in (4.1.23) is semi-classical but the coefficient is not. In particular, the existence of a threshold value equal to the Hardy constant $(n-2)^2/4$ which determines whether $\sigma_{\rm disc}(-\Delta+V)$ is finite or not, is a quantum effect which is not directly predictable by the semi-classical intuition.
Let us underline the importance of estimate (4.1.19) for the validity of the semi-classical Weyl law (4.1.23). In particular, (4.1.19) combined with (4.1.18) implies that the first-order derivatives of V decay at infinity faster than V itself. In order to construct a non-classical counterexample, let us consider the potential

$$W := \eta V$$

where $\eta \in C^\infty(\mathbb{T}^n; \mathbb{R})$ with $\mathbb{T}^n := \mathbb{R}^n/2\pi\mathbb{Z}^n$, and $V \in \Gamma_1^{-\gamma}$, $\gamma \in (0,2]$. Thus, $W \in \Gamma_0^{-\gamma}$ but if η is not constant, the derivatives of W do not decay faster than V. Denote by η_0 the mean value of η. Our first theorem treats the case where $\eta_0 \neq 0$.

Theorem 4.1.5 [158, Theorem 3.1] *Let* $W = \eta V$ *where* $\eta \in C^\infty(\mathbb{T}^n; \mathbb{R})$ *with* $\eta_0 \neq 0$, *and* $V \in \Gamma_1^{-\gamma}$, $\gamma \in (0,2]$.
(i) *Let* $\gamma \in (0,2)$. *Assume that*

$$\eta_0 V(\mathbf{x}) \leq -C\langle \mathbf{x}\rangle^{-\gamma}, \quad |\mathbf{x}| \geq R, \qquad (4.1.25)$$

with some constants $C > 0$ *and* $R > 0$. *Then we have*

$$N_{(-\infty,E)}(-\Delta+W) = N_{(-\infty,E)}(-\Delta+\eta_0 V)(1+o(1)), \quad E \uparrow 0. \qquad (4.1.26)$$

If, on the contrary,

$$\eta_0 V(\mathbf{x}) \geq C\langle \mathbf{x}\rangle^{-\gamma}, \quad |\mathbf{x}| \geq R, \qquad (4.1.27)$$

then (4.1.24) *holds true.*
(ii) *Let* $\gamma = 2$. *Assume that there exists* $\phi \in L^\infty(\mathbb{S}^{n-1}; \mathbb{R})$ *such that* (4.1.22) *holds true. Then we have*

$$\lim_{E\uparrow 0} |\ln|E||^{-1} N_{(-\infty,E)}(-\Delta+W) = \mathscr{C}_n(\eta_0\phi),$$

where $\mathscr{C}_n$ *is the constant defined in* (4.1.23). *If, moreover,* $\mu_1(\eta_0\phi) > -\frac{(n-2)^2}{4}$*, then* (4.1.24) *holds true.*

In our next theorem we suppose that $\eta_0 = 0$. In this case, we write the Fourier series $\eta(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n\setminus\{0\}} \eta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, set $\varphi(\mathbf{x}) := -\sum_{\mathbf{k}\in\mathbb{Z}^n\setminus\{0\}} |\mathbf{k}|^{-2}\eta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$,

$$\psi(\mathbf{x}) := |\nabla\varphi(\mathbf{x})|^2, \quad \mathbf{x} \in \mathbb{R}^n,$$

and denote by ψ_0 the mean value of ψ. Note that if η does not vanish identically, then $\psi_0 > 0$.

Theorem 4.1.6 [158, Theorem 3.2] *Let* $W = \eta V$ *where* $\eta \in C^\infty(\mathbb{T}^n;\mathbb{R})$ *does not vanish identically but* $\eta_0 = 0$*, and* $V \in \Gamma_1^{-\gamma}$*,* $\gamma \in (0,2]$.
(i) *Let* $\gamma \in (0,1)$*. Assume that*

$$V(\mathbf{x})^2 \geq C\langle \mathbf{x}\rangle^{-2\gamma}, \quad |\mathbf{x}| \geq R,$$

with some constants $C > 0$ *and* $R > 0$*. Then we have*

$$N_{(-\infty,E)}(-\Delta + W) = N_{(-\infty,E)}(-\Delta - \psi_0 V^2)(1 + o(1)) \asymp |E|^{-\frac{n}{2}(\frac{1}{\gamma}-1)}, \quad E \uparrow 0. \qquad (4.1.28)$$

(ii) *Let* $\gamma = 1$*. Assume that there exists* $\phi \in L^\infty(\mathbb{S}^{n-1};\mathbb{R})$ *such that*

$$\lim_{r\to\infty} r^2 V(r\omega)^2 = \phi(\omega)$$

uniformly with respect to $\omega \in \mathbb{S}^{n-1}$*. Then we have*

$$\lim_{E\uparrow 0} |\ln|E||^{-1} N_{(-\infty,E)}(-\Delta + W) = \mathscr{C}_n(-\psi_0\phi).$$

If, moreover, $\mu_1(-\psi_0\phi) > -\frac{(n-2)^2}{4}$*, then* (4.1.24) *holds true.*
(iii) *Let* $\gamma \in (1,2]$*. Then* (4.1.24) *holds true again.*

The results of Theorems 4.1.5 and 4.1.6 are not of semi-classical nature. For example, relation (4.1.25) is non-classical because there exist η and V which satisfy the general hypotheses of Theorem 4.1.5 as well as (4.1.25), but we have

$$\limsup_{E\uparrow 0} \int_{\mathbb{R}^n} (E - \eta_0 V(\mathbf{x}))_+^{n/2} d\mathbf{x} \Big/ \int_{\mathbb{R}^n} (E - W(\mathbf{x}))_+^{n/2} d\mathbf{x} < 1.$$

Similarly, (4.1.27) does not exclude the possibility that

$$\lim_{E\uparrow 0} \int_{\mathbb{R}^n} (E - W(\mathbf{x}))_+^{n/2} d\mathbf{x} = \infty,$$

but nevertheless (4.1.24) holds true. In the case of (4.1.28), even the asymptotic order of $N_{(-\infty,E)}(-\Delta + W)$ as $E \uparrow 0$ is non-semi-classical.
In fact, in Theorems 4.1.5 and 4.1.6, we observe the subtle effect of averaging of the

oscillating factor in the potentials ηV or $-\psi V^2$. Closely related phenomena which at first glance may seem fairly remote from our setting, are discussed in [192], [164, Section XI.8, Appendix 2], and [180].
In many applications of quantum mechanics, the spectrum of the unperturbed Hamiltonian H_0 is, similarly to (4.1.15), purely essential, and consists of finitely or infinitely many disjoint closed intervals called *spectral bands*. If we perturb such H_0 by a relatively compact operator, then discrete eigenvalues of the perturbed operator may appear in the *spectral gaps* of H_0, i.e. in the open intervals which constitute the complement $\mathbb{R} \setminus \sigma(H_0)$. We will consider in more detail such operators in Chapter 5 but here we note that the 2D Landau Hamiltonian $H_S(A,0)$ fits into this general scheme even though in this case the spectral bands are degenerated into points, namely the Landau levels $\Lambda_q = b(2q+1)$, $q \in \mathbb{Z}_+$, where $b > 0$ is the scalar constant magnetic field. Respectively, the spectral gaps are $(\Lambda_{q-1}, \Lambda_q)$, $q \in \mathbb{Z}_+$, with $\Lambda_{-1} := -\infty$. Assume that $V \in L^\infty(\mathbb{R}^2;\mathbb{R})$ and $\lim_{|\mathbf{x}|\to\infty} V(\mathbf{x}) = 0$, so that $VH_S(A,0)^{-1} \in \mathfrak{S}_\infty(L^2(\mathbb{R}^2))$. Thus the spectrum of $H_S(A,V)$ on $\mathbb{R} \setminus b(2\mathbb{Z}_+ + 1)$ is purely discrete.
In order to investigate the asymptotic distribution of the eigenvalues of $H_S(A,V)$ above (resp., below) a fixed Landau Level Λ_q, $q \in \mathbb{Z}_+$, we may pick numbers $\widetilde{\Lambda}_q \in (\Lambda_{q-1}, \Lambda_q)$, $q \in \mathbb{Z}_+$, set $\widetilde{\Lambda}_{-1} := -\infty$, introduce the function $N_{(E,\widetilde{\Lambda}_q)}(H_S(A,V))$ with $E \in (\Lambda_q, \widetilde{\Lambda}_q)$ (resp., $N_{(\widetilde{\Lambda}_{q-1},E)}(H_S(A,V))$ with $E \in (\widetilde{\Lambda}_{q-1}, \Lambda_q)$), and examine its asymptotics as $E \downarrow \Lambda_q$ (resp., as $E \uparrow \Lambda_q$). One of the central results of this chapter is that, generically, the effective Hamiltonian which governs these asymptotics is the compact Berezin-Toeplitz operator $p_q V p_q$ where, as usual, p_q is the orthogonal projection onto $\mathrm{Ker}\,(H_S(A,0) - \Lambda_q I)$. For example, if $V \geq 0$, then

$$N_{(\Lambda_q+\lambda,\widetilde{\Lambda}_q)}(H_S(A,V)) \sim N_{(\lambda,\infty)}(p_q V p_q), \tag{4.1.29}$$

$$N_{(\widetilde{\Lambda}_{q-1},\Lambda_q-\lambda)}(H_S(A,-V)) \sim N_{(\lambda,\infty)}(p_q V p_q), \tag{4.1.30}$$

as $\lambda \downarrow 0$ (see Proposition 4.4.1 for the precise formulation, and Proposition 4.4.2 for a related result in the case of a non-sign-definite V). That is why, in Section 4.2 we consider the eigenvalue asymptotics for the operator $p_q \mathscr{F} p_q$, $q \in \mathbb{Z}_+$, with sign-definite $\mathscr{F}$ of compact support or exponential decay at infinity, and with non-sign-definite $\mathscr{F}$ of power-like decay. The choice of this scale of decay of $\mathscr{F}$ is first motivated by the fact that even if $\mathrm{supp}\,\mathscr{F}$ is compact but $\mathscr{F} \geq 0$ and $\mathscr{F} \neq 0$, then $\mathrm{rank}\, p_q \mathscr{F} p_q = \infty$, $q \in \mathbb{Z}_+$, and hence $\lim_{\lambda\downarrow 0} N_{(\lambda,\infty)}(p_q \mathscr{F} p_q) = \infty$. Therefore, (4.1.29) - (4.1.30) imply that in contrast to the non-magnetic case, there is no threshold of the decay of $V \geq 0$ beyond which the discrete spectrum of $H_S(A, \pm V)$ near a given endpoint of $\sigma_{\mathrm{ess}}(H_S(A,0))$ is finite. Further, the Gaussian decay of V could be considered as the border-line case which separates the semi-classical from the non-classical spectral asymptotics for the operator $p_q \mathscr{F} p_q$. We allocate a special attention to this transition, and that is why we consider the entire scale of symbols $\mathscr{F}(\mathbf{x})$ which, roughly speaking, behave at infinity like $e^{-c|\mathbf{x}|^{2\beta}}$ with $c > 0$ and $\beta \in (0,\infty)$; thus, $\beta = 1$ corresponds to the Gaussian decay. The symbols $\mathscr{F}$ of power-like decay have been traditionally studied within the framework of the theory of compact ΨDOs. Accordingly, the spectral asymptotics near the Landau levels for 2D Landau Hamiltonians $H_S(A,V)$ with potentials V of power-like

decay was investigated in [153] and [101] (see, in particular, [101, Theorem 11.3.17]) and, recently, in [103] where also potentials V which decay not faster than $e^{-c|\mathbf{x}|}$, $c > 0$, were studied. The methods used in [101, 102, 103] are based on microlocal techniques, i.e. techniques arising in the theory of ΨDOs and Fourier integral operators, and Tauberian theorems. In [103, Remark 23.4.9], V. Ivrii wrote that the spectral asymptotics for the 2D Landau Hamiltonian $H_S(A,V)$ with V decaying faster than $e^{-c|\mathbf{x}|}$, $c > 0$, is out of reach of these methods. In our approach we apply variational arguments combined with pseudo-differential methods and techniques from the theory of Berezin-Toeplitz operators. As mentioned, this approach allows us to investigate successfully the spectral asymptotics of $H_S(A,V)$ with the whole range of V of power-like decay, exponential decay or compact support. Note, however, that in the cases of rapidly decaying V, we need the hypothesis that V has a definite sign.
We also obtain similar results for the 2D Pauli operators $H_P(A,\mathbf{V})$ with admissible magnetic fields b of mean value $b_0 \neq 0$, and decaying matrix-valued $\mathbf{V} : \mathbb{R}^2 \to \mathscr{M}_2$. More precisely, we examine the asymptotics as $\lambda \downarrow 0$ of the eigenvalue counting functions $N_{(-\infty,-\lambda)}(H_P(A,-\mathbf{V}))$ and $N_{(\lambda,\widetilde{\Lambda})}H_P((A,\mathbf{V}))$ with $\mathbf{V} \geq 0$ and appropriate $\widetilde{\Lambda} \in (0,\infty)$. If $b_0 > 0$ (resp., if $b_0 < 0$), then the effective Hamiltonian is $\mathfrak{p}_{ann}(b)V_{11}\mathfrak{p}_{ann}(b)$ (resp., $\mathfrak{p}_{cre}(b)V_{22}\mathfrak{p}_{cre}(b)$) where $\mathfrak{p}_{ann}(b)$ (resp., $\mathfrak{p}_{cre}(b)$) is the orthogonal projection onto $\operatorname{Ker} a(b)$ (resp., onto $\operatorname{Ker} a(b)^*$).
In Subsection 4.4.2 we also discuss the eigenvalue asymptotics for magnetic and geometric perturbations of the 2D Landau Hamiltonian and other 2D magnetic quantum Hamiltonians. These asymptotics near Landau levels are independent of the Landau level number, but the rate at which the discrete eigenvalues approach the qth Landau level tends to 0 as q tends to infinity. It justifies the terminology "eigenvalue clusters" which is studied in Section 4.5 where we distinguish different cases according to the decay rate of the potential. In the short-range case, the rate at which the discrete eigenvalues approach the qth Landau is independent of the potential and the Radon transform of the potential appears in the asymptotic distribution of eigenvalues when q tends to infinity. In the long-range case, a more refined study involves the mean-value transform of the main homogeneous part of the potential (introduced in Condition 4.5.2).
Finally, we mention some applications of the results described in this chapter. The precise spectral asymptotics for the Berezin-Toeplitz operator $p_q\mathscr{F}p_q$, $q \in \mathbb{Z}_+$, with compactly supported symbol $\mathscr{F}$ were applied in [51] to obtain a controllability result which played an important role in the proof of the Lipschitz continuity of the IDS for the 2D Landau Hamiltonian $H_S(A,V)$ with random ergodic alloy-type potential V (see [52]), and of the existence of dynamical delocalization for the same operator (see [80]). We reproduce this controllability result in Section 4.3.
The spectral asymptotics for the 2D Hamiltonians $H_S(A,V)$ and $H_P(A,\mathbf{V})$ are used in Chapter 5 in order to study respectively the threshold singularities of the spectral shift function, and the asymptotic distribution of the resonances near the thresholds, related to the 3D counterparts of $H_S(A,V)$ and $H_P(A,\mathbf{V})$.

4.2 Eigenvalue asymptotics for Berezin-Toeplitz operators

4.2.1 Notations and auxiliary results

Throughout the subsection $\mathfrak{H}$ and $\mathfrak{H}_j$, $j = 1, 2$, denote separable Hilbert spaces.
We start with a version of the *mini-max principle* known as the *variational Glazman lemma*.

Lemma 4.2.1 [20, Chapter 10, Section 2, Theorem 3] *Let* $T = T^*$ *be an operator lower bounded in* $\mathfrak{H}$*, and* τ *be its quadratic form with domain* $\mathfrak{D}(\tau) := \mathfrak{D}(|T|^{1/2})$*. Then for each* $s \in \mathbb{R}$ *we have*

$$N_{(-\infty,s)}(T) = \sup \dim \mathfrak{T}_s$$

where $\mathfrak{T}_s$ *is a subspace of* $\mathfrak{D}(\tau)$ *whose elements* $u \neq 0$ *satisfy*

$$\tau[u] < s\|u\|_{\mathfrak{H}}^2.$$

Lemma 4.2.1 immediately entails the following

Corollary 4.2.1 *Let* $T_j = T_j^*$, $j = 1, 2$, *be two operators lower-bounded in* $\mathfrak{H}$*. Let* τ_j *be the quadratic form of the operator* T_j, $j = 1, 2$*. Assume that* $T_1 \leq T_2$*, i.e. that* $\mathfrak{D}(\tau_2) \subset \mathfrak{D}(\tau_1)$ *and*

$$\tau_1[u] \leq \tau_2[u], \quad u \in \mathfrak{D}(\tau_2).$$

Then we have

$$N_{(-\infty,s)}(T_2) \leq N_{(-\infty,s)}(T_1), \quad s \in \mathbb{R}.$$

Corollary 4.2.2 *Let* T *be an operator self-adjoint in* $\mathfrak{H}$*, and* $-\infty < s < t < \infty$*. Then*

$$N_{(s,t)}(T) = \sup \dim \mathfrak{T}_{s,t}$$

where $\mathfrak{T}_{s,t}$ *is a subspace of* $\mathfrak{D}(T)$ *whose elements* $u \neq 0$ *satisfy*

$$\left\| \left(T - \frac{s+t}{2} I \right) u \right\|_{\mathfrak{H}}^2 < \frac{(s-t)^2}{4} \|u\|_{\mathfrak{H}}^2. \tag{4.2.1}$$

Proof. By the spectral theorem,

$$N_{(s,t)}(T) = N_{(-\infty,0)}((T - sI)(T - tI)).$$

By Lemma 4.2.1, the quantity $N_{(-\infty,0)}((T - sI)(T - tI))$ is equal to the maximal dimension of $\mathfrak{D}(T)$ whose elements $u \neq 0$ satisfy

$$\langle (T - tI)u, (T - sI)u \rangle_{\mathfrak{H}} < 0$$

which is equivalent to (4.2.1). □

Let $T \in \mathfrak{S}_\infty(\mathfrak{H}_1, \mathfrak{H}_2)$. For $s > 0$ set

$$n_*(s; T) := N_{(s^2,\infty)}(T^* T).$$

Thus $n_*(s;T)$ is the number of the singular values of the operator T larger than s, and counted with the multiplicities. Since the non-zero singular values of the operators T and T^* coincide together with the multiplicities (see e.g. [20, Chapter 11, Section 1, Theorem 1]), we have

$$n_*(s;T) = n_*(s;T^*), \quad s > 0. \tag{4.2.2}$$

If $T_j \in \mathfrak{S}_\infty(\mathfrak{H}_1, \mathfrak{H}_2)$, and $s_j > 0$, $j = 1,2$, then the *Ky Fan inequality*

$$n_*(s_1 + s_2; T_1 + T_2) \le n_*(s_1; T_1) + n_*(s_2; T_2) \tag{4.2.3}$$

holds true (see e.g. [20, Chapter 11, Section 1, Eq. (17)]).
Let now $T = T^* \in \mathfrak{S}_\infty(\mathfrak{H})$. For $s > 0$ set

$$n_\pm(s;T) := N_{(s,\infty)}(\pm T).$$

Thus, $n_+(s;T)$ (resp., $n_-(s;T)$) is the number of the eigenvalues of T larger than s (resp., smaller than $-s$), and counted with the multiplicities. In this case

$$n_*(s;T) = n_+(s;T) + n_-(s;T), \quad s > 0.$$

If $T_j = T_j^* \in \mathfrak{S}_\infty(\mathfrak{H})$, and $s_j > 0$, $j = 1,2$, then the *Weyl inequalities*

$$n_\pm(s_1 + s_2; T_1 + T_2) \le n_\pm(s_1; T_1) + n_\pm(s_2; T_2) \tag{4.2.4}$$

hold true (see e.g. [20, Chapter 9, Section 2, Theorem 9]).
Further, if $T = T^* \in \mathfrak{S}_\infty(\mathfrak{H})$, and $\mathbb{1}_{(0,\infty)}(T) \neq 0$ (resp., $\mathbb{1}_{(-\infty,0)}(T) \neq 0$), then $\left\{\nu_k^+(T)\right\}_{k\ge 0}$ (resp., $\left\{-\nu_k^-(T)\right\}_{k\ge 0}$) denotes the non-increasing (resp., non-decreasing) set of the positive (resp., negative) eigenvalues of the operator T, counted with the multiplicities. If $\operatorname{rank} \mathbb{1}_{(0,\infty)}(\pm T) = \infty$, then $\lim_{k\to\infty} \nu_k^\pm(T) = 0$, and $n_\pm(s;T) \to \infty$ as $s \downarrow 0$.
Next, we introduce special notations for the eigenvalues of the Berezin-Toeplitz operators $p_q \mathscr{F} p_q$, $q \in \mathbb{Z}_+$, and $p_\natural \mathscr{F} p_\natural$, $\natural = \text{ann}, \text{cre}$, under the assumption that

$$\mathscr{F} \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}), \quad \lim_{|\mathbf{x}|\to\infty} \mathscr{F}(\mathbf{x}) = 0. \tag{4.2.5}$$

Let at first $b \in (0,\infty)$ be constant. By Corollary 3.3.1 the operators $p_q(b)\mathscr{F}p_q(b)$ with $q \in \mathbb{Z}_+$ are compact and self-adjoint in $L^2(\mathbb{R}^2)$. Suppose that $\mathbb{1}_{(0,\infty)}(\pm p_q \mathscr{F} p_q) \neq 0$, and for brevity set

$$\nu_{k,q}^\pm(\mathscr{F}) = \nu_{k,q}^\pm(\mathscr{F}, b) := \nu_k^\pm(p_q(b)\mathscr{F}p_q(b)), \quad k \ge 0, \quad q \in \mathbb{Z}_+. \tag{4.2.6}$$

Assume now that b satisfies (3.3.18). By Corollary 3.3.2, the operators $\mathfrak{p}_\natural(b)\mathscr{F}\mathfrak{p}_\natural(b)$ with $\natural = \text{ann}, \text{cre}$ are self-adjoint and compact in $\operatorname{Ran} \mathfrak{p}_\natural$. If $\mathbb{1}_{(0,\infty)}(\pm \mathfrak{p}_\natural \mathscr{F} \mathfrak{p}_\natural) \neq 0$, then, similarly to (4.2.6), set

$$\nu_{k,\natural}^\pm(\mathscr{F}) = \nu_{k,\natural}^\pm(\mathscr{F}, b) := \nu_k^\pm(p_\natural(b)\mathscr{F}p_\natural(b)), \quad k \ge 0, \quad \natural = \text{ann}, \text{cre}.$$

4.2.2 Compactly supported symbols

In this subsection we assume that $\mathscr{F} \geq 0$, and that the support of $\mathscr{F}$ is compact and has a non-empty interior. Suppose, moreover, that the magnetic field $b > 0$ is constant. Then the operator $p_q(b)\mathscr{F}p_q(b)$, $q \in \mathbb{Z}_+$, is non-negative and compact, and $\operatorname{rank} p_q\mathscr{F}p_q = \infty$. Hence, it is relevant to study the asymptotics of the eigenvalue $\nu^+_{k,q}(\mathscr{F})$ as $k \to \infty$, $q \in \mathbb{Z}_+$, being fixed. As a warm-up, we note that if $\mathscr{F} = \mathbb{1}_{\mathscr{B}_R(0)}$ with $R \in (0,\infty)$, then Proposition 3.3.3 implies that the operator $p_q\mathbb{1}_{\mathscr{B}_R(0)}p_q$, $q \in \mathbb{Z}_+$, is diagonalized in the canonic basis $\left\{\varphi_{k,q}\right\}_{k\in\mathbb{Z}_+}$, and its eigenvalues can be calculated explicitly. More precisely, by (3.3.16), the set of these eigenvalues coincides with the set

$$\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right) = \frac{q!}{k!}\int_0^\rho L_q^{(k-q)}(t)^2 t^{k-q} e^{-t}\,dt, \quad k \in \mathbb{Z}_+. \tag{4.2.7}$$

where $\rho = \rho(R) := bR^2/2$. In particular,

$$\lambda_{k,0}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right) = \frac{1}{k!}\int_0^\rho t^k e^{-t}\,dt, \quad k \in \mathbb{Z}_+.$$

Note that in (4.2.7) we have $\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right) > 0$, $k \in \mathbb{Z}_+$, but still, generally speaking, we cannot claim that

$$\nu^+_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right) = \lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right) \tag{4.2.8}$$

for *every* $k \in \mathbb{Z}_+$ because the sequence $\left\{\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right)\right\}_{k\in\mathbb{Z}_+}$ could happen not to be non-increasing. However, it could be shown that the tail of this sequence is non-increasing so that (4.2.8) holds true for sufficiently large $k \in \mathbb{N}$.
For the formulation of our results we need the notion of *a logarithmic capacity* $\operatorname{Cap}(\mathscr{E})$ of a Borel set $\mathscr{E} \subset \mathbb{R}^2$. Let $M(\mathscr{E})$ be the set of probability measures, compactly supported on $\mathscr{E}$. Then

$$\operatorname{Cap}(\mathscr{E}) := e^{-\mathscr{I}(\mathscr{E})}$$

where

$$\mathscr{I}(\mathscr{E}) := \inf_{\mu\in M(\mathscr{E})}\int_{\mathscr{E}\times\mathscr{E}} \ln|x-y|^{-1}\,d\mu(x)d\mu(y),$$

(see e.g. [160, Definition 5.1.1, Definition 3.2.1]). Note that if $\Omega \subset \mathbb{R}^2$ is a bounded domain, then

$$0 < \operatorname{Cap}(\Omega) < \infty.$$

Theorem 4.2.1 *Assume that* $\mathscr{F} \in C(\mathbb{R}^2)$, $\operatorname{supp}\mathscr{F} = \overline{\Omega}$ *where* $\Omega \subset \mathbb{R}^2$ *is a bounded domain, and* $\mathscr{F} > 0$ *on* Ω. *Let* $b \in (0,\infty)$. *Fix* $q \in \mathbb{Z}_+$. *Then*

$$\operatorname{rank} p_q(b)\mathscr{F}p_q(b) = \infty, \tag{4.2.9}$$

and

$$\ln \nu^+_{k,q}(\mathscr{F}, b) = -k\ln k + \left(1 + \ln\left(\frac{b\operatorname{Cap}(\Omega)^2}{2}\right)\right)k + o(k) \tag{4.2.10}$$

as $k \to \infty$.

Remarks: (i) If we assume only that $\mathscr{F} \in L^\infty(\mathbb{R}^2;\mathbb{R})$, $\operatorname{supp}\mathscr{F}$ is compact, and for some $C > 0$, $r > 0$, and $\mathbf{x}_0 \in \mathbb{R}^2$ we have

$$\mathscr{F}(\mathbf{x}) \geq C\mathbb{1}_{\mathscr{B}_r(\mathbf{x}_0)}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{4.2.11}$$

then [159, Theorem 2.2, Proposition 4.1] imply

$$\ln \nu^+_{k,q}(\mathscr{F}, b) = -k \ln k\,(1 + o(1)), \quad k \to \infty, \tag{4.2.12}$$

which is a coarser version of (4.2.10), and is equivalent to

$$n_+(s; p_q \mathscr{F} p_q) = \varphi_\infty(s)\,(1 + o(1)), \quad s \downarrow 0, \tag{4.2.13}$$

with

$$\varphi_\infty(s) := (\ln|\ln s|)^{-1}\,|\ln s|, \quad s \in (0, e^{-1}). \tag{4.2.14}$$

The proof of (4.2.12) is based on the fact that due to the compactness of $\operatorname{supp}\mathscr{F}$, the boundedness of $\mathscr{F}$, estimate (4.2.11), and the unitary equivalence (2.7.27), there exist constants $c_\pm \in (0,\infty)$ and $r_\pm \in (0,\infty)$ such that

$$c_- \nu^+_{k,q}(\mathbb{1}_{\mathscr{B}_{r_-}(0)}) \leq \nu^+_{k,q}(\mathscr{F}) \leq c_+ \nu^+_{k,q}(\mathbb{1}_{\mathscr{B}_{r_+}(0)}), \quad k, q \in \mathbb{Z}_+,$$

and on the asymptotic analysis of $\nu^+_{k,q}(\mathbb{1}_{\mathscr{B}_r(0)})$ which for large k coincide with $\lambda^+_{k,q}(\mathbb{1}_{\mathscr{B}_r(0)})$, given in (4.2.7).
(ii) Let $\pi(\lambda)$ be the number of primes less that $\lambda > 0$. It is well known that

$$\pi(\lambda) = (\ln\lambda)^{-1}\lambda\,(1 + o(1)), \quad \lambda \to \infty,$$

(see e.g. [89, Section 1.8, Theorem 6]). Thus, (4.2.13) can be rewritten as

$$\lim_{s\downarrow 0} \frac{n_+(s; p_q \mathscr{F} p_q)}{\pi(|\ln s|)} = 1.$$

(iii) An obvious drawback of the hypotheses of Theorem 4.2.1 is the assumption $\mathscr{F} \geq 0$. Related partial results concerning fairly special Toeplitz-like operators with non-sign-definite symbols can be found in [152, 169].
(iv) Let $\Gamma \subset \mathbb{R}^2$ be a C^∞ simple closed curve of length $|\Gamma|$. Define the distribution $\delta_\Gamma \in \mathscr{E}'(\mathbb{R}^2)$ by

$$\left(\delta_\Gamma, u\right)_{\mathscr{E}'(\mathbb{R}^2)} = \frac{1}{|\Gamma|}\int_\Gamma u(t)ds(t), \quad u \in \mathscr{E}(\mathbb{R}^2).$$

By the remark after Proposition 3.3.2, we have $p_q\delta_\Gamma p_q \in \mathfrak{S}_p(L^2(\mathbb{R}^2))$ for any $p \in (0,\infty)$. Moreover, [151, Proposition 4.1 (ii)] implies $\operatorname{rank} p_q\delta_\Gamma p_q = \infty$, and

$$\ln \nu^+_k(p_q\delta_\Gamma p_q) = -k \ln k + \left(1 + \ln\left(\frac{b\,\mathrm{Cap}(\Gamma)^2}{2}\right)\right) k + o(k), \quad k \to \infty. \tag{4.2.15}$$

Before we prove Theorem 4.2.1, we would like to state analogous results concerning the operators $\mathfrak{p}_\natural \mathscr{F} \mathfrak{p}_\natural$ with $\natural = \text{ann}, \text{cre}$, admissible b with non-vanishing mean value b_0, and compactly supported $\mathscr{F} \geq 0$.

Corollary 4.2.3 *Let $\mathscr{F}$ satisfy the hypotheses of Theorem 4.2.1. Assume that* $b = b_0 + \widetilde{b}$ *is an admissible magnetic field.*
(i) *Let* $b_0 > 0$. *Then*

$$\operatorname{rank} \mathfrak{p}_{\mathrm{ann}}(b)\mathscr{F}\mathfrak{p}_{\mathrm{ann}}(b) = \infty, \tag{4.2.16}$$

and

$$\ln \nu^+_{k,\mathrm{ann}}(\mathscr{F}, b) = -k \ln k + \left(1 + \ln\left(\frac{b_0 \operatorname{Cap}(\Omega)^2}{2}\right)\right) k + o(k), \quad k \to \infty. \tag{4.2.17}$$

(ii) *Let* $b_0 < 0$. *Then*

$$\operatorname{rank} \mathfrak{p}_{\mathrm{cre}}(b)\mathscr{F}\mathfrak{p}_{\mathrm{cre}}(b) = \infty, \tag{4.2.18}$$

and

$$\ln \nu^+_{k,\mathrm{cre}}(\mathscr{F}, b) = -k \ln k + \left(1 + \ln\left(\frac{-b_0 \operatorname{Cap}(\Omega)^2}{2}\right)\right) k + o(k), \quad k \to \infty. \tag{4.2.19}$$

Proof. Let $\widetilde{\varphi} \in C^2_b(\mathbb{R}^2;\mathbb{R})$ be the solution of the Poisson equation $\Delta\widetilde{\varphi} = \widetilde{b}$. Then Lemma 4.2.1 easily implies

$$e^{-2\operatorname{osc}\widetilde{\varphi}}\, \nu^+_{k,0}(\mathscr{F}, b_0) \le \nu^+_{k,\mathrm{ann}}(\mathscr{F}, b) \le e^{2\operatorname{osc}\widetilde{\varphi}}\, \nu^+_{k,0}(\mathscr{F}, b_0), \quad k \ge 0, \tag{4.2.20}$$

(see [154, Proposition 3.2] for details). Therefore, if $b_0 > 0$, then (4.2.16) and (4.2.17) follow from Theorem 4.2.1. If $b_0 < 0$, then the results follow from the relation of the operators $a(b)$ and $a(b)^*$ under complex conjugation (see (2.8.8)) and the first part of the corollary. □

Let us now prove Theorem 4.2.1. We will divide the proof into several propositions. The first one concerns the model situation where $\mathscr{F}$ is the characteristic function of a bounded domain in $\mathbb{R}^2$; this result is the core of the proof of Theorem 4.2.1.

Proposition 4.2.1 [77, Lemmas 1, 2] *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Fix $q \in \mathbb{Z}_+$. Then* $\operatorname{rank} p_q \mathbb{1}_{\overline{\Omega}} p_q = \infty$, *and we have*

$$\lim_{k\to\infty} \nu^+_{k,q}(\mathbb{1}_{\overline{\Omega}}) = -k \ln k + \left(1 + \ln\left(\frac{b \operatorname{Cap}(\overline{\Omega})^2}{2}\right)\right) k + o(k), \quad k \to \infty. \tag{4.2.21}$$

Our next lemma contains some elementary properties of the logarithmic capacity.

Lemma 4.2.2 [160, Chapter 5] (i) *Let $\mathscr{E}_1, \mathscr{E}_2 \subset \mathbb{R}^2$ be Borel sets such that $\mathscr{E}_1 \subset \mathscr{E}_2$. Then we have*

$$\operatorname{Cap}(\mathscr{E}_1) \le \operatorname{Cap}(\mathscr{E}_2). \tag{4.2.22}$$

(ii) *Let $K \subset \mathbb{R}^2$ be a compact set. Then*

$$\operatorname{Cap}(K) = \lim_{\delta \downarrow 0} \operatorname{Cap}(K_\delta) \tag{4.2.23}$$

where $K_\delta := \{\mathbf{x} \in \mathbb{R}^2 \mid \operatorname{dist}(\mathbf{x}, K) \le \delta\}$, $\delta > 0$.

Corollary 4.2.4 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then there exists a sequence of bounded domains $\{\Omega_j\}_{j\in\mathbb{N}}$ with Lipschitz boundaries $\partial\Omega_j$, such that $\overline{\Omega} \subset \Omega_j$, $j \in \mathbb{N}$, and*

$$\lim_{j\to\infty} \mathrm{Cap}(\Omega_j) = \mathrm{Cap}(\overline{\Omega}). \tag{4.2.24}$$

Proof. Recall the notation

$$\mathscr{Q}_\delta(\mathbf{x}) := \mathbf{x} + (-\delta/2, \delta/2)^2, \quad \mathbf{x} \in \mathbb{R}^2, \quad \delta > 0.$$

Let $\{\delta_j\}_{j\in\mathbb{N}}$ be a decreasing sequence such that $\delta_j > 0$, $j \in \mathbb{N}$, and $\lim_{j\to\infty} \delta_j = 0$. Since $\overline{\Omega}$ is compact, there exists a finite set $\{\mathbf{x}_\ell\}_{\ell=1}^N$ of points $\mathbf{x}_\ell \in \overline{\Omega}$ such that $\overline{\Omega} \subset \bigcup_{\ell=1}^N \mathscr{Q}_{\delta_j}(\mathbf{x}_\ell)$. Set

$$\Omega_j := \left(\bigcup_{\ell=1}^N \overline{\mathscr{Q}_{\delta_j}(\mathbf{x}_\ell)}\right)^{\mathrm{Int}}, \quad j \in \mathbb{N}.$$

Evidently, Ω_j is a bounded domain with Lipschitz boundary, $\overline{\Omega} \subset \Omega_j$, and

$$\Omega_j \subset \left\{\mathbf{x} \in \mathbb{R}^2 \,|\, \mathrm{dist}(\mathbf{x}, \overline{\Omega}) \leq \delta_j\right\}, \quad j \in \mathbb{N}.$$

Hence, (4.2.23) and (4.2.22) imply (4.2.24). □

Within the context of the proof of Theorem 4.2.1, Corollary 4.2.4 will allow us to approximate supp$\mathscr{F}$ from outside by bounded domains with Lipschitz boundaries, while our next proposition will make possible such approximations from inside.

Proposition 4.2.2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then there exists a sequence of domains $\Omega_j \subset \mathbb{R}^2$ with Lipschitz boundaries $\partial\Omega_j$, such that $\overline{\Omega_j} \subset \Omega$ and* (4.2.24) *holds true.*

Proof. Let $\Gamma \subset \mathbb{R}^2$ be a Jordan curve, i.e. a simple closed curve. We will call it C^2-smooth if there exists a C^2-smooth diffeomorphism $\mathbf{x} : \mathbb{S}^1 \to \Gamma$. By [45, Proposition 5.6], there exists a sequence of C^2-smooth Jordan curves $\Gamma_j \subset \Omega$ such that

$$\lim_{j\to\infty} \mathrm{Cap}(\Gamma_j) = \mathrm{Cap}(\overline{\Omega}). \tag{4.2.25}$$

Let $\mathbf{n}(s) := \frac{(x_2'(s), -x_1'(s))}{|\mathbf{x}'(s)|}$, $s \in \mathbb{S}^1$, be a normal unit vector to Γ_j. Set

$$\Omega_j := \left\{\mathbf{x}(s) + t\mathbf{n}(s) \,|\, s \in \mathbb{S}^1, \ |t| < \varepsilon_j\right\}$$

where $\varepsilon_j > 0$ is so small that $\overline{\Omega}_j \subset \Omega$ and $\partial\Omega_j$ is Lipschitz-smooth. Evidently, Ω_j is a domain. Since $\Gamma_j \subset \Omega_j \subset \overline{\Omega}$, (4.2.24) follows from (4.2.25) and (4.2.22). □

Corollary 4.2.5 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then*

$$\mathrm{Cap}(\Omega) = \mathrm{Cap}(\overline{\Omega}). \tag{4.2.26}$$

Proof. Let $\{\Omega_j\}_j$ be sequence of domains satisfying the hypotheses of Proposition 4.2.2. Since $\Omega_j \subset \Omega \subset \overline{\Omega}$, (4.2.22) implies

$$\lim_{j\to\infty} \mathrm{Cap}(\Omega_j) \leq \mathrm{Cap}(\Omega) \leq \mathrm{Cap}(\overline{\Omega}). \tag{4.2.27}$$

Now (4.2.26) follows from (4.2.24) and (4.2.27). □

Assume now the hypotheses of Theorem 4.2.1 and pick a sequence of bounded domains $\Omega_j^+ \subset \mathbb{R}^2$ with Lipschitz boundaries such that $\overline{\Omega} \subset \Omega_j^+$, and

$$\lim_{j\to\infty} \mathrm{Cap}(\Omega_j^+) = \mathrm{Cap}(\overline{\Omega}); \tag{4.2.28}$$

the existence of such a sequence is ensured by Corollary 4.2.4. Pick another sequence of domains $\Omega_j^- \subset \mathbb{R}^2$ with Lipschitz boundaries such that $\overline{\Omega_j^-} \subset \Omega$ and

$$\lim_{j\to\infty} \mathrm{Cap}(\Omega_j^-) = \mathrm{Cap}(\overline{\Omega}); \tag{4.2.29}$$

the existence of such a sequence is guaranteed by Proposition 4.2.2. Set

$$m_j^- := \inf_{\mathbf{x}\in\Omega_j^-} \mathscr{F}(\mathbf{x}), \quad m_j^+ := \sup_{\mathbf{x}\in\Omega_j^+} \mathscr{F}(\mathbf{x}), \quad j \in \mathbb{N}.$$

Evidently, $0 < m_j^- \leq m_j^+ < \infty$. Moreover,

$$m_j^- \, \mathbb{1}_{\overline{\Omega_j^-}}(\mathbf{x}) \leq \mathscr{F}(\mathbf{x}) \leq m_j^+ \, \mathbb{1}_{\overline{\Omega_j^+}}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \quad j \in \mathbb{N}.$$

By the mini-max principle, these estimates imply

$$m_j^- \, \nu_{k,q}^+(\mathbb{1}_{\overline{\Omega_j^-}}) \leq \nu_{k,q}^+(\mathscr{F}) \leq m_j^+ \, \nu_{k,q}^+(\mathbb{1}_{\overline{\Omega_j^+}}), \quad k \in \mathbb{Z}_+. \tag{4.2.30}$$

By (4.2.21), (4.2.30), and (4.2.26), we get

$$1 + \ln\left(\frac{b\,\mathrm{Cap}(\Omega_j^-)^2}{2}\right) \leq \liminf_{k\to\infty} \frac{\ln \nu_{k,q}^+(\mathscr{F}) + k \ln k}{k} \leq$$

$$\limsup_{k\to\infty} \frac{\ln \nu_{k,q}^+(\mathscr{F}) + k \ln k}{k} \leq 1 + \ln\left(\frac{b\,\mathrm{Cap}(\Omega_j^+)^2}{2}\right), \tag{4.2.31}$$

for every j. Combining (4.2.31) and (4.2.26)-(4.2.29), we obtain (4.2.10). The proof of Theorem 4.2.1 is now complete.

4.2.3 Exponentially decaying symbols

Our next theorem concerns the asymptotics as $k \to \infty$ of $\nu_{k,q}^+(\mathscr{F}, b)$ with $\mathscr{F}$ decaying exponentially at infinity, and constant $b > 0$. Similarly to (4.2.7), we note that if

$$\mathscr{F}(\mathbf{x}) = e^{-\gamma|\mathbf{x}|^{2\beta}}, \quad \mathbf{x} \in \mathbb{R}^2,$$

with $\beta \in (0,\infty)$ and $\gamma \in (0,\infty)$, then Proposition 3.3.3 implies that the set of the eigenvalues of the operator $p_q \mathscr{F} p_q$, $q \in \mathbb{Z}_+$, coincides with

$$\lambda_{k,q}(\mathscr{F}) = \frac{q!}{k!}\int_0^\infty L_q^{(k-q)}(t)^2 t^{k-q} e^{-\mu t^\beta - t}\,dt, \quad k \in \mathbb{Z}_+.$$

where

$$\mu = \gamma(2/b)^\beta. \tag{4.2.32}$$

In particular,

$$\lambda_{k,0}(\mathscr{F}) = \frac{1}{k!}\int_0^\infty t^k e^{-\mu t^\beta - t}\,dt, \quad k \in \mathbb{Z}_+.$$

More generally, we will assume that $\mathscr{F} \in C(\mathbb{R}^2;\mathbb{R}_+)$, and there exist $\beta > 0$ and $\gamma > 0$ such that

$$\ln \mathscr{F}(\mathbf{x}) = -\gamma|\mathbf{x}|^{2\beta} + \mathscr{O}(\ln|\mathbf{x}|), \quad |\mathbf{x}| \to \infty, \tag{4.2.33}$$

uniformly with respect to $\frac{\mathbf{x}}{|\mathbf{x}|} \in \mathbb{S}^1$.

Theorem 4.2.2 *Assume that* $\mathscr{F} \in C(\mathbb{R}^2;\mathbb{R}_+)$ *satisfies* (4.2.33). *Fix* $q \in \mathbb{Z}_+$. *Then*

$$\operatorname{rank} p_q \mathscr{F} p_q = \infty. \tag{4.2.34}$$

Moreover:
(i) *If* $\beta \in (0,1)$, *then there exist constants* $f_j = f_j(\beta,\mu)$, $j \in \mathbb{N}$, *with* $f_1 = \mu$ *being defined in* (4.2.32), *such that*

$$\ln \nu_{k,q}^+(\mathscr{F}) = -\sum_{1 \le j < \frac{1}{1-\beta}} f_j k^{(\beta-1)j+1} + \mathscr{O}(\ln k), \quad k \to \infty. \tag{4.2.35}$$

(ii) *If* $\beta = 1$, *then*

$$\ln \nu_{k,q}^+(\mathscr{F}) = -(\ln(1+\mu))k + \mathscr{O}(\ln k), \quad k \to \infty. \tag{4.2.36}$$

(iii) *If* $\beta \in (1,\infty)$, *then there exist constants* $g_j = g_j(\beta,\mu)$, $j \in \mathbb{N}$, *such that*

$$\ln \nu_{k,q}^+(\mathscr{F}) =$$

$$-\frac{\beta-1}{\beta} k \ln k + \left(\frac{\beta - 1 - \ln(\mu\beta)}{\beta}\right) k - \sum_{1 \le j < \frac{\beta}{\beta-1}} g_j k^{(\frac{1}{\beta}-1)j+1} + \mathscr{O}(\ln k), \quad k \to \infty. \tag{4.2.37}$$

Remarks: (i) Let us describe explicitly the coefficients f_j and g_j, $j \in \mathbb{N}$, appearing in (4.2.35) and (4.2.37). Assume at first $\beta \in (0,1)$. For $s > 0$ and $\varepsilon \in \mathbb{R}$, $|\varepsilon| \ll 1$, introduce the function

$$F(s;\varepsilon) := s - \ln s + \varepsilon\mu s^\beta.$$

Denote by $s_<(\varepsilon)$ the unique positive solution of the equation $s = 1 - \varepsilon\beta\mu s^\beta$, so that $\frac{\partial F}{\partial s}(s_<(\varepsilon);\varepsilon) = 0$. Set

$$f(\varepsilon) := F(s_<(\varepsilon);\varepsilon).$$

Note that f is a real analytic function for small $|\varepsilon|$. Then $f_j := \frac{1}{j!}\frac{d^j f}{d\varepsilon^j}(0)$, $j \in \mathbb{N}$.
Let now $\beta \in (1,\infty)$. For $s > 0$ and $\varepsilon \in \mathbb{R}$, $|\varepsilon| \ll 1$, introduce the function

$$G(s;\varepsilon) := \mu s^{\beta} - \ln s + \varepsilon s.$$

Denote by $s_>(\varepsilon)$ the unique positive solution of the equation $\beta\mu s^{\beta} = 1 - \varepsilon s$ so that $\frac{\partial G}{\partial s}(s_>(\varepsilon);\varepsilon) = 0$. Define

$$g(\varepsilon) := G(s_>(\varepsilon);\varepsilon),$$

which is a real analytic function for small $|\varepsilon|$. Then $g_j := \frac{1}{j!}\frac{d^j g}{d\varepsilon^j}(0)$, $j \in \mathbb{N}$.
(ii) If we assume that instead of (4.2.33), the symbol $\mathscr{F}$ satisfies a more general condition

$$\ln \mathscr{F}(\mathbf{x}) = -\gamma|\mathbf{x}|^{2\beta}(1 + o(1)), \quad |\mathbf{x}| \to \infty, \tag{4.2.38}$$

then, by [159, Proposition 3.1], we have

$$\nu^{+}_{k,q}(\mathscr{F}, b) = \begin{cases} -\mu k^{\beta}\,(1 + o(1)) & \text{if } 0 < \beta < 1, \\ -(\ln(1+\mu))\,k\,(1 + o(1)) & \text{if } \beta = 1, \\ -\dfrac{\beta - 1}{\beta}\,k\,\ln k\,(1 + o(1)) & \text{if } 1 < \beta < \infty, \end{cases} \tag{4.2.39}$$

as $k \to \infty$, which is equivalent to

$$n_+(s; p_q \mathscr{F} p_q) = \varphi_\beta(s)(1 + o(1)), \quad s \downarrow 0, \tag{4.2.40}$$

where

$$\varphi_\beta(s) := \begin{cases} \mu^{-1/\beta}\,|\ln s|^{1/\beta} & \text{if } 0 < \beta < 1, \\ (\ln(1+\mu))^{-1}\,|\ln s| & \text{if } \beta = 1, \\ \dfrac{\beta}{\beta - 1}(\ln|\ln s|)^{-1}\,|\ln s| & \text{if } 1 < \beta < \infty. \end{cases} \tag{4.2.41}$$

(iii) As in the case of compactly supported symbols, it would be nice to extend Theorem 4.2.2 to the case of non-sign-definite symbols $\mathscr{F}$. Another drawback of the hypotheses of this theorem is that the asymptotics of $\mathscr{F}$ at infinity is radially symmetric. It is a challenging open problem to extend the theorem to the case where the constant γ in (4.2.33) is replaced by a non-constant function $0 < \gamma \in C(\mathbb{S}^1)$.

Before we prove Theorem 4.2.2, we formulate analogous results for the operators $\mathfrak{p}_\natural \mathscr{F} \mathfrak{p}_\natural$ with $\natural = \text{ann}, \text{cre}$, admissible b with non-vanishing mean value b_0, and exponentially decaying symbols $\mathscr{F} \geq 0$.

Corollary 4.2.6 *Let $\mathscr{F}$ satisfy the hypotheses of Theorem 4.2.2. Assume that* $b = b_0 + \tilde{b}$ *is an admissible magnetic field.*
(i) *Let* $b_0 > 0$. *Then* (4.2.34) *and* (4.2.35) - (4.2.37) *remain valid if we replace* $p_q(b)\mathscr{F}p_q(b)$ *by* $\mathfrak{p}_{\text{ann}}(b)\mathscr{F}\mathfrak{p}_{\text{ann}}(b)$ *on the left-hand sides, and* b *by* b_0 *on the right-hand sides.*
(ii) *Let* $b_0 < 0$. *Then* (4.2.34) *and* (4.2.35) - (4.2.37) *remain valid if we replace*

$p_q(b)\mathscr{F}p_q(b)$ *by* $\mathfrak{p}_{cre}(b)\mathscr{F}\mathfrak{p}_{cre}(b)$ *on the left-hand sides, and* b *by* $-b_0$ *on the right-hand sides.*

The proof is very similar to the one of Corollary 4.2.3 so that we omit the details.

Let us now prove Theorem 4.2.2. We follow the general lines of the proof of [124, Theorem 2.2] (see also the proof of [45, Theorem 5.2]). By (4.2.33), for any $r > 1$ there exist real numbers $\delta_<$ and $\delta_> \in \mathbb{R}$, such that $\delta_< \le \delta_>$, and

$$|\mathbf{x}|^{\delta_<} e^{-\gamma|\mathbf{x}|^{2\beta}} \mathbb{1}_{\mathbb{R}^2\setminus\mathscr{B}_r(0)}(\mathbf{x}) \le \mathscr{F}(\mathbf{x})$$

$$\le |\mathbf{x}|^{\delta_>} e^{-\gamma|\mathbf{x}|^{2\beta}} \mathbb{1}_{\mathbb{R}^2\setminus\mathscr{B}_r(0)}(\mathbf{x}) + m\,\mathbb{1}_{\mathscr{B}_r(0)}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{4.2.42}$$

with $m := \max_{\mathbf{y}\in\mathbb{R}^2} \mathscr{F}(\mathbf{y})$. Let $\eta_<, \eta_> \in C^\infty(\mathbb{R}^2;[0,1])$ be two radially symmetric functions such that $\eta_< = 1$ on $\mathbb{R}^2 \setminus \mathscr{B}_{r+1}(0)$, $\eta_< = 0$ on $\mathscr{B}_r(0)$, and $\eta_> = 1$ on $\mathbb{R}^2 \setminus \mathscr{B}_r(0)$, $\eta_> = 0$ on $\mathscr{B}_{r-1}(0)$. For $\mathbf{x} \in \mathbb{R}^2$ set

$$\mathscr{F}_{<,1}(\mathbf{x}) := |x|^{\delta_<} e^{-\gamma|\mathbf{x}|^{2\beta}} \eta_<(\mathbf{x}),$$

$$\mathscr{F}_{>,1}(x) := |\mathbf{x}|^{\delta_>} e^{-\gamma|\mathbf{x}|^{2\beta}} \eta_>(\mathbf{x}) + m\,\mathscr{F}(\mathbf{y})(1-\eta_<(\mathbf{x})).$$

Evidently, $\mathscr{F}_{<,1}, \mathscr{F}_{>,1} \in C_b^\infty(\mathbb{R}^2)$ and, by (4.2.42),

$$\mathscr{F}_{<,1}(\mathbf{x}) \le \mathscr{F}(\mathbf{x}) \le \mathscr{F}_{>,1}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2.$$

Therefore, the mini-max principle implies

$$\nu_{k,q}^+(\mathscr{F}_{<,1}) \le \nu_{k,q}^+(\mathscr{F}) \le \nu_{k,q}^+(\mathscr{F}_{>,1}), \quad k \in \mathbb{Z}_+. \tag{4.2.43}$$

Further, set

$$\mathscr{F}_{\substack{>\\<},2} := \mathscr{D}_{q,b}\mathscr{F}_{\substack{>\\<},1}, \quad q \in \mathbb{Z}_+,$$

where $\mathscr{D}_{q,b}$ is the operator defined in (3.4.53). Then Theorem 3.4.2 implies that

$$\nu_{k,q}^+\left(\mathscr{F}_{\substack{>\\<},1}\right) = \nu_{k,0}^+\left(\mathscr{F}_{\substack{>\\<},2}\right), \quad k \in \mathbb{Z}_+, \quad q \in \mathbb{Z}_+. \tag{4.2.44}$$

Next, it is tedious but straightforward to check that

$$\mathscr{F}_{\substack{>\\<},2}(\mathbf{x}) = \mathscr{F}_{\substack{>\\<},3}(\mathbf{x})(1+o(1)), \quad |\mathbf{x}| \to \infty, \tag{4.2.45}$$

where

$$\mathscr{F}_{\substack{>\\<},3}(\mathbf{x}) := C_{q,b}|\mathbf{x}|^{2\varepsilon_{\substack{>\\<}}} e^{-\gamma|\mathbf{x}|^{2\beta}}, \quad \mathbf{x} \in \mathbb{R}^2 \setminus \{0\},$$

$$\varepsilon_{\substack{>\\<}} := \begin{cases} \dfrac{\delta_{\substack{>\\<}}}{2} & \text{if } \beta \in (0,1/2], \\ \dfrac{\delta_{\substack{>\\<}}}{2} + q(2\beta - 1) & \text{if } \quad \beta \in (1/2,\infty), \end{cases}$$

and

$$C_{q,b} := \begin{cases} 1 & \text{if } \beta \in (0,1/2), \\ L_q\left(-\dfrac{\gamma}{2b}\right) & \text{if } \beta = 1/2, \\ \dfrac{(2\beta\gamma)^{2q}}{q!(2b)^q} & \text{if } \beta > 1/2. \end{cases}$$

Hence, by (4.2.45), there exists $R \in (0,\infty)$ such that for $\mathbf{x} \in \mathbb{R}^2$ we have

$$\mathscr{F}_{<,2}(\mathbf{x}) \geq \frac{1}{2}\mathscr{F}_{<,3}(\mathbf{x})\mathbb{1}_{\mathbb{R}^2\setminus\mathscr{B}_R(0)}(\mathbf{x}) - m_<\mathbb{1}_{\mathscr{B}_R(0)}(\mathbf{x}) =: \mathscr{F}_{<,4}(\mathbf{x}),$$

$$\mathscr{F}_{>,2}(\mathbf{x}) \leq \frac{3}{2}\mathscr{F}_{>,3}(\mathbf{x})\mathbb{1}_{\mathbb{R}^2\setminus\mathscr{B}_R(0)}(\mathbf{x}) + m_>\mathbb{1}_{\mathscr{B}_R(0)}(\mathbf{x}) =: \mathscr{F}_{>,4}(\mathbf{x}),$$

with $m_{\substack{>\\<}} := \max_{\mathbf{y}\in\mathbb{R}^2} |\mathscr{F}_{\substack{>\\<},2}(\mathbf{y})|$. Thus,

$$\nu^+_{k,0}(\mathscr{F}_{<,2}) \geq \nu^+_{k,0}(\mathscr{F}_{<,4}), \quad \nu^+_{k,0}(\mathscr{F}_{>,2}) \leq \nu^+_{k,0}(\mathscr{F}_{>,4}). \tag{4.2.46}$$

Now note that the functions $\mathscr{F}_{\substack{>\\<},4}$ are radially symmetric. By Proposition 3.3.3, we can calculate explicitly the eigenvalues of the operators $p_0\mathscr{F}_{\substack{>\\<},4}p_0$. More precisely, put $\rho := bR^2/2$, and for $c_0 \in (0,\infty)$, $c_1 \in \mathbb{R}$, and $\varepsilon \in \mathbb{R}$, set

$$\tau_{c_0,c_1,\varepsilon}(k) := \frac{1}{\Gamma(k+1)}\left((2/b)^{\varepsilon}c_0\int_\rho^\infty t^{k+\varepsilon}e^{-\mu t^\beta - t}dt + c_1\int_0^\rho e^{-t}t^k dt\right), \quad k \in (-1,\infty),$$

where Γ is the Euler gamma function. Then Proposition 3.3.3 implies that

$$\left\{\nu^+_{k,0}(\mathscr{F}_{>,4})\right\}_{k\in\mathbb{Z}_+} = \left\{\tau_{c_{0,>},c_{1,>},\varepsilon_>}(k)\right\}_{k\in\mathbb{Z}_+}, \tag{4.2.47}$$

with $c_{0,>} := 3C_{q,b}/2$ and $c_{1,>} := m_>$, while

$$\left\{\nu^+_{k,0}(\mathscr{F}_{<,4})\right\}_{k\in\mathbb{Z}_+} = \left\{\tau_{c_{0,<},c_{1,<},\varepsilon_<}(k)\right\}_{\{k\in\mathbb{Z}_+ \,|\, \tau(k)>0\}}. \tag{4.2.48}$$

with $c_{0,<} := C_{q,b}/2$ and $c_{1,<} := -m_<$. Arguing as in [124, Lemma 5.3], we find that for any $c_0 \in (0,\infty)$, $c_1 \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$, we get

$$\ln \tau_{c_0,c_1,\varepsilon}(k) = \begin{cases} -\sum_{1\leq j<\frac{1}{1-\beta}} f_j k^{(\beta-1)j+1} + \mathscr{O}(\ln k) & \text{if } \beta \in (0,1), \\ -(\ln(1+\mu))\, k + \mathscr{O}(\ln k) & \text{if } \beta = 1, \\ -\frac{\beta-1}{\beta} k \ln k + k \frac{\beta-1-\ln(\mu\beta)}{\beta} & \\ -\sum_{1\leq j<\frac{\beta}{\beta-1}} g_j k^{(\frac{1}{\beta}-1)j+1} + \mathscr{O}(\ln k) & \text{if } \beta \in (1,\infty), \end{cases} \tag{4.2.49}$$

as $k \to \infty$, where f_j and g_j are the coefficients introduced in the statement of Theorem 4.4.2. Moreover,

$$\tau'_{c_0,c_1,\varepsilon}(k) < 0 \tag{4.2.50}$$

for sufficiently large $k \in \mathbb{R}$. Combining (4.2.47) - (4.2.48) with (4.2.49) and (4.2.50), we find that for sufficiently large $k \in \mathbb{Z}_+$ we have

$$\nu^+_{k,0}(\mathscr{F}_{>,4}) = \tau_{c_{0,>},c_{1,>},\varepsilon_>}(k), \quad \nu^+_{k,0}(\mathscr{F}_{<,4}) = \tau_{c_{0,<},c_{1,<},\varepsilon_<}(k). \tag{4.2.51}$$

Now (4.2.35) – (4.2.37) follow from estimates (4.2.43), (4.2.44), (4.2.46), (4.2.51), and asymptotic relations (4.2.49).

4.2.4 Symbols of power-like decay

In this subsection we discuss the eigenvalue asymptotics for Berezin-Toeplitz operators with symbols $\mathscr{F}$ of power-like decay. More precisely, we suppose that $\mathscr{F} = \overline{\mathscr{F}} \in \Gamma_\rho^{-\gamma}(\mathbb{R}^2)$ with $\gamma > 0$ and $\rho \in (0,1]$, and impose supplementary conditions which guarantee that the order of decay at infinity of $\mathscr{F}$ is exactly equal to $-\gamma$ (see (4.2.55) below). An example of such $\mathscr{F}$ could be a symbol which is asymptotically homogeneous of order $-\gamma$, i.e. there exists $\phi \in C(\mathbb{S}^1)$ such that

$$\lim_{r\to\infty} r^\gamma \mathscr{F}(r\omega) = \phi(\omega), \quad \omega \in \mathbb{S}^1. \tag{4.2.52}$$

In order to formulate our first theorem, we need several notations and auxiliary facts. Let $f : (0,\infty) \to [0,\infty)$ be a non-increasing function. We will say that f satisfies the condition $\mathscr{C}$ if there exists $\lambda_0 \in (0,\infty)$ such that:

- f is derivable on $(0,\lambda_0)$;
- there exist numbers $0 < \gamma_1 < \gamma_2 < \infty$ such that for any $\lambda \in (0,\lambda_0)$ we have

$$\gamma_1 f(\lambda) < -\lambda f'(\lambda) < \gamma_2 f(\lambda). \tag{4.2.53}$$

By analogy with the volume function defined in (4.1.3), introduce the *area functions*

$$\mathfrak{A}^\pm(s;\mathscr{F}) := (2\pi)^{-1} \left|\left\{(x,\xi) \in \mathbb{R}^2 \mid \pm\mathscr{F}(x,\xi) > s\right\}\right|, \quad s > 0, \tag{4.2.54}$$

where $\mathscr{F} : \mathbb{R}^2 \to \mathbb{R}$ is a Lebesgue-measurable symbol.

Proposition 4.2.3 *Let* $\mathscr{F} = \Gamma_\rho^{-\gamma}(\mathbb{R}^2;\mathbb{R})$ *with* $\gamma > 0$ *and* $\rho \in (0,1]$. *Assume that the functions* $\mathfrak{A}^+(\cdot;\mathscr{F})$ *and* $\mathfrak{A}^-(\cdot;\mathscr{F})$ *satisfy the condition* $\mathscr{C}$, *and*

$$\liminf_{s\downarrow 0} s^{2/\gamma}\mathfrak{A}^{\pm}(s;\mathscr{F}) > 0. \tag{4.2.55}$$

Then there exists $\delta > 0$ *such that*

$$n_{\pm}(s;\mathrm{Op}^w(\mathscr{F})) = \mathfrak{A}^{\pm}(s;\mathscr{F})(1+\mathscr{O}(s^{\delta})), \quad s\downarrow 0. \tag{4.2.56}$$

Proposition 4.2.3 follows from a special case of the main theorem of [57] with $\varphi(x,\xi) = \phi(x,\xi) := (1+x^2+\xi^2)^{\rho/2}$ and $m(x,\xi) := (1+x^2+\xi^2)^{-\gamma/2}$, $(x,\xi)\in\mathbb{R}^2$. Under its assumptions we have

$$\mathfrak{A}^{\pm}(s;\mathscr{F}) \asymp s^{-2/\gamma}, \quad s\downarrow 0. \tag{4.2.57}$$

If $\mathscr{F}$ is asymptotically homogeneous of order $-\gamma$ (see (4.2.52)), then

$$\mathfrak{A}^{\pm}(s;\mathscr{F}) = \frac{1}{4\pi}\int_{\mathbb{S}^1}\phi(\omega)_{\pm}^{2/\gamma}d\omega\, s^{-2/\gamma}(1+o(1)), \quad s\downarrow 0.$$

Theorem 4.2.3 *Let* $b > 0$, *and* $\mathscr{F}\in\Gamma_\rho^{-\gamma}(\mathbb{R}^2;\mathbb{R})$ *with* $\gamma > 0$ *and* $\rho\in(0,1]$ *satisfy the hypotheses of Proposition 4.2.3. Fix* $q\in\mathbb{Z}_+$. *Then there exists* $\delta' > 0$ *such that*

$$n_{\pm}(s;p_q\mathscr{F}p_q)) = b\mathfrak{A}^{\pm}(s;\mathscr{F})(1+\mathscr{O}(s^{\delta'})), \quad s\downarrow 0. \tag{4.2.58}$$

Remark: Let $\mathfrak{I}_{\mathscr{F}}^{\pm}(k)$ be the function inverse to $\mathfrak{A}^{\pm}(\cdot;\mathscr{F})$, well defined for large $k > 0$. Then (4.2.58) is equivalent to

$$\nu_{k,q}^{\pm}(\mathscr{F}) = \mathfrak{I}_{\mathscr{F}}^{\pm}(k)(1+\mathscr{O}(k^{-\delta'})), \quad k\to\infty.$$

Let us now prove Theorem 4.2.3. According to Theorem 3.4.1, the operator $p_q\mathscr{F}p_q$ is unitarily equivalent to the operator $\pi_q\otimes\mathrm{Op}_q^{aw}(\mathscr{F}_b)$. Therefore,

$$n_{\pm}(s;p_q\mathscr{F}p_q) = n_{\pm}(s;\mathrm{Op}_q^{aw}(\mathscr{F}_b)), \quad s>0. \tag{4.2.59}$$

Further, by Corollary 3.4.2, we have

$$\mathrm{Op}_q^{aw}(\mathscr{F}_b) = \mathrm{Op}^w(\mathscr{F}_b*\Psi_q), \tag{4.2.60}$$

where Ψ_q is the Wigner function defined in (3.4.27). By Proposition 3.4.2,

$$\mathscr{F}_b*\Psi_q - \mathscr{F}_b\in\Gamma_\rho^{-\gamma-\rho}(\mathbb{R}^2). \tag{4.2.61}$$

Applying the Weyl inequalities (4.2.4), we get

$$\begin{gathered} n_{\pm}(s(1+\varepsilon);\mathrm{Op}^w(\mathscr{F}_b)) - n_{\mp}(s\varepsilon;\mathrm{Op}^w(\mathscr{F}_b*\Psi_q-\mathscr{F}_b)) \le \\ n_{\pm}(s;\mathrm{Op}^w(\mathscr{F}_b*\Psi_q)) \le \\ n_{\pm}(s(1-\varepsilon);\mathrm{Op}^w(\mathscr{F}_b)) + n_{\pm}(s\varepsilon;\mathrm{Op}^w(\mathscr{F}_b*\Psi_q-\mathscr{F}_b)) \end{gathered} \tag{4.2.62}$$

for any $s > 0$ and $\varepsilon \in (0,1)$. Making use of (4.2.62), (4.2.57), (4.2.56), and (4.2.61), and taking into account that

$$\mathfrak{A}^{\pm}(s;\mathscr{F}_b) = b\mathfrak{A}^{\pm}(s;\mathscr{F}), \quad b > 0,$$

we find that there exists a constant $C > 0$ such that

$$b\mathfrak{A}^{\pm}(s(1+\varepsilon)\,\mathscr{F})(1+\mathscr{O}(s^{\delta})) - C(s\varepsilon)^{-2/(\gamma+\rho)} \leq$$

$$n_{\pm}(s;\mathrm{Op}^{w}(\mathscr{F}_b * \Psi_q)) \leq$$

$$b\mathfrak{A}^{\pm}(s(1-\varepsilon);\mathscr{F})(1+\mathscr{O}(s^{\delta})) + C(s\varepsilon)^{-2/(\gamma+\rho)}, \quad s\downarrow 0, \quad \varepsilon \in (0,1). \tag{4.2.63}$$

Further, (4.2.53) implies

$$(1+\varepsilon)^{-\gamma_2}\mathfrak{A}^{\pm}(s;\mathscr{F}) \leq \mathfrak{A}^{\pm}(s(1+\varepsilon);\mathscr{F}), \quad \mathfrak{A}^{\pm}(s(1-\varepsilon);\mathscr{F}) \leq (1-\varepsilon)^{-\gamma_1}\,\mathfrak{A}^{\pm}(s) \tag{4.2.64}$$

for $s > 0$ small enough. Finally, pick $\varepsilon = s^{\theta}$ with $\theta := \frac{2\rho}{\gamma(2+\rho+\gamma)}$ so that

$$-\frac{2(1+\theta)}{\gamma+\rho} + \frac{2}{\gamma} = \theta.$$

Then (4.2.59), (4.2.60), (4.2.63), and (4.2.64) imply (4.2.58) with $\delta' = \min\{\delta,\theta\}$.

Let us now pass to the eigenvalue asymptotics for the operators $\mathfrak{p}\mathscr{F}\mathfrak{p}$ where $\mathfrak{p} = \mathfrak{p}_{\mathrm{ann}}(b)$, $b = b_0 + \widetilde{b}$ is an admissible magnetic field with $b_0 > 0$, and the real valued symbol $\mathscr{F}$ has a power-like decay at infinity. Since in this case the main asymptotic term of $n_{\pm}(s;p_0\mathscr{F}p_0)$ as $s\downarrow 0$, is not scale-invariant, i.e. it changes if we replace $\mathscr{F}$ by $c\mathscr{F}$ with a constant $c \neq 1$, estimates (4.2.20) yield now different lower and upper asymptotic bounds. Thus, in the case of power-like decay of $\mathscr{F}$, the asymptotic analysis of $n_{\pm}(s;p_0\mathscr{F}p_0)$ as $s\downarrow 0$ is more delicate than in Corollaries 4.2.3 and 4.2.6. Hence, we impose now more restrictive assumptions on b and $\mathscr{F}$.

Theorem 4.2.4 *Let* $b = b_0 + \widetilde{b}$ *with* $b_0 > 0$ *and let* $\tilde{b} \in \mathrm{WAP}(\mathbb{R}^2)$ *satisfy* (2.8.28). *Assume that* $\mathscr{F} \in C^1(\mathbb{R}^2)$*, and the estimates*

$$0 < \mathscr{F}(\mathbf{x}) \leq C\langle\mathbf{x}\rangle^{-\gamma}, \quad |\nabla\mathscr{F}(\mathbf{x})| \leq C\langle\mathbf{x}\rangle^{-\gamma-1}, \quad \mathbf{x} \in \mathbb{R}^2, \tag{4.2.65}$$

are fulfilled for some constants $\gamma > 0$ *and* $C \geq 0$. *Suppose, moreover, that there exists* $0 < \phi \in C(\mathbb{S}^1)$ *such that* (4.2.52) *holds true. Then for* $\mathfrak{p} = \mathfrak{p}_{\mathrm{ann}}(b)$ *we have*

$$\lim_{s\downarrow 0} s^{2/\gamma} n_{+}(s;\mathfrak{p}\mathscr{F}\mathfrak{p}) = \frac{b_0}{4\pi}\int_{\mathbb{S}^1}\phi(\omega)^{2/\gamma}\,d\omega. \tag{4.2.66}$$

Let us prove Theorem 4.2.4. If we try to mimic the proof of Theorem 4.2.3, we can show that, similarly to (4.2.59),

$$n_{+}(s;\mathfrak{p}\mathscr{F}\mathfrak{p}) = n_{+}(s;T), \quad s > 0,$$

where

$$T := \mathrm{Op}^{aw}(e^{-2\widetilde{\varphi}_{b_0}})^{-1/2}\mathrm{Op}^{aw}(\mathscr{F}_{b_0}e^{-2\widetilde{\varphi}_{b_0}})\mathrm{Op}^{aw}(e^{-2\widetilde{\varphi}_{b_0}})^{-1/2}, \tag{4.2.67}$$

$\widetilde{\varphi}$ is the solution of the Poisson equation $\Delta\widetilde{\varphi} = \widetilde{b}$, and

$$\mathscr{F}_{b_0} = O_{b_0}\mathscr{F}, \quad \widetilde{\varphi}_{b_0} = O_{b_0}\widetilde{\varphi},$$

in accordance with (3.4.46). However, even if we assume that $\mathscr{F} \in \Gamma_\rho^{-\gamma}(\mathbb{R}^2)$ with $\gamma > 0$ and $\rho \in (0,1]$, and $\varphi \in C^\infty(\mathbb{R}^2) \cap \mathrm{WAP}(\mathbb{R}^2)$, we would have only $\mathscr{F}_{b_0}e^{-2\widetilde{\varphi}_{b_0}} \in \Gamma_0^{-\gamma}(\mathbb{R}^2)$ and $e^{-2\widetilde{\varphi}_{b_0}} \in \Gamma_0^0(\mathbb{R}^2)$ because of the oscillations of $\widetilde{\varphi}$. Due to the absence of a convenient pseudo-differential calculus for the operator T, we will use an alternative approach based on results from [11, 105, 118, 93]. We will divide the proof of Theorem 4.2.4 into several propositions. Set

$$\mathfrak{N}_b(E) := \mathfrak{N}(E; a(b)^*a(b)), \quad E \in \mathbb{R}.$$

Thus, $\mathfrak{N}_b$ is the IDS for the operator

$$a(b)^*a(b) = (-i\nabla - A)^2 - b$$

(see (4.1.11) - (4.1.12)). Since $\tilde{b} \in \mathrm{WAP}(\mathbb{R}^2;\mathbb{R})$, the existence of this IDS follows from [154, Lemma 3.2]. If $b = b_0 > 0$ is constant, then (2.7.52) and (4.1.13) imply

$$\mathfrak{N}_{b_0}(E) = \frac{b_0}{2\pi}\sum_{q\in\mathbb{Z}_+}\mathbb{1}_{(0,\infty)}(E - 2b_0q), \quad E \in \mathbb{R}. \tag{4.2.68}$$

Set $E_0 := \mathrm{dist}(0, \sigma(a^*a) \setminus \{0\})$. By Proposition 2.8.3, we have $E_0 > 0$.

Proposition 4.2.4 [154, Lemma 3.2] *Let* b *satisfy the assumptions of Theorem 4.2.4. Then we have*

$$\mathfrak{N}_b(E) = \begin{cases} 0 & \text{if } E < 0, \\ \dfrac{b_0}{2\pi} & \text{if } E \in (0, E_0). \end{cases} \tag{4.2.69}$$

Proof. It follows from (4.1.12) that the IDS is constant on the gaps in $\sigma(a^*a)$. Therefore, (4.2.69) just tells us that the size of the jump of $\mathfrak{N}_b$ at $0 = \inf\sigma(a^*a)$ is equal to $\frac{b_0}{2\pi}$. In order to check this, set $b_s = b_0 + s\widetilde{b}$, $s \in [0,1]$, so that $b = b_1$. The operator $a(b_s)^*a(b_s)$ is norm resolvent continuous with respect to s. Hence, a gap-labeling result (see e.g. [11]) implies that the size of the jump of $\mathfrak{N}_{b_s}$ is independent of $s \in [0,1]$. Therefore,

$$\mathfrak{N}_b(+0) - \mathfrak{N}_b(-0) = \mathfrak{N}_{b_0}(+0) - \mathfrak{N}_{b_0}(-0) = \frac{b_0}{2\pi}$$

according to (4.2.68). □

Remark: The assumption about the almost periodicity of b in Theorem 4.2.3 is needed to ensure the presence of a gap $(0, E_0)$ in $\sigma(a(b)^*a(b))$, the existence of the IDS $\mathfrak{N}_b$, and the validity of (4.2.69). Evidently, all these properties are preserved also for more general admissible b with $b_0 > 0$.

As a by-product of Proposition 4.2.4, we obtain the following result which might be interesting within the context of weighted holomorphic spaces. Let $K_{ann,b}$ be the integral kernel of the orthogonal projection onto $\mathrm{Ker}\, a(b) = \mathrm{Ker}\, a(b)^* a(b)$.

Corollary 4.2.7 [154, Corollary 3.1] *Assume that* b *satisfies the assumptions of Theorem 4.2.4. Then for any* $\mathbf{x} \in \mathbb{R}^2$ *we have*

$$\lim_{T\to\infty} L^{-2} \int_{\mathscr{Q}_L(\mathbf{x})} K_{ann,b}(\mathbf{y},\mathbf{y}) d\mathbf{y} = \frac{b_0}{2\pi}. \tag{4.2.70}$$

Proof. Let $E \in (0, E_0)$. Then

$$\mathrm{Tr}\left(\mathbb{1}_{\mathscr{Q}_L(\mathbf{x})} \mathbb{1}_{(-\infty,E)}(a^*a)\mathbb{1}_{\mathscr{Q}_L(\mathbf{x})}\right) = \mathrm{Tr}\left(\mathbb{1}_{\mathscr{Q}_L(\mathbf{x})} \mathbb{1}_{\{0\}}(a^*a)\mathbb{1}_{\mathscr{Q}_L(\mathbf{x})}\right) =$$

$$\int_{\mathscr{Q}_L(\mathbf{x})} K_{ann,b}(\mathbf{y},\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^2, \quad L \in (0,\infty). \tag{4.2.71}$$

Now (4.2.70) follows from (4.2.71), (4.1.12), and (4.2.69). □

Proposition 4.2.5 [154, Lemma 3.3] *Assume the hypotheses of Theorem 4.2.4. Let* $E \in (0, E_0)$. *Then*

$$\lim_{s\downarrow 0} s^{2/\gamma} n_+(s; \mathscr{F}^{1/2}(E - a(b)^* a(b))^{-1} \mathscr{F}^{1/2}) = E^{-2/\gamma} \frac{b_0}{4\pi} \int_{\mathbb{S}^1} \phi(\omega)^{2/\gamma} d\omega. \tag{4.2.72}$$

Proof. Using the techniques developed in [118] and [93], we conclude that

$$\lim_{s\downarrow 0} s^{2/\gamma} n_+(s; \mathscr{F}^{1/2}(E - a(b)^* a(b))^{-1} \mathscr{F}^{1/2}) =$$

$$\int_{-\infty}^{E} \left| \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \phi(\mathbf{x}/|\mathbf{x}|)|\mathbf{x}|^{-\gamma} > E - t \right\} \right| d\mathfrak{N}_b(t),$$

and by (4.2.69) we have

$$\int_{-\infty}^{E} | \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \phi(\mathbf{x}/|\mathbf{x}|)|\mathbf{x}|^{-\gamma} > E - t \right\} | d\mathfrak{N}_b(t) = E^{-2/\gamma} \frac{b_0}{4\pi} \int_{\mathbb{S}^1} \phi(\omega)^{2/\gamma} d\omega.$$

□

Proposition 4.2.6 [105, Lemma 1.4] *Let* b *satisfy the hypotheses of Theorem 4.2.4, and* $\mathscr{G} \in C^1(\mathbb{R}^2)$ *satisfy*

$$|\mathscr{G}(\mathbf{x})| \le C\langle \mathbf{x} \rangle^{-m}, \quad |\nabla \mathscr{G}(\mathbf{x})| \le C\langle \mathbf{x} \rangle^{-m-1}, \tag{4.2.73}$$

with some constants $m > 0$ *and* $C \ge 0$. *Then the commutator* $\mathscr{C} := \mathfrak{p}\mathscr{G} - \mathscr{G}\mathfrak{p}$ *with* $\mathfrak{p} := \mathfrak{p}_{ann}(b)$ *admits the representation*

$$\mathscr{C} = \mathscr{C}_0 \langle \cdot \rangle^{-m-1} \tag{4.2.74}$$

where $\mathscr{C}_0 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ *is a bounded operator.*

Proof. Set $\mathfrak{c} := \{\zeta \in \mathbb{C} \,|\, |\zeta| = r\}$ with $r \in (0, E_0)$. Then we have

$$\mathfrak{p} = \frac{1}{2\pi i}\int_{\mathfrak{c}} (a^*a - \zeta)^{-1} d\zeta,$$

$\mathfrak{c}$ being run into the clockwise direction, and

$$\mathscr{C} = -\frac{1}{2\pi i}\int_{\mathfrak{c}} (a^*a - \zeta)^{-1}[a^*a, \mathscr{G}](a^*a - \zeta)^{-1} d\zeta =$$

$$\frac{1}{\pi}\int_{\mathfrak{c}} (a^*a - \zeta)^{-1}\left(a^*\frac{\partial \mathscr{G}}{\partial \bar{z}} + \frac{\partial \mathscr{G}}{\partial z} a\right)(a^*a - \zeta)^{-1} d\zeta. \quad (4.2.75)$$

Evidently the operators $(a^*a - \zeta)^{-1}a^*$ and $(a^*a - \zeta)^{-1}$ are uniformly bounded in $\zeta \in \mathfrak{c}$. Using the second inequality in (4.2.73), we easily find that the closures of the operators

$$\frac{\partial \mathscr{G}}{\partial \bar{z}}(a^*a - \zeta)^{-1}\langle\cdot\rangle^{m+1}, \quad \frac{\partial \mathscr{G}}{\partial z} a(a^*a - \zeta)^{-1}\langle\cdot\rangle^{m+1},$$

are also uniformly bounded in $\zeta \in \mathfrak{c}$. Thus, (4.2.74) follows from (4.2.75). □

Lemma 4.2.3 [154, Lemma 3.5] *Let* b *satisfy the hypotheses of Theorem 4.2.4. Suppose that the function* $\mathscr{G} : \mathbb{R}^2 \to \mathbb{R}$ *is Lebesgue-measurable and satisfies the first inequality in* (4.2.73) *with some* $m > 0$. *Then we have*

$$n_\pm(s; \mathfrak{p}\mathscr{G}\mathfrak{p}) = \mathscr{O}(s^{-2/m}), \quad s \downarrow 0. \quad (4.2.76)$$

Proof. By the mini-max principle and the analogue of (4.2.20) for $\mathscr{F} = \langle\cdot\rangle^{-m}$, it suffices to show that

$$n_\pm(s; p_0(b_0)\langle\cdot\rangle^{-m}p_0(b_0)) = \mathscr{O}(s^{-2/m}), \quad s \downarrow 0.$$

which follows from Theorem 4.2.3. □

Corollary 4.2.8 [154, Corollary 3.2] *Let* b *and* $\mathscr{G} : \mathbb{R}^2 \to \mathbb{R}$ *satisfy the assumptions of Proposition 4.2.6 with* $m > 0$. *Then we have*

$$n_*(s; \mathfrak{p}\mathscr{G}(I - \mathfrak{p})) = \mathscr{O}(s^{-2/(m+1)}), \quad s \downarrow 0. \quad (4.2.77)$$

Proof. Evidently,

$$n_*(s; \mathfrak{p}\mathscr{G}(I - \mathfrak{p})) = n_*(s; \mathfrak{p}\mathscr{C}(I - \mathfrak{p})) \le n_*(s; \mathfrak{p}\mathscr{C}) = n_*(s; \mathscr{C}\mathfrak{p}), \quad s > 0.$$

Applying Proposition 4.2.6, we get

$$n_*(s; \mathscr{C}\mathfrak{p}) = n_*(s; \mathscr{C}_0\langle\cdot\rangle^{-m-1}\mathfrak{p}) \le n_*(s; \|\mathscr{C}_0\|\langle\cdot\rangle^{-m-1}\mathfrak{p}) = n_*(s^2; \|\mathscr{C}_0\|^2\mathfrak{p}\langle\cdot\rangle^{-2m-2}\mathfrak{p}).$$

Bearing in mind (4.2.76), we obtain (4.2.77). □

Now we are in position to prove Theorem 4.2.4. We will establish a lower and an upper asymptotic bounds corresponding to (4.2.66). We start with the lower one. Fix $E \in (0, E_0)$. Since we have

$$E^{-1}\mathfrak{p} = \mathfrak{p}(E - a^*a)^{-1} \ge (E - a^*a)^{-1},$$

the mini-max principle and identity (4.2.2) imply

$$n_+(s;\mathfrak{p}\mathscr{F}\mathfrak{p}) = n_+(s;\mathscr{F}^{1/2}\mathfrak{p}\mathscr{F}^{1/2}) =$$

$$n_+(s/E;\mathscr{F}^{1/2}\mathfrak{p}(E-a^*a)^{-1}\mathscr{F}^{1/2}) \geq n_+(s/E;\mathscr{F}^{1/2}(E-a^*a)^{-1}\mathscr{F}^{1/2}), \quad s>0. \quad (4.2.78)$$

By (4.2.72),

$$\lim_{s\downarrow 0} s^{2/\gamma} n_+(s/E;\mathscr{F}^{1/2}(E-a(b)^*a(b))^{-1}\mathscr{F}^{1/2}) = \frac{b_0}{4\pi}\int_{\mathbb{S}^1}\phi(\omega)^{2/\gamma}d\omega. \quad (4.2.79)$$

Combining (4.2.78) and (4.2.79), we obtain

$$\liminf_{s\downarrow 0} s^{2/\gamma} n_+(s;\mathfrak{p}\mathscr{F}\mathfrak{p}) \geq \frac{b_0}{4\pi}\int_{\mathbb{S}^1}\phi(\omega)^{2/\gamma}d\omega. \quad (4.2.80)$$

Let us now handle the upper bound. The mini-max principle and the Weyl inequalities entail

$$n_+(s/E;\mathscr{F}^{1/2}(E-a^*a)^{-1}\mathscr{F}^{1/2}) \geq n_+(s/E;\mathfrak{p}\mathscr{F}^{1/2}\,(E-a^*a)^{-1}\mathscr{F}^{1/2}\mathfrak{p}) \geq$$

$$n_+((1+\varepsilon)s;\mathfrak{p}\mathscr{F}^{1/2}\mathfrak{p}\mathscr{F}^{1/2}\mathfrak{p}) - n_-(\varepsilon s/E;\mathfrak{p}\mathscr{F}^{1/2}(E-a^*a)^{-1}(I-\mathfrak{p})\mathscr{F}^{1/2}\mathfrak{p}), \quad \varepsilon>0. \quad (4.2.81)$$

Let us estimate the last term in (4.2.81). For $t>0$ we have

$$n_-(t;\mathfrak{p}\mathscr{F}^{1/2}(E-a^*a)^{-1}(I-\mathfrak{p})\mathscr{F}^{1/2}\mathfrak{p}) = n_*(t^{1/2};\mathfrak{p}\mathscr{F}^{1/2}|a^*a-E|^{-1/2}(I-\mathfrak{p})). \quad (4.2.82)$$

Since $\||a^*a-E|^{-1/2}(I-\mathfrak{p})\| = (E_0-E)^{-1/2}$, we have

$$n_*(t^{1/2};\mathfrak{p}\mathscr{F}^{1/2}|a^*a-E|^{-1}(I-\mathfrak{p})) \leq n_*(((E_0-E)t)^{1/2};\mathfrak{p}\mathscr{F}^{1/2}(I-\mathfrak{p})), \quad t>0. \quad (4.2.83)$$

Now note that the first estimate in (4.2.65), and relation (4.2.52) with $\phi>0$, imply in particular $\mathscr{F}(\mathbf{x}) \asymp \langle\mathbf{x}\rangle^{-\gamma}$, $\mathbf{x}\in\mathbb{R}^2$, so that $\mathscr{G} = \mathscr{F}^{1/2}$ satisfies the assumptions of Corollary 4.2.8 with $m = \gamma/2$. Putting together (4.2.82) and (4.2.83), and making use of Corollary 4.2.8, we get

$$n_-(\varepsilon s/E;\mathfrak{p}\mathscr{F}^{1/2}(E-a^*a)^{-1}(I-\mathfrak{p})\mathscr{F}^{1/2}\mathfrak{p}) = \mathscr{O}(s^{-2/(2+\gamma)}) = o(s^{-2/\gamma}), \quad s\downarrow 0. \quad (4.2.84)$$

Let us now estimate the last but one term in (4.2.81). By the Ky Fan inequalities, for $t>0$ and $\varepsilon>0$ we have

$$n_+(t;\mathfrak{p}\mathscr{F}^{1/2}\mathfrak{p}\mathscr{F}^{1/2}\mathfrak{p}) = n_+(t^{1/2};\mathfrak{p}\mathscr{F}^{1/2}\mathfrak{p}) = n_*(t^{1/2};\mathfrak{p}\mathscr{F}^{1/2}\mathfrak{p}) \geq$$

$$n_*((1+\varepsilon)t^{1/2};\mathfrak{p}\mathscr{F}^{1/2}) - n_*(\varepsilon t^{1/2};\mathfrak{p}\mathscr{F}^{1/2}(I-\mathfrak{p})). \quad (4.2.85)$$

Evidently,

$$n_*((1+\varepsilon)t^{1/2};\mathfrak{p}\mathscr{F}^{1/2}) = n_+((1+\varepsilon)^2t;\mathfrak{p}\mathscr{F}\mathfrak{p}), \quad t>0, \quad \varepsilon>0. \quad (4.2.86)$$

On the other hand, Corollary 4.2.8 implies

$$n_*(\varepsilon t^{1/2};\mathfrak{p}\mathscr{F}^{1/2}(I-\mathfrak{p})) = \mathscr{O}(t^{-2/(2+\gamma)}) = o(t^{-2/\gamma}), \quad t\downarrow 0. \quad (4.2.87)$$

Putting together (4.2.81), and (4.2.84) - (4.2.87), we obtain

$$n_+(s/E;\mathscr{F}^{1/2}(E-a^*a)^{-1}\mathscr{F}^{1/2}) \geq n_+((1+\varepsilon)^3 s;\mathfrak{p}\mathscr{F}\mathfrak{p}) - o(s^{-2/\gamma}), \quad s\downarrow 0,$$

which combined with (4.2.72) yields

$$\limsup_{s\downarrow 0} s^{2/\gamma} n_+(s;\mathfrak{p}\mathscr{F}\mathfrak{p}) \leq (1+\varepsilon)^{6/\gamma}\frac{b_0}{4\pi}\int_{\mathbb{S}^1}\phi(\omega)^{2/\gamma}d\omega \tag{4.2.88}$$

for any $\varepsilon > 0$. Letting $\varepsilon\downarrow 0$ in (4.2.88), and combining this upper bound with the lower bound (4.2.80), we obtain (4.2.66).

4.2.5 Semi-classical versus non-classical eigenvalue asymptotics

Let us discuss now which of the results on the asymptotics of $n_\pm(s;p_q\mathscr{F}p_q)$ as $s\downarrow 0$ are semi-classical and which are non-classical. In order to define these notions, let us recall that by Theorem 3.4.1 and Corollary 3.4.2 we have

$$n_\pm(s;p_q\mathscr{F}p_q) = n_\pm(s;\mathrm{Op}_q^{aw}(\mathscr{F}_b)) = n_\pm(s;\mathrm{Op}^w(\mathscr{F}_b*\Psi_q)), \quad s>0, \quad q\in\mathbb{Z}_+,$$

for fairly general symbols $\mathscr{F}$ which decay in a suitable sense at infinity. Hence we will say that the spectral asymptotics for the operator $p_q\mathscr{F}p_q$ is *semi-classical* if

$$\lim_{s\downarrow 0}\frac{n_\pm(s;p_q\mathscr{F}p_q)}{\mathfrak{A}^\pm(s;\mathscr{F}_b*\Psi_q)} = 1 \tag{4.2.89}$$

where $\mathfrak{A}^\pm$ are the area functions introduced in (4.2.54), and *non-classical* if (4.2.89) does not hold true. Thus, we will compare the asymptotics of the eigenvalue counting functions for the operator $\mathrm{Op}_q^{aw}(\mathscr{F}_b)$ and the area functions related to its Weyl symbol $\mathscr{F}_b*\Psi_q$. Since a priori it is not clear which quantization is the "correct" one, we will also take account of the asymptotics of the area functions related to the anti-Wick symbol $\mathscr{F}_b$ and the Wick symbol $\mathscr{F}_b*\Psi_q*\Psi_q$ of $\mathrm{Op}_q^{aw}(\mathscr{F}_b)$. In order to calculate the main asymptotic terms of these area functions, we will make use of the following simple analytic facts. First, we have $\Psi_q(\mathbf{w})>0$ for $|\mathbf{w}|$ large enough, and

$$\lim_{|\mathbf{w}|\to\infty}\frac{\ln\Psi_q(\mathbf{w})}{|\mathbf{w}|^2} = -1.$$

Thus, if $\mathscr{G}\in L^\infty(\mathbb{R}^2;\mathbb{R})$, $\mathscr{G}(\mathbf{w})>0$ for $|\mathbf{w}|$ large enough, and

$$\lim_{|\mathbf{w}|\to\infty}\frac{\ln\mathscr{G}(\mathbf{w})}{|\mathbf{w}|^{2\beta}} = -\mu$$

with some $\beta\in(0,\infty)$ and $\mu\in(0,\infty)$, then

$$\lim_{|\mathbf{w}|\to\infty}\frac{\ln(\mathscr{G}*\Psi_q)(\mathbf{w})}{|\mathbf{w}|^{2\beta}} = -\mu$$

if $\beta \in (0,1)$, and

$$\lim_{|\mathbf{w}|\to\infty} \frac{\ln(\mathscr{G} * \Psi_q)(\mathbf{w})}{|\mathbf{w}|^2} = \begin{cases} -\dfrac{\mu}{1+\mu} & \text{if } \beta = 1, \\ -1 & \text{if } \beta \in (1,\infty), \end{cases}$$

(see e.g. [97, Lemma 3.5]). Finally, if $\mathscr{G} \geq 0$ is compactly supported on a set of positive measure, then

$$\lim_{|\mathbf{w}|\to\infty} \frac{\ln(\mathscr{G} * \Psi_q)(\mathbf{w})}{|\mathbf{w}|^2} = -1.$$

Let us start the analysis of the asymptotics as $s \downarrow 0$ of $n_+(s; p_q \mathscr{F} p_q)$ with the case of compactly supported symbols $\mathscr{F} \geq 0$ which satisfy (4.2.11). Then

$$\mathfrak{A}^+(s; \mathscr{F}_b) = \mathscr{O}(1), \quad \mathfrak{A}^+(s; \mathscr{F}_b * \Psi_q) \sim |\ln s|, \quad \mathfrak{A}^+(s; \mathscr{F}_b * \Psi_q * \Psi_q) \sim 2|\ln s|$$

as $s \downarrow 0$, where we write $f(s) \sim g(s)$ if $\lim_{s\downarrow 0} f(s)/g(s) = 1$. On the other hand, (4.2.13) implies

$$n_+(s; p_q \mathscr{F} p_q) \sim \frac{|\ln s|}{\ln|\ln s|}, \quad s \downarrow 0.$$

Thus, in this case even the asymptotic order of $n_+(s; p_q \mathscr{F} p_q)$ is non-classical.
Assume now that $\mathscr{F} \geq 0$ decays exponentially at infinity, i.e. that $\mathscr{F}$ satisfies (4.2.38) with some $\beta \in (0,\infty)$ and $\gamma \in (0,\infty)$. If $\beta \in (1,\infty)$, then

$$\mathfrak{A}^+(s; \mathscr{F}_b) \sim \frac{b}{2\gamma^{1/\beta}} |\ln s|^{1/\beta}, \; \mathfrak{A}^+(s; \mathscr{F}_b * \Psi_q) \sim |\ln s|, \; \mathfrak{A}^+(s; \mathscr{F}_b * \Psi_q * \Psi_q) \sim 2|\ln s|,$$

while, by (4.2.41), we have

$$n_+(s; p_q \mathscr{F} p_q) \sim \frac{\beta}{\beta - 1} \frac{|\ln s|}{\ln|\ln s|}, \quad s \downarrow 0.$$

i.e. again the asymptotic order of $n_+(s; p_q \mathscr{F} p_q)$ is non-classical. Next, if $\beta = 1$ which corresponds to a Gaussian decay of $\mathscr{F}$, then

$$\mathfrak{A}^+(s; \mathscr{F}_b) \sim \frac{b}{2\gamma} |\ln s|, \tag{4.2.90}$$

$$\mathfrak{A}^+(s; \mathscr{F}_b * \Psi_q) \sim \left(\frac{b}{2\gamma} + 1\right) |\ln s|, \quad \mathfrak{A}^+(s; \mathscr{F}_b * \Psi_q * \Psi_q) \sim \left(\frac{b}{2\gamma} + 2\right) |\ln s|, \tag{4.2.91}$$

and, by (4.2.41),

$$n_+(s; p_q \mathscr{F} p_q) \sim \frac{1}{\ln(1 + 2\gamma/b)} |\ln s|, \quad s \downarrow 0, \tag{4.2.92}$$

i.e. the asymptotic order of $n_+(s; p_q \mathscr{F} p_q)$ is semi-classical, but the coefficient is not. Note that all the coefficients which appear in (4.2.90) - (4.2.92) have the same main asymptotic term $\frac{b}{2\gamma}$ in the strong-magnetic-field limit $b \to \infty$. Finally, if $\beta \in (0,1)$, then

$$\mathfrak{A}^+(s; \mathscr{F}_b) \sim \mathfrak{A}^+(s; \mathscr{F}_b * \Psi_q) \sim \mathfrak{A}^+(s; \mathscr{F}_b * \Psi_q * \Psi_q) \sim \frac{b}{2\gamma^{1/\beta}} |\ln s|^{1/\beta},$$

and, by (4.2.41),

$$n_+(s;p_q\mathscr{F}p_q) \sim \frac{b}{2\gamma^{1/\beta}}\,|\ln s|^{1/\beta}, \quad s\downarrow 0.$$

as well, i.e. in this case the asymptotics of $n_+(s;p_q\mathscr{F}p_q)$ is semi-classical.
Of course, the results of Theorem 4.2.3 which concern symbols $\mathscr{F}$ of regular power-like decay, are manifestly semi-classical. Under its hypotheses we have again

$$\mathfrak{A}^{\pm}(s;\mathscr{F}_b) \sim \mathfrak{A}^{\pm}(s;\mathscr{F}_b * \Psi_q) \sim \mathfrak{A}^{\pm}(s;\mathscr{F}_b * \Psi_q * \Psi_q) \asymp s^{-2/\gamma}, \quad s\downarrow 0,$$

due to Proposition 3.4.2.

Let us discuss a model situation which might shed light onto the cases of non-classical eigenvalue asymptotics for the operator $\mathrm{Op}^w(\mathscr{F} * \Psi_q)$. Suppose that $\mathscr{G} \in C^\infty(\mathbb{R}^2;\mathbb{R})$ is radially symmetric and

$$\mathscr{G}(\mathbf{w}) := e^{-|\mathbf{w}|^2} J(|\mathbf{w}|^2), \quad \mathbf{w} \in \mathbb{R}^2, \tag{4.2.93}$$

where $J : [0,\infty) \to \mathbb{R}$ satisfies

$$|J^{(k)}(t)| \le C_k e^{\alpha_k t}, \quad t \in [0,\infty), \quad k \in \mathbb{Z}_+,$$

with some constants $C_k \in [0,\infty)$ and $\alpha_k \in (-\infty,1]$ so that $\mathscr{G} \in \mathscr{S}'(\mathbb{R}^2)$. Note that under the hypotheses of Theorem 4.2.1 or Theorem 4.2.2 with $\beta \in [1,\infty)$, the asymptotic behaviour at infinity of the symbol $\mathscr{F}_b * \Psi_q$ is fairly similar to that of $\mathscr{G}$ of the form (4.2.93) with appropriate J. On the other hand, Proposition 3.4.4 implies that the set of the eigenvalues of $\mathrm{Op}^w(\mathscr{G})$ coincides with

$$\frac{(-1)^k}{2}\int_0^\infty J(t/2)L_k(t)e^{-t}\,dt = \frac{1}{k!}\int_0^\infty J^{(k)}(t)t^k e^{-2t}\,dt, \quad k \in \mathbb{Z}_+. \tag{4.2.94}$$

Thus, the function J defines completely the spectrum of $\mathrm{Op}^w(\mathscr{G})$ which may have fairly exotic non-classical asymptotic distribution near the origin.
Let us mention several classes of symbols $\mathscr{G}$ of the form (4.2.93), such that $\mathrm{Op}^w(\mathscr{G})$ has amusing spectral properties which may be of independent interest. If J is a polynomial, then (4.2.94) shows that the number of non-zero eigenvalues of $\mathrm{Op}^w(\mathscr{G})$ is not greater than $\deg J + 1$ so that $n_\pm(s;\mathrm{Op}^w(\mathscr{G})) = \mathscr{O}(1)$ as $s\downarrow 0$, while if the leading coefficient of J is, say, positive, then

$$\mathfrak{A}^+(s;\mathscr{G}) \sim \frac{1}{2}|\ln s|, \quad \mathfrak{A}^-(s;\mathscr{G}) = \mathscr{O}(1), \quad s\downarrow 0,$$

i.e. the behaviour of the positive eigenvalues of $\mathrm{Op}^w(\mathscr{G})$ is manifestly non-classical. If

$$\mathscr{G}(\mathbf{w}) = e^{-\mu|\mathbf{w}|^2}, \quad \mathbf{w} \in \mathbb{R}^2, \quad \mu \in (0,\infty),$$

then

$$\mathfrak{A}^+(s;\mathscr{G}) = \frac{\mathbb{1}_{(0,\infty)}(1-s)}{2\mu}\,|\ln s|, \quad \mathfrak{A}^-(s;\mathscr{G}) = 0, \quad s > 0. \tag{4.2.95}$$

On the other hand, in this case $J(t) = e^{(1-\mu)t}$, $t \in [0,\infty)$, so that, by (4.2.94), the eigenvalues of $\mathrm{Op}^w(\mathscr{G})$ are $\frac{1}{1+\mu}\left(\frac{1-\mu}{1+\mu}\right)^k$, $k \in \mathbb{Z}_+$, if $\mu \neq 1$, and if $\mu = 1$, then $\mathrm{Op}^w(\mathscr{G}) = \frac{1}{2}\langle\cdot,\psi_0\rangle_{L^2(\mathbb{R})}\psi_0$. Thus,

$$n_+(s;\mathrm{Op}^w(\mathscr{G})) = \begin{cases} \mathfrak{c}(\mu)\,|\ln s| + \mathscr{O}(1) & \text{if } \mu \in (0,1), \\ \mathbb{1}_{(0,\infty)}\left(\frac{1}{2} - s\right) & \text{if } \mu = 1, \\ \frac{1}{2}\mathfrak{c}(\mu)\,|\ln s| + \mathscr{O}(1) & \text{if } \mu \in (1,\infty), \end{cases}$$

and

$$n_-(s;\mathrm{Op}^w(\mathscr{G})) = \begin{cases} 0 & \text{if } \mu \in (0,1], \\ \frac{1}{2}\mathfrak{c}(\mu)\,|\ln s| + \mathscr{O}(1) & \text{if } \mu \in (1,\infty), \end{cases}$$

where

$$\mathfrak{c}(\mu) := \frac{1}{\left|\ln\left|\frac{1-\mu}{1+\mu}\right|\right|}, \quad \mu \in (0,\infty), \quad \mu \neq 1. \tag{4.2.96}$$

Note that if $\mu > 1$, then $\mathrm{Op}^w(\mathscr{G})$ has infinitely many negative eigenvalues even if $\mathscr{G} > 0$. Note also the coefficient $\mathfrak{c}(\mu)$ defined in (4.2.96) satisfies $\mathfrak{c}(\mu) \sim \frac{1}{2\mu}$ as $\mu \downarrow 0$, which is compatible with (4.2.95). Finally, if

$$J(t) = \cos(\omega t), \quad \omega > 0,$$

then the eigenvalues of $\mathrm{Op}^w(\mathscr{G})$ are

$$\frac{(\sin\theta)^k}{(\omega^2+4)^{1/2}}\cos((k+1)\theta + k\pi/2), \quad k \in \mathbb{Z}_+,$$

where $\theta := \arctan(\omega/2)$, so that the distribution of the positive and negative eigenvalues of $\mathrm{Op}^w(\mathscr{G})$ strongly depends on the arithmetic properties of θ.

The analysis of the nature of the asymptotics of $n_\pm(s;\mathfrak{p}_{\mathrm{ann}}\mathscr{F}\mathfrak{p}_{\mathrm{ann}})$ as $s \downarrow 0$ is not so conspicuous since we do not dispose of a convenient expression for the Weyl symbol of the operator T defined in (4.2.67). However, if we postulate that we should compare $n_\pm(s;\mathfrak{p}_{\mathrm{ann}}\mathscr{F}\mathfrak{p}_{\mathrm{ann}})$ with the area functions $\mathfrak{A}^\pm(s;\mathscr{F}_{b_0})$, then we can easily draw conclusions quite close to the ones concerning $n_\pm(s;p_q\mathscr{F}p_q)$.

4.3 A result from control theory

In this section we discuss an application of the explicit expressions (4.2.7) for the eigenvalues $\lambda_{k,q}(\mathbb{1}_{\mathscr{B}_R(0)})$, $k \in \mathbb{Z}_+$, of the operator $p_q\,\mathbb{1}_{\mathscr{B}_R(0)}\,p_q$, $q \in \mathbb{Z}_+$.
Let $u \in L^\infty(\mathbb{R}^2;\mathbb{R})$, and $\mathrm{supp}\,u$ be compact. Define the $\mathbb{Z}^2$-periodic symbol

$$\mathscr{G}(\mathbf{x}) := \sum_{\mathbf{j}\in\mathbb{Z}^2} u(\mathbf{x}-\mathbf{j}), \quad \mathbf{x} \in \mathbb{R}^2. \tag{4.3.1}$$

Theorem 4.3.1 [51, Theorem 4] *Let* $u \in L^\infty(\mathbb{R}^2;\mathbb{R})$ *be a compactly supported function which satisfies*

$$u(\mathbf{x}) \geq u_0 \mathbb{1}_{\mathscr{B}_\varepsilon(0)}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{4.3.2}$$

with some constants $\varepsilon > 0$ *and* $u_0 > 0$. *Let* $q \in \mathbb{Z}_+$. *Then there exists a constant* $C = C(q, b, u) > 0$ *such that*

$$p_q(b)\mathscr{G}p_q(b) \geq Cp_q(b), \tag{4.3.3}$$

where $\mathscr{G}$ *is the periodic symbol defined in* (4.3.1).

Remarks: (i) Evidently, estimate (4.3.3) is not trivial if the support of u is contained in the fundamental domain $(-1/2, 1/2)^2$ of the periodicity lattice $\mathbb{Z}^2$ for $\mathscr{G}$, which implies that $\varepsilon > 0$ in (4.3.2) should be small enough. Hence, Theorem 4.3.1 can be interpreted as a result from control theory because by (4.3.2) and (4.3.3), the positiveness of $\mathscr{G}$ on the set

$$\bigcup_{\mathbf{j}\in\mathbb{Z}^2} \left\{\mathbf{x} \in \mathbb{R}^2 \,|\, |\mathbf{x}-\mathbf{j}| < \varepsilon\right\}$$

with arbitrarily small $\varepsilon > 0$, implies the global positiveness of the Berezin-Toeplitz operator $p_q\mathscr{G}p_q$ on $\operatorname{Ran} p_q$, $q \in \mathbb{Z}_+$. In particular, (4.3.3) implies that no function $\varphi \in \operatorname{Ran} p_q$ which does not vanish identically, can be supported in $\mathbb{R}^2 \setminus \operatorname{supp}\mathscr{G}$. It is an important and challenging open problem to prove or disprove the analogue of (4.3.3) in the case where p_q is replaced by $\sum_{q=0}^n p_q$, $n \in \mathbb{N}$.
(ii) In [51], estimate (4.3.3) played a crucial role in the proof of a *Wegner estimate* for the 2D Landau Hamiltonian $H_S(A, V_\omega) = H_S(A, 0) + V_\omega$ (see (2.7.3)) with random alloy-type electric potential

$$V_\omega(\mathbf{x}) := \sum_{\mathbf{j}\in\mathbb{Z}^2} \eta_{\mathbf{j}}(\omega)u(\mathbf{x}-\mathbf{j}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{4.3.4}$$

where $u \in L^\infty(\mathbb{R})$ is a compactly supported function which satisfies (4.3.2), and $\{\eta_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^2}$ are independent and identically distributed random variables whose distribution has a compactly supported bounded density. More precisely, in [51, Theorem 3], it was shown that for each $\alpha \in (0, 1)$ and every bounded interval $\mathscr{I} \subset \mathbb{R}$ there exists a constant $C_{\mathscr{I},\alpha} \in (0, \infty)$ such that for any subinterval $\mathscr{O} \subset \mathscr{I}$ there is a length scale $L_{\mathscr{O}}$ such that $L > L_{\mathscr{O}}$ implies

$$\mathbb{E}\left(\operatorname{Tr}\left(\mathbb{1}_{\mathscr{Q}_L(\mathbf{0})} \mathbb{1}_{\mathscr{O}}(H_S(A, V_\omega)) \mathbb{1}_{\mathscr{Q}_L(\mathbf{0})}\right)\right) \leq C_{\mathscr{I},\alpha} |\mathscr{O}|^\alpha L^2, \tag{4.3.5}$$

where $\mathbb{E}$ denotes the mathematical expectation, and $|\mathscr{O}|$ is the length of $\mathscr{O}$. The Wegner estimate (4.3.5) implies the Hölder continuity of the integrated density of states (IDS) $\mathfrak{N}_{b,V}$ for the operator $H_S(A, V_\omega)$ (see (4.1.12)). The Wegner estimate (4.3.5) was extended in [52] to the case $\alpha = 1$ which implies the Lipschitz continuity of the IDS $\mathfrak{N}_{b,V}$, a result important for the applications in solid state physics since it is equivalent to the essential boundedness of the distributional derivative $\frac{d\mathfrak{N}_{b,V}}{dE}$ which is the *density of states* for the operator $H_S(A, V_\omega)$. Note that the IDS $\mathfrak{N}_{b,0}$ for the unperturbed

Landau Hamiltonian $H_S(A,0)$ is not even continuous since we have

$$\mathfrak{N}_{b,0}(E) = \frac{b}{2\pi} \sum_{q\in\mathbb{Z}_+} \mathbb{1}_{(0,\infty)}(E-\Lambda_q(b)), \quad E \in \mathbb{R},$$

(see (4.1.13), (2.7.52), and (4.2.68)), i.e. $\mathfrak{N}_{b,0}$ is a staircase function with jumps of size $b/(2\pi)$ at the Landau levels $\Lambda_q(b) = b(2q+1)$, $q \in \mathbb{Z}_+$, so that

$$\frac{d\mathfrak{N}_{b,0}}{dE}(E) = \frac{b}{2\pi} \sum_{q\in\mathbb{Z}_+} \delta(E-\Lambda_q(b)), \quad E \in \mathbb{R}.$$

Thus, the introduction of random impurities modeled by the potential V_ω leads to smearing and broadening of the peaks at the Landau levels.
(iii) The ideas of the proof of Theorem 4.3.1, and in particular Proposition 4.3.1 below, played an important role in the proof of the existence of *dynamical delocalization* for the operator $H_S(A,V_\omega)$ (see [80, Section 5]), a deep result in the dynamical counterpart of the Anderson localization theory.
(iv) A Wegner estimate similar to (4.3.5) was obtained in [203] in the case where the alloy-type potential defined in (4.3.4) is replaced by the *breather potential*

$$Q_\omega(\mathbf{x}) := \sum_{\mathbf{j}\in\mathbb{Z}^2} u\left((\mathbf{x}-\mathbf{j})/\eta_{\mathbf{j}}(\omega)\right), \quad \mathbf{x} \in \mathbb{R}^2,$$

where u is a compactly supported bounded function satisfying an appropriate positivity condition which replaces (4.3.2), while the distribution density for the i.i.d random variables $\eta_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{Z}^2$, is supported on a subinterval of $(0,1/2)$.

Let us now prove Theorem 4.3.1. First, recall that by Proposition 3.3.3, the operator $p_q \mathbb{1}_{\mathscr{B}_R(0)} p_q$ with any $R > 0$ is diagonalizable with respect to the canonic basis $\{\varphi_{k,q}\}_{k\in\mathbb{Z}_+}$, and its eigenvalues coincide with the set

$$\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right) = \langle \mathbb{1}_{\mathscr{B}_R(0)}\varphi_{k,q}, \varphi_{k,q}\rangle_{L^2(\mathbb{R}^2)} = \frac{q!}{k!}\int_0^\rho L_q^{(k-q)}(t)^2 t^{k-q} e^{-t}\, dt, \quad k \in \mathbb{Z}_+.$$

with $\rho = \rho(R) := bR^2/2$ (cf. (4.2.7)). Therefore, it is easy to check that

$$\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right) = \frac{e^{-\rho}\rho^{-q+1}}{q!}\frac{k^{2q-1}\rho^k}{k!}(1+o(1)), \quad k \to \infty, \tag{4.3.6}$$

uniformly with respect to $R \in [R_1,R_2]$ with $0 < R_1 < R_2 < \infty$ (see [51, Corollary 2]).

Proposition 4.3.1 [51, Lemma 2] *Let* $q \in \mathbb{Z}_+$. *Then for each* $R > 0$, $\varepsilon \in (0,R)$*, and* $\eta > 0$ *there exists a constant* $C_0 = C_0(q,R,\varepsilon,\eta)$ *such that for each* $s \in (1,2)$ *we have*

$$p_q \mathbb{1}_{\mathscr{B}_\varepsilon(0)} p_q \geq C_0\left(p_q \mathbb{1}_{\mathscr{B}_R(0)} p_q - \eta\, p_q \mathbb{1}_{\mathscr{B}_{sR}(0)} p_q\right). \tag{4.3.7}$$

Proof. We fix $\delta \in (0,1)$, and bearing in mind (4.3.6), pick $K \in \mathbb{Z}_+$ such that $k_0 \geq K$ implies

$$(1-\delta)\frac{e^{-\rho(R_0)}\rho(R_0)^{k-q+1}}{q!}\frac{k^{2q-1}}{k!} \leq \lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_{R_0}(0)}\right)$$
$$\leq (1+\delta)\frac{e^{-\rho(R_0)}\rho(R_0)^{k-q+1}}{q!}\frac{k^{2q-1}}{k!} \qquad (4.3.8)$$

for $R_0 \in [R/2, 2sR]$. We will show that if $\varepsilon \in (0,R)$, $s > 1$, and $k_0 \geq K$, then the operator inequality

$$p_q \mathbb{1}_{\mathscr{B}_\varepsilon(0)} p_q \geq C_1\left(p_q \mathbb{1}_{\mathscr{B}_R(0)} p_q - C_2 p_q \mathbb{1}_{\mathscr{B}_{sR}(0)} p_q\right) \qquad (4.3.9)$$

holds true with

$$C_1 = C_1(q,R,\varepsilon,k_0) := \min_{k\in\{0,\dots,k_0\}} \frac{\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_\varepsilon(0)}\right)}{\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right)}, \qquad (4.3.10)$$

and

$$C_2 := \frac{1+\delta}{1-\delta} s^{-2(k_0-q+1)} e^{-\rho(R)+\rho(sR)}. \qquad (4.3.11)$$

Since the operators $p_q \mathbb{1}_{\mathscr{B}_r(0)} p_q$ with any $r > 0$ are diagonalizable in the same basis, we find that in order to prove (4.3.9), it suffices to check the numerical inequalities

$$\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_\varepsilon(0)}\right) \geq C_1\left(\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right) - C_2\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_{sR}(0)}\right)\right) \qquad (4.3.12)$$

for each $k \in \mathbb{Z}_+$. If $k \leq k_0$, then (4.3.12) is valid because in this case (4.3.10) entails

$$\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_\varepsilon(0)}\right) \geq C_1\,\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right) \geq C_1\left(\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right) - C_2\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_{sR}(0)}\right)\right).$$

If $k > k_0$, then by (4.3.8) and (4.3.11), we have

$$\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right) - C_2\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_{sR}(0)}\right) \leq$$
$$(1+\delta)\frac{e^{-\rho(R)}}{q!}\frac{k^{2q-1}\rho(sR)^{k-q+1}}{k!} s^{2(q-1)}\left(s^{-2k} - s^{-2k_0}\right). \qquad (4.3.13)$$

Since $s > 1$ and $k > k_0$, we find that

$$\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_R(0)}\right) - C_2\lambda_{k,q}\left(\mathbb{1}_{\mathscr{B}_{sR}(0)}\right) < 0,$$

which again implies (4.3.12).
Now fix $\eta > 0$ and choose $k_0 = k_0(\eta,R,b) \geq K$ so large that $C_2 \leq \eta$. Then (4.3.7) with $C_0 = C_1(q,R.\varepsilon,k_0(\eta,R,b))$ follows from (4.3.12). □

Corollary 4.3.1 *Under the hypotheses of Proposition 4.3.1 we have*

$$p_q \mathbb{1}_{\mathscr{B}_\varepsilon(\mathbf{y})} p_q \geq C_0\left(p_q \mathbb{1}_{\mathscr{B}_R(\mathbf{y})} p_q - \eta\, p_q \mathbb{1}_{\mathscr{B}_{sR}(\mathbf{y})} p_q\right) \qquad (4.3.14)$$

for each $\mathbf{y} \in \mathbb{R}^2$.

Proof. By (2.7.27), we have

$$\mathscr{T}_{\mathbf{y}}^* p_q \mathscr{T}_{\mathbf{y}} = p_q, \quad \mathscr{T}_{\mathbf{y}}^* p_q \mathbb{1}_{B_r(0)} p_q \mathscr{T}_{\mathbf{y}} = p_q \mathbb{1}_{B_r(\mathbf{y})} p_q, \tag{4.3.15}$$

for each $r > 0$. Here $\mathscr{T}_{\mathbf{y}}$ are the magnetic translations defined in (2.7.8). Combining (4.3.7) with (4.3.15), we obtain (4.3.14). □

Now we are in position to prove (4.3.3). Choose $R > \sqrt{2}/2$ so that

$$\sum_{\mathbf{j}\in\mathbb{Z}^2} \mathbb{1}_{\mathscr{B}_R(\mathbf{j})}(\mathbf{x}) \geq 1, \quad \mathbf{x} \in \mathbb{R}^2. \tag{4.3.16}$$

Further, fix $s \in (1,2)$ so that we have

$$\sum_{\mathbf{j}\in\mathbb{Z}^2} \mathbb{1}_{\mathscr{B}_{sR}(\mathbf{j})}(\mathbf{x}) \leq K_0, \quad \mathbf{x} \in \mathbb{R}^2, \tag{4.3.17}$$

with some constant $K_0 > 0$. Finally, pick sufficiently ε small so that (4.3.2) holds true. Putting together (4.3.2), (4.3.14), (4.3.16), and (4.3.17), we find that for each $\eta > 0$ there exists $C_0 > 0$ such that

$$\begin{aligned} p_q \mathscr{G} p_q &\geq u_0 \sum_{\mathbf{j}\in\mathbb{Z}^2} p_q \mathbb{1}_{\mathscr{B}_\varepsilon(\mathbf{j})} p_q \\ &\geq u_0 C_0 \sum_{\mathbf{j}\in\mathbb{Z}^2} \left(p_q \mathbb{1}_{\mathscr{B}_R(\mathbf{j})} p_q - \eta p_q \mathbb{1}_{\mathscr{B}_{sR}(\mathbf{j})} p_q \right) \\ &\geq u_0 C_0 (1 - \eta K_0) p_q. \end{aligned}$$

Choosing $\eta = 1/(2K_0)$, we obtain (4.3.3) with $C := u_0 C_0/2$.

4.4 Eigenvalue asymptotics for 2D magnetic quantum Hamiltonians

In this section we apply the results on the eigenvalue asymptotics for Berezin-Toeplitz operators obtained in Section 4.2, to the investigation of the distribution of the discrete eigenvalues near the border points of the essential spectrum for relatively compact perturbations of magnetic quantum Hamiltonians. The leading examples of the unperturbed operator are the 2D Landau Hamiltonian and the 2D Pauli operator with admissible magnetic field $b = b_0 + \tilde{b}$, $b_0 \neq 0$. In Subsection 4.4.1 we consider electric perturbations of these unperturbed operators, i.e. perturbations by additive electric potentials, while in Subsection 4.4.2 we deal with magnetic and geometric perturbations of 2D magnetic quantum Hamiltonians.

4.4.1 Electric perturbations

Assume at first that the unperturbed operator H_0 is the 2D Landau Hamiltonian

$$H_0 := H_S(A,0) = \left(-i\frac{\partial}{\partial x} + \frac{by}{2}\right)^2 + \left(-i\frac{\partial}{\partial y} - \frac{bx}{2}\right)^2, \quad (x,y) \in \mathbb{R}^2,$$

with $A = \frac{b}{2}(-y,x)$ and constant $b > 0$ (see (2.7.3)). Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a Lebesgue-measurable function such that the operator $|V|^{1/2}H_0^{-1/2}$ is compact in $L^2(\mathbb{R}^2)$. Applying the Weyl theorem on the invariance of the essential spectrum, we find that

$$\sigma_{ess}(H_V) = \sigma_{ess}(H_0) = \sigma(H_0) = \bigcup_{q\in\mathbb{Z}_+} \left\{\Lambda_q\right\}, \tag{4.4.1}$$

where for brevity

$$H_V := H_S(A,V) = H_S(A,0) + V$$

denotes the perturbed operator, the sum being understood in the sense of the quadratic forms. Note that relation (4.4.1) just tells us that the essential spectra of the operators H_0 and H_V coincide as subsets of $\mathbb{R}$. However, the nature of the points of these essential spectra may change: if any Landau level Λ_q, $q \in \mathbb{Z}_+$, is necessarily an isolated eigenvalue of H_0 of infinite multiplicity, it could be an eigenvalue of H_V of infinite multiplicity or an accumulation point of the discrete spectrum of H_V, or both.
For notational convenience put $\Lambda_{-1} = \widetilde{\Lambda}_{-1} = -\infty$. For $q \in \mathbb{Z}_+$ choose $\widetilde{\Lambda}_q \in (\Lambda_q, \Lambda_{q+1})$ which is not an eigenvalue of H_V, and set

$$\mathscr{N}_q^+(\lambda;V) := N_{(\Lambda_q+\lambda,\widetilde{\Lambda}_q)}(H_V), \quad \lambda \in (0, \widetilde{\Lambda}_q - \Lambda_q),$$

$$\mathscr{N}_q^-(\lambda;V) := N_{(\widetilde{\Lambda}_{q-1},\Lambda_q-\lambda)}(H_V), \quad \Lambda_q - \lambda \in (\widetilde{\Lambda}_{q-1}, \Lambda_q),$$

where $N_{(s,t)}(T)$ is the eigenvalue counting function introduced in (4.1.1).
Further, for $q \in \mathbb{Z}_+$, let $\left\{\lambda_{k,q}^+(V)\right\}_{k\geq 0}$ (resp., $\left\{\lambda_{k,q}^-(V)\right\}_{k\geq 0}$) be the non-increasing (resp., non-decreasing) set of the eigenvalues of H_V lying on the interval $(\Lambda_q, \widetilde{\Lambda}_q)$ (resp., on $(\widetilde{\Lambda}_{q-1}, \Lambda_q)$), and counted with the multiplicities. *A priori*, any of these sets may be empty. We have

$$\mathscr{N}_q^\pm(\lambda;V) = \#\left\{k \in \mathbb{Z}_+ \mid \pm(\lambda_{k,q}^\pm(V) - \Lambda_q) > \lambda\right\}.$$

Set

$$m_q(V) := \#(\sigma(H_V) \cap (\Lambda_{q-1}, \Lambda_q)), \quad q \in \mathbb{Z}_+.$$

Assume now that $V \geq 0$. Then the discrete eigenvalues of H_V may accumulate to any Landau level Λ_q only from above. Accordingly, we choose $\widetilde{\Lambda}_q$, $q \in \mathbb{Z}_+$, so that $\sigma(H_V) \cap [\widetilde{\Lambda}_q, \Lambda_{q+1}) = \emptyset$, and hence

$$\mathscr{N}_q^+(\lambda;V) = N_{(\Lambda_q+\lambda,\Lambda_{q+1})}(H_V), \quad \mathscr{N}_q^-(\lambda;V) = 0,$$

and the set $\left\{\lambda_{k,q+1}^-(V)\right\}$ is empty. Similarly, the discrete eigenvalue of H_{-V} may accumulate to Λ_q, $q \in \mathbb{Z}_+$, only from below. Then we choose $\widetilde{\Lambda}_{q-1}$, $q \in \mathbb{N}$, so that $\sigma(H_{-V} \cap (\Lambda_{q-1}, \widetilde{\Lambda}_{q-1}] = \emptyset$, and therefore

$$\mathscr{N}_q^-(\lambda;-V) = N_{(\Lambda_{q-1},\Lambda_q-\lambda)}(H_{-V}), \quad \mathscr{N}_q^+(\lambda;-V) = 0,$$

and the set $\left\{\lambda_{k,q-1}^+(-V)\right\}$ is empty.

Compactly supported and exponentially decaying potentials

First, this subsection concerns the asymptotics of $\lambda^{\pm}_{k,q}(\pm V) - \Lambda_q$ as $k \to \infty$ for a fixed $q \in \mathbb{Z}_+$ and compactly supported $V \geq 0$.

Theorem 4.4.1 *Assume that* $V \in C(\mathbb{R}^2)$, $\operatorname{supp} V = \overline{\Omega}$ *where* $\Omega \subset \mathbb{R}^2$ *is a bounded domain, and* $V > 0$ *on* Ω. *Then for any* $q \in \mathbb{Z}_+$ *we have* $m_q(\pm V) = \infty$, *and*

$$\ln\left(\pm\left(\lambda^{\pm}_{k,q}(\pm V) - \Lambda_q\right)\right) = -k \ln k + \left(1 + \ln\left(\frac{b\,\mathrm{Cap}(\Omega)^2}{2}\right)\right) k + o(k) \tag{4.4.2}$$

as $k \to \infty$.

The theorem follows from Theorem 4.2.1, and Corollary 4.4.1 below.

Remarks: (i) If $V \in L^\infty(\mathbb{R}^2;\mathbb{R})$ is compactly supported and satisfies

$$V(\mathbf{x}) \geq C \mathbb{1}_{\mathscr{B}_r(\mathbf{x}_0)}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{4.4.3}$$

with some $C > 0$, $r > 0$, $\mathbf{x}_0 \in \mathbb{R}^2$, then (4.2.13), and (4.4.8) below imply

$$\mathscr{N}^{\pm}_q(\lambda;\pm V) = \varphi_\infty(\lambda)(1 + o(1)), \quad \lambda \downarrow 0,$$

where φ_∞ is the function defined in (4.2.14). This is a weaker version of (4.4.2), first obtained in [159, Theorem 2.2].
(ii) In [129, Theorem 1.2], the above result was extended to the 2d-dimensional case with $d > 1$, and constant magnetic field B of full rank. More precisely, it was shown that if $V \in L^\infty(\mathbb{R}^{2d};\mathbb{R}_+)$ is compactly supported and satisfies $V > 0$ on an open non-empty set, then

$$\mathscr{N}^{\pm}_q(\lambda;\pm V) = \frac{\kappa_q}{d!}\varphi_\infty(\lambda)^d(1 + o(1)), \quad \lambda \downarrow 0,$$

where κ_q is the multiplicity of the Landau level defined in (2.7.50). Moreover, [129, Theorem 1.3] contains a similar result for the 2d-dimensional *Dirac* operator with a constant full-rank B and compactly supported electric potential. The approach of [129] is based on the representation of $\mathscr{N}^{\pm}_q(\lambda)$ described in Corollary 4.2.2.
(iii) Another version of (4.4.2) can be found in [77, Theorem 2] which is the pioneering result containing a two-term asymptotics of $\ln\left(\pm\left(\lambda^{\pm}_{k,q}(\pm V) - \Lambda_q\right)\right)$ as $k \to \infty$. The assumptions are slightly different from ours, it is supposed that $V \in L^p(\mathbb{R}^2;\mathbb{R})$, $p > 1$, $\operatorname{supp} V = \overline{\Omega}$ where $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary, and $V(\mathbf{x}) \geq c > 0$ for $\mathbf{x} \in \Omega$. Note that our assumptions do not require that V has a jump at $\partial\Omega$, and are more convenient for further applications.
(iv) Theorem 4.4.1 admits an extension to singular potentials supported, for example, on a simple closed C^∞-curve $\Gamma \subset \mathbb{R}^2$. Let $v \in C^\infty(\Gamma;\mathbb{R}_+)$. Then the perturbation $H_S(A, \pm v\delta_\Gamma)$ of the 2D Landau Hamiltonian $H_S(A, 0)$ can be defined as the self-adjoint operator generated in $L^2(\mathbb{R}^2)$ by the closed quadratic form

$$\int_{\mathbb{R}^2} |i\nabla u + Au|^2\, d\mathbf{x} \pm \int_\Gamma v|u|^2\, ds, \quad u \in \mathfrak{D}(H_S(A,0)^{1/2}).$$

Due to the compactness of the embedding of $\mathfrak{D}(H(A,0)^{1/2})$ into $L^2(\Gamma)$, we have

$$\sigma_{\rm ess}(H_S(A,\pm v\delta_\Gamma)) = \sigma_{\rm ess}(H_S(A,0)) = \sigma(H_S(A,0)) = \bigcup_{q\in\mathbb{Z}_+} \left\{\Lambda_q\right\},$$

and again the discrete eigenvalues of $H_S(A, v\delta_\Gamma)$ (resp., $H(A,-v\delta_\Gamma)$) can accumulate at any given Landau level Λ_q, $q \in \mathbb{Z}_+$, only from above (resp., from below). Then we can define the eigenvalues $\lambda^{\pm}_{k,q}(\pm v\delta_\Gamma)$ by analogy with $\lambda^{\pm}_{k,q}(\pm V)$, $V \geq 0$ being a regular decaying potential. If, say, $v \geq c > 0$ on Γ, then [10, Theorem 6.5] implies that relation (4.4.2) remains valid if we replace in it $\lambda^{\pm}_{k,q}(\pm V)$ by $\lambda^{\pm}_{k,q}(\pm v\delta_\Gamma)$, and $\mathrm{Cap}(\Omega)$ by $\mathrm{Cap}(\Gamma)$. This result also follows easily from (4.2.12) and a suitable version of Proposition 4.4.1 below.

Our next theorem is the analogue of Theorem 4.4.1 in the case of exponentially decaying $V \geq 0$.

Theorem 4.4.2 *Assume that* $V \in C(\mathbb{R}^2;\mathbb{R}_+)$ *satisfies*

$$\ln V(\mathbf{x}) = -\gamma|\mathbf{x}|^{2\beta} + \mathscr{O}(\ln|\mathbf{x}|), \quad |\mathbf{x}| \to \infty, \tag{4.4.4}$$

for some constants $\gamma > 0$ *and* $\beta > 0$*, uniformly with respect to* $\frac{\mathbf{x}}{|\mathbf{x}|} \in \mathbb{S}^1$*. Then for any* $q \in \mathbb{Z}_+$ *we have* $m_q(\pm V) = \infty$*, and*

$$\ln\left(\pm\left(\lambda^{\pm}_{k,q}(\pm V) - \Lambda_q\right)\right) =$$

$$\begin{cases} -\displaystyle\sum_{1\leq j<\frac{1}{1-\beta}} f_j k^{(\beta-1)j+1} + \mathscr{O}(\ln k) & \text{if } \beta \in (0,1), \\ -(\ln(1+\mu))k + \mathscr{O}(\ln k) & \text{if } \beta = 1, \\ -\dfrac{\beta-1}{\beta} k\ln k + \left(\dfrac{\beta-1-\ln(\mu\beta)}{\beta}\right)k & \\ -\displaystyle\sum_{1\leq j<\frac{\beta}{\beta-1}} g_j k^{(\frac{1}{\beta}-1)j+1} + \mathscr{O}(\ln k) & \text{if } \beta \in (1,\infty), \end{cases} \tag{4.4.5}$$

as $k \to \infty$*, with* $\mu = \gamma(2/b)^\beta$*, and the same coefficients* f_j, g_j *as in Theorem 4.2.2.*

The theorem follows from Theorem 4.2.2, and Corollary 4.4.1 below.

Remark: If V satisfies

$$\ln V(\mathbf{x}) = -\gamma|\mathbf{x}|^{2\beta}(1+o(1)), \quad |\mathbf{x}| \to \infty,$$

instead of (4.4.4), then (4.2.40), and (4.4.8) below imply

$$\mathscr{N}^{\pm}_q(\lambda;\pm V) = \varphi_\beta(\lambda)(1+o(1)), \quad \lambda \downarrow 0,$$

φ_β being the function defined in (4.2.41). This is a weaker version of (4.4.5), first obtained in [159, Theorem 2.1].

In Proposition 4.4.1 and Corollary 4.4.1 below we reduce the asymptotic analysis of $\lambda^{\pm}_{k,q}(\pm V) - \Lambda_q$ as $k \to \infty$ needed in the proofs of Theorems 4.4.1 and 4.4.2, to that of $\nu^{+}_{k,q}(V)$, the kth eigenvalue of $p_q V p_q$, $q \in \mathbb{Z}_+$. The method we use is based on the Birman-Schwinger principle described in the following

Lemma 4.4.1 *Let* $T = T^*$ *be an operator lower bounded in the separable Hilbert space* $\mathfrak{H}$*, and* $K = K^*$ *be a an operator relatively form-compact with respect to* T.
(i) [15, Lemma 1.1] *Assume that* $s \in (-\infty, \inf \sigma(T))$. *Then*

$$N_{(-\infty,s)}(T+K) = n_-(1;(T-sI)^{-1/2}K(T-sI)^{-1/2}). \tag{4.4.6}$$

(ii) [16, Proposition 1.6] *Assume that* $K \geq 0$ *and* $s \in \mathbb{R} \setminus \sigma(T)$. *Then*

$$\sum_{0<g<1} \dim \mathrm{Ker}\,(T \pm gK - sI) = n_{\mp}(1; K^{1/2}(T-sI)^{-1}K^{1/2}). \tag{4.4.7}$$

Remarks: (i) The eigenvalues of the operators $T \pm gK$ in the spectral gaps of T are discrete. Moreover, the eigenvalues of $T + gK$ (resp, of $T - gK$) in any given spectral gap of T move monotonically upwards (resp., downwards) as the coupling constant g increases. Thus, $\dim \mathrm{Ker}\,(T \pm gK - sI) < \infty$ for any $g \in [0, \infty)$ and $s \in \mathbb{R} \setminus \sigma(T)$, and the number of $g \in (0,1)$ for which this quantity does not vanish, is not more than finite. Moreover, $\sum_{0<g<1} \dim \mathrm{Ker}\,(T \pm gK - sI)$ is equal to the number of the eigenvalues of $T \pm gK$ which pass through $s \in \mathbb{R} \setminus \sigma(T)$ as g as grows from 0 to 1.
(ii) If $s \in (-\infty, \inf \sigma(T))$ and $K \geq 0$, then $\dim \mathrm{Ker}\,(T + gK - sI) = 0$ for any $g \geq 0$, and

$$\sum_{0<g<1} \dim \mathrm{Ker}\,(T - gK - sI) = N_{(-\infty,s)}(T - K).$$

Note that in the latter case, identity (4.2.2) implies

$$n_+(1; K^{1/2}(T-sI)^{-1}K^{1/2}) = n_+(1;(T-sI)^{-1/2}K(T-sI)^{-1/2})$$

which is coherent with (4.4.6) - (4.4.7).

Proof of Lemma 4.4.1: (i) Let τ and κ be the quadratic forms of the operators T and K respectively. By the mini-max principle (see Lemma 4.2.1), the quantity $N_{(-\infty,s)}(T+K)$ coincides with the maximal dimension of the subspaces of $\mathfrak{D}(|T|^{1/2})$ whose elements $u \neq 0$ satisfy

$$\tau[u] + \kappa[u] < s\|u\|^2_{\mathfrak{H}},$$

or, equivalently,

$$\|(T-sI)^{1/2}u\|^2_{\mathfrak{H}} < -\kappa[u].$$

Changing the functional variable $(T-sI)^{1/2}u = v$, and noting that $(T-sI)^{1/2}$ is an isomorphism between $\mathfrak{D}(|T|^{1/2})$ and $\mathfrak{H}$, we find that $N_{(-\infty,s)}(T+K)$ is equal to the maximal dimension of the subspaces of $\mathfrak{H}$ whose elements $v \neq 0$ satisfy

$$\|v\|^2_{\mathfrak{H}} < -\kappa[(T-sI)^{-1/2}v],$$

which, by Lemma 4.2.1, coincides with $n_-(1;(T-sI)^{-1/2}K(T-sI)^{-1/2})$.
(ii) Relations (4.4.7) follow from the facts that if $K \geq 0$, then $s \in \mathbb{R}\setminus\sigma(T)$ is an eigenvalue of $T \pm gK$ with a given multiplicity, if and only if $\pm g^{-1}$ is an eigenvalue of $K^{1/2}(T-sI)^{-1}K^{1/2}$ with the same multiplicity, and that $K^{1/2}(T-sI)^{-1}K^{1/2}$ is an increasing operator-valued function of the variable Es running on a given spectral gap of T. □

Proposition 4.4.1 *Assume that* $V : \mathbb{R}^2 \to [0,\infty)$ *satisfies* $V^{1/2}H_0^{-1/2} \in \mathfrak{S}_\infty(L^2(\mathbb{R}^2))$. *Then for each* $\varepsilon \in (0,1)$ *we have*

$$n_+((1+\varepsilon)\lambda; p_q\, V\, p_q) + \mathscr{O}_{\varepsilon,q}(1) \leq$$
$$\mathscr{N}_q^{\pm}(\lambda;\pm V) \leq$$
$$n_+((1-\varepsilon)\lambda; p_q\, V\, p_q) + \mathscr{O}_{\varepsilon,q}(1), \quad \lambda \downarrow 0. \tag{4.4.8}$$

Proof. For $s \in \mathbb{R}\setminus\sigma(H_0)$ set

$$\mathfrak{n}^{\pm}(s) := \sum_{0<g<1} \dim \operatorname{Ker}(H_{\pm gV} - sI).$$

By (4.4.7),

$$\mathfrak{n}^{\pm}(s) = n_{\mp}(1; V^{1/2}(H_0 - sI)^{-1}V^{1/2}), \quad s \in \mathbb{R}\setminus\sigma(H_0). \tag{4.4.9}$$

Further, since $V \geq 0$ and $\widetilde{\Lambda}_q \notin \sigma(H_{\pm V})$, we have

$$\mathscr{N}_q^{+}(\lambda;V) = \mathfrak{n}^+(\Lambda_q + \lambda) - \mathfrak{n}^+(\widetilde{\Lambda}_q), \quad q \in \mathbb{Z}_+, \tag{4.4.10}$$

$$\mathscr{N}_q^{-}(\lambda;-V) = \begin{cases} \mathfrak{n}^-(\Lambda_q - \lambda) - \mathfrak{n}^-(\widetilde{\Lambda}_{q-1}) & \text{if } q \in \mathbb{N}, \\ \mathfrak{n}^-(\Lambda_0 - \lambda) & \text{if } q = 0. \end{cases} \tag{4.4.11}$$

Combining (4.4.9) with (4.4.10) - (4.4.11), we find that

$$\mathscr{N}_q^{\pm}(\lambda;\pm V) = n_{\mp}(1; V^{1/2}(H_0 - \Lambda_q \mp \lambda)^{-1}V^{1/2}) + \mathscr{O}(1), \quad q \in \mathbb{Z}_+. \tag{4.4.12}$$

Writing

$$(H_0 - \Lambda_q \mp \lambda)^{-1} = \mp\lambda^{-1}p_q + (I - p_q)(H_0 - \Lambda_q \mp \lambda)^{-1}, \tag{4.4.13}$$

bearing in mind that $(I-p_q)(H_0 - \Lambda_q \mp \lambda)^{-1}$ admits a uniform limit as $\lambda \downarrow 0$, and applying the Weyl inequalities (4.2.4), we conclude that for each $\varepsilon \in (0,1)$ we have

$$n_+((1+\varepsilon)\lambda; V^{1/2}p_qV^{1/2}) + \mathscr{O}_{\varepsilon,q}(1) \leq$$
$$n_{\mp}(1; V^{1/2}(H_0 - \Lambda_q \mp \lambda)^{-1}V^{1/2}) \leq$$
$$n_+((1-\varepsilon)\lambda; V^{1/2}p_qV^{1/2}) + \mathscr{O}_{\varepsilon,q}(1), \tag{4.4.14}$$

as $\lambda \downarrow 0$. By (4.2.2),

$$n_+(s; V^{1/2}p_qV^{1/2}) = n_+(s; p_qVp_q). \tag{4.4.15}$$

Putting together (4.4.12), (4.4.14), and (4.4.15), we obtain (4.4.8). □

Corollary 4.4.1 *Assume the hypotheses of Theorems 4.4.1 or 4.4.2. Then for any* $q \in \mathbb{Z}_+$ *we have*

$$m_q(\pm V) = \infty, \tag{4.4.16}$$

and for each $\varepsilon \in (0,1)$ *there exists* $k_0 \in \mathbb{Z}_+$ *such that the asymptotic relation*

$$\frac{1}{1+\varepsilon}\nu^+_{k+k_0,q}(V) \le \pm(\lambda^\pm_{k,q}(\pm V) - \Lambda_q) \le \frac{1}{1-\varepsilon}\nu^+_{k-k_0,q}(V) \tag{4.4.17}$$

holds true for $k \in \mathbb{N}$ *large enough.*

Proof. By (4.2.9) or (4.2.43), we have $\operatorname{rank}(p_q V p_q) = \infty$, and hence $n_+(s; p_q V p_q)$ tends to infinity as $s \downarrow 0$. By (4.4.8), the counting function $\mathcal{N}_q^\pm(\lambda; \pm V)$ also tends to infinity as $\lambda \downarrow 0$ which implies (4.4.16). Moreover, estimate (4.4.17) follows easily from (4.4.8).
□

Now Theorem 4.4.1 (resp., Theorem 4.4.2) follows directly from Corollary 4.4.1 and Theorem 4.2.1 (resp., Theorem 4.2.2).

Power-like decaying potentials

In our next theorem we deal with the case where V has a power-like decay at infinity, and its sign may change.

Theorem 4.4.3 *Let* $V \in \Gamma^{-\gamma}_\rho(\mathbb{R}^2; \mathbb{R})$ *with* $\gamma > 0$ *and* $\rho \in (0,1]$. *Assume that the area functions* $\mathfrak{A}^+(\cdot; V)$ *and* $\mathfrak{A}^-(\cdot; V)$ *satisfy the condition* $\mathscr{C}$*, and*

$$\liminf_{s\downarrow 0} s^{2/\gamma}\mathfrak{A}^\pm(s; V) > 0.$$

Then for any $q \in \mathbb{Z}_+$ *we have*

$$\mathcal{N}_q^\pm(\lambda) = b\mathfrak{A}^\pm(\lambda; V)(1 + o(1)), \quad \lambda \downarrow 0. \tag{4.4.18}$$

Remark: Let $\mathfrak{I}^\pm_V(k)$ be the function inverse to $\mathfrak{A}^\pm(\cdot; V)$, well defined for large $k > 0$. Then (4.4.18) is equivalent to

$$\pm\left(\lambda^\pm_{k,q}(V) - \Lambda_q\right) = \mathfrak{I}^\pm_V(k)(1 + o(1)), \quad k \to \infty.$$

In Proposition 4.4.2 below we reduce the asymptotic analysis of the eigenvalue counting function $\mathcal{N}_q^\pm(\lambda; V)$ as $\lambda \downarrow 0$ to that of $n_\pm(\lambda; p_q V p_q)$. However, since we do not assume that the perturbation has a definite sign, we cannot apply the Birman-Schwinger principle. That is why we need the following

Lemma 4.4.2 *Let* $\mathfrak{H}$ *be a separable Hilbert space,* T *be a linear operator, self-adjoint in* $\mathfrak{H}$*, and* $0 \le S = S^* \in \mathfrak{S}_\infty(\mathfrak{H})$. *Let* $(s,t) \subset \mathbb{R}$ *be an open non-empty interval. Then for any* $\varepsilon \in (0, t-s)$ *we have*

$$N_{(s+\varepsilon,t)}(T) - n_+(\varepsilon; S) \le N_{(s,t)}(T - S) \le N_{(s,t+\varepsilon)}(T) + n_+(\varepsilon; S), \tag{4.4.19}$$

$$N_{(s,t-\varepsilon)}(T) - n_+(\varepsilon; S) \le N_{(s,t)}(T + S) \le N_{(s-\varepsilon,t)}(T) + n_+(\varepsilon; S). \tag{4.4.20}$$

Proof. Let us prove the upper bound in (4.4.19). Let M be an operator self-adjoint in $\mathfrak{H}$, $K_1 = K_1^* \in \mathfrak{B}(\mathfrak{H})$ with $\sigma(K_1) \subset [\alpha, \beta]$, and $K_2 = K_2^* \in \mathfrak{B}(\mathfrak{H})$ with $\operatorname{rank} K_2 < \infty$. Then [20, Chapter 9, Section 4, Lemma 3] and [20, Chapter 9, Section 3, Theorem 3] imply

$$N_{(s,t)}(M) \leq N_{(s+\alpha,t+\beta)}(M + K_1 + K_2) + \operatorname{rank} K_2. \tag{4.4.21}$$

Now write $T = T - S + S_1 + S_2$ with

$$S_1 := S\mathbb{1}_{[0,\varepsilon]}(S), \quad S_2 := S\mathbb{1}_{(\varepsilon,\infty)}(S),$$

so that $\sigma(S_1) \subset [0, \varepsilon]$ and $\operatorname{rank} S_2 = n_+(\varepsilon; S)$. Then (4.4.21) with $M = T - S$, $K_j = S_j$, $j = 1, 2$, implies

$$N_{(s,t)}(T - S) \leq N_{(s,t+\varepsilon)}(T) + n_+(\varepsilon; S).$$

The other bounds in (4.4.19) and (4.4.20) can be proved in quite a similar manner. □

Proposition 4.4.2 *Let* $V : \mathbb{R}^2 \to \mathbb{R}$ *be a Lebesgue-measurable function such that the operator* VH_0^{-1} *is compact. Then for* $\eta \in (0,1)$ *and* $\varepsilon > 0$ *small enough we have*

$$\begin{aligned} n_-(\lambda(1+\eta); p_q V p_q) - n_+(\lambda^2\eta^2\varepsilon^{-2}; p_q V^2 p_q) + \mathscr{O}_{\varepsilon,q}(1) \leq \\ \mathscr{N}_q^-(\lambda; V) \leq \\ n_-(\lambda(1-\eta); p_q V p_q) + n_+(\lambda^2\eta^2\varepsilon^{-2}; p_q V^2 p_q) + \mathscr{O}_{\varepsilon,q}(1), \end{aligned} \tag{4.4.22}$$

$$\begin{aligned} n_+(\lambda(1+\eta); p_q V p_q) - n_+(\lambda^2\eta^2\varepsilon^{-2}; p_q V^2 p_q) + \mathscr{O}_{\varepsilon,q}(1) \leq \\ \mathscr{N}_q^+(\lambda; V) \leq \\ n_+(\lambda(1-\eta); p_q V p_q) + n_+(\lambda^2\eta^2\varepsilon^{-2}; p_q V^2 p_q) + \mathscr{O}_{\varepsilon,q}(1), \end{aligned} \tag{4.4.23}$$

as $\lambda \downarrow 0$.

Proof. We follow the general lines of [153, Section 5]. Set $S := p_q V(I - p_q)$. Then, $S \in \mathfrak{S}_\infty(L^2(\mathbb{R}^2))$, and

$$H_V = p_q H_V p_q + (I - p_q) H_V (I - p_q) + 2\operatorname{Re} p_q S(I - p_q).$$

Set $R := \operatorname{Re} S$, $M := \operatorname{Im} S$ so that $S = R + iM$. Moreover, $R = R_+ - R_-$ and $M = M_+ - M_-$ where, as usual, $R_\pm$ and $M_\pm$ are the positive and the negative parts of the self-adjoint operators R and M, respectively. Set

$$T := |R| + |M| = R_+ + R_- + M_+ + M_-.$$

Pick $\varepsilon > 0$. Completing the squares, we obtain the representations

$$H_V = \widetilde{H}_V + S_0 - S_1 = \widetilde{H}_V - S_0 + S_2, \tag{4.4.24}$$

where

$$\widetilde{H}_V := p_q H_V p_q + (I - p_q) H_V (I - p_q), \quad S_0 := \varepsilon p_q T p_q + \varepsilon^{-1}(I - p_q) T (I - p_q),$$

and

$$S_1 := |\varepsilon^{1/2} R_+^{1/2} p_q - \varepsilon^{-1/2} R_+^{1/2}(I - p_q)|^2 + |\varepsilon^{1/2} R_-^{1/2} p_q + \varepsilon^{-1/2} R_-^{1/2}(I - p_q)|^2 +$$
$$|i\varepsilon^{1/2} M_+^{1/2} p_q + \varepsilon^{-1/2} M_+^{1/2}(I - p_q)|^2 + |i\varepsilon^{1/2} M_-^{1/2} p_q - \varepsilon^{-1/2} M_-^{1/2}(I - p_q)|^2,$$

$$S_2 := |\varepsilon^{1/2} R_+^{1/2} p_q + \varepsilon^{-1/2} R_+^{1/2}(I - p_q)|^2 + |\varepsilon^{1/2} R_-^{1/2} p_q - \varepsilon^{-1/2} R_-^{1/2}(I - p_q)|^2 +$$
$$|i\varepsilon^{1/2} M_+^{1/2} p_q - \varepsilon^{-1/2} M_+^{1/2}(I - p_q)|^2 + |i\varepsilon^{1/2} M_-^{1/2} p_q + \varepsilon^{-1/2} M_-^{1/2}(I - p_q)|^2.$$

Evidently, the operators $S_j \geq 0$, $j = 1, 2$, are compact. Making use of Lemma 4.4.2 and representations (4.4.24), we find that the estimates

$$N_{(\Lambda_q+\lambda, \widetilde{\Lambda}_q-\varepsilon)}(\widetilde{H}_V - S_0) - n_+(\varepsilon; S_2) \leq$$
$$\mathscr{N}_q^+(\lambda; V) \leq$$
$$N_{(\Lambda_q+\lambda, \widetilde{\Lambda}_q+\varepsilon)}(\widetilde{H}_V + S_0) + n_+(\varepsilon; S_1), \tag{4.4.25}$$
$$N_{(\widetilde{\Lambda}_{q-1}+\varepsilon, \Lambda_q-\lambda)}(\widetilde{H}_V + S_0) - n_+(\varepsilon; S_1) \leq$$
$$\mathscr{N}_q^-(\lambda; V) \leq$$
$$N_{(\widetilde{\Lambda}_{q-1}-\varepsilon, \Lambda_q-\lambda)}(\widetilde{H}_V - S_0) + n_+(\varepsilon; S_2), \tag{4.4.26}$$

hold true as $\lambda \downarrow 0$ for any $q \in \mathbb{Z}_+$ and $\varepsilon > 0$ small enough. Further, for any interval $\mathscr{I} \subset \mathbb{R}$ we have

$$N_{\mathscr{I}}(\widetilde{H}_V \pm S_0) = N_{\mathscr{I}}(p_q(\Lambda_q I + V \pm \varepsilon T)p_q)$$
$$+ N_{\mathscr{I}}((I - p_q)(H_V \pm \varepsilon^{-1} T)(I - p_q)), \tag{4.4.27}$$

where$p_q(\Lambda_q I + V \pm \varepsilon T)p_q$ (resp., $(I - p_q)(H_V \pm \varepsilon^{-1} T)(I - p_q)$) with domain $p_q L^2(\mathbb{R}^2)$ (resp, $(I-p_q)\mathfrak{D}(H_0)$) is considered as a self-adjoint operator in the Hilbert space $p_q L^2(\mathbb{R}^2)$ (resp., $(I-p_q)L^2(\mathbb{R}^2)$). Moreover,

$$N_{(\Lambda_q+\lambda, \widetilde{\Lambda}_q\pm\varepsilon)}(p_q(\Lambda_q I + V \pm \varepsilon T)p_q) = n_+(\lambda; p_q(V \pm \varepsilon T)p_q)$$
$$- N_{[\widetilde{\Lambda}_q-\Lambda_q\pm\varepsilon, \infty)}(p_q(V \pm \varepsilon T)p_q)$$
$$= n_+(\lambda; p_q(V \pm \varepsilon T)p_q) + \mathscr{O}_{\varepsilon,q}(1). \tag{4.4.28}$$

Similarly,

$$N_{(\widetilde{\Lambda}_{q-1}\pm\varepsilon, \Lambda_q-\lambda)}(p_q(\Lambda_q I + V \pm \varepsilon T)p_q) = n_-(\lambda; p_q(V \pm \varepsilon T)p_q) + \mathscr{O}_{\varepsilon,q}(1). \tag{4.4.29}$$

Putting together (4.4.25) – (4.4.29), and taking into account that

$$\Lambda_q \notin \sigma_{\rm ess}\left(H_{0|(I-p_q)\mathfrak{D}(H_0)}\right),$$

and the operators $S_j, j = 1, 2,$ are compact, we obtain the estimates

$$n_+(\lambda; p_q(V - \varepsilon T)p_q) + \mathscr{O}_{q,\varepsilon}(1) \le \mathscr{N}_q^+(\lambda; V) \le n_+(\lambda; p_q(V + \varepsilon T)p_q) + \mathscr{O}_{q,\varepsilon}(1), \quad (4.4.30)$$

$$n_-(\lambda; p_q(V + \varepsilon T)p_q) + \mathscr{O}_{q,\varepsilon}(1) \le \mathscr{N}_q^-(\lambda; V) \le n_-(\lambda; p_q(V - \varepsilon T)p_q) + \mathscr{O}_{q,\varepsilon}(1), \quad (4.4.31)$$

valid as $\lambda \downarrow 0$ for $\varepsilon > 0$. Next, the Weyl inequalities (4.2.4) imply

$$n_\pm(\lambda; p_q(V \mp \varepsilon T)p_q) \ge n_\pm(\lambda(1+\eta); p_q V p_q) - n_+(\lambda\eta; \varepsilon T), \quad (4.4.32)$$

$$n_\pm(\lambda; p_q(S \pm \varepsilon T)p_q) \le n_\pm(\lambda(1-\eta); p_q V p_q) + n_+(\lambda\eta; \varepsilon T), \quad (4.4.33)$$

for any $\lambda > 0$, $\varepsilon > 0$, and $\eta \in (0,1)$. Finally, bearing in mind the mini-max principle, we easily find that the estimates

$$n_+(s; p_q T p_q) = n_+(s^2; p_q T p_q T p_q) \le n_+(s^2; p_q T^2 p_q) \le$$

$$n_+(s^2; 2p_q(R^2 + M^2)p_q) = n_+(s^2; p_q V(I - p_q)V p_q) \le n_+(s^2; p_q V^2 p_q) \quad (4.4.34)$$

are valid for any $s > 0$.
Combining (4.4.30) - (4.4.31) with (4.4.32) - (4.4.33) and (4.4.34), we arrive at (4.4.22) - (4.4.23). □

Let us now divide (4.4.22) by $b\mathfrak{A}^-(\lambda; V)$, and (4.4.23) by $b\mathfrak{A}^+(\lambda; V)$. Applying Theorem 4.2.3 and bearing in mind (4.2.58), we find that there exists a constant $C \in (0, \infty)$ such that

$$(1+\eta)^{-\gamma_2} - C(\varepsilon/\eta)^{2/\gamma} \le \liminf_{\lambda \downarrow 0} \frac{\mathscr{N}_q^\pm(\lambda)}{b\mathfrak{A}^\pm(\lambda; V)}$$

$$\le \limsup_{\lambda \downarrow 0} \frac{\mathscr{N}_q^\pm(\lambda)}{b\mathfrak{A}^\pm(\lambda; V)} \le (1-\eta)^{-\gamma_1} + C(\varepsilon/\eta)^{2/\gamma}$$

for every $\eta \in (0,1)$ and $\varepsilon > 0$ small enough. Choosing $\varepsilon = \eta^2$, and sending $\eta \downarrow 0$, we obtain (4.4.18). The proof of Theorem 4.4.3 is now complete.

Non-local perturbations

From the point of view of possible application in atomic and nuclear physics (see e.g. [54, 76, 181, 193, 48]), it is interesting to investigate also the spectral properties of the 2D Landau Hamiltonian $H_0 = H_S(A, 0)$, perturbed by a *non-local potential*, i.e. by a bounded self-adjoint Weyl ΨDO $\mathrm{Op}^w(\mathscr{V})$ such that $\mathrm{Op}^w(\mathscr{V})H_0^{-1}$ in compact in $L^2(\mathbb{R}^2)$. In [45] the eigenvalue asymptotics near the Landau levels of $H_\mathscr{V} := H_0 + \mathrm{Op}^w(\mathscr{V})$ was studied for symbols $\mathscr{V}$ of compact support, and of exponential or power-like decay at infinity. In the case of such non-local perturbations the effective Hamiltonian which governs the asymptotics of discrete eigenvalues of $H_\mathscr{V}$ which accumulate at a given Landau level Λ_q, $q \in \mathbb{Z}_+$, is $\mathrm{Op}^w(\mathscr{V}_{q,b}) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ where

$$(\mathscr{V}_{q,b})(y, \eta) = \int_{\mathbb{R}^2} (\mathscr{V} \circ \kappa_b)(x, y, \xi, \eta) \Psi_q(x, \xi) dx d\xi, \quad (y, \eta) \in \mathbb{R}^2,$$

κ_b being the symplectomorphism defined in (2.7.16), and Ψ_q being the Wigner function defined in (3.4.31).
Moreover, [45, Proposition 5.1] contains an explicit construction of a symbol $\mathscr{V} \in \mathscr{S}(\mathbb{R}^4)$ such that $\mathrm{Op}^w(\mathscr{V}) \geq 0$, and for any given $q \in Z_+$ and $m_q \in \mathbb{Z}_+ \cup \{\infty\}$, we have

$$\#\left(\sigma(H_{-\mathscr{V}}) \cap (\Lambda_{q-1}, \Lambda_q)\right) = m_q. \tag{4.4.35}$$

To this end, set

$$\mathscr{Z} := \left\{q \in \mathbb{Z}_+ \,|\, m_q \neq 0\right\}.$$

If $\mathscr{Z} = \emptyset$, it suffices to take $\mathscr{V} = 0$. Assume $\mathscr{Z} \neq \emptyset$. Let $\left\{c_{1,q}\right\}_{q\in\mathscr{Z}}$ be a decreasing set of numbers $c_{1,q} \in (0, 2b)$; if $0 \in \mathscr{Z}$, we can omit the condition $c_{1,0} < 2b$. If $\#\mathscr{Z} = \infty$, we assume that $\lim_{q\to\infty} q^m c_{1,q} = 0$ for any $m \in \mathbb{N}$. Fix $q \in \mathscr{Z}$. Let $\left\{c_{2,k}\right\}_{k=0}^{m_q-1}$ be a decreasing set of numbers $c_{2,k} \in (0,1)$. If $m_q = \infty$, we assume that $\lim_{k\to\infty} k^m c_{2,k} = 0$ for any $m \in \mathbb{N}$. Now put

$$C_{k,q} := c_{1,q} c_{2,k}, \quad k = 0, \dots, m_q - 1, \quad q \in \mathscr{Z},$$

$$\mathscr{V} := (2\pi)^2 \left(\sum_{q\in\mathscr{Z}} \sum_{k=0}^{m_q-1} C_{k,q} \Psi_q \otimes \Psi_k \right) \circ \kappa_b^{-1}.$$

Then, $\mathscr{V} \in \mathscr{S}(\mathbb{R}^4)$ (see [67, Theorem 2.5 (a)]), and, evidently, $\mathscr{V} \circ \kappa_b$ is radial. Moreover, by relation (2.7.40), Proposition 2.3.8, Lemma 3.4.3, and the orthogonality of the Laguerre polynomials (see (2.7.31)), we find that the integral kernel of the operator $\mathrm{Op}^w(\mathscr{V})$ can be written as

$$\sum_{(k,q)\in\mathbb{Z}_+^2} C_{k,q} \varphi_{k,q}(\mathbf{x}) \overline{\varphi_{k,q}(\mathbf{y})}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2,$$

where $\varphi_{k,q}$ are the functions defined in (2.7.39). Thus, $\varphi_{k,q}$ are eigenfunctions of the operator $\mathrm{Op}^w(\mathscr{V})$ with eigenvalues $C_{k,q}$, $k \in \mathbb{Z}_+$, $q \in \mathscr{Z}$; in particular, $\mathrm{Op}^w(\mathscr{V}) \geq 0$. Therefore, the operator $H_{-\mathscr{V}}$ is diagonalized in the orthonormal basis $\left\{\varphi_{k,q}\right\}_{k,q\in\mathbb{Z}_+^2}$, and

$$\sigma(H_{-\mathscr{V}}) \cap (\Lambda_{q-1}, \Lambda_q) = \begin{cases} \emptyset & \text{if} \quad q \notin \mathscr{Z}, \\ \cup_{k=0}^{m_q-1} \left\{\Lambda_q - C_{k,q}\right\} & \text{if} \quad q \in \mathscr{Z}. \end{cases} \tag{4.4.36}$$

By construction, all the eigenvalues $\Lambda_q - C_{k,q}$, $k = 0, \dots, m_q - 1$, lying in I_q^- with $q \in \mathscr{Z}$, are simple. Therefore, (4.4.35) holds true. In particular, we can have gaps with finitely many eigenvalues of $H_{-\mathscr{V}}$ or with no eigenvalues at all, which is in sharp contrast to the operators $H(A, V)$ with *local* V, considered in Theorems 4.4.1 - 4.4.3, which have infinitely many eigenvalues in every gap $(\Lambda_{q-1}, \Lambda_q)$.
Of course, it is possible to construct analogous positive compact perturbations of H_0 whose eigenvalues accumulate to Λ_q from above, or self-adjoint compact perturbation with non-trivial positive and negative parts whose eigenvalues accumulate to Λ_q both from above and from below.
It is easy to check that if for some $q \in \mathbb{Z}_+$ we have $m_q < \infty$, then the Landau level

remains an eigenvalue of infinite multiplicity of $H_{-\mathscr{V}}$ with $\mathrm{Op}^{w}(\mathscr{V}) \geq 0$.
At the same time, the accumulation of the eigenvalues of $H_S(A,V)$ at each Landau level Λ_q, established in Theorems 4.4.1 - 4.4.3 does not give an answer to the question whether Λ_q itself remains an eigenvalue of the perturbed operator $H_S(A,V)$, once it is an eigenvalue of infinite multiplicity for the unperturbed one $H_S(A,0)$. To fill in this gap, it was shown in [111, Theorem 1], that if $V \in L^{\infty}(\mathbb{R}^2;\mathbb{R})$ satisfies (4.4.3) which in particular implies $V \geq 0$, and

$$\|V\|_{L^{\infty}(\mathbb{R}^2)} < 2b, \tag{4.4.37}$$

then

$$\mathrm{Ker}\,(H_S(A,\pm V) - \Lambda_q I) = \{0\}, \quad q \in \mathbb{Z}_+; \tag{4.4.38}$$

if q = 0, and the perturbation is non-negative, then in fact condition (4.4.37) is not necessary for the validity of (4.4.38). Thus, the perturbation $\pm V$ completely destroys the eigenspaces of $H_S(A,0)$.
On the other hand, in [111, Theorem 2] it was proved that for every $q \in \mathbb{Z}_+$ there exists a compactly supported $V \in L^{\infty}(\mathbb{R}^2;\mathbb{R})$ with $\|V\|_{L^{\infty}(\mathbb{R}^2)} < b$, whose sign changes infinitely many times, such that

$$\dim \mathrm{Ker}\,(H_S(A,V) - \Lambda_q I) = \infty.$$

Pauli Hamiltonians

Let us consider now the 2D Pauli Hamiltonian $H_P(A,0)$ with admissible magnetic field $\mathrm{curl}\,A = b = b_0 + \widetilde{b}$ of non-zero mean value b_0. Then Proposition 2.8.3 tells us that $0 = \inf \sigma(H_P(A,0))$ is an eigenvalue of $H_P(A,0)$ of infinite multiplicity, and there exists a gap $(0,E_0)$ in $\sigma(H_P(A,0))$, adjoining the origin. Let $\mathbf{V} : \mathbb{R}^2 \to \mathscr{M}_2$ be a Hermitian Lebesgue-measurable function such that the operator $|\mathbf{V}|^{1/2}(H_P(A;0)+I)^{-1/2}$ is compact in $L^2(\mathbb{R}^2;\mathbb{C}^2)$. Then we have

$$\sigma_{\mathrm{ess}}(H_P(A,\mathbf{V})) = \sigma_{\mathrm{ess}}(H_P(A,0)).$$

In particular, $0 \in \sigma_{\mathrm{ess}}(H_P(A,\mathbf{V}))$. Pick $E \in (0,E_0)$, and set

$$\mathscr{N}_P^{+}(\lambda;\mathbf{V}) := N_{(\lambda,E)}(H_P(A,\mathbf{V})), \quad \lambda \in (0,E),$$

$$\mathscr{N}_P^{-}(\lambda;\mathbf{V}) := N_{(-\infty,-\lambda)}(H_P(A,\mathbf{V})), \quad \lambda > 0.$$

Further, let $\left\{\lambda_{k,P}^{+}(\mathbf{V})\right\}_{k\geq 0}$ (resp., $\left\{\lambda_{k,P}^{-}(\mathbf{V})\right\}_{k\geq 0}$) be the non-increasing (resp., non-decreasing) set of the eigenvalues of $H_P(A,\mathbf{V})$ lying on the interval $(0,E)$ (resp., on $(-\infty,0)$), and counted with the multiplicities. *A priori*, any of these sets can be empty. Set

$$m_-(\mathbf{V}) := \#(\sigma(H_P(A,\mathbf{V})) \cap (-\infty,0)), \quad m_+(\mathbf{V}) := \#(\sigma(H_P(A,\mathbf{V})) \cap (0,E_0)).$$

In our next two theorems we assume that the magnetic field b is admissible with $b_0 > 0$, $\mathbf{V} \geq 0$, and discuss the asymptotics of $\lambda_{k,P}(\pm\mathbf{V})$ as $k \to \infty$, or equivalently of $\mathscr{N}_P(\lambda;\pm\mathbf{V})$ as $\lambda \downarrow 0$. Under our assumptions about the sign of b_0 and $\mathbf{V}$, it turns

out that the effective Hamiltonian which governs this asymptotics, is the operator $a(b)^*a(b) \pm V_{11}$.
Our first theorem concerns the case where V_{11} decays rapidly at infinity, i.e.V_{11} has a compact support, or decays exponentially.

Theorem 4.4.4 *Let* $b = b_0 + \tilde{b}$ *with* $b_0 > 0$ *be an admissible magnetic field. Assume that* $\mathbf{V} \geq 0$*, and the operators* $|V_{j2}|^{1/2}(-\Delta + I)^{-1/2}$, $j = 1, 2$*, are compact in* $L^2(\mathbb{R}^2)$*. Suppose that* $V = V_{11}$ *satisfies the hypotheses of Theorem 4.4.1 (resp., Theorem 4.4.2). Then*

$$m_-(-\mathbf{V}) = \infty, \quad m_+(\mathbf{V}) = \infty,$$

and (4.4.2) *(resp.,* (4.4.5)*) remains valid if we replace* $\lambda_{k,q}(\pm V) - \Lambda_q$ *by* $\lambda_{k,P}(\pm\mathbf{V})$ *at the left-hand side, and* b *by* b_0 *at the right-hand side.*

Our second theorem handles the case where V_{11} has a power-like decay at infinity.

Theorem 4.4.5 *Let* $b = b_0 + \tilde{b}$ *with* $b_0 > 0$ *and let* $\tilde{b} \in \mathrm{WAP}(\mathbb{R}^2; \mathbb{R})$ *satisfy* (2.8.28). *Assume that* $V_{11} \in C^1(\mathbb{R}^2)$*, the estimates*

$$0 < V_{11}(\mathbf{x}) \leq C\langle \mathbf{x} \rangle^{-\gamma}, \quad |\nabla V_{11}(\mathbf{x})| \leq C\langle \mathbf{x} \rangle^{-\gamma-1}, \quad \mathbf{x} \in \mathbb{R}^2,$$

hold true with constants $\gamma > 0$ *and* $C \geq 0$*, and there exists* $0 < \phi \in C(\mathbb{S}^1)$ *such that*

$$\lim_{r \to \infty} r^{\gamma} V_{11}(r\omega) = \phi(\omega), \quad \omega \in \mathbb{S}^1.$$

Then we have

$$\lim_{\lambda \downarrow 0} \lambda^{2/\gamma} \mathscr{N}_P(\lambda; \pm\mathbf{V}) = \frac{b_0}{4\pi} \int_{\mathbb{S}^1} \phi(\omega)^{2/\gamma} d\omega.$$

Using relation (2.6.14) which establishes the unitary equivalence between $H_P(A, \mathbf{V})$ and $H_P(-A, \tilde{\mathbf{V}})$, we can easily obtain the analogues of Theorems 4.4.4 and 4.4.5 in the case where the mean value b_0 of the magnetic field b is negative.
The proofs of Theorems 4.4.4 and 4.4.5 are based on the following analogues of Proposition 4.4.1 and Corollary 4.4.1.

Proposition 4.4.3 [154, Proposition 3.1] *Assume that* b *and* $\mathbf{V}$ *satisfy the hypotheses of Theorem 4.4.4 or 4.4.5. Then for any* $\varepsilon \in (0, 1)$ *we have*

$$n_+(\lambda; \mathfrak{p}_{ann} V_{11} \mathfrak{p}_{ann}) \leq \mathscr{N}_P^-(\lambda; -\mathbf{V}) \leq n_+((1-\varepsilon)\lambda; \mathfrak{p}_{ann} V_{11} \mathfrak{p}_{ann}) + \mathscr{O}_\varepsilon(1), \quad (4.4.39)$$

$$n_+((1+\varepsilon)\lambda; \mathfrak{p}_{ann} V_{11} \mathfrak{p}_{ann}) + \mathscr{O}_\varepsilon(1) \leq \mathscr{N}_P^+(\lambda; \mathbf{V}) \leq n_+(\lambda; \mathfrak{p}_{ann} V_{11} \mathfrak{p}_{ann}), \quad (4.4.40)$$

as $\lambda \downarrow 0$*.*

Now Theorem 4.4.5 follows directly from Proposition 4.4.3 and Theorem 4.2.4.
Arguing as in the proof of Corollary 4.4.1, we find that Proposition 4.4.3 implies

Corollary 4.4.2 *Assume the hypotheses of Proposition 4.4.3. Then we have* $m_\pm(\pm\mathbf{V}) = \infty$*, and for each* $\varepsilon \in (0, 1)$ *there exists* $k_0 \in \mathbb{Z}_+$ *such that we have*

$$\frac{1}{1+\varepsilon} \nu^+_{k,ann}(V_{11}) \leq \lambda^+_{k,P}(\mathbf{V}) \leq \nu^+_{k-k_0,ann}(V_{11}),$$

$$\nu^+_{k+k_0,ann}(V_{11}) \leq -\lambda^+_{k,P}(-\mathbf{V}) \leq \frac{1}{1-\varepsilon} \nu^+_{k,ann}(V_{11}),$$

for sufficiently large $k \in \mathbb{Z}_+$*.*

Now Theorem 4.4.4 follows directly from Corollaries 4.4.2, 4.2.3, and 4.2.6.

4.4.2 Magnetic and geometric perturbations

Magnetic and metric perturbations

In this subsection we survey briefly results on the eigenvalue asymptotics for certain magnetic and metric perturbations for the 2D Landau operator and related magnetic quantum Hamiltonians. Let $G = \left\{g_{jk}\right\}_{j,k=1,2}$ be a measurable symmetric metric tensor such that

$$c_1|\boldsymbol{\xi}|^2 \leq \sum_{j,k=1,2} g_{jk}(\mathbf{x})\xi_k\xi_j \leq c_2|\boldsymbol{\xi}|^2, \quad \mathbf{x}, \boldsymbol{\xi} \in \mathbb{R}^2,$$

with some constants $0 < c_1 \leq c_2 < \infty$. Assume that $A \in L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$, $V_+ \in L^1_{loc}(\mathbb{R}^2;\mathbb{R}_+)$, $V_- \in \mathscr{L}_2(\mathbb{R}^2;\mathbb{R}_+)$ (see the notation before Lemma 2.5.1). Denote by $H_{G,A,V}$ the self-adjoint operator generated in $L^2(\mathbb{R}^2)$ by the closed quadratic form

$$\int_{\mathbb{R}^2} \left(\langle G\Pi(A)u, \Pi(A)u \rangle_{\mathbb{C}^2} + V|u|^2 \right) d\mathbf{x}, \quad u \in \mathfrak{D}(H_S(A,0)^{1/2}).$$

Of course, if $G = I$, then $H_{I;A,V} = H_S(A,V)$. If G and A are smooth then the *Bochner Laplacian*

$$g^{-1/2} \sum_{j,k=1,2} \Pi_j(A) \left(g^{1/2} g_{jk} \Pi_k(A) \right) \tag{4.4.41}$$

with $g := (\det G)^{-1}$, self-adjoint in $L^2(\mathbb{R}^2; \sqrt{g}d\mathbf{x})$ is unitarily equivalent by the mapping $u \mapsto g^{1/4}u$ to the operator $H_{G,A,Q}$ with

$$Q := \frac{1}{16} \sum_{j,k=1,2} \left(g_{jk} \frac{\partial \ln g}{\partial x_k} \frac{\partial \ln g}{\partial x_j} + 4 \frac{\partial}{\partial x_j} \left(g_{jk} \frac{\partial \ln g}{\partial x_k} \right) \right). \tag{4.4.42}$$

Set $M := \left\{m_{jk}\right\}_{j,k=1,2} = G - I$ and assume that

$$M \in C^\infty(\mathbb{R}^2; \mathscr{M}_2), \quad \lim_{|\mathbf{x}| \to \infty} M(\mathbf{x}) = 0.$$

Let, as usual, $b = \operatorname{curl} A$. Suppose that there exists $b_0 > 0$ such that

$$\widetilde{b} := b - b_0 \in C^\infty(\mathbb{R}^2; \mathbb{R}), \quad \lim_{|\mathbf{x}| \to \infty} \widetilde{b}(\mathbf{x}) = 0.$$

Set $A_0(\mathbf{x}) := \frac{b_0}{2}(x_2, -x_1)$ so that $\operatorname{curl} A_0 = b_0$, and denote again by H_0 the unperturbed Landau Hamiltonian $H_{I,A_0,0} = H_S(A_0, 0)$. By [104] and [124, Appendix] we have

$$\sigma_{ess}(H_{G,A,V}) = \sigma_{ess}(H_0) = \sigma(H_0) = \bigcup_{q \in \mathbb{Z}_+} \left\{\Lambda_q(b_0)\right\}$$

where, as usual $\Lambda_q(b_0) = b_0(2q+1)$, $q \in \mathbb{Z}_+$, are the Landau levels. Then, by analogy with $\left\{\lambda^\pm_{k,q}(V)\right\}_{k \geq 0}$ and $\mathscr{N}^\pm_q(\lambda; V)$, we can introduce the eigenvalue sets $\left\{\lambda^\pm_{k,q}(M, \widetilde{b}, V)\right\}_{k \geq 0}$

and the eigenvalue counting functions $\mathscr{N}_q^{\pm}(\lambda; M, \widetilde{b}, V)$. Our first theorem concerns the case of power-like decay of $M, \widetilde{b}$ and V.

Theorem 4.4.6 *Assume that*

$$m_{jk}, j, k = 1, 2,\ \widetilde{b},\ V \in \Gamma_1^{-\gamma}(\mathbb{R}^2; \mathbb{R}), \quad \gamma \in (0, \infty).$$

Fix $q \in \mathbb{Z}_+$, *set*

$$\widetilde{V}_q := V + \Lambda_q(b_0)(bg^{-1/2} - b_0),$$

and suppose that

$$\pm \widetilde{V}_q(\mathbf{x}) \geq c^{-1} \langle \mathbf{x} \rangle^{-\gamma}, \quad |\nabla \widetilde{V}_q(\mathbf{x})| \geq c^{-1} \langle \mathbf{x} \rangle^{-\gamma-1}$$

for $|\mathbf{x}| \geq c > 0$. *Then we have*

$$\mathscr{N}_q^{\pm}(\lambda; M, \widetilde{b}, V) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathbb{1}_{(0,\infty)}(\pm \widetilde{V}_q(\mathbf{x}) - \lambda) \sqrt{g}\, d\mathbf{x} + \mathscr{O}(1), \quad \lambda \downarrow 0.$$

Remarks: (i) Under the hypotheses of Theorem 4.4.6 we have

$$(bg^{-1/2})(\mathbf{x}) - b_0 = \widetilde{b}(\mathbf{x}) + \frac{1}{2} \mathrm{Tr}\, M(\mathbf{x}) + \mathscr{O}(|\mathbf{x}|^{-\gamma-1}), \quad |\mathbf{x}| \to \infty.$$

(ii) Theorem 4.4.6 was first announced as [100, Theorem 7], and later its more general version was formulated as [101, Theorem 11.3.17] and was claimed to be an immediate corollary of [101, Theorem 6.4.19] which is based on general results on propagation of singularities for hyperbolic systems. Moreover, Section 11.4 of [101] contains theorems on the local eigenvalue asymptotics for perturbations of the 2D Dirac operator with constant magnetic field. The results of [101] concerning $H_{G,A,V}$ and similar operators, were further extended in the volumes [102, 103] which form a part of an impressive five-volume monograph on microlocal analysis and precise eigenvalue asymptotics. However, as already mentioned in Section 4.1, in [103, Remark 23.4.9] it was noted that the methods developed and used in that monograph are not appropriate for handling perturbations which decay more rapidly than $e^{-c|\mathbf{x}|}$ with $c > 0$. On the other hand, [101, 102, 103] treat perturbations which decay like negative powers of $\ln |\mathbf{x}|$ or of the iterated logarithms. In this case the derivatives of the perturbations decay much faster than the perturbation itself, which considerably simplifies the analysis.
(iii) Up to details of the hypotheses, Theorem 4.4.6 contains Theorem 4.4.3 as a special case with $M = 0$ and $\widetilde{b} = 0$. We included the latter result in order to show that the machinery of Berezin-Toeplitz operators can be useful in the investigation of the asymptotics of $\mathscr{N}_q^{\pm}(\lambda; V)$ with non-sign-definite V of power-like decay, and provide relatively simple and fairly accessible approach to these problems. An alternative approach using Berezin-Toeplitz operators to the asymptotics of $\mathscr{N}_\lambda(0, \widetilde{b}, V)$ with $\widetilde{b}$ and V of power-lie decay can be found in [172].

Let us now pass to the operator $H_{G,A,V}$ with rapidly decaying perturbations. The asymptotics as $\lambda \downarrow \infty$ of $\mathscr{N}_q^{\pm}(\lambda; 0, \widetilde{b}, V)$ with compactly supported $\widetilde{b}$ and V is studied in [171]. In particular, in [171, Theorem 6.2] it is shown that if $\widetilde{b}, V \in C_0^2(\mathbb{R}^2)$, then

$$\limsup_{\lambda \downarrow 0} \frac{\mathcal{N}_q^{\pm}(\lambda, \widetilde{b}, V)}{\varphi_\infty(\lambda)} \leq \frac{1}{2}$$

where φ_∞ is the function defined in (4.2.14). Moreover, an effective compactly supported potential W depending on $\widetilde{b}, V$ and $\lambda > 0$ is introduced, and it is proved in [171, Theorem 6.3] that if W is non-negative for sufficiently small λ, then

$$\liminf_{\lambda \downarrow 0} \frac{\mathcal{N}_q^{\pm}(\lambda, \widetilde{b}, V)}{\varphi_\infty(\lambda)} \geq \frac{1}{2}.$$

Further, the asymptotics as $\lambda \downarrow 0$ of $\mathcal{N}_q^{\pm}(\lambda; \pm M, 0, 0)$ with $M \geq 0$ was examined in [124] under the assumption that M is compactly supported or has an exponential or power-like decay at infinity. The results obtained are quite close in spirit to Theorems 4.4.1 -4.4.3, so that we omit the details and just point that the effective Hamiltonian this time is $p_q \mathbb{A}^* U \mathbb{A} p_q$ where

$$\mathbb{A}u = \begin{pmatrix} a(b_0)^* u \\ a(b_0) u \end{pmatrix}, \quad u \in \mathfrak{D}(H_0^{1/2}),$$

and

$$U = \frac{1}{2} \begin{pmatrix} m_{11} + m_{22} & m_{11} - m_{22} - 2im_{12} \\ m_{11} - m_{22} + 2im_{21} & m_{11} + m_{22} \end{pmatrix}.$$

The results of [124] can be easily extended to operators $H_{I \pm M, 0, \pm V}$ provided that $M \geq 0$ and $V \geq 0$. Unfortunately, $G = I \pm M$ with $M \geq 0$ does not necessarily imply that Q defined in (4.4.42) as a function of G is sign-definite and has the same sign as the perturbation $\pm M$. Thus the extension of the results of [124] to operators $H_{I \pm M, 0, \pm V}$ is not sufficient to study the eigenvalue asymptotics for the Bochner Laplacian defined in (4.4.41) with $G = I + M$ with general rapidly decaying M, which remains a challenging open problem.

Perturbation by an obstacle

Finally, we would like to mention yet another geometric perturbation of the 2D Landau Hamiltonian $H_0 = H_S(A, 0)$ with constant magnetic field $b > 0$. Let $\Gamma \subset \mathbb{R}^2$ be a closed simple C^∞-curve, and Ω be the exterior of Γ, i.e. the unbounded component of $\mathbb{R}^2 \setminus \Gamma$. Denote by $H_\Omega^+(A)$ (resp., by $H_\Omega^-(A)$) the self-adjoint operator generated in $L^2(\Omega)$ by the closure of the quadratic form

$$\int_\Omega |i\nabla u + Au|^2 \, d\mathbf{x}$$

with domain $C_0^\infty(\Omega)$ (resp., $C_0^\infty(\overline{\Omega})$). Then, again,

$$\sigma_{ess}(H_\Omega^{\pm}(A)) = \sigma_{ess}(H_0) = \sigma(H_0) = \bigcup_{q \in \mathbb{Z}_+} \{\Lambda_q\}.$$

Moreover, the discrete eigenvalues of $H^+_\Omega(A)$ (resp., of $H^-_\Omega(A)$) can accumulate to any Landau level $\Lambda_q = b(2q+1)$, $q \in \mathbb{Z}_+$, only from above (resp., from below). By analogy with $\lambda^{\pm}_{k,q}(\pm V)$ with $V \geq 0$, for $q \in \mathbb{Z}_+$, let $\left\{\lambda^+_{k,q}(\Omega)\right\}_{k\geq 0}$ (resp., $\left\{\lambda^-_{k,q}(\Omega)\right\}_{k\geq 0}$) be the non-increasing (resp., non-decreasing) set of the eigenvalues of H^+_Ω (resp., H^-_Ω) lying on the interval $(\Lambda_q, \Lambda_{q+1})$ (resp., on $(\Lambda_{q-1}, \Lambda_q)$), and counted with the multiplicities, and set

$$\mathcal{N}^{\pm}_q(\lambda;\Omega) = \#\left\{k \in \mathbb{Z}_+ \mid \pm(\lambda^{\pm}_{k,q}(\Omega) - \Lambda_q) > \lambda\right\}, \quad \lambda > 0.$$

Theorem 4.4.7 *Let $\Gamma \subset \mathbb{R}^2$ be a closed simple C^∞-curve, and Ω be its exterior. Then for any $q \in \mathbb{Z}_+$ we have $\#(\sigma(H^{\pm}_\Omega) \cap (\Lambda_{q-1}, \Lambda_q)) = \infty$, and*

$$\ln\left(\pm\left(\lambda^{\pm}_{k,q}(\Omega) - \Lambda_q\right)\right) = -k\ln k + \left(1 + \ln\left(\frac{b\,\mathrm{Cap}(\Gamma)^2}{2}\right)\right)k + o(k) \qquad (4.4.43)$$

as $k \to \infty$.

Remarks: Under somewhat less restrictive assumptions relation (4.4.43) was obtained for the Dirichlet (resp, Neumann) case in [151, Theorem 1.3] (resp., in [141, Theorem 3.2 (a)] and [83, Theorem 1.2 (A)]). Moreover, in [141] and [83], a more general setting of a magnetic field of full rank in $\mathbb{R}^{2d}$, was considered; note, however, that all the positive eigenvalues of B are supposed to be the same which essentially simplifies the analysis. More precisely, in [141, Theorem 3.2 (b)] and [83, Theorem 1.2 (B)] it was shown that

$$\mathcal{N}^-_q(\lambda) = \frac{\kappa_q}{d!}\varphi_\infty(\lambda)^d(1 + o(1)), \quad \lambda \downarrow 0, \qquad (4.4.44)$$

where κ_q is the multiplicity of the Landau level Λ_q which is equal to $\binom{q+d-1}{d-1}$ if $b_1 = \ldots = b_d$. Moreover, in [83], it was shown that analogues of (4.4.43) and (4.4.44) hold true also in the case of Robin boundary conditions.

4.5 Asymptotic density of eigenvalue clusters

As in Section 4.4.1, we consider $H_V := H_S(A,V) = H_S(A,0) + V$, electric perturbations of Landau Hamiltonian:

$$H_0 := H_S(A,0) = \left(-i\frac{\partial}{\partial x} + \frac{by}{2}\right)^2 + \left(-i\frac{\partial}{\partial y} - \frac{bx}{2}\right)^2, \quad (x,y) \in \mathbb{R}^2,$$

with $A = \frac{b}{2}(-y,x)$. While the previous spectral asymptotics are independent of the Landau level number q, in this section we study the implication of the results in Section 3.5 on the distribution of the discrete eigenvalues near the qth Landau as q tends to infinity. We prove that the rate at which the discrete eigenvalues approach the qth Landau level tends to 0 as q tends to infinity and according to the decay rate of the potential, the asymptotic distribution of eigenvalues involves the Radon transform of the potential (short-range case) or the mean-value transform of the main homogeneous part of the potential (long-range case with the following Condition 4.5.2).

Let $V : \mathbb{R}^2 \to \mathbb{R}$ be continuous and bounded satisfying the following

Condition 4.5.1 *The potential* V *belongs to* $C(\mathbb{R}^2;\mathbb{R})$*, and the estimate*

$$|V(x)| \le C(1+|x|)^{-\gamma}, \quad x \in \mathbb{R}^2, \tag{4.5.1}$$

for some $\gamma \in (0,\infty)$.

Then as in Section 4.4.1, $|V|^{1/2}H_0^{-1/2}$ is compact in $L^2(\mathbb{R}^2)$ and the Weyl theorem yields:

$$\sigma_{\mathrm{ess}}(H_V) = \sigma_{\mathrm{ess}}(H_0) = \sigma(H_0) = \bigcup_{q\in\mathbb{Z}_+} \{\Lambda_q\}. \tag{4.5.2}$$

Moreover, for V of definite sign we know that H_V as discrete spectrum which accumulate at each Landau Level Λ_q, $q \in \mathbb{Z}_+$. The asymptotic behavior of the counting function given by the results of Section 4.4.1 does not depend on q, the Landau level number. In this Section we describe the behavior of the distribution of eigenvalues within the qth cluster, as q tends to infinity.

We will distinguish two different cases according to the decay rate of the potential. If V satisfies Condition 4.5.1 for some $\gamma > 1$, we will say that it is short-range. If, on the other hand, V verifies Condition 4.5.1 for some $\gamma \in (0,1)$ but not for $\gamma \ge 1$, we will classify it as being long-range.

In the case of long-range potentials, we will consider also some additional conditions on V. Let us write $u \in \mathscr{H}^{\sharp}_{-\gamma}(\mathbb{R}^2)$ if $u \in C^\infty(\mathbb{R}^2 \setminus \{0\})$, is a homogeneous function of order $-\gamma$ and use the class of symbols $\Gamma_1^{\gamma}(\mathbb{R}^2)$ introduced in Section 2.3.

Condition 4.5.2 *The potential* V *belongs to* $\Gamma_1^{-\gamma}(\mathbb{R}^2)$ *for some* $\gamma \in (0,1)$ *and there exists* $\mathbb{V} \in \mathscr{H}^{\sharp}_{-\gamma}(\mathbb{R}^2)$ *such that*

$$|V(x) - \mathbb{V}(x)| \le C|x|^{-\gamma-\varepsilon},\ x \in \mathbb{R}^2,\ |x| > 1, \tag{4.5.3}$$

with given constants C *and* $\varepsilon > 0$.

Notice that Condition 4.5.2 implies Condition 4.5.1 with $\gamma \in (0,1)$.

4.5.1 Eigenvalue clusters

Our first result concerns the location of the spectrum of $H_V = H_S(A,V)$. It provides an estimate on the rate at which the discrete eigenvalues approach the qth Landau level, as q tends to infinity, and justifies the terminology “eigenvalue clusters”.

Theorem 4.5.1 (i) ([150]) *Assume that* V *satisfies Condition 4.5.1 with* $\gamma > 1$. *Then*

$$\sigma(H_V) \subset \bigcup_{q\in\mathbb{Z}_+} \left(\Lambda_q - C_1\Lambda_q^{-1/2}, \Lambda_q + C_1\Lambda_q^{-1/2}\right) \tag{4.5.4}$$

with a constant $C_1 > 0$ *independent of* q.

(ii) ([123]) *Assume that* V *satisfies Condition 4.5.1 with* $\gamma \in (0,1)$. *Then*

$$\sigma(H_V) \subset \bigcup_{q\in\mathbb{Z}_+} \left(\Lambda_q - C_2\Lambda_q^{-\gamma/2}, \Lambda_q + C_2\Lambda_q^{-\gamma/2}\right) \tag{4.5.5}$$

with a constant $C_2 > 0$ *independent of* q.

Estimates (4.5.4) and (4.5.5) are sharp (see the proof below and the remark after Proposition 3.5.1). This will also follow from our main results, Theorems 4.5.2 and 4.5.3.

The proof of Theorem 4.5.1 is largely a corollary of the Birman–Schwinger principle and the norm estimates contained in Theorem 3.5.1 applied for the Berezin–Toeplitz operators $p_q V p_q$, where p_q is the orthogonal projection onto $\mathrm{Ker}(H_0 - \Lambda_q)$. Set $R_0(\lambda) := (H_0 - \lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \sigma(H_0)$. The Birman–Schwinger operator $|V|^{1/2} R_0(\lambda) V^{1/2}$ can be estimated in norm by the expression

$$\| |V|^{1/2} R_0(\lambda) V^{1/2} \| \le \sum_{k=q-m}^{q+m} |\Lambda_k - \lambda|^{-1} \|p_k |V| p_k\| + \| |V|^{1/2} \tilde{R}_0(\lambda; q, m) |V|^{1/2} \|, \quad (4.5.6)$$

where

$$\tilde{R}_0(\lambda; q, m) := R_0(\lambda) - \sum_{k=q-m}^{q+m} (\Lambda_k - \lambda)^{-1} p_k.$$

Now, taking $m \in \mathbb{Z}_+$ large enough such that $\|V\|_{L^\infty(\mathbb{R}^2)} \le \Lambda_m/2$ and

$$\tilde{C_1} \Lambda_q^{-1/2} < |\Lambda_q - \lambda| \le b$$

(resp. $\tilde{C_2} \Lambda_q^{-\gamma/2} < |\Lambda_q - \lambda| \le b$), the above estimate ensures that the norm of the Birman–Schwinger operator is strictly less than one for q large enough. Consequently for these λ, the operator $I + |V|^{1/2} R_0(\lambda) V^{1/2}$ is invertible and the invertibility of $(H_V - \lambda)$ follows from the Birman–Schwinger principle (see Lemma 4.4.1).

Let us mention that analogous sharp estimates on the size of the spectral clusters that depend only on an Lp-norm of V are also given in [55].

4.5.2 Asymptotic density for short-range electric perturbations

Our next goal is to give an asymptotic description of the distribution of eigenvalues within the qth cluster, as q tends to infinity. For $[\alpha, \beta] \subset \mathbb{R} \setminus \{0\}$ and $q \in \mathbb{Z}_+$ we define the eigenvalue counting measures by setting

$$\mu_q^{\mathrm{short}}([\alpha, \beta]) := \sum_{\Lambda_q + \alpha \Lambda_q^{-1/2} \le \lambda \le \Lambda_q + \beta \Lambda_q^{-1/2}} \dim \mathrm{Ker}(H_V - \lambda).$$

Notice that the interval $[\alpha, \beta]$ is rescaled in accordance with (4.5.4). For large values of q, we have that

$$[\Lambda_q + \alpha \Lambda_q^{-1/2},\ \Lambda_q + \beta \Lambda_q^{-1/2}] \cap \sigma_{\mathrm{ess}}(H_V) = \emptyset, \quad (4.5.7)$$

so that the above quantity is finite.

Note that if $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, then

$$\int_{\mathbb{R}} \varphi(\lambda) d\mu_q^{\mathrm{short}}(\lambda) = \mathrm{Tr}\ \varphi(\Lambda_q^{1/2}(H_V - \Lambda_q)). \quad (4.5.8)$$

Definition 4.5.1 (Radon transform) *Assume that* V *satisfies Condition 4.5.1 with* $\gamma > 1$*, and define its Radon transform*

$$\tilde{V}(\omega,s) := \frac{1}{2\pi}\int_{\mathbb{R}} V(s\omega + t\omega^{\perp})dt, \quad \omega = (\omega_1,\omega_2) \in \mathbb{S}^1,\ s \in \mathbb{R},$$

where $\omega^{\perp} = (-\omega_2,\omega_1) \in \mathbb{S}^1$.

Condition 4.5.1 with $\gamma > 1$ entails the following decay property of the Radon transform:

$$|\tilde{V}(\omega,s)| \le C(1+|s|)^{1-\gamma}, \quad \omega \in \mathbb{S}^1,\ s \in \mathbb{R}. \tag{4.5.9}$$

Theorem 4.5.2 ([150]) *Let* V *satisfy Condition 4.5.1 for some* $\gamma > 1$*. Then*

$$\lim_{q\to\infty} \Lambda_q^{-1/2}\,\mathrm{Tr}\,\varphi(\Lambda_q^{1/2}(H_V - \Lambda_q)) = \frac{1}{2\pi}\int_{\mathbb{S}^1}\int_{\mathbb{R}} \varphi(b\tilde{V}(\omega,s))\,ds\,d\omega \tag{4.5.10}$$

for each $\varphi \in C_0^\infty(\mathbb{R}\setminus\{0\})$.

For $[\alpha,\beta] \subset \mathbb{R}\setminus\{0\}$ define the measure

$$\mu_\infty^{\mathrm{short}}([\alpha,\beta]) := \frac{1}{2\pi}\left|\tilde{V}^{-1}([b^{-1}\alpha, b^{-1}\beta])\right|_{\mathbb{S}^1\times\mathbb{R}},$$

$|\cdot|_{\mathbb{S}^1\times\mathbb{R}}$ being the Lebesgue measure on $\mathbb{S}^1\times\mathbb{R}$. In terms of convergence of measures, Theorem 4.5.2 is equivalent to

$$\lim_{q\to\infty} \Lambda_q^{-1/2}\mu_q^{\mathrm{short}}([\alpha,\beta]) = \mu_\infty^{\mathrm{short}}([\alpha,\beta]), \tag{4.5.11}$$

for any α,β such that $\alpha\beta > 0$ and

$$\mu_\infty^{\mathrm{short}}(\{\alpha\}) = \mu_\infty^{\mathrm{short}}(\{\beta\}) = 0.$$

However, the above condition does not automatically hold. Indeed, the image of $C_0^\infty(\mathbb{R}\setminus\{0\})$ under the Radon transform is well-known (see, for instance, [92, Theorem 2.10]). According to this description, if $\phi \in C_0^\infty(\mathbb{R})$ is an even, real-valued function, then $\tilde{V}(\omega,s) := \phi(s)$ is the Radon transform of some $V \in C_0^\infty(\mathbb{R}\setminus\{0\})$. Of course, if the derivative $\phi'(s)$ vanishes on an open interval, the corresponding measure has an atom.

Let us outline the proof of Theorem 4.5.2. More detailed proofs and calculations can be found in [150].

An application of the Stone–Weierstrass theorem allows us to consider polynomials instead of $C_0^\infty(\mathbb{R}\setminus\{0\})$ functions. Then, a Cauchy formula representation yields

Proposition 4.5.1 *Let* $\ell > 1/(\gamma-1)$*. Then, for* q *large enough, the operators*

$$(H_V - \Lambda_q)^\ell \chi_{(\Lambda_q - b, \Lambda_q + b)}(H_V) \quad \text{and} \quad (p_q V p_q)^\ell$$

belong to $\mathfrak{S}_1$*, the trace class, and*

$$\mathrm{Tr}\,(H_V - \Lambda_q)^\ell \chi_{(\Lambda_q - b, \Lambda_q + b)}(H_V) = \mathrm{Tr}\,(p_q V p_q)^\ell + o(\Lambda_q^{-(\ell-1)/2}), \quad q \to \infty. \tag{4.5.12}$$

Once (4.5.12) is established, results in Section 3.5.2 allow us to consider the traces of the operators $\mathrm{Op^w}(\mathrm{V_b} * \delta_{\sqrt{2q+1}})^\ell$, with $\mathrm{V_b}$ defined by (3.4.46).

Proposition 4.5.2 *Assume that* V *belongs to* $\mathrm{C}_0^\infty(\mathbb{R}^2)$. *Then,*

$$\lim_{q\to\infty} \Lambda_q^{(\ell-1)/2}\,\mathrm{Tr}\,(\mathrm{Op^w}(\mathrm{V_b} * \delta_{\sqrt{2q+1}}))^\ell = \frac{b^\ell}{2\pi}\int_{\mathbb{S}^1}\int_{\mathbb{R}} \tilde{\mathrm{V}}(\omega,s)^\ell\,ds\,d\omega \tag{4.5.13}$$

for every integer $\ell \in \mathbb{N}$.

The proof of Proposition 4.5.2 relies chiefly on using the integral representation of the trace given by the composition formula for symbols of Weyl pseudodifferential operators (see Section 2.3). Thus,

$$\mathrm{Tr}\,(\mathrm{Op^w}(t_k))^\ell = (2\pi)^{-\ell+1}\int_{\mathbb{R}^{2(\ell-1)}} \hat{t}_k(-\xi_{\ell-1})\hat{t}_k(\xi_{\ell-1}-\xi_{\ell-2})\dots$$
$$\dots\hat{t}_k(\xi_1)e^{\frac{i}{2}\sum_{j=2}^{\ell-1}\sigma(\xi_j,\xi_{j-1})}d\xi, \tag{4.5.14}$$

with $t_k := \mathrm{V_b} * \delta_k$ and σ the standard canonical symplectic form (2.1.17), so that

$$\hat{t}_k(\xi) = \frac{1}{2\pi}\hat{\mathrm{V}}_b(\xi)\int_{\mathbb{S}^1} e^{-ik\omega\xi}\,d\omega, \quad \xi \in \mathbb{R}^2. \tag{4.5.15}$$

We obtain (4.5.13) by applying the stationary phase method.

Finally, a continuity argument allows us to extend the result, for $\ell > 1/(\gamma-1)$, from $\mathrm{C}_0^\infty(\mathbb{R}^2;\mathbb{R})$ to the set of functions verifying Condition 4.5.1 with $\gamma > 1$, which completes the proof of Theorem 4.5.2.

Let us mention that in the short range case, a Szegö limit theorem is stated in [94], for the eigenvalues in the clusters as the cluster index q and the field strength b tend to infinity with a fixed ratio $\mathscr{E}$. The result involves the averages of the potential over circles of radius $\sqrt{\mathscr{E}/2}$ (classical orbits). A related inverse spectral result is also discussed.

4.5.3 Asymptotic density for long-range electric perturbation

Similarly to the short-range case, we define for $[\alpha,\beta] \subset \mathbb{R}\setminus\{0\}$ and $q \in \mathbb{Z}_+$ the corresponding eigenvalue counting measures

$$\mu_q^{\mathrm{long}}([\alpha,\beta]) := \sum_{\Lambda_q+\alpha\Lambda_q^{-\gamma/2}\le\lambda\le\Lambda_q+\beta\Lambda_q^{-\gamma/2}} \dim\mathrm{Ker}(\mathrm{H_V}-\lambda).$$

Definition 4.5.2 (Mean-value transform) *Assume that* $u \in \mathrm{C}(\mathbb{R}^2\setminus\{0\})$, *and define its mean-value transform*

$$\mathring{u}(x) := \frac{1}{2\pi}\int_{\mathbb{S}^1} u(x-\omega)\,d\omega, \quad x \in \mathbb{R}^2\setminus\mathbb{S}^1.$$

We describe some elementary yet useful properties of the mean-value transform of functions belonging to given classes. If $u \in \Gamma_1^{-\gamma}(\mathbb{R}^2)$, $\gamma \in (0,\infty)$, then its mean-value transform $\mathring{u}$ extends to a function $\mathring{u} \in \Gamma_1^{-\gamma}(\mathbb{R}^2)$. If $u \in \mathscr{H}_{-\gamma}^{\sharp}(\mathbb{R}^2)$, $\gamma \in (0,\infty)$, then $\eta\mathring{u} \in \Gamma_1^{-\gamma}$ provided that $\eta \in \Gamma_1^0(\mathbb{R}^2)$ and $\operatorname{supp}\eta \cap \mathbb{S}^1 = \emptyset$. Moreover, if $\gamma \in (0,1)$, then the mean-value transform of $u \in \mathscr{H}_{-\gamma}^{\sharp}(\mathbb{R}^2)$ extends to a function $\mathring{u} \in C(\mathbb{R}^2)$. Finally, if $u \in \mathscr{H}_{-\gamma}^{\sharp}(\mathbb{R}^2)$, $\gamma \in (0,1)$, and $\mathring{u}(x) = 0$ for each $x \in \mathbb{R}^2$, then $u(x) = 0$ for each $x \in \mathbb{R}^2 \setminus \{0\}$.

Theorem 4.5.3 ([123]) *Let* V *satisfy Condition 4.5.2. Then*

$$\lim_{q\to\infty} \Lambda_q^{-1} \operatorname{Tr} \varphi(\Lambda_q^{\gamma/2}(H_V - \Lambda_q)) = \frac{1}{2\pi B} \int_{\mathbb{R}^2} \varphi(bn^{\gamma}\mathring{\mathbb{V}}(x))\, dx \tag{4.5.16}$$

for each $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$.

Again, defining for $q \in \mathbb{Z}_+$ the measure

$$\mu_\infty^{\text{long}}([\alpha,\beta]) := \frac{1}{2\pi B} \left| \mathring{\mathbb{V}}^{-1}([b^{-\gamma}\alpha, b^{-\gamma}\beta]) \right|_{\mathbb{R}^2}, \quad [\alpha,\beta] \subset \mathbb{R} \setminus \{0\},$$

with $|\cdot|_{\mathbb{R}^2}$ the Lebesgue measure on $\mathbb{R}^2$, we find that (4.5.16) is equivalent to

$$\lim_{q\to\infty} \Lambda_q^{-1} \mu_q^{\text{long}}([\alpha,\beta]) = \mu_\infty^{\text{long}}([\alpha,\beta]), \tag{4.5.17}$$

for any α, β such that $\alpha\beta > 0$ and such that α and β are not atoms of the measure μ_∞^{long}. We remark that if, for instance, $\mathbb{V}$ is radially symmetric, the corresponding measure has no atoms.

Remark 4.5.1 *The case* $\gamma = 1$ *differs from both the short-range and the long-range cases and will not be considered here. In fact, for, say,* V *asymptotically homogeneous of order* -1*, the Radon transform* $\tilde{V}$ *is not well defined. Further, since* $\mathring{\mathbb{V}}$ *can generically have a logarithmic singularity at* $\mathbb{S}^1$*, given* $\mathbb{V} \in \mathscr{H}_{-1}^{\sharp}(\mathbb{R}^2)$*, the support of the limiting measure* μ_∞^{long} *need not be compact, which would mean that (4.5.5) fails to hold in this case.*

Let us give ideas of the proof of Theorem 4.5.3. Methodology used for the short-range case is not readily extensible to the long-range scenario. Conversely, most of the results and methods exposed below do not adapt well to the previous scenario.

The first step in the proof is the passage to Berezin–Toeplitz operators.

Proposition 4.5.3 *Assume that* V *verifies Condition 4.5.2 for some* $\gamma \in (0,1)$*. Then,*

$$\operatorname{Tr} \varphi(\Lambda_q^{\gamma/2}(H_V - \Lambda_q)) = \operatorname{Tr} \varphi(\Lambda_q^{\gamma/2} p_q V p_q)) + o(\Lambda_q), \quad q \to \infty. \tag{4.5.18}$$

Note that, in contrast to the short-range case, no approximation by polynomials is used, instead, we work directly with the function φ. Proposition 4.5.3 amounts essentially to approximating H_V by its Weinstein average $\langle H_V \rangle$.

Definition 4.5.3 (Weinstein average) *Assume that* $V \in L^\infty$ *and set*

$$\langle V \rangle := \frac{b}{\pi}\int_0^{\pi/b} e^{-itH_0} V e^{itH_0} dt = \sum_{s\in\mathbb{Z}_+} p_s V p_s, \tag{4.5.19}$$

$$\langle H_V \rangle := H_0 + \langle V \rangle. \tag{4.5.20}$$

Note that a sufficient condition for the uniform convergence of the series defining $\langle V \rangle$ in (4.5.19), is that the norm of $p_q V p_q$ tends to zero as q goes to infinity, which holds true according to Theorem 3.5.1.

Proposition 4.5.4 *Under the hypotheses of Condition 4.5.2, we have*

$$\operatorname{Tr} \varphi(\Lambda_q^{\gamma/2}(H_V - \Lambda_q)) = \operatorname{Tr} \varphi(\Lambda_q^{\gamma/2}(\langle H_V \rangle - \Lambda_q)) + o(\Lambda_q), \quad q \to \infty \tag{4.5.21}$$

for each $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$.

The main ingredients in the proof of this proposition are the Helffer–Sjöstrand formula used to express the difference of both operators in terms of the difference of their resolvents, and the Schur–Feshbach formula to estimate this difference accordingly. Namely, let $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^2)$ be a suitable almost-analytic continuation of φ, let $\psi := \partial\tilde{\varphi}/\partial\bar{z}$ and set

$$\psi_q(x,y) := \Lambda_q^{\gamma/2}\psi(\Lambda_q^{\gamma/2}(x-\Lambda_q), \Lambda_q^{\gamma/2}y), \quad q \in \mathbb{Z}_+,$$

so that

$$\varphi(\Lambda_q^{\gamma/2}(H_V - \Lambda_q)) = \frac{1}{\pi}\int_{\mathbb{R}^2} \psi_q(x,y)(H_V - z)^{-1}\, dxdy, \quad z := x+iy \in \mathbb{C}. \tag{4.5.22}$$

Iterating the resolvent identity, we can write the above expression as

$$\varphi(\Lambda_q^{\gamma/2}(H_V - \Lambda_q)) = \frac{(-1)^\ell}{\pi}\int_{\mathbb{R}^2} \psi_q(x,y)((H_0-z)^{-1}V)^\ell (H_V - z)^{-1}\, dxdy, \tag{4.5.23}$$

so that we are able to extract the terms which contribute to the main asymptotic term of the trace of the operator. Indeed, writing

$$((H_0-z)^{-1}V)^\ell = (\Lambda_q - z)^{-\ell}(p_q V)^\ell + r_{q,\ell}(z), \tag{4.5.24}$$

and taking ℓ to be the smallest integer strictly greater than $\gamma/2$, we find that the trace of the part of the operator corresponding to $r_{q,\ell}$ is of order $o(\Lambda_q)$, as $q \to \infty$.

We analogously express the trace of the operator obtained by replacing H_V by $\langle H_V \rangle$ and we are left with estimating the trace of the difference

$$\frac{(-1)^\ell}{\pi}\int_{\mathbb{R}^2} \psi_q(x,y)(\Lambda_q - z)^{-\ell}(p_q V)^\ell \left((H_V - z)^{-1} - p_q(\langle H_V \rangle - z)^{-1}\right) dxdy.$$

Application of the Schur–Feshbach formula and estimates of Section 3.5 imply that this quantity is of order $o(\Lambda_q)$ as q tends to infinity.

Once the approximation by the Weinstein average is established, a simple argument relying on the fact that φ has a compact support shows that for large enough q, the trace of this operator reduces to that of a single Berezin–Toeplitz operator:

Proposition 4.5.5 *Assume that* V *verifies Condition 4.5.2. Then, for each* $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ *there exists* $q_0 \in \mathbb{Z}_+$ *such that*

$$\operatorname{Tr} \varphi(\Lambda_q^{\gamma/2}(\langle H_V \rangle - \Lambda_q)) = \operatorname{Tr} \varphi(\Lambda_q^{\gamma/2} p_q V p_q), \quad q \geq q_0. \tag{4.5.25}$$

Finally, as q tends to infinity, V_b is integrated over larger and larger circles, so that it can be well approximated by its behavior far from the origin. This is expressed by the following result:

Proposition 4.5.6 *Assume that* V *verifies Condition 4.5.2. Then,*

$$\operatorname{Tr} \varphi(\Lambda_q^{\gamma/2} p_q V p_q) = \operatorname{Tr} \varphi(\Lambda_q^{\gamma/2} \operatorname{Op}^w(\mathbb{V}_b * \delta_{\sqrt{2q+1}})) + o(\Lambda_q), \quad q \to \infty, \tag{4.5.26}$$

for each $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$.

The proof of this fact makes use of the technical assumption in (4.5.3), and is based on certain estimates of trace-class norms of operators of the type $\varphi(T+Q) - \varphi(T)$, where $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ and T and Q are appropriate self-adjoint compact operators. The main tools used to obtain these trace-class estimates are the results of [**?**].

Proposition 4.5.7 *Assume that* V *verifies Condition 4.5.2. Then, we have that*

$$\lim_{q \to \infty} \Lambda_q^{-1} \operatorname{Tr} \varphi(\Lambda_q^{\gamma/2} \operatorname{Op}^w(\mathbb{V}_b * \delta_{\sqrt{2q+1}})) = \frac{1}{2\pi b} \int_{\mathbb{R}^2} \varphi(b^\gamma \mathring{\mathbb{V}}(x)) dx. \tag{4.5.27}$$

The proof of Proposition 4.5.7 relies on the unitary equivalence between the operator $\Lambda_q^{\gamma/2} \operatorname{Op}^w(\mathbb{V}_b * \delta_{\sqrt{2q+1}})$ and the pseudodifferential operator whose Weyl symbol is given by

$$s_\hbar(x, \xi) := b^\gamma \mathring{\mathbb{V}}_1(x, \hbar\xi), \quad (x, \xi) \in \mathbb{R}^2,$$

with $\hbar := (2q+1)^{-1}$. If the symbol were regular, standard semiclassical results would then imply (4.5.27) (see, for instance, [63, Theorem 9.6]). However, $\mathring{\mathbb{V}}_1$ has a singularity at $\mathbb{S}^1$, so a suitable approximation by smooth symbols is required to establish the result. For this, estimates for weak Schatten–von Neumann classes, as (2.3.17), are used and the desired result is finally obtained.

4.5.4 Semiclassical interpretation

For $(x, \xi) \in T^*\mathbb{R}^2$, consider the Hamiltonian function (see (2.1.14)):

$$\mathscr{H}(x, \xi) := \left(\xi_1 + \frac{1}{2} b x_2\right)^2 + \left(\xi_2 - \frac{1}{2} b x_1\right)^2.$$

The projections of the orbits of the Hamiltonian flow of $\mathscr{H}$ onto the configuration space are circles of radius $\sqrt{E}/b$; here $E > 0$ is the energy corresponding to the orbit (see Section 2.1). The classical particles move around these circles with period $T_b = \pi/b$. The orbits can be parametrized by the energy $E > 0$ and the center $c \in \mathbb{R}^2$ of the circle.

Let $\gamma(c,E,t)$, $t \in [0,T_b)$, be the path in the configuration space corresponding to such an orbit. Set

$$\mathrm{Av}(V)(c,E) := \frac{1}{T_b}\int_0^{T_b} V(\gamma(c,E,t))dt.$$

Under Condition 4.5.1 with $\gamma > 1$, we have the following identity concerning the short-range case:

$$\frac{1}{2\pi}\lim_{E\to\infty}\frac{1}{E^{1/2}}\int_{\mathbb{R}^2}\varphi(E^{1/2}\,\mathrm{Av}(V)(c,E))\,b\,dc = \frac{1}{2\pi}\int_{\mathbb{S}^1}\int_{\mathbb{R}}\varphi(b\tilde{V}(\omega,s))ds\,d\omega. \quad (4.5.28)$$

The basis of this calculation is the fact that as E tends to infinity, so does the radius $\sqrt{E}/b$ of the orbits. Hence, these orbits approximately look like straight lines on any compact domain of the configuration space. Under this consideration, Th. 4.5.2 can be restated as

$$\lim_{q\to\infty}\frac{1}{\Lambda_q^{1/2}}\mathrm{Tr}\,\varphi(\Lambda_q^{1/2}(H_V-\Lambda_q)) = \frac{1}{2\pi}\lim_{E\to\infty}\frac{1}{E^{1/2}}\int_{\mathbb{R}^2}\varphi(E^{1/2}\,\mathrm{Av}(V)(c,E))\,b\,dc. \quad (4.5.29)$$

Notice that if we consider the set M_E of orbits of given energy $E > 0$, parametrized by $c \in \mathbb{R}^2$, the measure $b\,dc$ appearing in the right hand side of (4.5.29) coincides with the restriction of the Lebesgue measure on $\mathbb{R}^4$ onto the quotient of the constant-energy surface

$$\left\{(x,\xi) \in \mathbb{R}^4 \mid \mathscr{H}(x,\xi) = E\right\}$$

with respect to the flow of $\mathscr{H}$.

In an analogous fashion, in the long-range regime we have

$$\begin{aligned}\frac{1}{2\pi}\lim_{E\to\infty}\frac{1}{E}\int_{\mathbb{R}^2}\varphi(E^{\gamma/2}\,\mathrm{Av}(V)(c,E))\,b\,dc &= \frac{1}{2\pi b}\int_{\mathbb{R}^2}\varphi(b^{\gamma}\mathring{V}(x))dx \quad (4.5.30)\\ &= \lim_{q\to\infty}\frac{1}{\Lambda_q}\mathrm{Tr}\;\varphi(\Lambda_q^{\gamma/2}(H_V-\Lambda_q)),\end{aligned}$$

provided that Condition 4.5.2 holds true.

Equations (4.5.29) and (4.5.30) are in agreement with the semiclassical intuition and can be interpreted in the spirit of the “averaging principle” for systems close to integrable ones. According to this principle, a good approximation is obtained if the original perturbation is replaced by its average along the orbits of the free dynamics (see for instance [5, Section 82]).

Chapter 5

Asymptotics for the Spectral Shift Function and the resonances

Abstract: We introduce the notions of Spectral Shift Function (SSF) and of the resonances in Section 5.1. These quantities are used to quantify phenomena of spectral concentration even in presence of continuous spectrum. By analogy with the counting function of eigenvalues, the blow up of the SSF, or of the counting function of resonances near the real axis, will express a phenomenon of spectral concentration. For perturbations, by an electric potential of definite sign, of 3D magnetic magnetic Schrödinger operators, we study the SSF and the resonances near Landau levels in Section 5.2. The cases of perturbations by obstacles is treated in Section 5.3. Other situations where continuous spectrum appears are 2D magnetic Schrödinger operators with unbounded boundaries. In Section 5.4 we discuss the cases of magnetic Schrödinger operators in the half-plane and in the strip for which several questions remain open both concerning the SSF and the resonances.

5.1 Perturbation of the continuous spectrum

As described in Sections 2.7 and 2.8, the quantum magnetic hamiltonians may have essential spectrum not reduced to a point spectrum. For example the spectrum of the 3D Schrödinger operator $\mathrm{H_S}(\mathrm{A},0)$, with constant magnetic field having a non-trivial kernel, is $[\Lambda_0,+\infty)$ (see (2.7.61)). Any relatively compact perturbation of $\mathrm{H_S}(\mathrm{A},0)$ have the same essential spectrum with a possible discret spectrum below Λ_0. Thus the influence of the perturbation can be analyzed below Λ_0, with the study of the discret spectrum, but above Λ_0, the perturbation of the continuous part of the spectrum is less obvious. For the analysis of the perturbation of the continuous spectrum, we will study the *Spectral Shift Function* and the *Resonances*.

The *Spectral Shift Function*, now denoted SSF introduced in the early fifties by the Physicist IM Lifshits measures an energy distribution. First, it was defined for a pair of bounded selfadjoint operators (A, A_0) such that $(A_1 - A_0) \in \mathfrak{S}_1$ by a trace perturbation formula :

$$\xi(A_1, A_0; \lambda) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im} \ln(D_{A_1,A_0}(\lambda + i\varepsilon)). \tag{5.1.1}$$

where D_{A_1,A_0} is the perturbation (or relative) determinant:

$$D_{A_1,A_0}(z) := \det\left(I + (A_1 - A_0)(A_0 - zI)^{-1}\right), \qquad z \in C_\pm := \{z \in \mathbb{C} \mid \pm \operatorname{Im} z > 0\}.$$

Then, the definition was extended to relatively trace class perturbations of self-adjoint operators (see for instance [216] for more details). In particular for semi-bounded selfadjoint operators it is defined thanks to the following Theorem.

Theorem 5.1.1 ([121, 113, 216]) *Let* H_1 *and* H_0 *be two semi-bounded selfadjoint operators on a Hilbert space* $\mathscr{H}$ *such that for some* $k > 0$ *and* $c > 0$*,*

$$(H_1 + cI)^{-k} - (H_0 + cI)^{-k} \in \mathfrak{S}_1(\mathscr{H}).$$

Then there exists a unique Spectral Shift Function $\xi(H_1, H_0; .)$*, such that:*

1. $\lambda \mapsto (1 + |\lambda|)^{-k-1} \xi(H_1, H_0; \lambda)$ *is integrable on* $\mathbb{R}$
2. $\xi(H_1, H_0; \lambda) = 0$ *for* $\lambda < \inf(\sigma(H_1) \cup \sigma(H_0))$
3. $\operatorname{Tr}((H_1 + cI)^{-k} - (H_0 + cI)^{-k}) = -k \int_{\mathbb{R}} \xi(H_1, H_0; .)(\lambda + c)^{-k-1} d\lambda$
4. *For any* $f \in \mathscr{S}(\mathbb{R})$*,*

$$\operatorname{Tr}\Big(f(H_1) - f(H_0)\Big) = \int_{\mathbb{R}} \xi(H_1, H_0; \lambda) f'(\lambda) d\lambda.$$

By the *Birman–Krein formula*, almost everywhere on the absolutely continuous spectrums, the SSF $\xi(H_1, H_0; \cdot)$ coincides with *the scattering phase* for the operator pair (H_1, H_0) (see the original work [17] or [216, Chapter 8]). Further, if H_1 has discrete spectrum below $\sigma(H_0)$, then for almost every $E \in \sigma_d(H_1) \cap (-\infty, \inf \sigma(H_0)]$, we have

$$-\xi(H_1, H_0; E) = \operatorname{Tr} \mathbb{1}_{(-\infty, E)}(H_1), \tag{5.1.2}$$

the number of the eigenvalues of H_1 less than E, counted with their multiplicities.

There exists several interpretation and representation formulas of the SSF (see for instance the reviews [167, 33]). One that will be particularly useful later in this Chapter is the following due to A. Pushnitski [148].

Proposition 5.1.1 *([148, Theorem 1.2]) Let* $H_\pm$ *and* H_0 *be two semi-bounded selfadjoint operators on a Hilbert space* $\mathscr{H}$ *such that* $H_\pm = H_0 \pm V$ *with* V *a nonnegative operator satisfying, for some* $k > 0$*,* $m > 0$ *and* $c > 0$*:*

$$V^{\frac{1}{2}}(H_0 + cI)^{-\frac{1}{2}} \in \mathfrak{S}_\infty; \quad V^{\frac{1}{2}}(H_0 + cI)^{-m} \in \mathfrak{S}_2(\mathscr{H}); \quad (H_\pm + cI)^{-k} - (H_0 + cI)^{-k} \in \mathfrak{S}_1(\mathscr{H}).$$

Then for almost every $E \in \mathbb{R}$ *the following limit exists in* $\mathfrak{S}_\infty$

$$T_0(E) := \lim_{\varepsilon \to 0^+} V^{\frac{1}{2}}(H_0 - E - i\varepsilon)^{-1} V^{\frac{1}{2}}, \tag{5.1.3}$$

and we have the representation formula of the Spectral Shift Function $\xi(H_\pm, H_0; .)$*:*

$$\xi(H_\pm, H_0; E) = \pm\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\mathbb{1}_{(1,\infty)}(\mp(\mathrm{Re}\,T_0(E) + t\,\mathrm{Im}\,T_0(E)))\frac{dt}{1+t^2}, \tag{5.1.4}$$

where $\mathrm{Re}\,T := \frac{1}{2}(T+T^*) \in \mathfrak{S}_\infty$*,* $\mathrm{Im}\,T := \frac{1}{2i}(T-T^*) \in \mathfrak{S}_1$.

This representation of the SSF is an extension to continous spectrum, of the Birman-Schwinger principle. Indeed, for $E \in \mathbb{R} \cap \rho(H_0)$ (where $H_\pm$ may only have eigenvalues), $\mathrm{Im}\,T_0(E) = 0$, and the formula becomes:

$$\xi(H_\pm, H_0; E) = \pm \mathrm{Tr}\mathbb{1}_{(1,\infty)}(\mp(T_0(E)),$$

which corresponds to the *Birman-Schwinger principle* (Lemma 4.4.1). An extension of this formula to perturbations of nondefinite sign is given in [149] (it involves Index of Fredholm pair of operators).

The previous result gives an abstract setting in which $T_0(\lambda + i0)$ given by (5.1.3) exists. As we will see below, for some V (typically for compactly supported potentials), the operator-valued function $z \mapsto T_0(z) := |V|^{\frac{1}{2}}(H_0 - z)^{-1}V^{\frac{1}{2}}$, first analytic on $\mathbb{C}_+$ with value in $\mathfrak{S}_\infty$, could admit an analytic extension to a Riemannian surface $\mathscr{M}$. Then, from the Analytic Fredholm Theorem [162, Theorem VI.14] and the resolvent equation

$$T_1(z) := |V|^{\frac{1}{2}}(H_1 - z)^{-1}V^{\frac{1}{2}} = I - \big(I + T_0(z)\big)^{-1}, \tag{5.1.5}$$

we deduce that the operator-valued function $z \mapsto T_1(z)$, first analytic on $\mathbb{C}_+$, admits a meromorphic extension to $\mathscr{M}$ with poles $w \in \mathscr{M}$ such that $\big(I + T_0(w)\big)$ is not invertible. Thus, we define the resonances on the following way.

Proposition 5.1.2 *([24]) Let* H_1 *and* H_0 *be two semi-bounded selfadjoint operators on a Hilbert space* $\mathscr{H}$ *such that* $H_1 = H_0 + V$ *with* $V = |V|^{\frac{1}{2}}V^{\frac{1}{2}}$ *a symmetric bounded operator such that* $z \mapsto T_0(z) := |V|^{\frac{1}{2}}(H_0 - z)^{-1}V^{\frac{1}{2}}$*, admits an analytic extension to a Riemannian surface* $\mathscr{M}$*, in the class of compact operators* $\mathfrak{S}_\infty$*.*

Then the operator-valued function $z \mapsto T_1(z) := |V|^{\frac{1}{2}}(H_1 - z)^{-1}V^{\frac{1}{2}}$*, admits a meromorphic extension to* $\mathscr{M}$ *whose poles are the numbers* $w \in \mathscr{M}$ *such that* (-1) *is an eigenvalue of* $T_0(w)$*. Such* w *is called a* Resonance *of* H_1 *with multiplicity*

$$\mathrm{mult}(w) := \mathrm{Rank}\frac{1}{2i\pi}\int_{|z-w|=r} T_1(z)dz = \frac{1}{2i\pi}\mathrm{Tr}\int_{|z-w|=r} T_0'(z)\big(I + T_0(z)\big)^{-1}dz, \tag{5.1.6}$$

where $r > 0$ *is sufficiently small such that* $\{w;\ |z - w_0| \le r\}$ *contains a unique pole* w.

The set of resonances of H_1 will be denoted $\mathrm{Res}(H_1)$. For an overview on the theory of resonances, we refer for instance to [68].

5.2 Electric perturbation of 3D magnetic Hamiltonians

In this section we show how the results on the distribution of the eigenvalues described in Section 4.4.1 can be adapted for electric perturbations of the 3D magnetic Schrödinger operator, on the singularities of the Spectral Shift Function as well as on the distribution of the resonances.

5.2.1 The 3D magnetic Hamiltonians

Let us consider the 3D Schrödinger operator $H_S(A,0)$ with constant magnetic field of strength $b > 0$, pointing at the x_3-direction, corresponding to (2.7.57) with $d = 2$, $k = 1$:

$$H_0 := H_S(A,0) := H_\perp \otimes I_\| + I_\perp \otimes H_\| \tag{5.2.1}$$

with $H_\perp$ (resp. $H_\|$) the Landau hamiltonian (resp. the 1D Laplacian):

$$H_\perp := \left(D_1 + \frac{b}{2}x_2\right)^2 + \left(D_2 - \frac{b}{2}x_1\right)^2, \qquad H_\| = D_3^2, \qquad D_j := -i\frac{\partial}{\partial x_j}.$$

and $I_\|$, $I_\perp$, the identities in $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R})$ respectively.

The spectrum of H_0, purely a.c., is given by (2.7.61):

$$\sigma(H_0) = \bigcup_{q\in\mathbb{Z}_+} [\Lambda_q, +\infty) = [\Lambda_0, +\infty).$$

Assume that V is a bounded, real-valued, Lebesgue-measurable function and denote again by V the multiplier by this electric potential V. On the domain of H_0 the perturbed operator

$$H_V := H_S(A,V) = H_0 + V \tag{5.2.2}$$

is self-adjoint in $L^2(\mathbb{R}^3)$.

As in Section 2.7, for $x \in \mathbb{R}^3$ we write $x = (x_\perp, x_\|)$ where $x_\perp = (x_1, x_2) \in \mathbb{R}^2$ are the variables in the plane perpendicular to the magnetic field, and $x_\| = x_3 \in \mathbb{R}$ is the variable along the magnetic field. In the following sections, we will suppose that V satisfies one of the following estimates:

- D (anisotropic decay): $V(x) = O(\langle x_\perp\rangle^{-m_\perp}\langle x_\|\rangle^{-m_\|})$ with $m_\perp > 2$, $m_\| > 1$;
- D_0 (isotropic decay): $V(x) = O(\langle x\rangle^{-m_0})$ with $m_0 > 3$;
- D_{exp} (fast decay with respect to $x_\|$): $V(x) = O(\langle x_\perp\rangle^{-m_\perp}\exp(-N|x_\||))$ with some $m_\perp > 0$ and any $N > 0$.

Note that assumption D_0 implies D. Moreover, evidently, assumption D_{exp} with $m_\perp > 2$ again implies D. Thanks to the Diamagnetic inequality (see Section 2.5), as soon as V satisfies D, the operator VH_0^{-1} is compact in $L^2(\mathbb{R}^3)$ and applying the Weyl theorem on the invariance of the essential spectrum, we find that

$$\sigma_{ess}(H_V) = \sigma_{ess}(H_0) = \sigma(H_0) = [\Lambda_0, +\infty). \tag{5.2.3}$$

Thus H_V may have discret spectrum below Λ_0 and even if the essential spectra of the operators H_0 and H_V coincide as subsets of $\mathbb{R}$, their nature may be different. Note that for these properties, it suffices to assume that $m_\perp$ and $m_\|$ are positive constants. Before to quantify these differences through the study of the SSF and of the resonances, let us give particular cases showing that the presence of the constant magnetic field changes the nature of the essential spectrum (in comparison with $-\Delta+V$). In particular, embedded eigenvalues may exist even for compactly supported perturbations.

For instance, if V satisfies the estimate

$$V(x) \leq -C\mathbb{1}_U(x), \qquad x \in \mathbb{R}^3, \tag{5.2.4}$$

where $C > 0$ and $U \subset \mathbb{R}^3$ is an open non empty set, then the operator H_V has an infinite negative discrete spectrum (see e.g. [7, Theorem 1.5]). Next, if V is axisymmetric, i.e. depends only on $|x_\perp|$ and $x_\|$, and satisfies (5.2.4), then below each Landau level Λ_q, $q \in \mathbb{Z}_+$, the operator H_V has at least one eigenvalue which for all sufficiently large q is embedded in the essential spectrum (see [7, Theorem 1.5]). Finally, if V is axisymmetric and satisfies

$$V(x) \leq -C\mathbb{1}_{U_\perp}(x_\perp)\langle x_\| \rangle^{-m_\|}, \tag{5.2.5}$$

where $C > 0$, $m_\| \in (0,2)$ and $U_\perp \subset \mathbb{R}^2$ is an open non empty set, then there exists an infinite series of eigenvalues of H_V below each Landau level (see [155], [156]).

In the following sections, we show that such spectral concentration at the Landau levels is still true for a large class of electric potentials V. This general phenomena will be expressed by the blow up of the SSF or by the accumulation of resonances near Landau levels. As in the 2D case (see Section 4.4), near a fixed Landau level Λ_q the main contribution of the perturbation will be governed by the singular part of the Birman-Schwinger operator $T_0(z) := |V|^{\frac{1}{2}}(H_0-z)^{-1}V^{\frac{1}{2}}$. By analogy with the 2D case (see (4.4.13)), a key ingredient is the following spectral decomposition of the free resolvent. For p_q, the orthogonal projection onto $\mathrm{Ker}(H_\perp-\Lambda_q)$ introduced in Section 3.3, we have

$$(H_0-z)^{-1} = p_q \otimes (H_\| + \Lambda_q - z)^{-1} + (H_0-z)^{-1}(I - p_q \otimes I_\|). \tag{5.2.6}$$

Then, using that for $k^2 \in \mathbb{C} \setminus [0,+\infty)$, $\mathrm{Im}k > 0$, the integral kernel of $(H_\| - k^2)^{-1}$ is

$$-\frac{e^{ik|x_\|-x'_\||}}{2ik}, \quad x_\|, x'_\| \in \mathbb{R}, \tag{5.2.7}$$

for $z_q(k) = \Lambda_q + k^2$, $q \in \mathbb{Z}_+$, in the resolvent set of the operator H_0, we have

$$(H_0-\Lambda_q-k^2)^{-1} = -\frac{p_q \otimes r(ik)}{ik} + (H_0-\Lambda_q-k^2)^{-1}(I-p_q\otimes I_\|) \tag{5.2.8}$$

where r(z) is the 1D operator with the integral kernel $\frac{e^{z|x_\|-x'_\||}}{2}$. Hence, as we will see in the following sections, near each Landau level Λ_q, the singularities of the SSF and the distribution of the resonances will be governed by the operator $|V|^{\frac{1}{2}}\big(p_q \otimes r(0)\big)|V|^{\frac{1}{2}}$.

Thanks to the following abstract lemma, it will be related to the Berezin-Toeplitz operator $p_q W p_q$, with

$$W(x_\perp) := \frac{1}{2}\int_{\mathbb{R}} |V(x_\perp, x_\parallel)| dx_\parallel. \tag{5.2.9}$$

More precisely, applying the following Lemma 5.2.1 with appropriate L, we immediately find that for each $s > 0$

$$\mathrm{Tr}\, \mathbb{1}_{(s,\infty)} \left(|V|^{1/2} \left(p_q \otimes r_0 \right) |V|^{1/2} \right) = \mathrm{Tr}\, \mathbb{1}_{(s,\infty)} \left(p_q W p_q \right), \tag{5.2.10}$$

where $r_0 = r(0)$ denotes the 1D operator with constant integral kernel $\frac{1}{2}$.

Lemma 5.2.1 [20, Theorem 8.1.4] *Let* L *be a linear compact operator acting between two, possible different, Hilbert spaces. Then for each* $s > 0$ *we have*

$$\mathrm{Tr}\, \mathbb{1}_{(s,\infty)}(L^* L) = \mathrm{Tr}\, \mathbb{1}_{(s,\infty)}(L L^*).$$

5.2.2 Singularities of the spectral shift function near Landau levels

Let V satisfy D. Then the diamagnetic inequality easily implies that the operator $V^{1/2}(H_0 + 1)^{-1}$ is Hilbert–Schmidt (see Section 2.5), and hence, for $c > 0$ large enough, the resolvent difference $(H + cI)^{-1} - (H_0 + cI)^{-1}$ is a trace-class operator. Therefore, according to Theorem 5.1.1, the spectral shift function (SSF) for the operator pair (H_V, H_0),

$$\xi(H_V, H_0; \cdot) \in L^1(\mathbb{R}; (1+\lambda^2)^{-1} d\lambda)$$

exists, satisfies the Lifshits-Krein trace formula

$$\mathrm{Tr}\,(f(H_V) - f(H_0)) = \int_{\mathbb{R}} \xi(H_V, H_0; \lambda) f'(\lambda) d\lambda$$

for each $f \in C_0^\infty(\mathbb{R})$ and is unique with the normalization condition $\xi(H_V, H_0; \lambda) = 0$ for $\lambda \in (-\infty, \inf \sigma(H_V))$. By [39, Proposition 2.5], the SSF possesses the following more particular features:

- $\xi(H_V, H_0; \cdot)$ is bounded on every compact subset of $\mathbb{R} \setminus \cup_{q \in \mathbb{Z}_+} \{\Lambda_q\}$;
- $\xi(H_V, H_0; \cdot)$ is continuous on $\mathbb{R} \setminus (\cup_{q \in \mathbb{Z}_+} \{\Lambda_q\} \cup \sigma_{pp}(H_V))$ where $\sigma_{pp}(H_V)$ is the set of the eigenvalues of H.

In order to describe the asymptotic behavior of the SSF $\xi(H_V, H_0; \lambda)$ as $\lambda \to \Lambda_q$, $q \in \mathbb{Z}_+$, let us introduce some notations.

For V satisfying D, W denotes the function defined by (5.2.9) and for $x_\perp \in \mathbb{R}^2$, $\lambda \geq 0$, we introduce:

$$\mathscr{W}_\lambda = \mathscr{W}_\lambda(x_\perp) := \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

where

$$w_{11} := \frac{1}{2}\int_{\mathbb{R}} |V(x_\perp, x_3)| \cos^2(\sqrt{\lambda} x_3) dx_3, \quad w_{22} := \frac{1}{2}\int_{\mathbb{R}} |V(x_\perp, x_3)| \sin^2(\sqrt{\lambda} x_3) dx_3,$$

$$w_{12} = w_{21} := \frac{1}{2}\int_{\mathbb{R}} |V(x_\perp, x_3)| \cos(\sqrt{\lambda}x_3)\sin(\sqrt{\lambda}x_3)dx_3.$$

It follows from results of Section 4.2.2, that unless $V = 0$ almost everywhere, we have

$$\operatorname{rank} p_q W p_q = \infty, \quad \operatorname{rank} p_q \mathscr{W}_\lambda p_q = \infty, \; \lambda \geq 0.$$

In the following, if $F_j(V;\lambda)$, $j = 1,2$, are two real non decreasing functionals of V, depending on $\lambda > 0$, we write

$$F_1(V;\lambda) \sim F_2(V;\lambda), \quad \lambda \downarrow 0,$$

if for each $\varepsilon \in (0,1)$ we have

$$F_2((1-\varepsilon)V;\lambda) + O_\varepsilon(1) \leq F_1(V;\lambda) \leq F_2((1+\varepsilon)V;\lambda) + O_\varepsilon(1).$$

We also use analogous notations for non increasing functionals $F_j(V;\lambda)$ of V.

Theorem 5.2.1 [75, Theorems 3.1, 3.2] *Let* V *satisfy* D_0*, and* $V \geq 0$ *or* $V \leq 0$*. Then, below each Landau level* Λ_q*,* $q \in \mathbb{Z}_+$*, we have:*

$$\text{for} \quad V \geq 0, \qquad \xi(H_V, H_0; \Lambda_q - \lambda) = O(1), \quad \lambda \downarrow 0, \tag{5.2.11}$$

$$\text{for} \quad V \leq 0, \qquad \xi(H_V, H_0; \Lambda_q - \lambda) \sim -\operatorname{Tr} \mathbb{1}_{(\sqrt{\lambda},\infty)}\left(p_q W p_q\right), \; \lambda \downarrow 0. \tag{5.2.12}$$

Moreover, when approaching each Landau level Λ_q *from above,*

$$\text{for} \quad V \geq 0, \qquad \xi(H_V, H_0; \Lambda_q + \lambda) \sim \frac{1}{\pi}\operatorname{Tr} \arctan\left(\frac{p_q \mathscr{W}_\lambda p_q}{\sqrt{\lambda}}\right), \; \lambda \downarrow 0, \tag{5.2.13}$$

$$\text{for} \quad V \leq 0, \qquad \xi(H_V, H_0; \Lambda_q + \lambda) \sim -\frac{1}{\pi}\operatorname{Tr} \arctan\left(\frac{p_q \mathscr{W}_\lambda p_q}{\sqrt{\lambda}}\right), \; \lambda \downarrow 0. \tag{5.2.14}$$

Note that in the case $q = 0$ asymptotic relation (5.2.12) concerns the distribution of the discrete eigenvalues of the operator H_V with $V \leq 0$ near the first Landau level Λ_0 which coincides with the infimum of its essential spectrum. Such results on the discrete spectrum have been known for a long time, and could be found in:

- [194, 195, 202, 153, 101] in the case of *a power-like decay of* V;
- [159] in the case of *an exponential decay of* V;
- [159, 129] in the case of *compactly supported potentials* V.

As for the 2D case (Section 4.4), inserting the results of Theorems 4.2.1, 4.2.2, or 4.2.3, concerning the counting function for the Berezin-Toeplitz operators, into (5.2.12), (5.2.13), and (5.2.14), we could obtain the main asymptotic term of the SSF, $\xi(H_V, H_0; E)$ as $E \to \Lambda_q$. We omit here these explicit formulae referring the reader to the original work (see [75, Corollary 3.1]), and prefer to state here only the following intriguing

Corollary 5.2.1 ([156]) *Let* V *satisfy* D_0*, and* $V \leq 0$*. Fix* $q \in \mathbb{Z}_+$*. Then*

$$\lim_{\lambda\downarrow 0}\frac{\xi(H_V,H_0;\Lambda_q+\lambda)}{\xi(H_V,H_0;\Lambda_q-\lambda)} = \frac{1}{2\cos\frac{\pi}{\gamma}} \tag{5.2.15}$$

if W *satisfies the assumptions of Theorem 4.2.3, i.e. if* W *admits a power-like decay with decay rate* $\gamma > 2$*, or*

$$\lim_{\lambda\downarrow 0}\frac{\xi(H_V,H_0;\Lambda_q+\lambda)}{\xi(H_V,H_0;\Lambda_q-\lambda)} = \frac{1}{2} \tag{5.2.16}$$

if W *satisfies the assumptions of Theorem 4.2.2 or Theorem 4.2.1, i.e. if* W *decays exponentially*[1] *or has a compact support.*

Relations (5.2.15)–(5.2.16) could be interpreted as *generalized Levinson formulae*. We recall that the classical Levinson formula relates the number of the negative eigenvalues of $-\Delta+V$ with V which decays sufficiently fast at infinity, and $\lim_{\lambda\downarrow 0}\xi(-\Delta+V,-\Delta;\lambda)$ (see the original work [119] or the survey article [167]). The proof of such a Levinson formulae exploits high energy asymptotics of the SSF (see for instance [167]) and that the SSF has no singularities on $(0,+\infty)$. For magnetic Hamiltonians a high energy asymptotic also holds but we need to avoid neighborhoods of the Landau levels (see [40]).

Sketch of the proof of Theorem 5.2.1

We prove Theorem 5.2.1 by using the representation formula given by Proposition 5.1.1. Thus for V satisfying D, the norm limit

$$T_0(E) := \lim_{\delta\downarrow 0}|V|^{1/2}(H_0-E-i\delta)^{-1}|V|^{1/2}$$

exists for every $E \in \mathbb{R}\setminus\cup_{q\in\mathbb{Z}_+}\{\Lambda_q\}$, $T_0(E)$ is compact, $0 \leq \operatorname{Im} T_0(E) \in \mathfrak{S}_1$ (see [39, Lemma 4.2]), and the formula (5.1.4) holds for $\xi(H_V,H_0;E)$, $\pm V \geq 0$.

The first important step in the proof of Theorem 5.2.1 is the estimate

$$\pm\xi(H_V,H_0;E) \sim \frac{1}{\pi}\int_{\mathbb{R}} \operatorname{Tr} \mathbb{1}_{(1,\infty)}(\mp(\operatorname{Re} T_{0,q}(E)+t\operatorname{Im} T_{0,q}(E)))\frac{dt}{1+t^2}, \quad E\to\Lambda_q \tag{5.2.17}$$

where

$$T_{0,q}(E) := \lim_{\delta\downarrow 0}|V|^{1/2}(p_q\otimes(H_\parallel-E-i\delta)^{-1})|V|^{1/2} \qquad E\neq\Lambda_q.$$

It uses the decomposition (5.2.6) and combines the Weyl inequalities (4.2.4) with the fact that $|V|^{1/2}(H_0-E-i\delta)^{-1}(I-p_q\otimes I_\parallel)|V|^{1/2}$ admits a uniform limit as $\delta\downarrow 0$, for E near Λ_q.

Now, if $E=\Lambda_q-\lambda$ with $\lambda>0$, then $T_{0,q}(E)=T_{0,q}(E)^*$, and (5.2.17) implies

$$\pm\xi(H_V,H_0;E) \sim \operatorname{Tr} \mathbb{1}_{(1,\infty)}(\mp T_{0,q}(E)), \quad E\to\Lambda_q. \tag{5.2.18}$$

[1]In the case of exponential decay of W we should also suppose that V satisfies D with $m_\parallel > 2$.

Moreover, we have $T_{0,q}(E) \geq 0$, i.e. $\mathrm{Tr}\,\mathbb{1}_{(1,\infty)}(-T_{0,q}(E)) = 0$. Then (5.2.18) with the upper sign implies

$$\xi(E;H,H_0) = O(1), \quad E \uparrow \Lambda_q,$$

provided that $V \geq 0$, i.e. we obtain (5.2.11).

Assume now that $V \leq 0$. The second important step in the proof of Theorem 5.2.1 exploites the decomposition (5.2.8) which, with the Weyl inequalities (4.2.4), yields the estimate

$$\mathrm{Tr}\,\mathbb{1}_{(1,\infty)}(T_{0,q}(\Lambda_q - \lambda)) \sim \mathrm{Tr}\,\mathbb{1}_{(1,\infty)}\left(\frac{1}{\sqrt{\lambda}}|V|^{1/2}\left(p_q \otimes r_0\right)|V|^{1/2}\right), \quad \lambda \downarrow 0, \quad (5.2.19)$$

where r_0 denotes the operator with constant integral kernel $\frac{1}{2}$. Since

$$\mathrm{Tr}\,\mathbb{1}_{(1,\infty)}\left(\frac{1}{\sqrt{\lambda}}|V|^{1/2}\left(p_q \otimes r_0\right)|V|^{1/2}\right) = \mathrm{Tr}\,\mathbb{1}_{(\sqrt{\lambda},\infty)}\left(|V|^{1/2}\left(p_q \otimes r_0\right)|V|^{1/2}\right),$$

putting together (5.2.18), (5.2.19), and (5.2.10), we obtain (5.2.12).

Now for $E = \Lambda_q + \lambda$ with $\lambda \downarrow 0$, the main contribution of the singular term in (5.2.8) is given by the imaginary part. Then we obtain the estimate

$$\begin{aligned} \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\,\mathbb{1}_{(1,\infty)}(\mp(\mathrm{Re}\,T_{0,q}(E) + t\,\mathrm{Im}\,T_{0,q}(E)))\frac{dt}{1+t^2} &\sim \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\,\mathbb{1}_{(1,\infty)}(\mp t\,\mathrm{Im}\,T_{0,q}(E))\frac{dt}{1+t^2} \\ = \frac{1}{\pi}\mathrm{Tr}\arctan\left(\mathrm{Im}\,T_{0,q}(E)\right) &= \frac{1}{\pi}\mathrm{Tr}\arctan\left(|V|^{1/2}\left(p_q \otimes r_+(\lambda)\right)|V|^{1/2}\right) \qquad (5.2.20) \end{aligned}$$

where $r_+(\lambda)$ is the operator with integral kernel $\frac{\cos\sqrt{\lambda}(x_\parallel - x'_\parallel)}{2\sqrt{\lambda}}$, $x_\parallel, x'_\parallel \in \mathbb{R}$. Applying Lemma 5.2.1 with appropriate L, we get

$$\frac{1}{\pi}\mathrm{Tr}\arctan\left(|V|^{1/2}\left(p_q \otimes r_+(\lambda)\right)|V|^{1/2}\right) = \frac{1}{\pi}\mathrm{Tr}\arctan\left(\frac{p_q \mathscr{W}_\lambda p_q}{\sqrt{\lambda}}\right). \qquad (5.2.21)$$

Now the combination of (5.2.17), (5.2.20), and (5.2.21), yields (5.2.13)–(5.2.14). □

Theorem 5.2.1 admits extensions to Pauli and Dirac operators with admissible non constant magnetic fields (see definition in Section 2.8). In the case of the Pauli operator, the role of the Landau levels is played by the origin (see Proposition 2.8.3). The analogue of Theorem 5.2.1 could be found in [157]. Related results for negative energies (when the SSF is proportional to the eigenvalue counting function) are contained in [105].

In the case of the Dirac operator, the role of the Landau levels is played by the points $\pm m$ where $m > 0$ is the mass of the relativistic quantum particle. The analogue of Theorem 5.2.1 could be found in [209].

Similar singularities of the SSF also occur outside the Landau level for electric perturbations of magnetic Schrödinger operators having eigenvalues of infinite multiplicities embedded in the continuous spectrum (see [6]).

5.2.3 Distribution of resonances near Landau levels

In this section we still consider the magnetic Schrödinger operators introduced in Section 5.2.1. In order to define the resonances in the spirit of the Proposition 5.1.2, we assume that V satisfies D_{exp} (exponential fast decay with respect to $x_{\|}$).

First, using the decomposition (5.2.8) and the explicit expression of the integral kernel of the resolvent of $H_{\|}$, we have:

Proposition 5.2.1 ([23, Proposition 1]) *For* V *satisfying* D_{exp} *and* $q \in \mathbb{Z}_+$, *the operator valued function*

$$\widetilde{T}_{0,q} : k \longmapsto T_0(\Lambda_q + k^2) := J|V|^{\frac{1}{2}} \left(H_0 - \Lambda_q - k^2\right)^{-1} |V|^{\frac{1}{2}}, \qquad J := \mathrm{sign} V,$$

defined in $]0, \sqrt{2b}[e^{i]0,\pi/2[}$, *has an analytic extension to the set* $\mathscr{D} \setminus \{0\}$ *where* $\mathscr{D} := \{k \in \mathbb{C};\ 0 \leq |k| < \min(\sqrt{2b}, N)\}$.

From the above analytic extension, we deduce the holomorphic extension of T_0 on, $\mathscr{M}$, the infinite-sheeted Riemann surface of the countable family

$$\left\{\sqrt{z - \Lambda_q}\right\}_{q \in \mathbb{Z}_+}. \tag{5.2.22}$$

This Riemann surface $\mathscr{M}$, described in more details in [23, Section 2], can be viewed as the quotient of the universal covering of $\mathbb{C} \setminus \cup_{q \in \mathbb{Z}_+} \{\Lambda_q\}$ by the equivalence relation which identifies two points connected by a path going an even number of times round each Landau level. The global structure of $\mathscr{M}$ is quite complicated and may make difficult the analysis of the resonances. The investigation of their asymptotic distribution near a fixed Landau level Λ_q, $q \in \mathbb{Z}_+$, however is facilitated by the fact that in this case we are concerned with the local properties of $\mathscr{M}$. Thus, in a domain analytically diffeomorphic to a vicinity of Λ_q, the surface $\mathscr{M}$ resembles the two-sheeted Riemann surface of the square root $\sqrt{z - \Lambda_q}$. Namely, for z_q in a vicinity of Λ_q in $\mathscr{M}$, and $\mathscr{P}_G : \mathscr{M} \to \mathbb{C} \setminus \cup_{q \in \mathbb{Z}_+} \{\Lambda_q\}$ the covering corresponding to $\mathscr{M}$, there exists an analytic bijection $k \mapsto z_q(k) \in \mathscr{M}$, such that $\mathscr{P}_G(z_q(k)) = \Lambda_q + k^2$, $k \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, $|k| \ll 1$. Near a fixed Landau level Λ_q, we identify a point $z_q \in \mathscr{M}$ with $\Lambda_q + k^2$, $0 < |k| \ll 1$, and for $\varepsilon > 0$, we denote $\mathscr{M}_\varepsilon$ the set of points $z \in \mathscr{M}$ such that for each $q \in \mathbb{Z}_+$, we have $\mathrm{Im}(\sqrt{z - \Lambda_q}) > -\varepsilon$ (i.e. near each Λ_q, up to the identification $z_q(k) = \Lambda_q + k^2$, $\mathrm{Im}(k) > -\varepsilon$). We have $\cup_{\varepsilon > 0} \mathscr{M}_\varepsilon = \mathscr{M}$ and the holomorphic extension of the free resolvent given in Proposition 5.2.1 follows from the formula (5.2.8) (see [23, Proposition 1]).

Then, using the resolvent equation (5.1.5), from the Analytic Fredholm Theorem [162, Theorem VI.14], we deduce the following characterization of the resonances of H_V near each Landau level Λ_q, $q \in \mathbb{Z}_+$.

Proposition 5.2.2 ([23, Propositions 2-3], [24, Proposition 6.2]) *Let* V *satisfies* D_{exp}. *The complex number* $w_0 = \Lambda_q + k_0^2$ *is a resonance of* H_V *near* Λ_q, $q \in \mathbb{Z}_+$, *if and only if* $I + \widetilde{T}_{0,q}(k_0)$ *is not invertible, and the multiplicity of this resonance is given by* (5.1.6)*:*

$$\mathrm{mult}(w_0) = \frac{1}{2i\pi} \mathrm{Tr} \int_{|k-k_0|=r} \left(\frac{d}{dk} \widetilde{T}_{0,q}(k)\right) \left(I + \widetilde{T}_{0,q}(k)\right)^{-1} dk, \tag{5.2.23}$$

with $r > 0$ *sufficiently small.*

Moreover, it can be checked that it is equivalent to the definition of resonances as the poles of the meromorphic extension of the resolvent

$$R_V(z) = (H_V - z)^{-1} : e^{-\varepsilon\langle x_\parallel\rangle} L^2(\mathbb{R}^3) \longrightarrow e^{\varepsilon\langle x_\parallel\rangle} L^2(\mathbb{R}^3),$$

with multiplicity given by the rank of their residues, for all $\varepsilon < N$.

The previous results imply in particular that in any compact set which contains no Landau levels, the number of resonances is finite. However resonances can accumulate at the Landau levels. Before to prove that for perturbations of definite sign there is accumulation of resonances near each Λ_q, $q \in \mathbb{Z}_+$, let us give an upper bound on the number of resonances in an annulus centered at a Landau Level with inner (resp. outer) radius $r > 0$ (resp. $2r$) with $r \to 0$.

Theorem 5.2.2 ([23, Theorem 1]) *Let* V *satisfies* $D_{\exp}$ *and* W *defined by* (5.2.9). *Then near each Landau Level* Λ_q, *we have the following upper bound on the number of resonances in an annulus that contracts to* Λ_q*:*

$$\#\{z = \Lambda_q + k^2 \in \mathrm{Res}(H_V); r < |k| \leq 2r\} = O\left(|\ln r| \,\mathrm{Tr}\, \mathbb{1}_{(r,\infty)}\left(p_q W p_q\right)\right), \qquad r \downarrow 0.$$

We prove this upper bound by using a Jensen inequality (see [23, Lemma 6]). Thanks to the above definition of the multiplicity (introduced in [24]), the assumption $m_\perp > 2$ (made in [23, Theorem 1] in order to use the regularized determinant $\det_2$) is not necessary.

In order to give a result of existence of a infinite number of resonances and their asymptotic distributions near each Landau Level, let us introduce the following assumptions:

- $\mathscr{C}_{1,\pm}$: The potential V satisfies $D_{\exp}$ and is of definite sign, $\pm V \geq 0$;
- $\mathscr{C}_{2,q}$: The potential V *does not produce an isolated resonance at* Λ_q in the sense that the following limit exists for z in a sector $S_\delta := \{z \in \mathbb{C};\ \mathrm{Im}(z) > \delta|\mathrm{Re}(z) - \Lambda_q|\}$, $\delta > 0$:

$$\lim_{S_\delta \ni z \longrightarrow \Lambda_q} |V|^{\frac{1}{2}}(I + ik(p_q \otimes r_0))(H_V - z)^{-1}|V|^{\frac{1}{2}}, \tag{5.2.24}$$

  where $k = \sqrt{z - \Lambda_q}$, $\mathrm{Im}(k) > 0$, $\mathrm{Re}(k) > 0$.

Let us comment this second assumption $\mathscr{C}_{2,q}$, $q \in \mathbb{Z}_+$ fixed. From (5.2.8), $z = \Lambda_q$ is a singularity of $|V|^{\frac{1}{2}}(H_0 - z)^{-1}|V|^{\frac{1}{2}}$ given, for $z = \Lambda_q + k^2$ by the formula

$$|V|^{\frac{1}{2}}(H_0 - z)^{-1}|V|^{\frac{1}{2}} = -\frac{1}{ik}|V|^{\frac{1}{2}}(p_q \otimes r_0)|V|^{\frac{1}{2}} + |V|^{\frac{1}{2}}(I + ik(p_q \otimes r_0))(H_0 - z)^{-1}|V|^{\frac{1}{2}}.$$

for which we simply use that

$$|V|^{\frac{1}{2}}(p_q \otimes r_0)(H_0 - z)^{-1}|V|^{\frac{1}{2}} = -\frac{1}{k^2}|V|^{\frac{1}{2}}(p_q \otimes r_0)|V|^{\frac{1}{2}}.$$

Thus, the operator valued function $z \mapsto |V|^{\frac{1}{2}}(I + ik(p_q \otimes r_0))(H_V - z)^{-1}|V|^{\frac{1}{2}}$ is analytic near Λ_q for $H_V = H_0$ and the assumption $\mathscr{C}_{2,q}$ requires that Λ_q remains a regular point under perturbation by V. As we will see in Remark 5.2.1, this assumption holds for generic V.

Under the above assumptions, we obtain existence of resonances near each Landau Level, their concentration to a semi-axis and we have the asymptotic behavior of the distribution of resonances in an annulus centered at Λ_q with inner radius $r \downarrow 0$, in terms of the Berezin-Toeplitz operators $p_q W p_q$.

Theorem 5.2.3 ([24, Theorem 6.5]) *Let* $q \in \mathbb{Z}_+$. *Suppose* V *satisfies* $\mathscr{C}_{1,\pm}$ *and* $\mathscr{C}_{2,q}$ *and defined* W *by* (5.2.9). *Then, for* $0 < r_0 < \min(\sqrt{2b}, N)$ *fixed,*

i) *The resonances* $z_q(k) = \Lambda_q + k^2$ *of* H_V *with* $|k|$ *sufficiently small satisfy*

$$\pm \mathrm{Im}\, k \leq 0, \qquad \mathrm{Re}\, k = o(|k|).$$

ii) *There exists a sequence* $(r_\ell)_\ell \in \mathbb{R}$ *which tends to* 0 *such that*

$$\#\{z = \Lambda_q + k^2 \in \mathrm{Res}(H_V);\ r_\ell < |k| \leq r_0\} = \mathrm{Tr}\, \mathbb{1}_{(r_\ell, \infty)}\left(p_q W p_q\right)(1 + o(1)), \qquad \ell \to +\infty.$$

iii) *Eventually, if* W *satisfies the assumption of the Theorem 4.2.1, or 4.2.2, or 4.2.3, then*

$$\#\{z = \Lambda_q + k^2 \in \mathrm{Res}(H_V);\ r < |k| \leq r_0\} = \mathrm{Tr}\, \mathbb{1}_{(r, \infty)}\left(p_q W p_q\right)(1 + o(1)), \qquad r \searrow 0.$$

As for the SSF, the results on the counting function for the Berezin-Toeplitz operators (see Theorems 4.2.1, 4.2.2, or 4.2.3) imply asymptotic behaviors of the counting functions of resonances near Landau Levels.

Sketch of the proof of Theorem 5.2.3

The proof of Theorem 5.2.3 uses again the decomposition (5.2.8) and applies the following abstract results (Propositions 5.2.3 and 5.2.4). Let $\mathscr{D}$ be a domain of $\mathbb{C}$ containing 0, and let $\mathfrak{H}$ be a separable Hilbert space. Consider an analytic function

$$A : \mathscr{D} \longrightarrow \mathfrak{S}_\infty(\mathfrak{H}),$$

and $\mathscr{P}(A)$ the orthogonal projection onto $\mathrm{Ker}\, A(0)$.
In the sequel we will suppose that the following assumptions are fulfilled:

- $\widetilde{\mathscr{C}_1}$: The operator $A(0)$ is self-adjoint;
- $\widetilde{\mathscr{C}_2}$: The operator $I - A'(0)\mathscr{P}(A)$ is invertible.

Let $\Omega \subset \mathscr{D} \setminus \{0\}$. Define the characteristic values of $I - A(z)/z$ on Ω as the points $z \in \Omega$ for which the operator $I - A(z)/z$ is not invertible. We will denote the characteristic values of $I - A(z)/z$ on Ω by $\mathscr{Z}_A(\Omega)$. Since $z \mapsto (I - A(z)/z)$ is a finite meromorphic Fredholm function on Ω, thanks to an Analytic Fredholm Theorem (see for instance

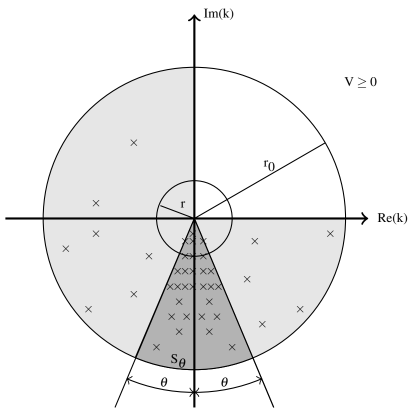


Figure 5.1: **Localization of the resonances in variable** k**:** For r_0 sufficiently small, the resonances $z_q(k) = \Lambda_q + k^2$ of the operators $H_0 + V$, $\pm V \geq 0$, near a Landau level Λ_q, $q \in \mathbb{Z}_+$, are concentrated in the sectors $S_\theta := \{k \in \mathbb{C};\, |\mathrm{Re}k| < \tan(\theta)\mathrm{Im}k\}$. For positive potential V, they are concentrated near the semi-axis $-i]0,+\infty)$ in both sides, while they are concentrated near the semi-axis $i]0,+\infty)$ on the left for negative V.

[84, Proposition 4.1.4]) $\mathscr{Z}_A(\Omega)$ is a discrete set and $z \mapsto (I - A(z)/z)^{-1}$ is finite meromorphic on $\Omega \setminus \mathscr{Z}_A(\Omega)$. The multiplicity of $z_0 \in \mathscr{Z}_A(\Omega)$ is defined by

$$\mathrm{Mult}(z_0) := \frac{1}{2\pi i}\mathrm{Tr}\int_\gamma \Big(I - \frac{A(z)}{z}\Big)'\Big(I - \frac{A(z)}{z}\Big)^{-1} dz$$

where γ is an appropriate circle centered at z_0.

Obviously, if $A(z) = A(0)$ (i.e. A is constant), then $\mathscr{Z}_A(\Omega) = \sigma(A) \cap \Omega$. The aim of the following propositions is to state that under the assumptions $\widetilde{\mathscr{C}_1}$ and $\widetilde{\mathscr{C}_2}$, $\mathscr{Z}_A(\mathscr{D} \setminus \{0\})$ is close to $\sigma(A(0))$, the spectrum of the compact self-adjoint operator $A(0)$.

Proposition 5.2.3 ([24, Corollary 3.4]) *Assume* $\widetilde{\mathscr{C}_1}$ *and* $\widetilde{\mathscr{C}_2}$. *Suppose that the origin is an accumulation point of* $\mathscr{Z}_A(\mathscr{D} \setminus \{0\})$. *Then we have*

$$|\mathrm{Im}\, z_0| = o(|z_0|), \quad z_0 \in \mathscr{Z}_A(\mathscr{D} \setminus \{0\}),$$

as $z_0 \to 0$. *If, moreover,* $\pm A(0) \geq 0$, *then* $\pm \mathrm{Re}\, z_0 \geq 0$ *for* $z_0 \in \mathscr{Z}_A(\mathscr{D} \setminus \{0\})$ *with* $|z_0|$ *small enough.*

Set

$$\mathscr{N}_A(\Omega) := \sum_{z_0 \in \mathscr{Z}_A(\Omega)} \mathrm{Mult}(z_0).$$

If $\partial\Omega$ is sufficiently regular, and $\mathscr{Z}_A(\Omega) \cap \partial\Omega = \emptyset$, then we have

$$\mathscr{N}_A(\Omega) = \mathrm{ind}_{\partial\Omega}\Big(I - \frac{A(z)}{z}\Big) := \frac{1}{2\pi i}\mathrm{Tr}\int_{\partial\Omega}\Big(I - \frac{A(z)}{z}\Big)'\Big(I - \frac{A(z)}{z}\Big)^{-1} dz.$$

The index $\mathrm{ind}_{\partial\Omega}\left(I-\frac{A(z)}{z}\right)$ plays a central role in the proof of Theorem 5.2.3. More information about its properties could be found in [85], [84, Section 4], and [24, Section 2] (see also [191] where the notion of index allows to define generalized determinants). For $0 < a < b < \infty$ and $\theta > 0$ define the domain

$$C_\theta(a,b) := \{x+iy \in \mathbb{C} \,|\, a < x < b, \quad |y| < \theta x\}. \tag{5.2.25}$$

Proposition 5.2.4 ([24, Corollary 3.11]) *Assume $\widetilde{\mathscr{C}_1}$ and $\widetilde{\mathscr{C}_2}$. Suppose moreover that*

$$\mathrm{Tr}\,\mathbf{1}_{(r,\infty)}(A(0)) = \Phi(r)(1+o(1)), \quad r \downarrow 0,$$

where Φ satisfies $\Phi(r) \to \infty$ as $r \downarrow 0$, and

$$\Phi(r(1\pm\delta)) = \Phi(r)(1+o(1)+O(\delta)), \quad r \downarrow 0, \tag{5.2.26}$$

for each sufficiently small $\delta > 0$. Then we have

$$\mathscr{N}_A(C_\theta(r,1)) = \Phi(r)(1+o(1)), \quad r \downarrow 0,$$

for any $\theta > 0$.

It is easy to check that the functions $\Phi(r) = Cr^{-\gamma}$, $\Phi(r) = C|\ln r|^\gamma$, or $\Phi(r) = C\frac{|\ln r|}{\ln|\ln r|}$, with some $\gamma, C > 0$, satisfy asymptotic relation (5.2.26). Hence, for W satisfying the assumption of the Theorem 4.2.1, (resp. Theorem 4.2.2, resp. Theorem 4.2.3), the main contribution in the asymptotic expansion of $\mathrm{Tr}\,\mathbb{1}_{(r,\infty)}(p_q W p_q)$ in (4.2.13) (resp. (4.2.40), resp. (4.2.58)) satisfies it as well.
Now the proof of Theorem 5.2.3 consists to apply Propositions 5.2.3 and 5.2.4 using that $z_q(k_0) = \Lambda_q + k_0^2$ is a resonance of H_V if and only if (ik_0) is a characteristic value of $I + A_q(z)/z$, with the same multiplicity (i.e. $\mathrm{mult}(z_q(k_0)) = \mathrm{Mult}(ik_0)$), where

$$A_q(z) = J|V|^{\frac12} p_q \otimes r(z)|V|^{\frac12} + zJ|V|^{\frac12} R_0(\Lambda_q - z^2)(I - p_q \otimes I_{\|})|V|^{\frac12}. \tag{5.2.27}$$

Indeed, thanks to the decomposition (5.2.8), for $k \in \mathscr{D}\setminus\{0\}$, $\mathscr{D} := \{k \in \mathbb{C};\ 0 \le |k| < \min(\sqrt{2b}, N)\}$, we have

$$I + \widetilde{T}_{0,q}(k) = I + \frac{A_q(ik)}{ik},$$

where $z \longmapsto A_q(z) \in \mathfrak{S}_\infty(L^2(\mathbb{R}^3))$ is holomorphic on $\mathscr{D}$ and $A_q(0) = J|V|^{\frac12} p_q \otimes r(0)|V|^{\frac12}$ is self-adjoint as soon as $J = \pm I$ (i.e. $\widetilde{\mathscr{C}_1}$ is fullfield), with $\mathrm{Tr}\,\mathbb{1}_{(r,\infty)}(JA_q(0)) = \mathrm{Tr}\,\mathbb{1}_{(r,\infty)}(p_q W p_q)$ (see (5.2.10)). To conclude, we just have to prove that V satifies $\mathscr{C}_{2,q}$ if and only if $I + JA_q'(0)\mathscr{P}(JA_q)$ is invertible (i.e. $\widetilde{\mathscr{C}_2}$ is fullfield for $-JA_q$). For this technical point, we refer to [42, Section 4.4].

□

Remark 5.2.1 *In general,* $\ker A_q(0)$ *is not trivial. Nevertheless, the assumption $\mathscr{C}_{2,q}$ holds for generic V. More precisely, if the potential V is fixed, there exists a finite or infinite discrete set $\mathscr{E} = \{e_n\}$ such that the operator $H_{eV} = H_0 + eV$ satisfies $\mathscr{C}_{2,q}$ for* *all* $e \in \mathbb{R} \setminus \mathscr{E}$. *The numbers* $1/e_n$ *are in fact the real non vanishing eigenvalues of the compact operator* $A_q'(0)\mathscr{P}(A_q)$. *To check this, it is enough to remark that* $\mathscr{P}(A_q)$ *is independent of* e *and* $A_q'(0)_{|eV} = eA_q'(0)_{|V}$ *for* $e \neq 0$. *Note also that, for* $|e|$ *small enough,* H_{eV} *satisfies always* $\mathscr{C}_{2,q}$.

Remark 5.2.2 *These results can be generalized to the case of constant magnetic fields of non full rank* $2d$ *in an arbitrary dimension* n *(see definition in Section 2.7). More precisely, the situation* $n-2d=1$ *is close to the one treated here. Whereas, if* $n-2d \geq 3$ *is odd, it is expected that there is no accumulation of resonances at the Landau levels since the corresponding* $A(z)$ *is analytic near these thresholds. The case* $n-2d$ *even is different since the weighted resolvent has a logarithmic singularity at the Landau levels.*

Remark 5.2.3 *Since eigenvalues embedded in the continuous spectrum are among the real resonances, let us remark that a consequence of Theorem 5.2.3* i) *at each Landau level is that for non-negative potentials* V *(satisfying assumption of Theorem 5.2.3), the (embedded) eigenvalues of* H_V *form a discrete set. On the other hand it is known that for small potentials* $V \geq 0$ *satisfying* D *with* $m_{\|} > 0$ *there are no eigenvalues outside of the Landau levels (see [23, Proposition 7]).*

Recall that the setting is very different for non-positive perturbations. Indeed, for a large class of non-positive potentials, there is an accumulation of embedded eigenvalues at each Landau level (see (5.2.5) *and the references [155], [156]).*

Let us mention that for potentials which are dilation-analytic with respect to the variable along the magnetic field it is possible to study the resonances near the real axis. It is done in [213] in the strong magnetic field regime when the electric potential has a high barrier. In [6], near eigenvalues of infinite multiplicities embedded in the continuous spectrum of a reference operator H_0, the resonances of H_0+V are also defined and studied using two approaches: the classical analytic dilation and a dynamical one based on appropriate Mourre estimates (as in [46]). For this model, the distribution of the resonances near the embedded eigenvalues of infinite multiplicities is studied in [108].

An extension of Theorem 5.2.3 to Pauli and Dirac operators with admissible non constant magnetic fields (defined in Section 2.8) is given in [174] near the origin for the Pauli operator and near the points $\pm m$ (m is the mass of the relativistic quantum particle) for the Dirac operator. Similar technics provide also results on the discrete spectrum for non-selfadjoint perturbations (i.e. complex-valued potentials), see the works [175, 176, 177, 178, 179].

5.2.4 Link between the SSF and the resonances

The SSF and the resonances are usually connected by the Breit–Wigner formula. Such a formula, proved in [143, 144, 145, 37] in the non-magnetic setting, represents the derivative of the SSF as a sum of a harmonic measure associated to the resonances, and the imaginary part of a holomorphic function. It can be exploited to obtain asymptotic expansions of the SSF or to localize resonances. Eventually, it can imply local trace formulas in the spirit of [190]. Such a Breit–Wigner formula also holds near the Landau levels. We have the following result which shows that on small intervals $\Lambda_q \pm (r, 2r)$ the derivative of the SSF is related to the resonances in a sector $\pm C_\theta(r, 2r)$ (defined by (5.2.25)):

Theorem 5.2.4 [23, Theorem 3] *Let* V *satisfies* $D_{\exp}$ *with* $m_\perp > 2$. *Then for* $q \in \mathbb{Z}_+$ *and* $\varepsilon, \theta > 0$, *there exist* $r_0 > 0$ *and functions* $g_\pm(\cdot, r)$ *holomorphic in* $\pm C_\theta(1,2)$, *such that, for* $\lambda \in r[1+\varepsilon, 2-\varepsilon]$, *we have*

$$\xi'(\Lambda_q \pm \lambda; H_V, H_0) = \sum_{\substack{\Lambda_q \pm w \in \mathrm{Res}(H_V) \\ w \in rC_\theta(1,2)\setminus\mathbb{R}}} \frac{\operatorname{Im} w}{\pi|\lambda - w|^2} - \sum_{\substack{\Lambda_q \pm w \in \mathrm{Res}(H_V) \\ w \in r[1,2]}} \delta(\lambda - w) + \frac{1}{r} \operatorname{Im} g'_\pm\Big(\frac{\lambda}{r}, r\Big),$$

where $g_\pm(z, r)$ *satisfies the estimate*

$$g_\pm(z, r) = O\Big(|\ln r|\, r^{-\frac{1}{m_\perp}}\Big),$$

uniformly with respect to $0 < r < r_0$ *and* $z \in C_\theta(1+\varepsilon, 2-\varepsilon)$.

The proof of this formula and more general statement with some applications can be found in [23, Section 5]. Note that this Breit–Wigner approximation implies that the SSF is analytic outside of the resonances (including the embedded eigenvalues) and their complex conjugate.

5.3 Obstacle perturbation of the 3D magnetic Schrödinger operator

In this section, we still consider the 3D magnetic Schrödinger operator, but now the perturbation is a bounded obstacle with Dirichlet or Neumann boundary condition. As in the previous section, we study the singularities of the Spectral Shift Function and the distribution of the resonances.

5.3.1 The Dirichlet and Neumann realizations of the 3D magnetic Schrödinger operators

We consider the free 3D Schrödinger operator $H_S(A, 0)$ with constant magnetic field of strength $b > 0$, pointing at the x_3-direction, corresponding to (2.7.57) with $d = 2$, $k = 1$:

$$H_S(A,0) := \sum_{j=1}^{3} \Pi_j(A)^2 = \Big(-i\frac{\partial}{\partial x_1} + \frac{bx_2}{2}\Big)^2 + \Big(-i\frac{\partial}{\partial x_2} - \frac{bx_1}{2}\Big)^2 - \frac{\partial^2}{\partial x_3^2} \qquad (5.3.1)$$

where we use the notations of Section 2.5:

$$\Pi(A) = (\Pi_1(A), \Pi_2(A), \Pi_3(A)) := -i\nabla - A,$$

with the magnetic potential $A(x) := \frac{b}{2}(-x_2, x_1, 0)$.

In order to define realizations of $H_S(A,0)$ on domains of $\mathbb{R}^3$, let us define the magnetic Sobolev spaces on Ω, an open non-empty subset of $\mathbb{R}^3$:

$$H^s_A(\Omega) := \left\{ u \in \mathscr{D}'(\Omega) \,|\, \Pi(A)^\alpha u \in L^2(\mathbb{R}^3),\ \alpha \in \mathbb{Z}^3_+, 0 \le |\alpha| \le s \right\}, \quad s \in \mathbb{Z}_+.$$

Denote by $H^s_{A,0}(\Omega)$ the closure of $C^\infty_0(\Omega)$ in the norm of $H^s_A(\Omega)$ defined by

$$\|u\|^2_{H^s_A(\Omega)} := \sum_{\alpha\in\mathbb{Z}^3_+:0\le|\alpha|\le s} \int_\Omega |\Pi(A)^\alpha u|^2 \, dx.$$

Then, as in the previous section, the operator $H_0 := H_S(A,0)$ with domain $\mathfrak{D}(H_0) := H^2_A(\mathbb{R}^3)$ is self-adjoint in $L^2(\mathbb{R}^3)$, and essentially self-adjoint on $C^\infty_0(\mathbb{R}^3)$ with purely a.c. spectrum, given by (2.7.61): $\sigma(H_0) = [\Lambda_0, +\infty) = [b, +\infty)$.

Let us introduce the obstacle perturbation. We define a *domain* in $\mathbb{R}^d$, $d \ge 1$, as an open, connected, non-empty subset of $\mathbb{R}^d$. Let $\Omega_{in} \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega_{in} \in C^\infty$. Set

$$\Gamma := \partial\Omega_{in}, \quad \Omega_{ex} := \mathbb{R}^3 \setminus \overline{\Omega_{in}}.$$

Then the operator $H_{+,\natural} := H_S(A,0)$, $\natural = ex, in$, with domain

$$\mathfrak{D}(H_{+,\natural}) := \left\{ u \in H^2_A(\Omega_\natural) \,|\, u_{|\Gamma} = 0 \right\},$$

is the *Dirichlet* realization of $H_S(A,0)$ on $\Omega_\natural$. Similarly, if ν is the unit normal vector at Γ, outward looking with respect to Ω_{in}, then the operator $H_{-,\natural} := H_S(A,0)$, $\natural = ex, in$, with domain

$$\mathfrak{D}(H_{-,\natural}) := \left\{ u \in H^2_A(\Omega_\natural) \,|\, \nu \cdot \Pi(A) u_{|\Gamma} = 0 \right\},$$

is the *Neumann* realization of $H_S(A,0)$ on $\Omega_\natural$. The operators $H_{\pm,\natural}$, $\natural = ex, in$, are self-adjoint in $L^2(\Omega_\natural)$. Moreover, $H_{+,\natural}$ (resp., $H_{-,\natural}$) corresponds to the closed quadratic form

$$\int_{\Omega_\natural} |\Pi(A) u|^2 \, dx \tag{5.3.2}$$

with domain $H^1_{A,0}(\Omega_\natural)$ (resp., $H^1_A(\Omega_\natural)$).

The operators $H_{\pm,in}$ are second-order elliptic partial differential operators acting in a bounded domain with smooth boundary, then their spectrums $\sigma(H_{\pm,in})$ are discrete. The exterior problems can be considered as compactly supported perturbations of H_0 and then the essential spectrum of $H_{\pm,ex}$ coincides with that of H_0:

$$\sigma_{ess}(H_{+,ex}) = \sigma_{ess}(H_{-,ex}) = \sigma_{ess}(H_0) = \sigma(H_0) = [b, +\infty).$$

In the following sections, we show that spectral concentration at the Landau levels is still true for $H_{\pm,ex}$. As for the electric perturbations, it will be expressed by the blow up of the SSF or by the accumulation of resonances near Landau levels. The results for $H_{\pm,ex}$ will be close to those for the Schrödinger operators $H_0 \pm \mathbb{1}_{\Omega_{in}}$, but a first difficulty is to define the SSF and the resonances for pairs $(H_0, H_{\pm,ex})$ whose operators are not defined on the same domains. To overcome this difficulty, using the orthogonal decomposition $L^2(\mathbb{R}^3) = L^2(\Omega_{in}) \oplus L^2(\Omega_{ex})$, we introduce the auxiliary operators:

$$H_\pm := H_{\pm,in} \oplus H_{\pm,ex}.$$

By the Dirichlet-Neumann bracketing and the non-negativeness of the quadratic forms (5.3.2), we have

$$H_+ \geq H_0 \geq H_- \geq 0. \tag{5.3.3}$$

Even if these operators act in $L^2(\mathbb{R}^3)$ their domains are different and $(H_\pm - H_0)$ is not well defined as a selfadjoint operator. An adaptation of the general outline of Section 5.1 consists to work with the inverses (or resolvents) of the operators $H_\pm$ and H_0. Since $b > 0$, the invertibility of the operators H_0 and H_+ is a consequence of (5.3.3). It is not difficult to see that H_- is invertible as well. To this end, we use that $\sigma_{ess}(H_-) = \sigma_{ess}(H_0) = [b, +\infty)$, then $0 \in \sigma(H_-)$ should be a discrete eigenvalue of H_-. Let $u \in \mathfrak{D}(H_-)$ such that $H_- u = 0$. By (5.3.2), we have

$$\Pi(A)u_{|\Omega_{in}} = 0, \quad \Pi(A)u_{|\Omega_{ex}} = 0. \tag{5.3.4}$$

Taking into account the explicit expression for A, we find that the only element $u \in \mathfrak{D}(H_-)$ which satisfies (5.3.4), is $u = 0$, and hence $0 \notin \sigma(H_-)$.
Further, (5.3.3) implies

$$H_-^{-1} \geq H_0^{-1} \geq H_+^{-1}, \tag{5.3.5}$$

and the operators

$$V_+ := H_0^{-1} - H_+^{-1}, \quad V_- := H_-^{-1} - H_0^{-1},$$

are non-negative, bounded, selfadjoint operators on $L^2(\mathbb{R}^3)$. Then, having in mind that λ is in the spectrum of an operator if and only if λ^{-1} is in the spectrum of its inverse, we introduce the Birman-Schwinger operators associated to $H_\pm^{-1} = H_0^{-1} \mp V_\pm$ at z^{-1}:

$$T^\pm(z) := V_\pm^{\frac{1}{2}}(H_0^{-1} - z^{-1})^{-1} V_\pm^{\frac{1}{2}}, \qquad z \in \mathbb{C}_-. \tag{5.3.6}$$

Obviously, combining the relation:

$$T^\pm(z) = -zV_\pm^{\frac{1}{2}} H_0 (H_0 - z)^{-1} V_\pm^{\frac{1}{2}} = -zV_\pm - z^2 V_\pm^{\frac{1}{2}} (H_0 - z)^{-1} V_\pm^{\frac{1}{2}}, \tag{5.3.7}$$

with the formula (5.2.8), we see that the Landau levels are singularities of $T^\pm$. As for the electric perturbations, these singularities will be the origin of the following spectral concentrations for the obstacle problem.

5.3.2 The spectral shift function for the exterior problems

In order to define the spectral shift function $\xi(H_\pm, H_0; \cdot)$, by exploiting that $H_\pm$ coincide with H_0 outside the boundary of Ω_{in}, we prove that $V_\pm$ contains terms localized near $\partial\Omega_{in}$, then that $V_\pm \in \mathfrak{S}_2$ and $H_\pm^{-2} - H_0^{-2} \in \mathfrak{S}_1$ (see [41, Proposition 2.1]).

Then, we define the spectral shift function $\xi(H_\pm, H_0; \cdot)$ as

$$\xi(H_\pm, H_0; E) := \begin{cases} -\xi(H_\pm^{-2}, H_0^{-2}; E^{-2}) & \text{if} \quad E > \inf \sigma(H_\pm), \\ 0 \quad \text{if} \quad E < \inf \sigma(H_\pm), \end{cases}$$

where, for almost every $E > 0$,

$$\xi(H_\pm^{-2}, H_0^{-2}; E^{-2}) := \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \arg \det \left(\left(H_\pm^{-2} - E^{-2} - i\varepsilon \right) \left(H_0^{-2} - E^{-2} - i\varepsilon \right)^{-1} \right), \qquad (5.3.8)$$

the branch of the argument being fixed by the condition

$$\lim_{\mathrm{Im}\, z \to \infty} \arg \det \left(\left(H_\pm^{-2} - z \right) \left(H_0^{-2} - z \right)^{-1} \right) = 0.$$

The SSF $\xi(H_\pm, H_0; \cdot)$ is the unique element of $L^1_{loc}(\mathbb{R})$ which satisfies the *Lifshits-Krein identity*

$$\mathrm{Tr}\, (f(H_\pm) - f(H_0)) = \int_{\mathbb{R}} f'(E)\, \xi(H_\pm, H_0; E)\, dE, \quad f \in C_0^\infty(\mathbb{R}),$$

and the normalization condition

$$\xi(H_\pm, H_0; E) = 0, \quad E < \inf \sigma(H_\pm).$$

Morever, for almost every $E \in (0, b)$, $\xi(E; H_-, H_0) = -\mathrm{Tr}\, \mathbb{1}_{(-\infty, E)}(H_-)$.

Our next goal is to introduce a canonic representative of the class of equivalence $\xi(H_\pm, H_0; \cdot)$ analogous to (5.1.4). Using (5.3.7) and localization properties of $V_\pm$, we have:

Proposition 5.3.1 ([41, Proposition 2.2]) *Let* $E \in (0, \infty) \setminus \cup_{q \in \mathbb{Z}_+} \{\Lambda_q\}$. *Then there exists a norm limit*

$$T^\pm(E) := n - \lim_{\mathbb{C}_- \ni z \to E} T^\pm(z) \in \mathfrak{S}_2, \qquad (5.3.9)$$

and

$$\mathrm{Im}\, T^\pm(E) \in \mathfrak{S}_1. \qquad (5.3.10)$$

Moreover, $\mathrm{Re}\, T^\pm(E)$ (*resp.,* $\mathrm{Im}\, T^\pm(E)$) *depends continuously in* $\mathfrak{S}_2$ *(resp., in* $\mathfrak{S}_1$*) on* $E \in (0, \infty) \setminus \cup_{q \in \mathbb{Z}_+} \{\Lambda_q\}$.

For $E \in (0, \infty) \setminus \cup_{q \in \mathbb{Z}_+} \{\Lambda_q\}$ set

$$\tilde{\xi}(E; H_\pm, H_0) := \pm \frac{1}{\pi} \int_{\mathbb{R}} \mathrm{Tr} \mathbb{1}_{(1, \infty)} \left(\pm \left(\mathrm{Re}\, T^\pm(E) + t\, \mathrm{Im}\, T^\pm(E) \right) \right) \frac{dt}{1 + t^2}. \qquad (5.3.11)$$

We can not apply directly the results used in Section 5.2.2 (because the operators have different domains), but following arguments of [148] we show that we can identify $\xi(H_\pm, H_0; E)$ with $\tilde{\xi}(H_\pm, H_0; E)$, defined for every $E \in (0, \infty) \setminus \cup_{q \in \mathbb{Z}_+} \{\Lambda_q\}$.

Proposition 5.3.2 ([41, Proposition 2.3]) *The function* $\tilde{\xi}(\cdot;H_{\pm},H_0)$ *is well defined on* $(0,\infty)\setminus\cup_{q\in\mathbb{Z}_+}\{\Lambda_q\}$*, bounded on every compact subset of* $(0,\infty)\setminus\cup_{q\in\mathbb{Z}_+}\{\Lambda_q\}$*, and continuous on* $(0,\infty)\setminus(\sigma_p(H_{\pm})\cup_{q\in\mathbb{Z}_+}\{\Lambda_q\})$ *where* $\sigma_p(H_{\pm})$ *denotes the set of the eigenvalues of* $H_{\pm}$.
Moreover, for almost every $E\in(0,\infty)$ *we have*

$$\xi(H_{\pm},H_0;E)=\tilde{\xi}(H_{\pm},H_0;E). \tag{5.3.12}$$

As for the electric perturbations, we study the asymptotic behavior of the SSF at the Landau levels.

For $x\in\mathbb{R}^3$, we write $x=(x_{\perp},x_{\parallel})$ where $x_{\perp}=(x_1,x_2)\in\mathbb{R}^2$ are the variables on the plane perpendicular to the magnetic field B while $x_{\parallel}=x_3\in\mathbb{R}$ is the variable along B. For $x=(x_{\perp},x_{\parallel})\in\mathbb{R}^3$ define the projections $\pi_{\perp}(x):=x_{\perp}$, $\pi_{\parallel}(x):=x_{\parallel}$. Note that if $\Omega\subset\mathbb{R}^3$ is a (bounded) domain, then $\pi_{\perp}(\Omega)\subset\mathbb{R}^2$ is a (bounded) domain as well. Set

$$\mathcal{O}_{in}:=\pi_{\perp}(\Omega_{in}).$$

Thus, $\mathcal{O}_{in}$ is the projection of the obstacle Ω_{in} onto the plane perpendicular to the magnetic field B. The following result involves $\mathrm{Cap}(\mathcal{O}_{in})$, the Logarithmic Capacity of $\mathcal{O}_{in}$ (introduced in Section 4.2.2).
For $\lambda>0$ small enough, and $C\in\mathbb{R}$ set

$$\ln_2(\lambda):=\ln|\ln\lambda|,\quad \ln_3(\lambda):=\ln\ln_2(\lambda),$$

and

$$\Phi_0(\lambda):=\frac{|\ln\lambda|}{\ln_2(\lambda)},\quad \Phi_1(\lambda;C):=\Phi_0(\lambda)\left(1+\frac{\ln_3(\lambda)}{\ln_2(\lambda)}+\frac{C}{\ln_2(\lambda)}\right).$$

Theorem 5.3.1 ([41, Theorem 3.1]) *Let* Ω_{in} *be a bounded domain with* $\partial\Omega_{in}\in C^{\infty}$. *Fix* $q\in\mathbb{Z}_+$. *Then, as* $\lambda\downarrow 0$,

$$\xi(H_+,H_0;\Lambda_q-\lambda)=O(1), \tag{5.3.13}$$

$$\xi(H_-,H_0;\Lambda_q-\lambda)=-\frac{1}{2}\Phi_1(\lambda;\mathrm{Cap}(\mathcal{O}_{in}))+o\left(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\right), \tag{5.3.14}$$

$$\xi(H_{\pm},H_0;\Lambda_q+\lambda)=\pm\frac{1}{4}\Phi_1(\lambda;\mathrm{Cap}(\mathcal{O}_{in}))+o\left(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\right). \tag{5.3.15}$$

Remark 5.3.1 *From* (5.3.14) *and* (5.3.15) *with sign "–" we deduce the following generalized Levinson formula (analogue of* (5.2.16)*):*

$$\lim_{\lambda\downarrow 0}\frac{\xi(;H_-,H_0;\Lambda_q+\lambda)}{\xi(H_-,H_0;\Lambda_q-\lambda)}=\frac{1}{2}. \tag{5.3.16}$$

Remark 5.3.2 *By the so-called telescopic property of the SSF, we have*

$$\xi(H_+,H_-;E)=\xi(H_+,H_0;E)-\xi(H_-,H_0;E),\quad E\in(0,\infty)\setminus\cup_{q\in\mathbb{Z}_+}\{\Lambda_q\}.$$

Therefore, (5.3.13) - (5.3.14) *imply*

$$\xi(\mathrm{H}_+, \mathrm{H}_-; \Lambda_q - \lambda) = \frac{1}{2}\Phi_1(\lambda;\ \mathrm{Cap}(\mathscr{O}_{\mathrm{in}})) + o\left(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\right), \quad \lambda \downarrow 0,$$

while (5.3.15) *implies*

$$\xi(\mathrm{H}_+, \mathrm{H}_-; \Lambda_q + \lambda) = \frac{1}{2}\Phi_1(\lambda; \mathrm{Cap}(\mathscr{O}_{\mathrm{in}})) + o\left(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\right), \quad \lambda \downarrow 0.$$

In particular, similarly to (5.3.16),

$$\lim_{\lambda\downarrow 0} \frac{\xi(\mathrm{H}_+, \mathrm{H}_-; \Lambda_q - \lambda)}{\xi(\mathrm{H}_+, \mathrm{H}_-; \Lambda_q + \lambda)} = 1.$$

Remark 5.3.3 *Since, we have*

$$\xi(\mathrm{H}_-, \mathrm{H}_0; \Lambda_0 - \lambda) = -\mathrm{Tr}\, \mathbb{1}_{(-\infty, \Lambda_0 - \lambda)}(\mathrm{H}_-) = \tag{5.3.17}$$

$$-\mathrm{Tr}\, \mathbb{1}_{(-\infty, \Lambda_0 - \lambda)}(\mathrm{H}_{-,\mathrm{ex}}) - \mathrm{Tr}\, \mathbb{1}_{(-\infty, \Lambda_0 - \lambda)}(\mathrm{H}_{-,\mathrm{in}}), \quad \lambda > 0.$$

and

$$\mathrm{Tr}\, \mathbb{1}_{(-\infty, \Lambda_0 - \lambda)}(\mathrm{H}_{-,\mathrm{in}}) = O(1), \quad \lambda \downarrow 0,$$

($\sigma(\mathrm{H}_{-,\mathrm{in}})$ *being discrete), then* (5.3.14) *with* $q = 0$ *implies*

$$\mathrm{Tr}\, \mathbb{1}_{(-\infty, \Lambda_0 - \lambda)}(\mathrm{H}_{-,\mathrm{ex}}) = \frac{1}{2}\Phi_1(\lambda; \mathrm{Cap}(\mathscr{O}_{\mathrm{in}})) + o\left(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\right), \quad \lambda \downarrow 0.$$

It describes the accumulation of the discrete spectrum of the exterior Neumann magnetic Laplacian $\mathrm{H}_{-,\mathrm{ex}}$ *at* $\Lambda_0 = \inf \sigma_{\mathrm{ess}}(\mathrm{H}_{-,\mathrm{ex}})$.

Let us compare Theorem 5.3.1 with Theorem 5.2.1 the similar result for $(\mathrm{H}_0 + \mathrm{V}, \mathrm{H}_0)$, V a real-valued fast decaying electric potential. Formally, our Theorem 5.3.1 resembles the results of Theorem 5.2.1 for compactly supported V (more explicitely described in [75]), which however are less precise than (5.3.14) and (5.3.15): the right-hand side of the analogue of (5.3.14) (resp., of (5.3.15)) in [75] is $-\frac{1}{2}\Phi_0(\lambda)(1 + o(1))$ (resp., $\pm\frac{1}{4}\Phi_0(\lambda)(1 + o(1))$).

Idea of the proof of Theorem 5.3.1

Following the arguments (and notations) of Section 5.2.2 (for electric perturbation) and using the relation (5.3.7), the singularity of the SSF is essentially governed by the distribution of the eigenvalues of the operator $\Lambda_q^2 \mathrm{V}_\pm^{\frac{1}{2}}(p_q \otimes r_0)\mathrm{V}_\pm^{\frac{1}{2}}$. Then we can use properties of localization near $\partial\Omega_{\mathrm{in}}$ for $\mathrm{V}_\pm$ in order to justify that the operator $\Lambda_q^2 \mathrm{V}_\pm^{\frac{1}{2}}(p_q \otimes r_0)\mathrm{V}_\pm^{\frac{1}{2}}$ can be replaced mainly by $\pm\mathbb{1}_{\Omega_{\mathrm{in}}}(p_q \otimes r_0)\mathbb{1}_{\Omega_{\mathrm{in}}}$. It is not so obvious because $V_{\pm}$ contains some non local operators like H_0^{-1}. One one hand, using that in some sense $H_0(p_q \otimes r_0) = \Lambda_q p_q \otimes r_0$ we replace $\Lambda_q H_0^{-1}$ by the identity. One the other hand, on a subspace of Range($p_q \otimes r_0$), we show that $\Lambda_q^2 V_{\pm}$ is comparable to $\mathbb{1}_{\Omega_{in}}$ (on a space of finite co-dimension). This last point is justified in [42, Section 5] using ellipticity properties of Dirichlet-to-Neumann operators and in [41, Section 5] by variational arguments.

Finally, we exploit (as in Section 5.2.2) that the distribution of the eigenvalues of the operator $\mathbb{1}_{\Omega_{in}}(p_q \otimes r_0)\mathbb{1}_{\Omega_{in}}$ coincides with that of $p_q W_{in} p_q$ where

$$W_{in}(x_\perp) := \frac{1}{2}\int_{\mathbb{R}} \mathbb{1}_{\Omega_{in}}(x_\perp, x_\parallel)dx_\parallel$$

which is comparable to $\frac{1}{2}p_q \mathbb{1}_{\mathscr{O}_{in}} p_q$. Then we conclude using results of Section 4.2.2.

5.3.3 Resonances for the exterior problems

In this section we define the resonances of $H_{\pm,ex}$ and study their distribution near the Landau levels. As for the SSF, we show that the logarithmic capacity of the projection of Ω_{in} onto the plane perpendicular to the magnetic field appears in the asymptotic expansion of the counting function of resonances near the Landau levels.

As in Section 5.2.3 we define the resonances on the the infinite-sheeted Riemann surface $\mathscr{M}$, where near a fixed Landau level Λ_q, we identify a point $z_q \in \mathscr{M}$ with $\Lambda_q + k^2$, $k \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, $|k| \ll 1$.

Combining the relation (5.3.7) with Proposition 5.2.1 and using that $H_{\pm}$ coincide with H_0 outside the obstacle (which gives cut off properties of $V_{\pm}$) we obtain that the operator-valued function $z \mapsto T^{\pm}(z)$ is analytic on $\mathbb{C}_+$ with value in $\mathfrak{S}_\infty(L^2(\mathbb{R}^3))$. Moreover using the analytic Fredholm theorem, we can define resonances of $H_{\pm}$ as follows (see the proof of [42, Proposition 3.3]).

Definition 5.3.1 *We define resonances of* $H_{\pm}$ *as the discret set of values of* $z \in \mathscr{M}$ *for which* $\big(I \mp T^{\pm}(z)\big)$ *is not invertible. The multiplicity of a resonance* z_0 *is defined by*

$$\mathrm{mult}(z_0) := \frac{1}{2i\pi}\mathrm{tr}\Big(\int_{\mathscr{C}} \big(\mp T^{\pm}\big)'(z)\,\big(I \mp T^{\pm}(z)\big)^{-1}dz\Big), \tag{5.3.18}$$

where $\mathscr{C}$ *is a small contour positively oriented, containing* z_0 *as the unique point* z *satisfying* $\big(I \mp T^{\pm}(z)\big)$ *is not invertible.*
Theses resonances are also the poles of the meromorphic extension of the resolvent

$$R_{\pm}(z) = (H_{\pm} - z)^{-1} : e^{-\varepsilon\langle x_\parallel\rangle}L^2(\mathbb{R}^3) \longrightarrow e^{\varepsilon\langle x_\parallel\rangle}L^2(\mathbb{R}^3),$$

with multiplicity given by the rank of their residues.

In particular since

$$(H_{\pm} - z)^{-1} = (H_{\pm,in} - z)^{-1} \oplus (H_{\pm,ex} - z)^{-1},$$

it gives the following definition of resonances of $H_{\pm,ex}$.

Definition 5.3.2 *We define the resonances of* $H_{\pm,ex}$ *as the poles of the meromorphic extension of the resolvent*

$$(H_{\pm,ex}-z)^{-1} : e^{-\varepsilon\langle x_\parallel\rangle}L^2(\Omega_{ex}) \to e^{\varepsilon\langle x_\parallel\rangle}L^2(\Omega_{ex}).$$

These poles (i.e. the resonances) are resonances of $H_\pm$*, excepted the poles given by the eigenvalues of* $H_{\pm,in}$.

Near a fixed Landau level Λ_q, $q \in \mathbb{Z}_+$, using the identification described above we parametrize the resonances z_q by $z_q(k) = \Lambda_q + k^2$ with $|k| << 1$.

Then as for electric perturbation (Theorem 5.2.3) we have the localization of the resonances of $H_{\pm,ex}$ near the Landau levels Λ_q, $q \in \mathbb{Z}_+$, together with an asymptotic expansion of the resonances counting function in small annulus adjoining Λ_q, $q \in \mathbb{N}$:

Theorem 5.3.2 ([42, Theorem 2.1][41]) *Let* $q \in \mathbb{Z}_+$. *Suppose* Ω_{in} *satisfies* $\mathscr{C}_{2,q}$ *(in the sense that* (5.2.24) *holds true by replacing* V *and* H_V *by* $V_\pm$ *and* $H_\pm$*). Then, for* $0 < r_0 < \sqrt{2b}$ *fixed, the resonances* $z_q(k) = \Lambda_q + k^2$ *of* $H_\pm$ *with* $|k|$ *sufficiently small satisfy*

$$\pm \mathrm{Im} k \leq 0, \qquad \mathrm{Re} k = o(|k|),$$

and

$$\#\{z = \Lambda_q + k^2 \in \mathrm{Res}(H_\pm);\ r < |k| \leq r_0\} = \Phi_1(\lambda; \mathrm{Cap}(\mathscr{O}_{in}))(1+o(1)), \qquad r \searrow 0, \tag{5.3.19}$$

with the notations of Section 5.3.2.

Idea of the proof of Theorem 5.3.2

The proof combines results of [42] with those of [41]. As for the proof of Theorem 5.2.3, we apply the Propositions 5.2.3 and 5.2.4 to an appropriated meromorphic Fredholm function. This time the analytic function A is:

$$A_q^\pm(ik) = \pm z_q(k)^2 |V_\pm|^{\frac12} p_q \otimes r(ik) |V_\pm|^{\frac12} \mp ik |V_\pm|^{\frac12} \Big(z_q(k) + z_q(k)^2 R_0(z_q(k))(I - p_q \otimes I_\parallel) \Big) |V_\pm|^{\frac12},$$

being such that $T_\pm(\Lambda_q + k^2) = A_q^\pm(ik)/(ik)$. Thanks to Proposition 5.2.4, the main contribution is given by $A_q^\pm(0) = \pm\Lambda_q^2 |V_\pm|^{\frac12} p_q \otimes r(0) |V_\pm|^{\frac12}$, the same as for the SSF. Then as in [42] we obtain the first order asymptotic (with $\Phi_0(\lambda) = \frac{|\ln\lambda|}{\ln|\ln\lambda|}$). For the third order asymptotic expansion, we exploit the more refined results of [41] as for the SSF.

5.4 2D magnetic hamiltonians on domains with unbounded boundaries

In this section, we discuss 2D magnetic Schrödinger operators, for which the presence of unbounded boundaries produces continuum spectrum. Our reference operators are mainly the magnetic Schrödinger operators in the half-plane $\mathscr{O} := \mathbb{R}_+ \times \mathbb{R}$ and in the strip $\mathscr{O}_L := [-L, L] \times \mathbb{R}$, $L > 0$. As we will see, even for electric perturbations of these fibered operators, new difficulties appear.

5.4.1 Dirichlet and Neumann magnetic Schrödinger operators on the half-plane

On $\mathscr{O} := (0,+\infty)\times\mathbb{R}$, we consider the free 2D Schrödinger operator $H_S(A,0)$ with constant magnetic field corresponding to (2.7.3) (the kernel of the magnetic field is reduced to zero):

$$H_S(A,0) := \Pi_1(A)^2+\Pi_2(A)^2 = -\frac{\partial^2}{\partial x^2}+\left(-i\frac{\partial}{\partial y}-bx\right)^2, \qquad (x,y)\in\mathscr{O}, \tag{5.4.1}$$

with the magnetic potential $A(x,y) = b(0,x)$, $b > 0$ being the strength of the magnetic field.

As in Section 5.3.1, we define $H_{0,D}$ (resp., $H_{0,N}$) the Dirichlet (resp., Neumann) realization of (5.4.1) with domains

$$\mathfrak{D}(H_{0,D}) := \left\{u\in H^2_A(\mathscr{O})\,|\,u_{|\{0\}\times\mathbb{R}}=0\right\}; \qquad \mathfrak{D}(H_{0,N}) := \left\{u\in H^2_A(\mathscr{O})\,|\,\partial_x u_{|\{0\}\times\mathbb{R}}=0\right\}.$$

Hamiltonians of this type arise in various areas of mathematical and theoretical physics: for instance, $H_{0,D}$ and its perturbations are important models in the theory of the quantum Hall effect (see e.g. [59]), while the spectral properties of $H_{0,N}$ play a central role in the contemporary theory of superconductivity (see [60]).

Exploiting the translation invariance in the variable y, let us analyze these operators by using $\mathscr{F}$, the following partial Fourier transform with respect to y:

$$(\mathscr{F}u)(x,k) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-iyk}u(x,y)dy, \quad (x,k)\in\mathscr{O}.$$

Then we have

$$\mathscr{F}H_{0,\ell}\mathscr{F}^* = \int_{\mathbb{R}}^{\oplus} h_\ell(k)dk, \quad \ell = D,N,$$

where $h_D(k)$ (resp., $h_N(k)$) is the Friedrichs extension in $L^2(\mathbb{R}_+)$ of the operator

$$-\frac{d^2}{dx^2}+(bx-k)^2, \quad k\in\mathbb{R}, \tag{5.4.2}$$

defined originally on $C_0^\infty(\mathbb{R}_+)$ (resp., on $C_0^\infty(\overline{\mathbb{R}_+})$). Thus $(h_D,\mathfrak{D}(h_D))$ (resp., $(h_N,\mathfrak{D}(h_N))$) is the Dirichlet (resp., Neumann) realization of (5.4.2). Note that the operators h_ℓ, $\ell = D,N$, are Kato analytic families (see [107]). Moreover, for each $k\in\mathbb{R}$ the operators $h_\ell(k)$ have discrete and simple spectra. Let $\{E_{q,\ell}(k)\}_{q\in\mathbb{Z}_+}$, $k\in\mathbb{R}$, be the (simple) eigenvalues of $h_\ell(k)$, $\ell = D,N$. By the Kato analytic perturbation theory, for all $q\in\mathbb{Z}_+$, $E_{q,\ell}(\cdot)$ are real analytic functions of $k\in\mathbb{R}$.

Let us give some of the properties of the functions $E_{q,\ell}$ which we will need in the sequel. In both cases $\ell = D,N$, using that $h_\ell(k)$ is unitarily equivalent to $-\frac{d^2}{dx^2}+(bx)^2$ on $[-kb^{-1},+\infty)$, we have

$$\lim_{k\to+\infty} E_{q,\ell}(k) = \Lambda_q = b(2q+1),$$

(see e.g. [59] for $\ell = \mathrm{D}$, and e.g. [58] for $\ell = \mathrm{N}$). Moreover, the mini-max principle easily implies that

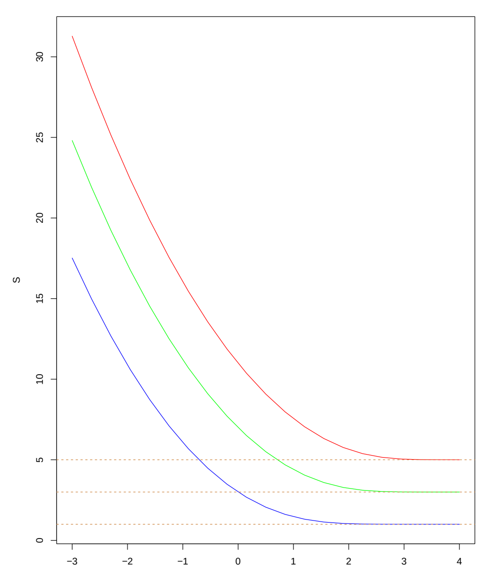
$$E_{q,\ell}(k) = k^2(1+o(1)), \quad k \to -\infty.$$

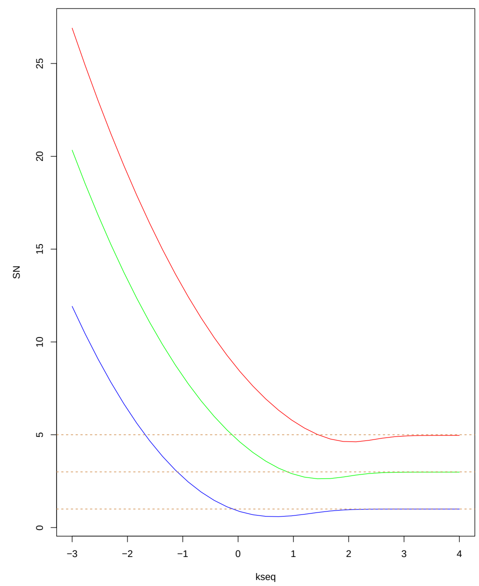
However, $E'_{q,D}(k) < 0$ for any $k \in \mathbb{R}$ (see [59]) while there exists $k_{q,*} \in (0,\infty)$, a non-degenerate minimum of $E_{q,N}$, such that $E'_{q,N}(k) < 0$ for $k \in (-\infty, k_{q,*})$, $E'_{q,N}(k) > 0$ for $k \in (k_{q,*},\infty)$, $E''_{q,N}(k_{q,*}) > 0$, and $E_{q,N}(k_{q,*}) \in (0,b)$ (see [58]). Thus the bottom of the band functions $E_{q,\ell}$, $\ell = D,N$ are given by :

$$\mathscr{E}_{q,D} := \Lambda_q = \lim_{k\to\infty} E_{q,D}(k), \quad \mathscr{E}_{q,N} := E_{q,N}(k_{q,*}),$$

and the spectrum $\sigma(H_{0,\ell})$ of $H_{0,\ell}$, $\ell = D,N$, is absolutely continuous, and we have

$$\sigma(H_{0,\ell}) = \cup_{q\in\mathbb{Z}_+} E_{q,\ell}(\mathbb{R}) = \cup_{q\in\mathbb{Z}_+} [\mathscr{E}_{q,\ell},+\infty) = [\mathscr{E}_{0,\ell},+\infty).$$





It is known that the extrema of the band functions play a significant role in the description of the spectral properties of fibered operators (see [79]). In the following, under an electric perturbation, we will analyze these roles and distinguish the particular case where these extrema are reached and non-degenerate (e.g. the points $\mathscr{E}_{q,N}$, $q \in \mathbb{Z}_+$, for the Neumann magnetic operator), from the case of the not reached extrema which are the limits of the band functions at infinity (the Landau levels Λ_q, $q \in \mathbb{Z}_+$, minima for the Dirichlet magnetic operator and maxima for the Neumann magnetic operator).

Near a reached non-degenerate extremum, we will apply a well know procedure to obtain an effective Hamiltonian. Near the Landau levels, extremal points which are only the limits on the band functions, the procedure is less classical. In some cases, an effective Hamiltonian will involve Anti-Wick-type operators, but more complicated situations may arise, for example when the considered energy level crosses the band function at infinity.

Assume that V is a bounded, real-valued, Lebesgue-measurable function and denote again by V the multiplier by this electric potential. For $\ell = D, N$, on the domain of $H_{0,\ell}$ the perturbed operator

$$H_\ell := H_S(A, V) = H_{0,\ell} + V \tag{5.4.3}$$

is self-adjoint in $L^2(\mathscr{O})$.
In the following sections, we will suppose that V satisfies one of the following properties on $\mathscr{O}$:

- D_0 (null at infinity) : $V \in L^\infty_0(\mathscr{O}) := \left\{ u \in L^\infty(\mathscr{O}) \mid \lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0 \right\}$;
- D_m (polynomial decay): $V(x, y) = O(\langle x, y\rangle^{-m})$ with $m > 2$;

- D_{exp} (fast decay with respect to y): $V(x) = O(\langle x\rangle^{-m_0} \exp(-N|y|))$ with some $m_0 > 0$ and any $N > 0$.

Evidently, assumption D_{exp} with $m_0 > 2$ implies D_m. Moreover D_{exp} and D_m imply D_0. Thanks to the Diamagnetic inequality (see Section 2.5), as soon as V satisfies D_0, the operator $VH_{0,\ell}^{-1}$ is compact in $L^2(\mathscr{O})$ and therefore

$$\sigma_{ess}(H_\ell) = \sigma_{ess}(H_{0,\ell}) = \sigma(H_{0,\ell}) = [\mathscr{E}_{0,\ell}, +\infty), \quad \ell = D, N.$$

However, the interval $(-\infty, \mathscr{E}_{0,\ell})$ may contain discrete eigenvalues of the operator H_ℓ whose number could be finite or infinite; in the latter case they could accumulate only at $\mathscr{E}_{0,\ell}$. In the following section, for negative $V \in L_0^\infty(\mathscr{O})$, we study the counting function of the eigenvalues of H_ℓ, below $\mathscr{E}_{0,\ell}$, $\ell = D, N$. We will see that the effective Hamiltonians in the Dirichlet and the Neumann cases are quite different due to the different nature of the infima $\mathscr{E}_{0,D}$ and $\mathscr{E}_{0,N}$ of the spectra of $H_{0,D}$ and $H_{0,N}$ respectively. Then, in the other sections, under the assumption D_m, we will define the SSF for the pairs $(H_\ell, H_{0,\ell})$, $\ell = D, N$ and analyze its asymptotic properties near the *thresholds*: the Landau levels Λ_q, $q \in \mathbb{Z}_+$ and the headed non-degenerated minima $\mathscr{E}_{q,N}$ (for the Neumann hamiltonian). Up to small modifications, we will find again the previous effective Hamiltonians corresponding to each type of extremum.

The study of resonances of H_ℓ is still little explored. To our knowledge, for the moment, for V satisfying D_{exp}, there are ways to define and study the resonances of these operators only near the real axis outside the Landau levels (see [78, Chapter 1]), but further investigations deserve to be carried out. In particular, near the Landau levels, given the exponential behavior of the band functions $E_{q,\ell}$, a first difficulty is to describe the complex surface on which the resonances are defined.

5.4.2 Counting function of eigenvalues below the bottom of the essential spectrum

For $\lambda > 0$ let us introduce the counting function of the eigenvalues of the operator H_ℓ lying on the interval $(-\infty, \mathscr{E}_{0,\ell} - \lambda)$, and counted with the multiplicities:

$$\mathscr{N}_\ell(\lambda) := \operatorname{Tr} \mathbb{1}_{(-\infty, \mathscr{E}_{0,\ell} - \lambda)}(H_\ell), \quad \ell = D, N.$$

In this section, we exhibit the effective 1D hamiltonians for each type of extrema: a Schrödinger operator for the non-degenerated minimum $\mathscr{E}_{0,N}$ (Neumann case) and an Anti-Wick-type operator perturbed by the band function $E_{0,D}$ (view as an operator of multiplication) for the minimum $\mathscr{E}_{0,D}$ given by the limit of the band function $E_{0,D}$ (Dirichlet case). These effective Hamiltonians will appear after restrictions to the lower band function $E_{0,\ell}$ and the projection onto the associated real valued eigenfunctions $\psi_\ell(\cdot; k)$ satisfying

$$h_\ell(k)\psi_\ell(\cdot; k) = E_{0,\ell}(k)\psi_\ell(\cdot; k), \quad \|\psi_\ell(\cdot; k)\|_{L^2(\mathbb{R}_+)} = 1, \quad k \in \mathbb{R}, \quad \ell = D, N,$$

such that the mappings $\mathbb{R} \ni k \mapsto \psi_\ell(\cdot; k) \in \mathfrak{D}(h_\ell(k))$ are analytic.

Spectral accumulation for Dirichlet Hamiltonians

The bottom of the essential spectrum of H_D is given by $\mathscr{E}_{0,D}$ the limit, as k tends to infinity of the first (decreasing) band function $E_{0,D}$. We show that this band function, near $+\infty$, provides the main contribution of $H_{0,D}$ in the study of the eigenvalues below $\mathscr{E}_{0,D} = b$. Moreover, we exploit that, as $k \to +\infty$, the associated eigenfunction $\psi_D(\cdot;k)$ is approximated by $\psi_{D,\infty}(\cdot;k)$ defined by

$$\psi_{D,\infty}(x;k) = b^{1/4}\varphi(b^{1/2}x - b^{-1/2}k); \quad \varphi(x) := \pi^{-1/4}e^{-x^2/2} \qquad x \in \mathbb{R}, \quad k \in \mathbb{R}. \tag{5.4.4}$$

This function satisfies

$$\Big(-\frac{d^2}{dx^2} + (bx-k)^2\Big)\psi_{D,\infty}(x;k) = b\psi_{D,\infty}(x;k), \quad \|\psi_{D,\infty}(\cdot;k)\|_{L^2(\mathbb{R})} = 1, \tag{5.4.5}$$

and for

$$\Psi_{x,\xi}(k) := b^{-1/2}e^{-i\xi k}\psi_{D,\infty}(x;k), \quad k \in \mathbb{R}, \quad (x,\xi) \in T^*\mathbb{R},$$

the system $\left\{\Psi_{x,\xi}\right\}_{(x,\xi)\in T^*\mathbb{R}}$ being overcomplete with respect to the measure $\frac{b}{2\pi}dxd\xi$ (see Sections 3.1 and 3.4), we can introduce the orthogonal projection

$$\mathscr{P}_{x,\xi} := |\Psi_{x,\xi}\rangle\langle\Psi_{x,\xi}|, \quad (x,\xi) \in T^*\mathbb{R},$$

acting in $L^2(\mathbb{R})$, as well as the anti-Wick-type operator $\mathscr{V} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined as the weak integral

$$\mathscr{V} := \frac{b}{2\pi}\int_{\mathscr{O}} V(x,\xi)\,\mathscr{P}_{x,\xi}\,dxd\xi.$$

Since $V \in L^\infty_0(\mathscr{O})$, the operator $\mathscr{V}$ is compact (see Corollary 3.1.1). Then we show that the effective Hamiltonian which governs the asymptotics of $\mathscr{N}_D(\lambda)$ as $\lambda \downarrow 0$ is $E_{0,D} + \mathscr{V} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ where $E_{0,D}$ should be interpreted here as the multiplier by the function $E_{0,D}$. More precisely, we have the following

Theorem 5.4.1 ([36, Theorem 2.1]) *Assume $V \in L^\infty_0(\mathscr{O})$ is negative. Then we have*

$$\mathscr{N}_D(\lambda) \sim \mathrm{Tr}\,\mathbb{1}_{(-\infty,\mathscr{E}_{0,D}-\lambda)}(E_{0,D} + \mathscr{V}), \quad \lambda \downarrow 0, \tag{5.4.6}$$

where we use the notation "$\sim$" introduced just before Theorem 5.2.1.

Remark 5.4.1 *Due to the compactness of the operator $\mathscr{V}$ we have $\sigma_{ess}(E_{0,D} + s\mathscr{V}) = \sigma_{ess}(E_{0,D}) = [\mathscr{E}_{0,D}, \infty)$ for any $s \in \mathbb{R}$ so that $\mathrm{Tr}\,\mathbb{1}_{(-\infty,\mathscr{E}_{0,D}-\lambda)}(E_{0,D} + s\mathscr{V}) < \infty$ for any $s \in \mathbb{R}$ and $\lambda > 0$. Moreover, as we will see in the proof, the main contribution of $E_{0,D}$ is when it is close to $\mathscr{E}_{0,D}$ (as $k \to +\infty$), in particular, in the above result, $E_{0,D}$ can be replaced by any function $E^{eff}_{0,D}$ which coincides with $E_{0,D}$ for $k \geq A >> 1$ and is larger than $E_{0,D}(A)$ for $k \leq A$.*

Remark 5.4.2 *The operator $E_{0,D} + \mathscr{V}$ is quite similar to the effective Hamiltonians which arose in [35] where is studied the asymptotic distributions of the discrete spectrum in the gaps of the essential spectrum of the operator $H_0 \pm V$ self-adjoint in $L^2(\mathbb{R}^2)$ where*

$$H_0 := -\frac{\partial^2}{\partial x^2} + \left(-i\frac{\partial}{\partial y} - bx\right)^2 + W(x), \tag{5.4.7}$$

with a bounded monotone W. *There are many other similarities of the spectral properties of the perturbations of the operator* H_0 *in* (5.4.7)*, and of the operator* $H_{0,D}$*, but also several essential differences due to the existence of a boundary in the case of* $H_{0,D}$.
(iii) Another model where such effective operator $E_{0,D} + \mathscr{V}$ *also appears, is the Iwatsuka model corresponding to non-constant magnetic field along one direction (see [132]).*

In Corollary 5.4.1 below we will show that even if a non vanishing identically V has a compact support, $\mathscr{N}_D(\lambda)$ blows up as $\lambda \downarrow 0$, i.e. the operator H_D has infinitely many eigenvalues. In order to formulate this corollary, we need the following notations. Let Ω be a bounded domain. Denote by $\mathbf{c}_-(\Omega)$ the maximal length of the vertical segments contained in $\overline{\Omega}$. Further, let $B_R(\zeta) \subset \mathbb{C}$ be a disk of radius $R > 0$ centered at $\zeta \in \mathbb{C}$. Identifying $\mathbb{C}$ with $\mathbb{R}^2$, set

$$K(\Omega) := \{(\xi, R) \in \mathbb{R} \times \mathbb{R}_+ \,|\, \exists \eta \in \mathbb{R} \quad \text{such that} \quad \Omega \subset B_R(\xi + i\eta)\},$$

and

$$\mathbf{c}_+(\Omega) = \inf_{(\xi,R)\in K(\Omega)} R\kappa\left(\frac{\xi_+}{eR}\right),$$

where $\xi_+ := \max\{\xi, 0\}$, $\kappa(s) := |\{t > 0 \,|\, t\ln t < s\}|$, $\quad s \in [0,\infty)$, and $|\cdot|$ denotes the Lebesgue measure.

Corollary 5.4.1 ([36, Corollary 2.2]) *Assume that* V *satisfies*

$$c_- \mathbb{1}_{\Omega_-}(x,y) \le -V(x,y) \le c_+ \mathbb{1}_{\Omega_+}(x,y), \quad (x,y) \in \mathscr{O}, \tag{5.4.8}$$

where $\Omega_\pm \subset \mathscr{O}$ *are bounded domains, and* $0 < c_- \le c_+ < \infty$ *are some constants. Then we have*

$$\mathscr{C}_- |\ln\lambda|^{1/2}(1 + o(1)) \le \mathscr{N}_D(\lambda) \le \mathscr{C}_+ |\ln\lambda|^{1/2}(1 + o(1)), \quad \lambda \downarrow 0, \tag{5.4.9}$$

with $\mathscr{C}_- := (2\pi)^{-1}\sqrt{b}\,\mathbf{c}_-(\Omega_-)$ *and* $\mathscr{C}_+ := e\sqrt{b}\,\mathbf{c}_+(\Omega_+)$. *In particular,*

$$\lim_{\lambda \downarrow 0} \frac{\ln \mathscr{N}_D(\lambda)}{\ln|\ln\lambda|} = \frac{1}{2}.$$

Remark 5.4.3 *The constants* $\mathscr{C}_\pm$ *already appeared in [35, Theorem 6.1]. As it is indicated there, we have* $\mathscr{C}_- < \mathscr{C}_+$ *(in fact,* $\frac{\mathscr{C}_+}{\mathscr{C}_-} > e\pi$*).*

The main interest of this corollary is to show that H_D has infinitely many discrete eigenvalues and that the distribution is of order $\sqrt{|\ln\lambda|}$. The rather technical proof consists in extracting the main properties of the operator $E_{0,D} + \mathscr{V}$, in which there is competition between $E_{0,D}$ and $\mathscr{V}$ (none of both operators dominates). A difficulty is in particular the spectral study of an intermediate operator of integral kernel

$$\frac{\sin(k-k')}{k-k'}\,\frac{2\sqrt{kk'}}{k+k'}, \quad (k,k') \in I \times I$$

on I, a bounded interval of $\mathbb{R}_+$.

Counting function of discrete spectrum of Neumann Hamiltonians

The bottom of the essential spectrum of H_N is given by $\mathscr{E}_{0,N}$ the non-degenerated minimum of the band function $E_{0,N}$ at the point $k_{0,*}$. This setup is more standard and it is not so difficult to show that the main contribution of $H_{0,N}$ in the study of the eigenvalues below $\mathscr{E}_{0,N} \in (0,b)$ is given by the behavior of this band function $E_{0,N}$, near $k_{0,*}$, that is $\mathscr{E}_{0,N} + \mu(k-k_{0,*})^2$ with

$$\mu := \frac{1}{2} E''_{0,N}(k_{0,*}). \tag{5.4.10}$$

Moreover, the contribution of the potential V will be given by its restriction to the eigenspace $\mathrm{Ker}\big(h_N(k_{0,*}) - \mathscr{E}_{0,N}\big)$ generated by $\psi_N(\cdot;k_{0,*})$. We introduce the effective potential

$$v(y) := \int_0^\infty V(x,y)\psi_N(x;k_{0,*})^2 dx, \quad y \in \mathbb{R}. \tag{5.4.11}$$

Then the effective Hamiltonian which governs the asymptotics of $\mathscr{N}_N(\lambda)$ as $\lambda \downarrow 0$ is the self-adjoint operator

$$-\mu \frac{d^2}{dy^2} + v \tag{5.4.12}$$

defined originally on $C_0^\infty(\mathbb{R})$ and then closed in $L^2(\mathbb{R})$. More precisely, we have the following Theorem (using the notation "$\sim$" introduced just before Theorem 5.2.1).

Theorem 5.4.2 ([36, Theorem 2.3]) *Assume $V \in L_0^\infty(\mathscr{O})$ is negative. Then we have*

$$\mathscr{N}_N(\lambda) \sim \mathrm{Tr}\, \mathbb{1}_{(-\infty,-\lambda)} \left(-\mu \frac{d^2}{dy^2} + v\right), \quad \lambda \downarrow 0. \tag{5.4.13}$$

Remark 5.4.4 *Since $v \in L_0^\infty(\mathbb{R})$, the multiplier by v is an operator relatively compact with respect to $-\mu \frac{d^2}{dy^2}$ in $L^2(\mathbb{R})$, and we have $\sigma_{\mathrm{ess}}\left(-\mu \frac{d^2}{dy^2} + sv\right) = [0,\infty)$ for any $s \in \mathbb{R}$ so that $\mathrm{Tr}\, \mathbb{1}_{(-\infty,-\lambda)} \left(-\mu \frac{d^2}{dy^2} + sv\right) < \infty$ for any $s \in \mathbb{R}$ and $\lambda > 0$.*

Remark 5.4.5 *Effective Hamiltonians quite similar to* (5.4.12) *arose also for magnetic Schrödinger operators on the strip $\mathscr{O}_L = (-L,L) \times \mathbb{R}$ where the band functions also have non-degenerated minima (see [30] and Section 5.4.4).*

Remark 5.4.6 *In [134], an infinite-matrix-valued version of* (5.4.12) *appears as an effective Hamiltonian in the study of the asymptotic distributions of the discrete spectrum in the gaps of the essential spectrum of the operator $H_0 \pm V$ self-adjoint in $L^2(\mathbb{R}^2)$ with*

$$H_0 := -\frac{\partial^2}{\partial x^2} + \left(-i\frac{\partial}{\partial y} - bx\right)^2 + W_{per}(x), \tag{5.4.14}$$

and W_{per} a periodic potential. The band functions associated to this fibered operator are periodic with a finite number of extrema on one period. The periodicity yields an *infinite number of extrema for each band function which justifies the infinite-matrix-value of the effective Hamiltonian (under additional assumptions as non-crossing of band functions and non-degeneracy of extrema).*

Applying well-known results concerning the counting function for 1D Schrödinger operators (see e.g. [165, Theorem XIII.82], and [110]), in Corollary 5.4.2 below we establish sufficient conditions for the infiniteness and the finiteness of $\sigma_{\rm disc}(H_N)$. We also use the Hardy inequality

$$\int_0^\infty |u'(y)|^2 dy \geq \frac{1}{4}\int_0^\infty y^{-2}|u(y)|^2 dy, \quad u \in C_0^\infty(\mathbb{R}_+),$$

and the result of [165, Problem 22, Chapter XIII].

Corollary 5.4.2 ([36, Corollary 2.4]) *Let* $V \in L_0^\infty(\mathscr{O})$ *negative.*
(i) Assume that there exists $\alpha \in (0,2)$ *and constants* $\omega_\pm \geq 0$ *such that*

$$\lim_{y\to\pm\infty} |y|^\alpha v(y) = -\omega_\pm. \tag{5.4.15}$$

Then we have

$$\lim_{\lambda\downarrow 0} \lambda^{\frac{1}{\alpha}-\frac{1}{2}} \mathscr{N}_N(\lambda) = \frac{B\left(\frac{3}{2}, \frac{1}{\alpha}-\frac{1}{2}\right)}{\pi\alpha\sqrt{\mu}}\left(\omega_-^{1/\alpha} + \omega_+^{1/\alpha}\right), \tag{5.4.16}$$

B *being the Euler beta function.*
(ii) Assume now that (5.4.15) *holds with* $\alpha = 2$. *Then we have*

$$\lim_{\lambda\downarrow 0} |\ln\lambda|^{-1}\mathscr{N}_N(\lambda) = \frac{1}{\pi}\left(\left(\frac{\omega_-}{\mu} - \frac{1}{4}\right)_+^{1/2} + \left(\frac{\omega_+}{\mu} - \frac{1}{4}\right)_+^{1/2}\right). \tag{5.4.17}$$

(iii) Finally, assume that

$$\limsup_{|y|\to\infty} y^2|v(y)| < \frac{\mu}{4}. \tag{5.4.18}$$

Then we have

$$\mathscr{N}_N(\lambda) = O(1), \quad \lambda \downarrow 0. \tag{5.4.19}$$

Remark 5.4.7 *If at least one of the constants* $\omega_\pm$ *in* (5.4.15) *with* $\alpha \in (0,2)$ *is positive, then* (5.4.16) *implies that the operator* H_N *has infinitely many discrete eigenvalues. Similarly, if at least of the constants* $\omega_\pm$ *in* (5.4.15) *with* $\alpha = 2$ *is greater than* $\mu/4$, *then* (5.4.17) *again implies that* $\sigma_{\rm disc}(H_N)$ *is infinite. Finally,* (5.4.19) *shows that under assumption* (5.4.18)*, the operator* H_N *has at most finitely many discrete eigenvalues. Note that the estimate*

$$\limsup_{|y|\to\infty} y^2 \sup_{x\in\mathbb{R}_+} |V(x,y)| < \frac{\mu}{4}$$

evidently implies (5.4.18).

Remark 5.4.8 *Relation* (5.4.16) *can be written in a semiclassical form, namely*

$$\mathscr{N}_N(\lambda) = \frac{1}{2\pi}\left|\left\{(y,\eta) \in T^*\mathbb{R} \mid \mu\eta^2 + v(y) < -\lambda\right\}\right|(1+o(1)), \quad \lambda \downarrow 0.$$

The following sections concern the continuous spectrum. We analyze spectral concentration phenomena near the thresholds.

5.4.3 The spectral shift function near Landau levels

In this section, we still consider the operators H_ℓ, $\ell = D, N$ defined by (5.4.3), but now we assume that V satisfies D_m. Then thanks to the Diamagnetic inequality (see Section 2.5), the operator $VH_{0,\ell}^{-1}$ is Hilbert–Schmidt, and hence, for $c > 0$ large enough, the resolvent difference $(H_\ell + cI)^{-1} - (H_{0,\ell} + cI)^{-1}$ is a trace-class operator. Therefore, according to Theorem 5.1.1, the spectral shift function (SSF) for the operator pair $(H_\ell, H_{0,\ell})$,

$$\xi(H_V, H_0; \cdot) \in L^1(\mathbb{R}; (1+\lambda^2)^{-1} d\lambda)$$

exists, satisfies the Lifshits-Krein trace formula

$$\mathrm{Tr}\,(f(H_\ell) - f(H_{0,\ell})) = \int_{\mathbb{R}} \xi(H_\ell, H_{0,\ell}; \lambda) f'(\lambda) d\lambda$$

for each $f \in C_0^\infty(\mathbb{R})$ and is unique with the normalization condition $\xi(H_\ell, H_{0,\ell}; \lambda) = 0$ for $\lambda \in (-\infty, \inf \sigma(H_\ell))$. In particular, for almost every $\lambda > 0$ we have

$$-\xi(H_\ell, H_{0,\ell}; \mathscr{E}_{0,\ell} - \lambda) = \mathscr{N}_\ell(\lambda)$$

which has been studied in the previous section, for $V \leq 0$.

Now, we extend this investigation to the continuous spectrum where the thresholds $\mathscr{E}_{q,\ell}$, $q \in \mathbb{Z}_+$, $\ell = D, N$ will play particular roles. First, we can show that for V of definite sign the SSF is bounded on compact sets not containing the thresholds, that is (see [34, Corollary 2.2]):

- $\xi(H_D, H_{0,D}; \cdot)$ is bounded on every compact subset of $\mathbb{R} \setminus \cup_{q \in \mathbb{Z}_+} \{\Lambda_q\}$;
- $\xi(H_N, H_{0,N}; \cdot)$ is bounded on every compact subset of $\mathbb{R} \setminus \cup_{q \in \mathbb{Z}_+} \{\Lambda_q, \mathscr{E}_{q,N}\}$;

In this section we study the asymptotic behavior of these SSF (in the Dirichlet and Neumann cases) near the Landau levels Λ_q, $q \in \mathbb{Z}_+$, while for the Neumann case, the thresholds $\mathscr{E}_{q,N}$, $q \in \mathbb{Z}_+$, will be studied in the next section.

Since the Landau levels are limits, as $k \to +\infty$, of the band functions, by mimicking the study of the counting function of Section 5.4.2, a natural effective hamiltonian for the behavior of $\xi(H_\ell, H_{0,\ell}; \Lambda_q + \lambda)$, $\ell = D, N$, as $\lambda \to 0$ would be $E_{q,\ell} + \mathscr{V}_q$ with $\mathscr{V}_q$ the Anti-Wick operator $\mathscr{V}_q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined as the weak integral:

$$\mathscr{V}_q := \frac{b}{2\pi} \int_{\mathscr{O}} V(x,\xi)\, \mathscr{P}_{q;x,\xi}\, dx d\xi,$$

with

$$\mathscr{P}_{q;x,\xi} := |\Psi_{q;x,\xi}\rangle\langle\Psi_{q;x,\xi}|, \quad \Psi_{q;x,\xi}(k) := b^{-1/2} e^{-i\xi k} \psi_{q,\infty}(x;k), \quad k \in \mathbb{R}, \quad (x,\xi) \in T^*\mathbb{R},$$

and $\psi_{q,\infty}$ satisfying

$$\Big(-\frac{d^2}{dx^2} + (bx-k)^2\Big)\psi_{q,\infty}(x;k) = \Lambda_q \psi_{q,\infty}(x;k), \quad \|\psi_{q,\infty}(\cdot;k)\|_{L^2(\mathbb{R})} = 1. \qquad (5.4.20)$$

That is $\psi_{q,\infty}(x;k) = b^{1/4}\psi_q(b^{1/2}x - b^{-1/2}k)$ where $\{\psi_q\}_{q\in\mathbb{Z}_+}$ are the eigenfunctions of the harmonic oscillator given by (2.7.14) (see Section 2.7).

For $\ell = D, N$, $q \in \mathbb{Z}_+$, we can show that these functions $\psi_{q,\infty}(\cdot;k)$ are good approximations, when $k \to +\infty$, of $\psi_{q,\ell}(x;k)$, the normalized, real-valued eigenfunctions of $h_\ell(k)$ associated to $E_{q,\ell}(k)$:

$$h_\ell(k)\psi_{q,\ell}(x;k) = E_{q,\ell}(k)\,\psi_{q,\ell}(x;k), \quad \psi_{q,\ell} \in \mathfrak{D}(h_\ell(k)), \quad \|\psi_{q,\ell}(\cdot;k)\|_{L^2(\mathbb{R}_+)} = 1. \tag{5.4.21}$$

Fix $k \in \mathbb{R}$ and $q \in \mathbb{Z}_+$, let us denote by $\pi_{q,\ell}(k)$ the one-dimensional orthogonal projection onto $\mathrm{Ker}(h_\ell(k) - E_{q,\ell}(k))$:

$$\pi_{q,\ell}(k) = |\psi_{q,\ell}(\cdot;k)\rangle\langle\psi_{q,\ell}(\cdot;k)|. \tag{5.4.22}$$

Fix $q \in \mathbb{Z}_+$, $\ell = D$ or N. For $z \in \mathbb{C}_+$ let us introduce the operator

$$T_{q,\ell}(z) := |V|^{1/2}\mathscr{F}^* \int_{\mathbb{R}}^{\oplus} (E_{q,\ell}(k) - z)^{-1}\pi_{q,\ell}(k)\,dk\,\mathscr{F}|V|^{1/2}. \tag{5.4.23}$$

By definition, the sum:

$$\sum_{q\in\mathbb{Z}_+} T_{q,\ell}(z) = |V|^{1/2}(H_{0,\ell} - z)^{-1}|V|^{1/2},$$

is the Birman-Schwinger operator and we can prove ([34, Proposition 3.6]) that the limit $\lim_{\delta\downarrow 0} T_{q,\ell}(E + i\delta) := T_{q,\ell}(E)$ exists in the trace class-norm for energies $E \in \mathbb{R}$, $E \neq \Lambda_q$ (and for $\ell = N$, $E \neq \mathscr{E}_{q,N}$).

Then as in the proof of Theorem 5.2.1, we can use the representation formula given by Proposition 5.1.1. Thus applying the Weyl inequalities (4.2.4) in the formula (5.1.4) we have the analogue of (5.2.17) (replacing $T_{0,q}$ by $T_{q,\ell}$), and exploiting that $\mathrm{Im}\,T_{q,\ell}(E)$ is of finite rank, we deduce (with still the notations introduced before Theorem 5.2.1):

Theorem 5.4.3 ([34, Theorem 2.1]) *Fix* $q \in \mathbb{Z}_+$, $\ell = D$ *or* N *and* $\pm V \geq 0$ *satisfying* D_m. *Then, as* $E \to \Lambda_q$,

$$\pm\xi(H_\ell, H_{0,\ell}; E) \sim \mathrm{Tr}\,\mathbb{1}_{(1,\infty)}\Big(\mp \mathrm{Re}\,T_{q,\ell}(E)\Big). \tag{5.4.24}$$

Consequently, the analysis of the SSF near Landau levels becomes close to the study of the counting function made in Section 5.4.2 (Dirichlet case). Indeed it works on the same way below each Landau level for the Dirichlet case, but new difficulties appear in the other cases because band functions $E_{q,\ell}$ cross the energy level $\Lambda_q + \lambda$, $\lambda > 0$ (resp. $\lambda < 0$) both for $\ell = D$ and $\ell = N$ (resp. for $\ell = N$). When the crossing is at a finite value of k (when $\lambda \to 0$), as in the case $\ell = N$ for $\Lambda_q + \lambda$, $\lambda > 0$, the contribution of the crossing is negligible but in the other cases (when ${E_{q,\ell}}^{-1}(\Lambda_q + \lambda)$ is not uniformly bounded w.r.t. $\lambda \to 0$) the spectral analysis of the operator $T_{q,\ell}(E)$, as $E \to \Lambda_q$, is more difficult.

Thus, we will be able to extend results of Theorem 5.4.1 when $\Lambda_q + \lambda$ doesn't cross the band function $E_{q,\ell}(k)$ for large values of k, and for the other cases, we will have asymptotic expansion of the SSF near Landau levels only when V decrease moderately (power like decreasing) at infinity. In this case, the operator $\mathscr{V}_q$ is a classical pseudodifferential operator and the contribution of $E_{q,\ell}(k)$ is negligible thanks to the following behavior ([102, Corollary 15.A.6]):

$$E_{q,\ell}(k) - \Lambda_q \asymp k^{2q-1} e^{-b^{-1}k^2}, \quad \text{as} \quad k \to +\infty,$$

where for two functions F and G defined on some interval I, we write

$$F(x) \asymp G(x) \quad \text{if} \quad c_- G(x) \leq F(x) \leq c_+ G(x), \tag{5.4.25}$$

for all x in I and for positive constants $c_\pm$.

Results for Dirichlet boundary conditions

We have the following extension to the SSF, of Theorem 5.4.1.

Theorem 5.4.4 ([34]) *Fix* $q \in \mathbb{Z}_+$ *and* $\pm V \geq 0$ *satisfying* D_m. *Then,*

$$\pm\xi(H_D, H_{0,D}; \Lambda_q - \lambda) \sim \operatorname{Tr} \mathbb{1}_{(-\infty, \Lambda_q - \lambda)}\left(E_{q,D}^{\text{eff}} + \mathscr{V}_q\right), \qquad \lambda \downarrow 0, \tag{5.4.26}$$

where we use the notation "∼" introduced just before Theorem 5.2.1 and $E_{q,D}^{\text{eff}}$ *stands for any function which coincides with* $E_{q,D}$ *on* $[A, +\infty)$*,* $A \in \mathbb{R}$ *fixed and such that* $E_{q,D}^{\text{eff}}(k) \geq E_{q,D}(A)$ *for* $k \leq A$.

Let us remark that since $E_{q,D}$ is decreasing, we can take $E_{q,D}^{\text{eff}} = E_{q,D}$. Moreover, on one hand exploiting that $E_{q,D} \geq \Lambda_q$ (and $\mathscr{V}_q$ has the sign of V), on the other hand following the proof of Corollary 5.4.1, we deduce:

Corollary 5.4.3 *Fix* $q \in \mathbb{Z}_+$.

1. *If* $V \geq 0$ *satisfies* D_m*, then*

$$\xi(H_D, H_{0,D}; \Lambda_q - \lambda) = O(1), \quad \lambda \downarrow 0.$$

2. *On the other side, if* $V \leq 0$ *is compactly supported (i.e. satisfies* (5.4.8)*), then*

$$\xi(H_D, H_{0,D}; \Lambda_q - \lambda) \asymp |\ln \lambda|^{\frac{1}{2}}, \quad \lambda \downarrow 0.$$

In order to give a result when we approach the Landau level from above (i.e. the energy level $\Lambda_q + \lambda > \Lambda_q$ crosses the band function $E_{q,D}$) let us introduce the assumptions corresponding to moderately decreasing potentials. To measure this decaying rate it is typically considered the following volume function:
For $a : \mathbb{R}^2 \to \mathbb{R}$ measurable and $\lambda > 0$, set

$$N(\lambda, a) := \frac{1}{2\pi} |\{(x, \xi) \in \mathbb{R}_+ \times \mathbb{R};\ a(x, \xi) > \lambda\}|, \tag{5.4.27}$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^2$. If $a \in \Gamma_1^m(\mathbb{R}^2)$, $m > 0$, the class of symbols introduced in Section 2.3, we have that $N(\lambda, a) = O(\lambda^{-\frac{2}{m}})$, for $\lambda \downarrow 0$. Some control "from below" and a regularity of the volume function are also introduced by supposing that V satisfies the following conditions.

There exist $m > 2$ such that:

$$\begin{cases} \text{a) } V \text{ is the restriction on } \mathbb{R}_+ \times \mathbb{R} \text{ of a function in } \Gamma_1^m(\mathbb{R}^2) \\ \text{b) } N(\lambda, |V|) \geq C\lambda^{-2/m}, \text{ for some } C > 0 \text{ and } 0 < \lambda < \lambda_0 \\ \text{c) } \lim_{\varepsilon \downarrow 0} \limsup_{\lambda \downarrow 0} \lambda^{2/m} \left(N(\lambda(1-\varepsilon), |V|) - N(\lambda(1+\varepsilon), |V|) \right) = 0. \end{cases} \tag{5.4.28}$$

Conditions (5.4.28) are commonly assumed in the study of the distribution of eigenvalues of some pseudodifferential operators (see for instance [57, 153, 105, 172]). A typical situation of V satisfying (5.4.28) is when $\lim_{(x,y)\to\infty} \langle x, y\rangle^m |V|(x, y) = \omega\left(\frac{(x,y)}{|(x,y)|}\right)$, where $\omega : S^1 \to [\varepsilon, \infty)$ is smooth and $\varepsilon > 0$.

As written above, under this conditions (5.4.28), we can prove that the contribution of $E_{q,D}^{\text{eff}}$ is negligible compared to that of $\mathscr{V}_q$ which has pseudodifferential properties for moderately decreasing potentials. Then standard results like [57, Theorem 1.3, Lemma 4.7] give the following theorem.

Theorem 5.4.5 ([34, Theorem 2.3, Corollary 2.4]) *Fix* $q \in \mathbb{Z}_+$.

1. *If* $\pm V \geq 0$ *satisfies* (5.4.28)*, then the following asymptotic formulas for the SSF*

$$\xi(H_D, H_{0,D}; \Lambda_q \pm \lambda) = \pm b N(\lambda, |V|)(1 + o(1)), \quad \lambda \downarrow 0, \tag{5.4.29}$$

hold true. This implies in particular that

$$\xi(H_D, H_{0,D}; \Lambda_q \pm \lambda) \asymp \lambda^{-2/m}, \quad \lambda \downarrow 0.$$

2. *On the other side, if* $V \leq 0$ *satisfies* (5.4.28)*, then*

$$\xi(H_D, H_{0,D}; \Lambda_q + \lambda) = o(\lambda^{-2/m}), \quad \lambda \downarrow 0.$$

From Theorem 5.4.5 we can compare the behavior of the SSF as a Landau level is approached from different sides. For $\pm V \geq 0$ satisfying (5.4.28), we obtain:

$$\lim_{\lambda \downarrow 0} \frac{\xi(H_D, H_{0,D}; \Lambda_q \mp \lambda)}{\xi(H_D, H_{0,D}; \Lambda_q \pm \lambda)} = 0. \tag{5.4.30}$$

This result is different to the results obtained for 3D magnetic Hamiltonians for which the behavior of the SSF was studied at the thresholds (see Corollary 5.2.1 and formula (5.3.16)) For those models the corresponding limit (5.4.30) is a constant different from zero, at least for negative perturbations, which gives a generalization of the Levinson's formula.

Now, let us mention some analog results for the half-plane magnetic Schrödinger operator with a Neumann boundary condition.

Results for Neumann boundary conditions

In the Neumann case, the band function $E_{q,N}$ tends to the Landau level Λ_q from below, then the behaviors of the SSF above Λ_q for Neumann with $\pm V \geq 0$ corresponds to these below Λ_q for Dirichlet with $\mp V \geq 0$. Concerning the behaviors of the SSF from below Λ_q, since the energy level $\Lambda_q - \lambda$, as $\lambda \to 0$, crosses $E_{q,N}$ at infinity, we only have a result for moderately decreasing potentials (assumption (5.4.28)). These analogies between Dirichlet and Neumann are also due to the fact that the behavior of the band functions and of the associated eigenfunctions are the same in both cases. Thus we have:

Theorem 5.4.6 ([34]) *The statement 1. of Theorem 5.4.5 hold true for the Neumann boundary conditions. On the other hand, for the Neumann boundary conditions, the results of Theorem 5.4.4 and of Corollary 5.4.3, as well as the statement 2. of Theorem 5.4.5, have to be replaced by:*

1. *If* $\pm V \geq 0$ *satisfies* D_m*, then, as* $\lambda \searrow 0$*,*

$$\pm\xi(H_N, H_{0,N}; \Lambda_q + \lambda) \sim \operatorname{Tr} \mathbb{1}_{(\Lambda_q+\lambda,+\infty)}\left(E^{\text{eff}}_{q,N} + \mathscr{V}_q\right), \qquad \lambda \downarrow 0, \tag{5.4.31}$$

where $E^{\text{eff}}_{q,N}$ *stands for any function which coincides with* $E_{q,N}$ *on* $[A, +\infty)$*,* $A \in \mathbb{R}$ *fixed and such that* $E^{\text{eff}}_{q,N}(k) \leq E_{q,N}(A)$ *for* $k \leq A$.

2. *If* $V \leq 0$ *satisfies* D_m*, then*

$$\xi(H_N, H_{0,N}; \Lambda_q + \lambda) = O(1), \quad \lambda \downarrow 0.$$

3. *If* $V \geq 0$ *is compactly supported (i.e. satisfies* (5.4.8)*), then*

$$\xi(H_N, H_{0,N}; \Lambda_q + \lambda) \asymp |\ln \lambda|^{\frac{1}{2}}, \quad \lambda \downarrow 0.$$

4. *If* $V \geq 0$ *satisfies* (5.4.28)*, then*

$$\xi(H_N, H_{0,N}; \Lambda_q - \lambda) = o(\lambda^{\frac{2}{m}}), \quad \lambda \downarrow 0.$$

Remark 5.4.9 *Some of the above results are also established for Iwatsuka Hamiltonians (i.e. with non-constant magnetic field along one direction) in [133]. It seems also reasonable that these results could be extended without mayor changes to other 2D magnetic models like the Robin boundary problem in the half-plane.*

5.4.4 The spectral shift function near a non-degenerate extrema

Let us recall that for the Neumann boundary condition in the half-plane each band function, $E_{q,N}$ has a non-degenerate minimum (see Section 5.4.1). In this case, as in Theorem 5.4.2 the behavior of the SSF below the minimum $\mathscr{E}_{q,N} = E_{q,N}(k_{q,*})$ is governed by the counting function of the negative eigenvalues of the selfadjoint operator (in $L^2(\mathbb{R})$):

$$h^{\text{eff}}_q := -\mu_q \frac{d^2}{dy^2} + v_q$$

where $\mu_q := E''_{q,N}(k_{q,*}) > 0$ and

$$v_q(y) := \int_0^\infty V(x,y)\psi_{q,N}(x;k_{q,*})^2 dx, \quad y \in \mathbb{R}, \tag{5.4.32}$$

$\psi_{q,N}(\cdot;k)$ being the eigenfunction satisfying:

$$h_N(k)\psi_{q,N}(\cdot;k) = E_{q,N}(k)\psi_{q,N}(\cdot;k), \quad \psi'_{q,N}(0;k) = 0, \quad \|\psi_{q,N}(\cdot;k)\|_{L^2(\mathbb{R}_+)} = 1, \quad k \in \mathbb{R}.$$

It means that, if $V \geq 0$, then the SSF is uniformly bounded below each threshold $\mathscr{E}_{q,N}$, and if $V \leq 0$, we have

$$\xi(H_N, H_{0,N}; \mathscr{E}_{q,N} - \lambda) \sim -\mathrm{Tr}\,\mathbb{1}_{(-\infty,-\lambda)}\left(h_q^{\mathrm{eff}}\right), \quad \lambda \downarrow 0.$$

On the other hand, the behavior of $\xi(H_N, H_{0,N}; \mathscr{E}_{q,N} + \lambda)$ as $\lambda \downarrow 0$ (i.e. above $\mathscr{E}_{q,N}$) will be governed by the behavior of $\xi(h_q^{\mathrm{eff}}, -\mu_q \frac{d^2}{dy^2}; \lambda)$, the SSF for the 1D pair $(h_q^{\mathrm{eff}}, -\mu_q \frac{d^2}{dy^2})$.

Such result is proved in [30] for the magnetic Schrödinger operator in a strip (whose each band function has a unique non-degenerated minimum). Let us now describe these results obtained for perturbations by a potential whose the sign can change.

The magnetic Schrödinger operator in a strip

By replacing the half-plane $\mathscr{O} := \mathbb{R}_+ \times \mathbb{R}$ by the strip $\mathscr{O}_L := [-L, L] \times \mathbb{R}$, $L > 0$, in Section 5.4.1, we define H_0, the magnetic Schrödinger operator in the strip $\mathscr{O}_L$ as the Dirichlet realization of (5.4.1) with domain

$$\mathfrak{D}(H_0) := \left\{ u \in H^2(\mathscr{O}_L) \,|\, u_{|\{\pm L\}\times\mathbb{R}} = 0 \right\}.$$

Exploiting again the translation invariance in the variable y, for $\mathscr{F}$, the partial Fourier transform with respect to y, we have

$$\mathscr{F} H_0 \mathscr{F}^* = \int_{\mathbb{R}}^{\oplus} h_0(k)dk,$$

where $h_0(k)$ is the Friedrichs extension in $L^2(-L, L)$ of the operator

$$-\frac{d^2}{dx^2} + (bx - k)^2, \quad k \in \mathbb{R}, \tag{5.4.33}$$

with domain $\mathfrak{D}(h_0(k)) := \left\{ w \in H^2(-L, L) | w(-L) = w(L) = 0 \right\}$.

The operators h_0 is a Kato analytic family and for each $k \in \mathbb{R}$ the operators $h_0(k)$ have discrete and simple spectra. Let $\left\{E_q(k)\right\}_{q=0}^{\infty}$ be the increasing sequence of the eigenvalues of $h_0(k)$, which are even real analytic functions of $k \in \mathbb{R}$. The minimax principle easily implies

$$E_q(k) = k^2(1 + o(1)), \quad k \to \pm\infty. \tag{5.4.34}$$

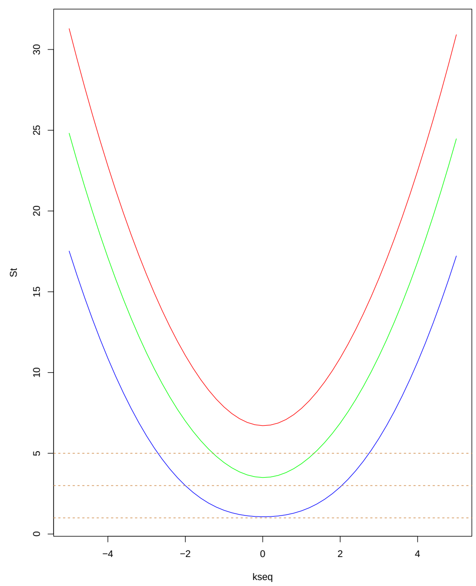
By [81, Theorem 2] we have

$$kE_q'(k) > 0, \quad k \neq 0, \tag{5.4.35}$$

$$E_q(k) = \mathscr{E}_q + \mu_q k^2 + O(k^4), \quad k \to 0, \tag{5.4.36}$$

with

$$\mathscr{E}_q := E_q(0) > \Lambda_q := b(2q+1), \quad \mu_q := \frac{1}{2}E_q''(0) > 0. \tag{5.4.37}$$



Thus

$$\sigma(H_0) = \sigma_{ac}(H_0) = \cup_{q\in\mathbb{Z}_+} E_q(\mathbb{R}) = \cup_{q\in\mathbb{Z}_+}[\mathscr{E}_q, +\infty) = [\mathscr{E}_0, +\infty),$$

and $\mathscr{E}_q$, $q \in \mathbb{Z}_+$ are thresholds in $\sigma(H_0)$ corresponding to non-degenerated minima of the band functions.

For $V : \mathscr{O}_L \to \mathbb{R}$ a bounded, real-valued, Lebesgue-measurable function, on the domain of H_0, the perturbed operator $H := H_S(A,V) = H_0 + V$ is self-adjoint in $L^2(\mathscr{O}_L)$. Moreover, let us assume that V satisfies the following polynomial decay property:

- $\mathscr{D}_m$: $V(x,y) = O(\langle y\rangle^{-m})$, $(x,y) \in \mathscr{O}_L$, with $m > 0$.

Then the Diamagnetic inequality (see Section 2.5), implies that the operator VH_0^{-1} is compact in $L^2(\mathscr{O}_L)$ and therefore

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = \sigma(H_0) = [\mathscr{E}_0, +\infty).$$

However, the interval $(-\infty, \mathscr{E}_0)$ may contain discrete eigenvalues of the operator H whose number could be finite or infinite; in the latter case they could accumulate only at $\mathscr{E}_0$.

As in Section 5.4.3, we also define $\xi(H, H_0; \cdot)$, the SSF for the pair (H, H_0), using that for $c > 0$ large enough, the resolvent difference $(H + cI)^{-1} - (H_0 + cI)^{-1}$ is a trace-class operator when V satisfies $\mathscr{D}_m$ with $m > 1$.

As in the previous cases, it can be proved (see [30, Proposition 2.1]) that the SSF $\xi(H, H_0; \cdot)$ is bounded on every compact subset of $\mathbb{R} \setminus \{\mathscr{E}_q;\ q \in \mathbb{Z}_+\}$ and continuous on $\mathbb{R} \setminus \big(\{\mathscr{E}_q;\ q \in \mathbb{Z}_+\} \cup \sigma_p(H)\big)$. Thus the thresholds $\mathscr{E}_q$, $q \in \mathbb{Z}_+$ are the only points where the SSF can be unbounded.

Set

$$J(x, y) = \operatorname{sign} V(x, y) := \begin{cases} 1 & \text{if} \quad V(x, y) \geq 0, \\ -1 & \text{if} \quad V(x, y) < 0. \end{cases}$$

For $q \in \mathbb{Z}_+$ fixed, let $\psi_q(\cdot; k) : (-L, L) \to \mathbb{R}$, $k \in \mathbb{R}$, be the real-valued normalized in $L^2(-L, L)$ eigenfunction of the operator $h_0(k)$ corresponding to the eigenvalue $E_q(k)$. For $\varepsilon \in (-1, 1)$ introduce *the effective potential*

$$w_{q,\varepsilon}(y) := \int_{-L}^{L} |V(x, y)| (J(x, y) - \varepsilon)^{-1} \psi_q(x; 0)^2 dx, \quad y \in \mathbb{R},$$

so that $w_{q,0}(y) = \int V(x, y) \psi_q(x; 0)^2 dx$ is the analog of the effective potential (5.4.32). We introduce *the effective Hamiltonians*

$$h_{0,q} := -\mu_q \frac{d^2}{dy^2}, \quad h_q(\varepsilon) := h_{0,q} + w_{q,\varepsilon},$$

the number μ_q being defined in (5.4.37). Note if V satisfies $\mathscr{D}_m$ with $m > 1$, then the SSFs $\xi(h_q(\varepsilon), h_{0,q}; \cdot)$, $q \in \mathbb{Z}_+$, $\varepsilon \in (-1, 1)$, are well defined and as stated below, their behavior near 0 will govern the behavior of the SSF $\xi(H, H_0; \cdot)$ near $\mathscr{E}_q$.
For $\lambda > 0$ set

$$\theta_\beta(\lambda) := \begin{cases} 1 & \text{if} \quad \beta > 1/2, \\ |\ln \lambda| & \text{if} \quad \beta = 1/2, \\ \lambda^{-\frac{1}{2} + \beta} & \text{if} \quad 0 < \beta < 1/2 \end{cases} \tag{5.4.38}$$

and for $\lambda < 0$, set $\theta_\beta(\lambda) := 1$ for all $\beta > 0$.

Theorem 5.4.7 ([30, Theorem 2.2]) *Assume that* V *satisfies* $\mathscr{D}_m$ *with* $m > 1$. *Fix* $q \in \mathbb{Z}$+. *Then for each* $\varepsilon \in (0, 1)$ *we have*

$$\xi(h_q(-\varepsilon), h_{0,q}; \lambda) + O(\theta_{2\gamma}(\lambda)) \leq \xi(H, H_0; \mathscr{E}_q + \lambda) \leq \xi(h_q(\varepsilon), h_{0,q}; \lambda) + O(\theta_{2\gamma}(\lambda)), \tag{5.4.39}$$

as $\lambda \to 0$, *for any* $\gamma \in (0, (\alpha - 1)/2)$, $\gamma \leq 1$.

Consequently, as in Corollary 5.4.2, from well-known results concerning the asymptotic behaviour of the SSF $\xi(h_q(\varepsilon), h_{0,q}; \lambda)$ as $\lambda \to 0$ (see e.g. [165, Theorem XIII.82], [215] and [110]), more explicit asymptotic behaviour of the SSF $\xi(H, H_0; \mathscr{E}_q + \lambda)$ can be deduce. In particular it provides conditions ensuring the boundedness (or not) of the SSF $\xi(H, H_0; \cdot)$ near $\mathscr{E}_q$.

Corollary 5.4.4 ([30, Corollaries 2.1, 2.2, 2.3]) *Let* V *satisfy* $\mathscr{D}_m$ *with* $m > 1$ *and fix* $q \in \mathbb{Z}_+$.

1. *If* $m > 2$, *then*
$$\xi(H, H_0; \mathscr{E}_q + \lambda) = O(1), \quad \lambda \to 0. \tag{5.4.40}$$

2. *If* $m \in (1,2)$ *and for each* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ *(for some* $\varepsilon_0 \in (0,1)$*) there exist real numbers* $\omega_{q,\pm}(\varepsilon)$ *such that*
$$\lim_{y \to \pm\infty} |y|^m w_{q,\varepsilon}(y) = \omega_{q,\pm}(\varepsilon) \tag{5.4.41}$$
uniformly with respect to ε, *then we have*
$$\lim_{\lambda \downarrow 0} \lambda^{\frac{1}{m}-\frac{1}{2}} \xi(H, H_0; \mathscr{E}_q - \lambda) = -\mu_q^{-1/2} \mathscr{C}_m \Omega_q^-, \tag{5.4.42}$$
$$\lim_{\lambda \downarrow 0} \lambda^{\frac{1}{m}-\frac{1}{2}} \xi(H, H_0; \mathscr{E}_q + \lambda) = -\mu_q^{-1/2} \mathscr{C}_m \left(\csc(\pi/m) \Omega_q^- + \cot(\pi/m) \Omega_q^+ \right), \tag{5.4.43}$$
where $\mathscr{C}_m := \frac{1}{\pi} \int_0^1 (t^{-m} - 1)^{1/2} dt$, *and* $\Omega_q^\pm := \sum_{\varsigma=+,-} \omega_{q,\varsigma}(0)_\pm^{1/m}$, *while* $\omega_{q,\varsigma}(0)_+$ *and* $\omega_{q,\varsigma}(0)_-$ *denote the positive and the negative part of* $\omega_{q,\varsigma}(0)$ *respectively.*

3. *If* $m = 2$ *and* $w_{q,\varepsilon}$ *satisfies* (5.4.41), *then we have*
$$\lim_{\lambda \downarrow 0} |\ln \lambda|^{-1} \xi(H, H_0; \mathscr{E}_q - \lambda) = -\frac{1}{2\pi} \sum_{\varsigma=+,-} \left(\frac{\omega_{q,\varsigma}(0)}{\mu_q} + \frac{1}{4} \right)_-^{1/2}.$$
Moreover, if $\omega_{q,\pm}(0) > -\mu_q/4$, *then* $\xi(H, H_0; \mathscr{E}_q - \lambda) = O(1)$ *as* $\lambda \downarrow 0$.

Remark 5.4.10 *If* $q = 0$ *and* $\lambda > 0$, *we have (cf.* (5.1.2)*):*
$$-\xi(H, H_0; \mathscr{E}_0 - \lambda) = \mathscr{N}_H(\lambda) := \operatorname{Tr} \mathbb{1}_{(-\infty, \mathscr{E}_0 - \lambda)}(H),$$
the number of the eigenvalues of the operator H *lying on the interval* $(-\infty, \mathscr{E}_0 - \lambda)$, *counted with the multiplicities. This quantity is defined for* $m > 0$ *and as in* (5.4.42), *we have*
$$\lim_{\lambda \downarrow 0} \lambda^{\frac{1}{m}-\frac{1}{2}} \mathscr{N}_H(\lambda) = \mu_q^{-1/2} \mathscr{C}_m \Omega_0^- \tag{5.4.44}$$
for all $m \in (0,2)$ *and not only for* $m \in (1,2)$.

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