SPECTRAL ESTIMATES FOR RUELLE OPERATORS WITH TWO PARAMETERS AND SHARP LARGE DEVIATIONS

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Abstract. We obtain spectral estimates for the iterations of Ruelle operator $L_{f+(a+\mathbf{i}b)\tau+(c+\mathbf{i}d)g}$ with two complex parameters and Hölder functions f, g generalizing the case $\Pr(f) = 0$ studied in [9]. As an application we prove a sharp large deviation theorem concerning exponentially shrinking intervals which improves the result in [8].

1. Introduction

Let M be a C^2 complete Riemannian manifold, let $\varphi_t : M \longrightarrow M$ $(t \in \mathbb{R})$ be a C^2 flow on M and let $\varphi_t : M \longrightarrow M$ be a C^2 weak mixing Axiom A flow ([5], [7]). Let Λ be a *basic set* for φ_t, that is, Λ is a compact locally maximal invariant subset of M and φ_t is hyperbolic and transitive on Λ.

As in [9], we will use a symbolic coding of the flow on Λ provided by a a fixed Markov family $\{R_i\}_{i=1}^k$. More precisely, we consider a Markov family of *pseudo-rectangles* $R_i = [U_i, S_i] = \{[x, y] : x \in U_i, y \in S_i\}$ (see section 2 for more details). Denote by $\mathcal{P} : R = \cup_{i=1}^k R_i \longrightarrow R$ the related Poincaré map, by $\tau(x) > 0$ the first return time function on R, and by $\sigma : U = \cup_{i=1}^k U_i \longrightarrow U$ the *shift map* given by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \longrightarrow U$ is the *projection* along stable leaves. The flow φ_t on Λ is naturally related to the suspension flow σ_t^τ on the suspension space R^τ (see section 2 for details). There exists a natural semi-conjugacy projection $\pi(x, t) : R^\tau \longrightarrow \Lambda$ which is one-to-one on a residual set (see [2]). For $x \in R$ set

$$\tau^n(x) := \tau(x) + \tau(\sigma(x)) + ... + \tau(\sigma^{n-1}(x)).$$

Given Hölder continuous functions $F, G : \Lambda \longrightarrow \mathbb{R}$, define $f, g : R \longrightarrow \mathbb{R}$ by

$$f(x) = \int_0^{\tau(x)} F(\pi(x,t))dt \quad , \quad g(x) = \int_0^{\tau(x)} G(\pi(x,t))dt.$$

The main object of study in this paper are the Ruelle transfer operators of the form

$$L_{f-s\tau+zg}v(x) = \sum_{\sigma y = x} e^{f(y)-s\tau(y)+zg(y)}v(y) \quad , \quad s, z \in \mathbb{C} \quad , x \in U,$$

depending on two complex parameters s and z. Under certain assumptions, strong spectral estimates for such operators have been established in [9] and some significant applications to the study of zeta functions depending on two complex parameters have been made. We denote by m_H the equilibrium state corresponding to H in R^τ and by μ_k the equilibrium state corresponding to k in

R. More precisely,

$$\mathrm{Pr}_\sigma(k) = h(\sigma, \mu_k) + \int_R k d\mu_k = \sup_{\mu \in \mathcal{M}_\sigma} \Big\{ h(\sigma, \mu) + \int_R k d\mu \Big\},$$

$$\mathrm{Pr}_{\sigma_\tau}(H) = h(\sigma_\tau^t, m_H) + \int_{R^\tau} H dm_H = \sup_{m \in \mathcal{M}_{\sigma_\tau}} \Big\{ h(\sigma_\tau^t, m) + \int_{R^\tau} H dm \Big\},$$

where $h(\sigma, \mu)$ is the metric entropy of σ with respect to μ and $h(\sigma_\tau^t, m)$ is the metric entropy of the suspended flow σ_τ^t with respect to m. Let $P = \mathrm{Pr}_\sigma(f)$.

Let $\|h\|_0$ denote the *standard* sup *norm* of h on U. For $|b| \geq 1$, and $\beta > 0$, as in [4], define the norm $\|h\|_{\beta,b} = \|h\|_\infty + \frac{|h|_\beta}{|b|}$ on the space $C^\beta(U)$ of β-Hölder functions on U.

Our first aim in this paper is to prove the following theorem.

Theorem 1. *Let $\varphi_t : M \longrightarrow M$ satisfy the Standing Assumptions (see Sect. 4) over the basic set Λ, and let $0 < \beta < \alpha$. Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for φ_t over Λ as in section 2. Then for any real-valued functions $f, g \in C^\alpha(\widehat{U})$ and any constants $\epsilon > 0$ and $B > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C = C(B, \epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$ then*

$$\|L^m_{f-(a+\mathbf{i}b)\tau+(c+\mathbf{i}w)g} h\|_{\beta,b} \leq C\, e^{Pm}\, \rho^m\, |b|^\epsilon\, \|h\|_{\beta,b} \tag{1.1}$$

for all $h \in C^\beta(U)$, all integers $m \geq 1$ and all $b, w \in \mathbb{R}$ with $|b| \geq b_0$ and $|w| \leq B\,|b|$.

In Theorem 5.1 in [9] the above estimate has been proved in the case $P = 0$ assuming $|w| \leq B\,|b|^\nu$ for some constant $\nu \in (0,1)$. The present results is significantly stronger. See also Remark 1 below.

In the proof of Theorem 1 we will use some arguments from the proof of Theorem 5.1 in [9] with necessary modifications.

Remark 1. *Notice that in Theorem 1 above we do not assume that pressure P of f is zero, unlike what has been done in previous papers. This contributes the term e^{Pm} in the right-hand-side of (1.1) which is significant especially in the case $P < 0$ which occurs in the applications concerning large deviations (see Section 3). In previous papers the authors consider the case $P = 0$ and remark that the general case follows from this. However a more careful argument shows that an estimate of the form (1.1) does not follow immediately from a similar estimate[1] with $P = 0$.*

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[1]Indeed, assume we have proved (1.1) in the case $P = 0$, and then deal with the general case using the standard approach. Given a, b as in the theorem and $h \in \mathcal{F}_\theta(\widehat{U})$, we have

$$(L^m_{f-(a+\mathbf{i}b)\tau} h)(u) = \sum_{\sigma^m(v)=u} e^{(f-(a+\mathbf{i}b)\tau)^m(v)} h(v) = (L^m_{f-(P+a+\mathbf{i}b)\tau}(e^{P\tau^m(v)}\, h))(u). \tag{1.2}$$

We can now apply (1.1) in the case $P = 0$ replacing h by $e^{P\tau^m(v)}\, h$. Since $0 < c \leq \tau(u) \leq c_1$ for some constants c and c_1, assuming e.g. $P \leq 0$ (the other case is similar), we get $\|e^{P\tau^m}\, h\|_\infty \leq e^{mPc}\|h\|_\infty$, and $|e^{P\tau^m}\, h|_\theta \leq e^{mPc}\,|h|_\theta + |e^{P\tau^m}|_\theta\,\|h\|_\infty$. Given $u, v \in U_i$ we have (assuming e.g. $e^{P\tau^m(u)} > e^{P\tau^m(v)}$)

$$\left| e^{P\tau^m(u)} - e^{P\tau^m(v)} \right| \leq e^{P\tau^m(u)}\, |P\tau^m(u) - P\tau^m(v)| \leq |P|\, e^{mPc} \sum_{j=0}^{m-1} \frac{D_\theta(u,v)}{\theta^j} \leq \mathrm{Const}\frac{e^{mPc}}{\theta^m}\, D_\theta(u,v).$$

Taking θ closer to 1 and replacing c by some $c_0 < c$, we get $|e^{P\tau^m}|_\theta \leq \mathrm{Const}\; e^{mPc_0}$. This implies

$$\|e^{P\tau^m}\, h\|_{\theta,b} \leq e^{mPc}\|h\|_\infty + \frac{1}{|b|}\left(e^{mPc}\,|h|_\theta + \mathrm{Const}\; e^{mPc_0}\; \|h\|_\infty \right) \leq \mathrm{Const}\; e^{mPc_0}\; \|h\|_{\theta,b}.$$

Combining the latter with (1.2) gives $\|L^m_{f-(a+\mathbf{i}b)\tau} h\|_{\theta,b} \leq C\; \rho^m\; \|e^{P\tau^m(v)}\, h\|_{\theta,b} \leq C\; e^{mPc_0}\; \rho^m\; \|h\|_{\theta,b}$. As one can see this estimate is a bit worse than (1.1), since $c > c_0 > 0$ (and also c_1 in the case $P > 0$) can be rather small constants.

Remark 2. *In the proof of Theorem* 2 *in Section* 3 *we apply Theorem* 1 *with* $b = Cw$ *for some constant* $C > 0$*; then* $|w| = \frac{1}{C}|b|$*. The relevant part of Theorem* 5.1 *in* [9] *assumes* $|w| \leq B|b|^{\nu}$ *for some* $\nu \in (0, 1)$ *and this is clearly not sufficient for the proof of Theorem* 2 *below.*

Let G be a Hölder function on Λ such that $G > 0$ everywhere on Λ. Consider a number

$$0 < a = \int_{R^\tau} G dm_{F+\xi(a)G} \in \Big\{ \int_{R^\tau} G dm_{F+tG},\ t \in \mathbb{R} \Big\},$$

where $\xi(a)$ is determined by the equation

$$\frac{d\mathrm{Pr}_{\sigma_\tau}(F + tG)}{dt}\Big|_{t=\xi(a)} - a = 0.$$

Let $0 < \rho < 1$ be the constant from Theorem 1, and let $0 < \alpha_0 = -\frac{\log \rho}{2}$. Fix an arbitrary $0 < \delta \leq \alpha_0$ and consider the sequence $\{\delta_n\}_{n\in\mathbb{N}}$, where

$$\delta_n = e^{-\delta n}.$$

Set $g_a = g - \tau a$. Then

$$g_a^n(x) = g^n(x) - \tau^n(x)a = \int_0^{\tau^n(x)} G(\pi(t, x))dt - \tau^n(x)a.$$

Clearly the property

$$\frac{g^n(x)}{\tau^n(x)} - a \in \Big(-\frac{\delta_n}{\tau^n(x)}, \frac{\delta_n}{\tau^n(x)} \Big),$$

is equivalent to

$$g^n(x) - \tau^n(x)a \in (-\delta_n, \delta_n).$$

On the other hand, since $cn \leq \tau^n(x) \leq c_1 n$, $\forall x \in R, \forall n \in \mathbb{N}$ with some constants $0 < c \leq c_1$ for every x, the interval $\Big(-\frac{\delta_n}{\tau^n(x)}, \frac{\delta_n}{\tau^n(x)}\Big)$ is exponentially shrinking to 0 as $n \to \infty$. Let $\mu = \mu_f$ be the equilibrium state of f.

Our second problem concerns the analysis of the asymptotic of

$$\mu\{x : g^n(x) - \tau^n(x)a \in (-\delta_n, \delta_n)\},\ n \to \infty$$

and for $a \neq \int_{R^\tau} G dm_F$ we obtain a large deviation result. On the other hand, as in the previous paper [8], we examine the measure of points $x \in R$ for which the difference $\frac{g^n(x)}{\tau^n(x)} - a$ stays in an exponentially shrinking interval. Next we state two definitions from [6] and [14].

Definition 1. *Two functions* f_1, f_2 *are called* $\sigma-$*independent if whenever there are constants* $t_1, t_2 \in \mathbb{R}$ *such that* $t_1 f_1 + t_2 f_2$ *is cohomologous to a function in* $C(R : 2\pi\mathbb{Z})$ *, we have* $t_1 = t_2 = 0$.

For a function $G \in C^\beta(R^\tau : \mathbb{R})$ consider the skew product flow S_t^G on $\mathbb{S}^1 \times R^\tau$ defined by

$$S_t^G(e^{2\pi \mathbf{i}\alpha}, y) = \Big(e^{2\pi\mathbf{i}(\alpha + \int_0^t G(\varphi_\tau^s y)ds)}, \sigma_\tau^t(y)\Big).$$

Definition 2. *Let* $G \in C^\beta(R^\tau : \mathbb{R}))$*. Then* G *and* σ_τ^t *are flow independent if the following condition is satisfied: if* $t_0, t_1 \in \mathbb{R}$ *are constants such that the skew product flow* S_t^H *with* $H = t_0 + t_1 G$ *is not topologically ergodic, then* $t_0 = t_1 = 0$.

Notice that if G and σ_τ^t are flow independent, the flow σ_τ^t is topologically weak mixing and the function G is not cohomologous to a constant function. This implies that the set

$$\Big\{\int_{R^\tau} G dm_{F+tG},\, t\in\mathbb{R}\Big\}$$

has a non-empty interior and setting $\beta(t) = \mathrm{Pr}_{\sigma_\tau}(F+tG)$, one has

$$\beta''(t) = \frac{d^2\mathrm{Pr}_{\sigma_\tau}(F+tG)}{dt^2} = \sigma^2_{m_{F+tG}}(G)$$

with

$$\sigma_m^2(G) = \lim_{T\to\infty}\frac{1}{T}\Big\{\int_{R^\tau}\int_0^T G(\sigma_\tau^t(y))dtdm - T\int_{R^\tau} Gdm\Big\}^2 < \infty.$$

Moreover, $\beta'(\xi(a)) = a$ and $\xi(a)$ is differentiable with $\xi'(a) = \frac{1}{\beta''(\xi(a))} > 0$. Without loss of generality by adding a constant, we may assume that $\mathrm{Pr}_{\sigma_\tau}(F) = 0$. Then m_F and $\mathrm{Pr}_{\sigma_\tau}(F+tG)$ don't change and $\mathrm{Pr}_\sigma(f - \mathrm{Pr}_{\sigma_\tau}(F)\tau) = 0$ yields $\mathrm{Pr}_\sigma(f) = 0$. Introduce the rate function

$$\gamma(a) =: \mathrm{Pr}_{\sigma_\tau}(F+\xi(a)G) - \xi(a)a.$$

Then

$$\gamma'(a) = \beta'(\xi(a))\xi'(a) - \xi'(a)a - \xi(a) = -\xi(a),$$

and $\gamma(a) \le 0$ is a concave function with strict maximum 0 at $a = \int_{R^\tau} Gdm_F$. In the following we assume that G and σ_τ^t are flow independent, which guarantees that $g(x)$ and $\tau(x)$ are $\sigma-$independent. Consequently, the function $g_a = g - a\tau$ is not cohomologous to a function in $C(R : 2\pi\mathbb{Z})$, and this yields

$$\frac{d^2\mathrm{Pr}(f+tg_a)}{dt^2}\Big|_{t=\xi(a)} = \omega(a) > 0. \tag{1.3}$$

From now on for simplicity of the notation we will write Pr instead of Pr_σ. Consider the rate function

$$J(a) =: \inf_{t\in\mathbb{R}}\{\mathrm{Pr}(f+t(g-\tau a)\} = \mathrm{Pr}(f+\eta(a)(g-\tau a)),$$

where $\eta(a)$ is the unique real number such that

$$0 = \frac{d\mathrm{Pr}(f+t(g-\tau a))}{dt}\Big|_{t=\eta(a)} = \int_R g dm_{f+\eta(a)(g-\tau a)} - a\int_R \tau dm_{f+\eta(a)(g-\tau a)}.$$

Notice that

$$a = \frac{\int_R\int_0^{\tau(x)} G(\pi(x,t))dtdm_{f+\eta(a)g_a}}{\int_R \tau dm_{f+\eta(a)g_a}} = \int_{R^\tau} G(\pi(x,t))dm_{F+\eta(a)(G-a)}$$

$$= \int_{R^\tau} G(\pi(x,t))dm_{F+\eta(a)G} = \frac{d\mathrm{Pr}_{\sigma_\tau}(F+tG)}{dt}\Big|_{t=\eta(a)}.$$

Here we have used the fact that $F+\eta(a)G$ and $F+\eta(a)G-\eta(a)a$ have the same equilibrium state in R^τ. Since $\frac{d\mathrm{Pr}_{\sigma_\tau}(F+tG)}{dt}$ is increasing, there exists an unique $\xi(a)$ such that

$$\frac{d\mathrm{Pr}_{\sigma_\tau}(F+\xi(a)G)}{dt} = a,$$

therefore $\xi(a) = \eta(a)$. Hence $J(a) = \mathrm{Pr}(f+\xi(a)(g-a\tau))$. In Section 2 we show that

$$J(a) = \gamma(a)\int_R \tau d\mu_{f+\xi(a)(g-a\tau)}. \tag{1.4}$$

This implies $J(a) \leq 0$ and $J(a) = 0$ if only if $a = \int_{R^\tau} G dm_F$ and $\xi(a) = 0$. We prove the following large deviation result.

Theorem 2. *Let the assumptions of Theorem 1 be satisfied. Assume that $G : \Lambda \longrightarrow (0, \infty)$ is a Hölder continuous function for which there exists a Markov family $\mathcal{R} = \{R_i\}_{i=1}^k$ for the flow φ_t on Λ such that G is constant on the stable leaves of all "rectangular boxes"*

$$B_i = \{\varphi_t(x) : x \in R_i, 0 \leq t \leq \tau(x)\},$$

$i = 1, \dots, k$. Assume in addition that G and σ_τ^t are flow independent. Let $0 < \rho < 1$ be the constant in Theorem 1 and let $\delta_n = e^{-\delta n}$, $0 < \delta \leq -\frac{\log \rho}{2}$. Then

$$\mu\{x : g^n(x) - \tau^n(x)a \in (-\delta_n, \delta_n)\} \sim \frac{2\delta_n}{\sqrt{2\pi\omega(a)n}} e^{nJ(a)},\ n \to \infty. \tag{1.5}$$

A similar result for the measure of points $x \in R$ for which the difference

$$\frac{1}{n}\int_0^{\tau^n(x)} G(\pi(t,x))dt - p$$

stays in a exponentially shrinking interval has been obtained in [8] under the conditions that G is a Lipschitz function on Λ and $\frac{\mathrm{Lip}\, G}{\min G} < \mu_0$ with a suitable positive constant μ_0. In the present paper we improve the result in [8] assuming G only Hölder. Moreover, here one examines a more natural difference $\frac{1}{\tau^n(x)}\int_0^{\tau^n(x)} G(\pi(t,x))dt - a$. This progress is essentially based on the spectral estimates for the Ruelle operator with two complex parameters established in Theorem 1. A further improvement will be the analysis of the asymptotic of

$$\mu\Big\{x : \frac{1}{T}\int_0^T G(\pi(t,x))dt - a \in \Big(-\frac{e^{\delta T}}{T}, \frac{e^{-\delta T}}{T}\Big)\Big\},\ \delta > 0,$$

as $T \to +\infty$ and this is an interesting open problem. On the other hand, the case when the interval $\Big(-\frac{e^{\delta T}}{T}, \frac{e^{-\delta T}}{T}\Big)$ is replaced by an interval $\Big(\frac{\alpha}{T}, \frac{\beta}{T}\Big)$, $\alpha < \beta$, has been studied in [14]. Comparing (1.5) with Theorem 1 in [14], one observes that in the case we deal with the variable tending to $+\infty$ by a scaling can take the form $T_n = n\int_R \tau d\mu_{\tau+\xi(a)(g-a\tau)}$. Setting

$$\omega(a) = \frac{1}{C^2(a)}\beta''(\xi(a))\int_R \tau d\mu_{\tau+\xi(a)(g-a\tau)},\ C(a) \neq 0,$$

one may write the leading term in (1.5) as

$$\frac{2\delta_n C(a)}{\sqrt{2\pi\beta''(\xi(a))T_n}} e^{T_n\gamma(a)}$$

which modulo the constant $C(a)$ is similar to the asymptotic in [14] with $T_n \to \infty$, where the rate function is precisely $\gamma(a)$.

Remark 3. *The result stated in Theorem 2 holds if we assume that G is non-lattice and g and τ are $\sigma-$independent. The condition $G > 0$ is not a restriction since we can replace G by $G + C > 0$ for some large constant $C > 0$. Then $a = \int_{R^\tau} G dm_F + C$, and the asymptotic (1.5) is independent on the constant C. The assumption that G is constant on stable leaves of rectangular boxes B_i is significant, however it seems difficult to remove when very sensitive asymptotics such as (1.5) are obtained. For "standard" large deviation results, this assumption is not necessary, since one can use Sinai's Lemma (see e.g. Proposition 1.2 in [7]) to replace an arbitrary Hölder G by a cohomologous* *function which is constant on stable leaves. In* [14] *and* [11], *where instead of* $(-e^{-\delta n}, e^{-\delta n})$ *the authors deal with significantly larger intervals, however still smaller than* $(-c/n, c/n)$ *for a constant* $c > 0$, *claims have been made that the general case of Hölder functions on two-sided shifts is easily derived from the one for one-sided shifts. However in both papers there are no proofs of these claims. For sharp estimates similar to* (1.5), *it is tempting to believe that such claims would be difficult to justify.*

2. Preliminaries

As in section 1, let $\varphi_t : M \longrightarrow M$ be a C^2 Axiom A flow on a Riemannian manifold M, and let Λ be a basic set for φ_t. The restriction of the flow on Λ is a hyperbolic flow [7]. For any $x \in M$ let $W^s_\epsilon(x), W^u_\epsilon(x)$ be the local stable and unstable manifolds through x, respectively (see [2], [5], [7]). When M is compact and M itself is a basic set, φ_t is called an *Anosov flow*. It follows from the hyperbolicity of Λ that if $\epsilon_0 > 0$ is sufficiently small, there exists $\epsilon_1 > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \epsilon_1$, then $W^s_{\epsilon_0}(x)$ and $\varphi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$ intersect at exactly one point $[x, y] \in \Lambda$ (cf. [5]). That is, there exists a unique $t \in [-\epsilon_0, \epsilon_0]$ such that $\varphi_t([x, y]) \in W^u_{\epsilon_0}(y)$. Setting $\Delta(x, y) = t$, defines the so called *temporal distance function*.

We will use the set-up and some arguments from [12] and [9]. As in these papers, fix a *(pseudo) Markov family* $\mathcal{R} = \{R_i\}_{i=1}^k$ of *pseudo-rectangles* $R_i = [U_i, S_i] = \{[x, y] : x \in U_i, y \in S_i\}$. Set $R = \cup_{i=1}^k R_i, U = \cup_{i=1}^k U_i$. Consider the *Poincaré map* $\mathcal{P} : R \longrightarrow R$, defined by $\mathcal{P}(x) = \varphi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\varphi_{\tau(x)}(x) \in R$ (*first return time* function). The *shift map* $\sigma : U \longrightarrow U$ is given by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \longrightarrow U$ is the *projection* along stable leaves.

The hyperbolicity of the flow on Λ implies the existence of constants $c_0 \in (0, 1]$ and $\gamma_1 > \gamma_0 > 1$ such that

$$c_0\gamma_0^m\, d(u_1, u_2) \le d(\sigma^m(u_1), \sigma^m(u_2)) \le \frac{\gamma_1^m}{c_0} d(u_1, u_2) \tag{2.1}$$

whenever $\sigma^j(u_1)$ and $\sigma^j(u_2)$ belong to the same U_{i_j} for all $j = 0, 1 \dots, m$.

Define a $k \times k$ matrix $A = \{A(i, j)\}_{i,j=1}^k$ by

$$A(i, j) = \begin{cases} 1 \text{ if } \mathcal{P}(\operatorname{Int} R_i) \cap \operatorname{Int} R_j \neq \emptyset, \\ 0 \text{ otherwise.} \end{cases}$$

It is possible to construct a Markov family $\mathcal{R}$ so that A is irreducible and aperiodic (see [2]).

Consider the suspension space $R^\tau = \{(x, t) \in R \times \mathbb{R} : 0 \le t \le \tau(x)\}/\sim$, where by $\sim$ we identify the points $(x, \tau(x))$ and $(\sigma x, 0)$. The corresponding suspension flow is defined by $\sigma^\tau_t(x, s) = (x, s+t)$ on R^τ taking into account the identification $\sim$. For a Hölder continuous function f on R, the *topological pressure* $\Pr(f)$ with respect to σ is defined as

$$\Pr(f) = \sup_{m \in \mathcal{M}_\sigma} \{h(\sigma, m) + \int f dm\},$$

where $\mathcal{M}_\sigma$ denotes the space of all σ-invariant Borel probability measures and $h(\sigma, m)$ is the entropy of σ with respect to m. We say that f and g are *cohomologous* and we denote this by $f \sim g$ if there exists a continuous function w such that $f = g + w \circ \sigma - w$.

The proof of (1.4) follows from the following computation:

$$\gamma(a) = \Pr_{\sigma_\tau}(F + \xi(a)G) - \xi(a)a = h(\sigma_\tau, m_{F+\xi(a)G}) + \int_{R^\tau} (F + \xi(a)G - \xi(a)a) dm_{F+\xi(a)G}$$

$$= h(\sigma_\tau, m_{F+\xi(a)G-\xi(a)a}) + \int_{R^\tau} (F + \xi(a)G - \xi(a)a) dm_{F+\xi(a)G-\xi(a)a}$$

$$= \frac{h(\sigma, \mu_{f+\xi(a)(g-a\tau)})}{\int_R \tau d\mu_{f+\xi(a)(g-a\tau)}} + \frac{\int_R (f + \xi(a)(g - a\tau)) d\mu_{f+\xi(a)(g-a\tau)}}{\int_R \tau d\mu_{f+\xi(a)(g-a\tau)}}$$

$$= \frac{\Pr(f + \xi(a)(g - a\tau))}{\int_R \tau d\mu_{f+\xi(a)(g-a\tau)}} = \frac{J(a)}{\int_R \tau d\mu_{f+\xi(a)(g-a\tau)}}.$$

3. Proof of Theorem 2

In this section we prove Theorem 2 exploiting the spectral estimates obtained in Theorem 1. We work under the assumptions of Theorem 2, in particular, G is constant on stable leaves of rectangular boxes B_i for a certain Markov family $\mathcal{R} = \{R_i\}_{i=1}^k$. Then the function $g(x)$ depends only on $x \in U$.

We may replace f by a Hölder function $\tilde{f}$ depending only on $x \in U$ so that with some Hölder function $z(x)$ we have

$$f(x) = \tilde{f}(x) + z(\sigma(x)) - z(x).$$

Therefore for all $t \in \mathbb{R}$ we have $\Pr(f + t(g - a\tau)) = \Pr(\tilde{f} + t(g - a\tau))$ and $\mu_f = \mu_{\tilde{f}}$. Below we use again the notation f assuming that $f(x)$ depends only on $x \in U$.

We will examine the sequence

$$\rho(n) = \int_U \chi_n(g_a^n(x))\, d\mu\ , \tag{3.1}$$

where $\chi \in C_0^\infty(\mathbb{R} : \mathbb{R}^+)$ is a *fixed cut-off function* and

$$\chi_n(t) = \chi(\delta_n^{-1} t) \quad , \quad x \in \mathbb{R}\ . \tag{3.2}$$

Proposition 1. *Under the assumptions of Theorem 2 we have the asymptotic*

$$\rho(n) \sim \frac{\delta_n}{\sqrt{2\pi\omega(a)n}} \Big(\int \chi(t)dt\Big) e^{nJ(a)},\ n \to \infty. \tag{3.3}$$

Proof. The Ruelle operator $\mathcal{L}_{f+\xi(a)g_a}$ has a simple eigenvalue

$$\lambda_a = e^{\Pr(f+\xi(a)g_a)} = e^{J(a)},$$

and so for all sufficiently small $u \in \mathbb{C}$ the operator $\mathcal{L}_{f+(\xi(a)+iu)g_a}$ has a simple eigenvalue $e^{\Pr(f+(\xi(a)+\mathbf{i}u)g_a)}$ and the rest of the spectrum of $\mathcal{L}_{f+(\xi(a)+iu)g_a}$ is contained in a disk of radius $\theta\lambda_a$ with some $0 < \theta < 1$. Note that

$$\frac{d^2 \Pr(f + (\xi(a) + \mathbf{i}u))g_a)}{d^2 u}\Big|_{u=0} = -\omega(a) < 0.$$

Clearly for the Fourier transform $\hat{\chi}_n$ of χ_n we get $\hat{\chi}_n(u) = \delta_n \hat{\chi}(\delta_n u)$. Set $\omega_n(y) = e^{-\xi(a)y}\chi_n(y)$. Since $\Pr(f) = 0$, the Ruelle operator L_f has a simple eigenvalues 1 and the adjoint operator L_f^* satisfies

$$L_f^* \mu = \mu,$$

where we denote $\mu = \mu_f$ as in Section 1.

Using this property and applying the Fourier transform, we have

$$\begin{aligned}\rho(n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int e^{iug_a^n(x)}d\mu(x)\Big)\hat{\chi}_n(u)du\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int e^{(\xi(a)+iu)g_a^n(x)}d\mu(x)\Big)\hat{\omega}_n(u)du\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int L^n_{f+(\xi(a)+iu)g_a}1(x)d\mu(x)\Big)\hat{\omega}_n(u)du\\ &= \frac{\delta_n}{2\pi}\int_{-\infty}^{\infty}\Big(\int L^n_{f+(\xi(a)+iu)g_a}1(x)\, d\mu(x)\Big)\hat{\chi}(\delta_n(u-i\xi(a)))du.\end{aligned}$$

By Taylor expansion for small $|u|$ one gets

$$e^{\Pr(f+(\xi(a)+iu)g_a)} = \lambda_a\Big(1-\frac{\omega(a)}{2}u^2+\mathcal{O}(|u|^3)\Big).$$

We choose $\epsilon_0 > 0$ sufficiently small and changing the coordinates on $(-\epsilon_0, \epsilon_0)$ by $u = \frac{\sqrt{2}v}{\sqrt{\omega(a)}}$, we write

$$I_1(n) = \frac{\delta_n}{\sqrt{2\omega(a)}\pi}\lambda_a^n\int_{-\epsilon_1}^{\epsilon_1}\Big((1-v^2+iQ(v))^n(1+\mathcal{O}(v))(\hat{\chi}(-i\delta_n\xi(a))+\mathcal{O}(\delta_n v))\Big)dv + \delta_n\mathcal{O}(\lambda_a^n\theta^n)$$

with $\epsilon_1 = \sqrt{\frac{\omega(a)}{2}}\epsilon_0$ and real valued function $Q(v) = \mathcal{O}(|v|^3)$. The analysis of the asymptotic of this integral is given in section 4.1 in [11]. The leading term has the form

$$\frac{\delta_n}{\sqrt{2\omega(a)}\pi}\hat{\chi}(0)\lambda_a^n\int_{-\epsilon_1}^{\epsilon_1}(1-v^2)^n dv = \frac{\delta_n\lambda_a^n}{\sqrt{2\omega(a)}\pi}\hat{\chi}(0)\sqrt{\frac{\pi}{n}} + \delta_n\mathcal{O}\Big(\frac{\lambda_a^n}{n}\Big) \quad , \quad n\to\infty.$$

Thus we deduce

$$I_1(n) \sim \frac{\delta_n}{\sqrt{2\pi\omega(a)n}}\Big(\int\chi(t)dt\Big)e^{nJ(a)} + \mathcal{O}\Big(\frac{\delta_n e^{nJ(a)}}{n}\Big). \tag{3.4}$$

Next consider the integral

$$I_2(n) = \frac{\delta_n}{2\pi}\int_{\epsilon_0<|u|\leq\frac{c}{a}}\Big(\int L^n_{f+(\xi(a)+iu)g_a}1(x)d\mu(x)\Big)\hat{\chi}(\delta_n(u-i\xi(a)))du$$

with $c \gg 1$ sufficiently large. Since g_a is non-lattice, for $0 < \epsilon_0 \leq |u| \leq \frac{c}{a}$ the operator $L_{f+(\xi(a)+iu)g_a}$ has no eigenvalues λ with $|\lambda| = \lambda_a$ and the spectral radius of $L_{f+(\xi(a)+iu)g_a}$ is strictly less than λ_a. Thus, there exists $\alpha = \alpha(a,c)$, $0 < \alpha < 1$, such that for $n \geq N(a,c)$ we have

$$\|L^n_{f+(\xi(a)+iu)g_a}\| \leq \alpha^n\lambda_a^n. \tag{3.5}$$

On the other hand,

$$|\hat{\chi}(\delta_n(u-i\xi(a)))| \leq C_k\frac{e^{c_0\delta_n|\xi(a)|}}{\delta_n^k|u|^k},\ |u|\geq\epsilon_0,\ \forall k\in\mathbb{N}\,, \tag{3.6}$$

with $c_0 > 0$ depending on the support of χ. Applying (3.5) and (3.6) with $k = 0$, for large n we get

$$I_2(n) = \mathcal{O}\Big(\frac{\delta_n e^{nJ(a)}}{n}\Big).$$

Now consider

$$I_3(n) = \frac{\delta_n}{2\pi} \int_{|u|>\frac{c}{a}} \Big(\int L^n_{f+(\xi(a)+iu)g_a} 1(x) d\mu(x) \Big) \hat{\chi}(\delta_n(u - i\xi(a))) du \ .$$

We are going to use the spectral estimates established in Theorem 1 for the Ruelle operator

$$L_{f-a(\xi(a)+iu)\tau+(\xi(a)+iu)g} = L_{f+\xi(a)(g-a\tau)-iau\tau+iug}.$$

Then $|u| \leq \frac{1}{|a|}|au|$ and for sufficiently large $|u| \geq \frac{c}{a}$ and for every $\epsilon > 0$ we are in situation to apply the spectral estimates

$$\begin{aligned} & \Big\| L^n_{f-\xi(a)(g-a\tau)-iau\tau+iug} 1 \Big\|_\infty \\ \leq \ & C_\epsilon e^{n\mathrm{Pr}(f+\xi(a)(g-a\tau))} \rho^n |au|^\epsilon, \ 0 < \rho < 1, |au| \geq c, \ n \in \mathbb{N}. \end{aligned} \tag{3.7}$$

Fix $0 < \epsilon \leq 1/2$ and apply the estimate (3.6) with $k = 2$ and (3.7) for ϵ. This gives

$$|I_3(n)| \leq \delta_n \lambda_a^n A_\epsilon e^{c_0 \delta_n |\xi(a)|} \frac{\rho^n |a|^\epsilon}{\delta_n^2} \int_{|u|>\frac{c}{a}} |u|^{\epsilon-2} du = D\delta_n \lambda_a^n \Big(\frac{\rho^n}{\delta_n^2} \Big).$$

Recall that we have the condition

$$0 < \delta \leq \alpha_0 \leq -\frac{\log \rho}{2}$$

and one deduces the inequality

$$n \log \rho + 2\delta n - \log n \leq 0,$$

which leads to

$$\frac{\rho^n}{\delta_n^2} \leq \frac{1}{n}, \ n \geq 1 \ .$$

Thus, we conclude that

$$I_3(n) = \mathcal{O}\Big(\frac{\delta_n e^{nJ(a)}}{n} \Big).$$

Consequently,

$$\rho(n) = I_1(n) + I_2(n) + I_3(n) = \frac{\delta_n}{\sqrt{2\pi\omega(a)n}} \Big(\int \chi(t) dt \Big) e^{nJ(a)} \Big(1 + \mathcal{O}(1/\sqrt{n}) \Big) \ , n \to \infty$$

and this completes the proof of Proposition 1. ■

To establish Theorem 2, as in [11], [8], we approximate the characteristic function $\mathbf{1}_{[-1,1]}$ of the interval $[-1, 1]$ by cut-off functions.

4. Ruelle operators – definitions and assumptions

Assume as in Sect. 1 that $\varphi_t : M \longrightarrow M$ is a C^2 weak mixing Axiom A flow and Λ be a basic set for φ_t. Here we work under the same assumptions as these in [9]. One of these is:

(LNIC): *There exist $z_0 \in \Lambda$, $\epsilon_0 > 0$ and $\theta_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, any $\hat{z} \in \Lambda \cap W^u_\epsilon(z_0)$ and any tangent vector $\eta \in E^u(\hat{z})$ to Λ at $\hat{z}$ with $\|\eta\| = 1$ there exist $\tilde{z} \in \Lambda \cap W^u_\epsilon(\hat{z})$, $\tilde{y}_1, \tilde{y}_2 \in \Lambda \cap W^s_\epsilon(\tilde{z})$ with $\tilde{y}_1 \neq \tilde{y}_2$, $\delta = \delta(\tilde{z}, \tilde{y}_1, \tilde{y}_2) > 0$ and $\epsilon' = \epsilon'(\tilde{z}, \tilde{y}_1, \tilde{y}_2) \in (0, \epsilon]$ such that*

$$|\Delta(\exp^u_z(v), \pi_{\tilde{y}_1}(z)) - \Delta(\exp^u_z(v), \pi_{\tilde{y}_2}(z))| \geq \delta \, \|v\|$$

for all $z \in W^u_{\epsilon'}(\tilde{z}) \cap \Lambda$ and $v \in E^u(z; \epsilon')$ with $\exp^u_z(v) \in \Lambda$ and $\langle \frac{v}{\|v\|}, \eta_z \rangle \geq \theta_0$, where η_z is the parallel translate of η along the geodesic in $W^u_{\epsilon_0}(z_0)$ from $\hat{z}$ to z.

The above condition may seem complicated at a first glance, however a careful look at it shows that it is just a rather natural non-integrability condition.

Given $x \in \Lambda$, $T > 0$ and $\delta \in (0, \epsilon]$ set

$$B^u_T(x, \delta) = \{y \in W^u_\epsilon(x) : d(\varphi_t(x), \varphi_t(y)) \leq \delta \ , \ 0 \leq t \leq T\}.$$

We will say that φ_t has a *regular distortion along unstable manifolds* over the basic set Λ if there exists a constant $\epsilon_0 > 0$ with the following properties:

(a) For any $0 < \delta \leq \epsilon \leq \epsilon_0$ there exists a constant $R = R(\delta, \epsilon) > 0$ such that

$$\mathrm{diam}(\Lambda \cap B^u_T(z, \epsilon)) \leq R\, \mathrm{diam}(\Lambda \cap B^u_T(z, \delta))$$

for any $z \in \Lambda$ and any $T > 0$.

(b) For any $\epsilon \in (0, \epsilon_0]$ and any $\rho \in (0, 1)$ there exists $\delta \in (0, \epsilon]$ such that for any $z \in \Lambda$ and any $T > 0$ we have $\mathrm{diam}(\Lambda \cap B^u_T(z, \delta)) \leq \rho\, \mathrm{diam}(\Lambda \cap B^u_T(z, \epsilon))$.

In this paper we work under the following **Standing Assumptions:**

(A) φ_t has Lipschitz local holonomy maps over Λ,

(B) the local non-integrability condition (LNIC) holds for φ_t on Λ,

(C) φ_t has a regular distortion along unstable manifolds over the basic set Λ.

A rather large class of examples satisfying the conditions (A) – (C) is provided by imposing the following *pinching condition*:

(P): *There exist constants $C > 0$ and $\beta \geq \alpha > 0$ such that for every $x \in M$ we have*

$$\frac{1}{C}\, e^{\alpha_x t}\, \|u\| \leq \|d\varphi_t(x) \cdot u\| \leq C\, e^{\beta_x t}\, \|u\| \quad , \quad u \in E^u(x) \ , t > 0$$

for some constants $\alpha_x, \beta_x > 0$ with $\alpha \leq \alpha_x \leq \beta_x \leq \beta$ and $2\alpha_x - \beta_x \geq \alpha$ for all $x \in M$.

We should note that (P) holds for geodesic flows on manifolds of strictly negative sectional curvature satisfying the so called $\frac{1}{4}$-pinching condition. (P) always holds when $\dim(M) = 3$.

Simplifying Assumptions: φ_t is a C^2 contact Anosov flow satisfying the condition (P).

By [13], the pinching condition (P) implies that φ_t has Lipschitz local holonomy maps and regular distortion along unstable manifolds. This and Proposition 6.1 in [13] show that:
the Simplifying Assumptions imply the Standing Assumptions.

Throughout we work under the Standing Assumptions. In what follows we will use arguments similar to those in section 4 in [9], however technically they will be more complicated, since the numbers of parameters involved will increase. E.g. where we had functions f_{at}, h_{at}, etc., depending on two parameters, now we have to deal with functions f_{atc}, h_{atc}, etc., depending on three parameters. While some of the arguments we use here are almost the same as corresponding arguments in [9] (and we omit them), there are others that require more significant modification and we do them in some detail.

Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a **fixed Markov family** for the flow ϕ_t on Λ consisting of rectangles $R_i = [U_i, S_i]$ and let $U = \cup_{i=1}^k U_i$ (see section 2). Then (2.1) hold for some constants $c_0 \in (0, 1]$ and $\gamma_1 > \gamma_0 > 1$. Let $\widehat{U}$ be the set of those points $x \in U$ such that $\mathcal{P}^m(x)$ is not a boundary point of a rectangle for any integer m. In a similar way define $\widehat{R}$.

Fix a number $\alpha > 0$ **and two real-valued functions** f **and** g **in** $C^\alpha(\widehat{U})$. Let $P = P_f$ be the unique real number so that $\mathrm{Pr}(f - P\,\tau) = 0$, where Pr is the topological pressure with respect to σ. For any $t \in \mathbb{R}$ with $t \geq 1$, let f_t be the *average of* f *over balls in* U *of radius* $1/t$ obtained as follows: fix an arbitrary extension $f \in C^\alpha(V)$ (with the same Hölder constant), where V is an open neighborhood of U in M, and then take the averages in question. Then $f_t \in C^\infty(V)$ and:
(a) $\|f - f_t\|_\infty \leq |f|_\alpha / t^\alpha$;
(b) $\mathrm{Lip}(f_t) \leq \mathrm{Const}\ \|f\|_\infty t$;
(c) For any $\beta \in (0, \alpha)$ we have $|f - f_t|_\beta \leq 2\,|f|_\alpha / t^{\alpha-\beta}$.

Let $G : \Lambda \longrightarrow \mathbb{R}$ be a **fixed** α**-Hölder function which is constant on the stable leaves** of all "rectangular boxes" $B_i = \{\varphi_t(x) : x \in R_i, 0 \leq t \leq \tau(x)\}$, $i = 1, \ldots, k$.

Given a large parameter $t > 0$, define G_t as above, so that G_t is again constant on the stable leaves of all rectangular boxes B_i and
(a') $\|G - G_t\|_\infty \leq |G|_\alpha / t^\alpha$;
(b') $\mathrm{Lip}(G_t) \leq \mathrm{Const}\ \|G\|_\infty t$;
(c') For any $\beta \in (0, \alpha)$ we have $|G - G_t|_\beta \leq 2\,|G|_\alpha / t^{\alpha-\beta}$.
In particular, for some constant $C_0 > 0$ we have $\mathrm{Lip}(G_t) \leq C_0 t$.

Then define $g_t : R \longrightarrow \mathbb{R}$ by

$$g_t(x) = \int_0^{\tau(x)} G_t(\pi(x, s))\, ds. \tag{4.1}$$

Clearly g_t is α-Hölder and constant on stable leaves, so we can regard g_t as a function on U. Thus, $g_t \in C^\alpha(U)$.

Let $\lambda_0 > 0$ be the *largest eigenvalue of* L_f, i.e. $\lambda_0 = e^P$, and let $\hat{\nu}_0$ be the (unique) probability measure on U with $L_f^* \hat{\nu}_0 = \lambda_0\, \hat{\nu}_0$. Fix a corresponding (positive) eigenfunction $h_0 \in \hat{C}^\alpha(U)$ such that $\int_U h_0\, d\hat{\nu}_0 = 1$. Then $d\nu_0 = h_0\, d\hat{\nu}_0$ defines a σ*-invariant probability measure* ν_0 on U. Setting

$$f^{(0)} = f + \ln h_0(u) - \ln h_0(\sigma(u)) - \ln \lambda_0,$$

we have $L_{f^{(0)}}^* \nu_0 = \nu_0$, i.e. $\displaystyle\int_U L_{f^{(0)}} H\, d\nu_0 = \int_U H\, d\nu_0$ for any $H \in C(U)$ and $L_{f^{(0)}} 1 = 1$.

Given real numbers a, c and t (with $|a| + \frac{1}{|t|}$ small and $c \in I$), denote by λ_{atc} the *largest eigenvalue* of $L_{f_t - a\tau + cg_t}$ on $C^{\mathrm{Lip}}(U)$ and by h_{atc} the corresponding (positive) eigenfunction such that $\int_U h_{atc}\, d\nu_{atc} = 1$, where $\hat{\nu}_{atc}$ is the unique probability measure on U with $L_{f_t - a\,\tau + cg_t}^* \hat{\nu}_{atc} = \lambda_{atc}\, \hat{\nu}_{atc}$. Setting $d\nu_{atc} = h_{atc}\, d\hat{\nu}_{atc}$ defines a σ*-invariant probability measure* ν_{atc} on U.

Given $\theta \in (0, 1)$, consider the metric d_θ on $\widehat{U}$ defined by $d_\theta(x, x) = 0$ and $d_\theta(x, y) = \theta^m$, where m is the largest integer such that $x \neq y$ belong to the same cylinder of length m. Taking $\theta \in (0, 1)$ sufficiently close to 1 and $\beta \in (0, \alpha)$ sufficiently close to 0 we have $(d(x, y))^\alpha \leq \mathrm{Const}\ d_\theta(x, y)$ and $d_\theta(x, y) \leq \mathrm{Const}\ (d(x, y))^\beta$ for all $x, y \in \widehat{U}$. In what follows we assume that θ and β satisfy these assumptions.

By the properties of the approximations f_t and g_t stated above, there exists a constant $C_0 > 0$, depending on f and α but independent of β, such that

$$\|[f_t - a\tau + cg] - f\|_\beta \leq C_0\, [|a| + |c| + 1/t^{\alpha-\beta}] \tag{4.2}$$

for all $|a|, |c| \leq 1$ and $t \geq 1$. Next, the analyticity of pressure and the eigenfunction projection corresponding to the *maximal eigenvalue* $\lambda_{atc} = e^{\mathrm{Pr}(f_t - a\tau + cg_t)}$ of the Ruelle operator $L_{f_t - a\tau + cg_t}$ on

$C^\beta(U)$ (cf. e.g. Ch. 3 in [7] or Appendix 1 in [6]) that there exists a constant $a_0 > 0$ such that, taking $C_0 > 0$ sufficiently large, we have

$$|\Pr(f_t - a\tau + cg_t) - P| \leq C_0 \left(|a| + |c| + \frac{1}{t^{\alpha-\beta}}\right) \quad , \quad \|h_{atc} - h_0\|_\beta \leq C_0 \left(|a| + |c| + \frac{1}{t^{\alpha-\beta}}\right) \quad (4.3)$$

for $|a|, |c| \leq a_0$ and $1/t \leq a_0$. We take $C_0 > 0$ and $a_0 > 0$ so that

$$\lambda_0/C_0 \leq \lambda_{atc} \leq C_0\, \lambda_0,$$

$\|f_t\|_\infty \leq C_0$ and $1/C_0 \leq h_{atc}(u) \leq C_0$ for all $u \in U$ and all $|a|, |c|, 1/t \leq a_0$.

Given real numbers a, c and t with $|a|, |c|, 1/t \leq a_0$ consider the functions

$$f_{atc} = f_t - a\tau + cg_t + \ln h_{atc} - \ln(h_{atc} \circ \sigma) - \ln \lambda_{atc}$$

and the operators $\mathcal{M}_{atc} = L_{f_{atc}} : C(U) \longrightarrow C(U)$. One checks that $\mathcal{M}_{atc}\, 1 = 1$.

Taking the constant $C_0 > 0$ sufficiently large, we may assume that

$$\|f_{atc} - f^{(0)}\|_\beta \leq C_0 \left[|a| + |c| + \frac{1}{t^{\alpha-\beta}}\right] \quad , \quad |a|, |c|, 1/t \leq a_0. \quad (4.4)$$

The proof of the following lemma is given in [9] when $c = 0$. In the case with three parameters the proof is almost the same, so we omit it.

Lemma 1. *Taking the constant $a_0 > 0$ sufficiently small, there exists a constant $T' > 0$ such that for all $a, t, c \in \mathbb{R}$ with $|a|, |c| \leq a_0$ and $t \geq 1/a_0$ we have $h_{atc} \in C^{\mathrm{Lip}}(\widehat{U})$ and $\mathrm{Lip}(h_{atc}) \leq T't$.*

Consequently, assuming $a_0 > 0$ is chosen sufficiently small, there exists a constant $T > 0$ (depending on $|f|_\alpha$ and a_0) such that

$$\|f_{atc}\|_\infty \leq T \quad , \quad \|g_t\|_\infty \leq T \quad , \quad \mathrm{Lip}(h_{atc}) \leq T\, t \quad , \quad \mathrm{Lip}(f_{atc}) \leq T\, t \quad (4.5)$$

for $|a|, |c|, 1/t \leq a_0$. In what follows **we assume that** a_0**,** C_0**,** $T \geq \max\{\, \|\tau\|_0\, , \mathrm{Lip}(\tau_{|\widehat{U}})\,\}$ $1 < \gamma_0 < \gamma_1$ **are fixed constants** with (2.1) and (4.2) – (4.5).

Next, Ruelle operators of the form $L_{f - s\tau + zg}$, where $s = a + \mathbf{i}b$ and $z = c + \mathbf{i}w$, $a, b, c, w \in \mathbb{R}$, and $|a|, |c| \leq a_0$ for some constant $a_0 > 0$, will be studied approximating them by Ruelle operators of the form

$$\mathcal{L}_{abtz} = L_{f_{atc} - \mathbf{i}\, b\tau + zg_t} : C^\alpha(\widehat{U}) \longrightarrow C^\alpha(\widehat{U}).$$

Since $f_{atc} - \mathbf{i}b\tau + zg_t$ is Lipschitz, the operators $\mathcal{L}_{abtz}$ preserve each of the spaces $C^{\alpha'}(\widehat{U})$ for $0 < \alpha' \leq 1$ including the space $C^{\mathrm{Lip}}(\widehat{U})$ of Lipschitz functions $h : \widehat{U} \longrightarrow \mathbb{C}$. For such h we will denote by $\mathrm{Lip}(h)$ the Lipschitz constant of h. For $|b| \geq 1$, define the norm $\|.\|_{\mathrm{Lip},b}$ on $C^{\mathrm{Lip}}(\widehat{U})$ by $\|h\|_{\mathrm{Lip},b} = \|h\|_0 + \frac{\mathrm{Lip}(h)}{|b|}$. Recall the norm $\|h\|_{\beta,b} = \|h\|_\infty + \frac{|h|_\beta}{|b|}$ on $C^\beta(U)$ defined in section 1.

The main step in proving Theorem 1 is the following.

Theorem 3. *Under the assumptions in Theorem 1 there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$, $A_0 > 0$ and $C = C(B, \epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then*

$$\|L^m_{f_{atc} - \mathbf{i}b\tau + (c + \mathbf{i}w)g_t} h\|_{\mathrm{Lip},b} \leq C\, \rho^m\, \|h\|_{\mathrm{Lip},b}$$

for all $h \in C^{\mathrm{Lip}}(\widehat{U})$, all integers $m \geq 1$ and all $b, w, t \in \mathbb{R}$ with $|b| \geq b_0$, $te^{A_0 t} \leq |b|$ and $|w| \leq B\,|b|$.

Throughout we work under the Standing Assumptions made above and with fixed real-valued functions $f, g \in C^\alpha(\widehat{U})$ as in section 1, where $\alpha > 0$ is a fixed number. Another fixed number $\beta \in (0, \alpha)$ will be used later.

Assuming that all rectangles R_i are sufficiently small we have $\mathrm{diam}(U_i) < 1$ for all i. Recall the metric D on $\widehat{U}$ defined in [12]: $D(x,y) = \min\{\mathrm{diam}(\mathcal{C}) : x, y \in \mathcal{C}\ ,\ \mathcal{C}$ a cylinder contained in $U_i\}$ if $x, y \in U_i$ for some $i = 1, \dots, k$, and $D(x,y) = 1$ otherwise. As shown in [12], $d(x,y) \leq D(x,y)$ for $x, y \in \widehat{U}_i$ for some i, and for any cylinder $\mathcal{C}$ in U the characteristic function $\chi_{\widehat{\mathcal{C}}}$ of $\widehat{\mathcal{C}}$ on $\widehat{U}$ is Lipschitz with respect to D and $\mathrm{Lip}_D(\chi_{\widehat{\mathcal{C}}}) \leq 1/\mathrm{diam}(\mathcal{C})$. Let $C_D^{\mathrm{Lip}}(\widehat{U})$ be the *space of all Lipschitz functions* $h : \widehat{U} \longrightarrow \mathbb{C}$ *with respect to the metric* D and let $\mathrm{Lip}_D(h)$ be the *Lipschitz constant* of h with respect to D.

Given $A > 0$, denote by $K_A(\widehat{U})$ *the set of all functions* $h \in C_D^{\mathrm{Lip}}(\widehat{U})$ such that $h > 0$ and $\frac{|h(u)-h(u')|}{h(u')} \leq A\, D(u,u')$ for all $u, u' \in \widehat{U}$ that belong to the same $\widehat{U}_i$ for some $i = 1, \dots, k$. For $h \in K_A(\widehat{U})$ we have $|\ln h(u) - \ln h(v)| \leq A\ D(u,v)$ and so $e^{-A\, D(u,v)} \leq \frac{h(u)}{h(v)} \leq e^{A\, D(u,v)}$ for all $u, v \in \widehat{U}_i$, $i = 1, \dots, k$.

Fix an arbitrary constant $\hat{\gamma}$ **with** $1 < \hat{\gamma} < \gamma_0$. The following lemma is similar to Lemma 5.2 in [9], hoverer some technical details are different, so we sketch its proof in the Appendix.

Lemma 2. *Assuming* $a_0 > 0$ *is chosen sufficiently small, there exists a constant* $A_0 > 0$ *such that for all* $a, c, t \in \mathbb{R}$ *with* $|a|, |c| \leq a_0$ *and* $t \geq 1$ *the following hold:*

(a) If $H \in K_Q(\widehat{U})$ *for some* $Q > 0$, *then*

$$\frac{|(\mathcal{M}_{atc}^m H)(u) - (\mathcal{M}_{atc}^m H)(u')|}{(\mathcal{M}_{atc}^m H)(u')} \leq A_0 \left[\frac{Q}{\hat{\gamma}^m} + e^{A_0 t}\, t\right] D(u,u')$$

for all $m \geq 1$ *and all* $u, u' \in U_i$, $i = 1, \dots, k$.

(b) If the functions h *and* H *on* $\widehat{U}$ *and* $Q > 0$ *are such that* $H > 0$ *on* $\widehat{U}$ *and*

$$|h(v) - h(v')| \leq t\, Q\, H(v')\, D(v,v')$$

for any $v, v' \in \widehat{U}_i$, $i = 1, \dots, k$, *then for any integer* $m \geq 1$ *and any* $b, w \in \mathbb{R}$ *with* $|b|, |w| \geq 1$, *for* $z = c + \mathbf{i}w$ *we have*

$$|\mathcal{L}_{abtz}^m h(u) - \mathcal{L}_{abtz}^m h(u')| \leq A_0 \left(\frac{t\, Q}{\hat{\gamma}^m} (\mathcal{M}_{atc}^m H)(u') + (|b| + e^{A_0 t} t + t|w|)(\mathcal{M}_{atc}^m |h|)(u')\right) D(u,u')$$

whenever $u, u' \in \widehat{U}_i$ *for some* $i = 1, \dots, k$. *In particular, if* $e^{A_0 t} t \leq |b|$ *and* $|w| \leq B|b|$ *for some constant* $B > 0$, *then*

$$|\mathcal{L}_{abtz}^m h(u) - \mathcal{L}_{abtz}^m h(u')| \leq A_1 \left(\frac{t\, Q}{\hat{\gamma}^m} (\mathcal{M}_{atc}^m H)(u') + t\, |b| (\mathcal{M}_{atc}^m |h|)(u')\right) D(u,u').$$

for some constant $A_1 > 0$, *depending on* B.

From now on we will assume that a_0 **and** A_0 **are fixed with the properties in Lemma 2 above and** $a, b, c, w, t \in \mathbb{R}$ **are such that** $|a|, |c| \leq a_0$**,** $|b|, t, |w| \geq 1$ **and** $|w| \leq B|b|$**. As before, set** $z = c + \mathbf{i}d$**.**

As in [9], we need the entire set-up and notation from section 4 in [12], so we will now recall some of it.

Following section 4 in [12], **fix an arbitrary point** $z_0 \in \Lambda$ **and constants** $\epsilon_0 > 0$ **and** $\theta_0 \in (0,1)$ **with the properties described in** (LNIC). Assume that $z_0 \in \mathrm{Int}_\Lambda(U_1)$, $U_1 \subset \Lambda \cap W^u_{\epsilon_0}(z_0)$ and $S_1 \subset \Lambda \cap W^s_{\epsilon_0}(z_0)$. **Fix an arbitrary constant** θ_1 such that $0 < \theta_0 < \theta_1 < 1$.

Next, fix an arbitrary orthonormal basis $e_1, \ldots, e_n$ in $E^u(z_0)$ and a C^1 parameterization $r(s) = \exp^u_{z_0}(s)$, $s \in V'_0$, of a small neighborhood W_0 of z_0 in $W^u_{\epsilon_0}(z_0)$ such that V'_0 is a convex compact neighborhood of 0 in $\mathbb{R}^n \approx \mathrm{span}(e_1, \ldots, e_n) = E^u(z_0)$. Then $r(0) = z_0$ and $\frac{\partial}{\partial s_i} r(s)_{|s=0} = e_i$ for all $i = 1, \ldots, n$. Set $U'_0 = W_0 \cap \Lambda$. Shrinking W_0 (and therefore V'_0 as well) if necessary, we may assume that $\overline{U'_0} \subset \mathrm{Int}_\Lambda(U_1)$ and $\left| \left\langle \frac{\partial r}{\partial s_i}(s), \frac{\partial r}{\partial s_j}(s) \right\rangle - \delta_{ij} \right|$ is uniformly small for all $i, j = 1, \ldots, n$ and $s \in V'_0$, so that $\frac{1}{2}\langle \xi, \eta \rangle \leq \langle\ dr(s) \cdot \xi\ ,\ dr(s) \cdot \eta\ \rangle \leq 2\,\langle \xi, \eta \rangle$ for all $\xi, \eta \in E^u(z_0)$ and $s \in V'_0$, and $\frac{1}{2}\,\|s - s'\| \leq d(r(s), r(s')) \leq 2\,\|s - s'\|$ for all $s, s' \in V'_0$.

Definitions ([12]): (a) For a cylinder $\mathcal{C} \subset U'_0$ and a unit vector $\xi \in E^u(z_0)$ we will say that a *separation by a ξ-plane occurs* in $\mathcal{C}$ if there exist $u, v \in \mathcal{C}$ with $d(u,v) \geq \frac{1}{2}\,\mathrm{diam}(\mathcal{C})$ such that $\left\langle \frac{r^{-1}(v) - r^{-1}(u)}{\|r^{-1}(v) - r^{-1}(u)\|}\ ,\ \xi \right\rangle \geq \theta_1$.

Let $\mathcal{S}_\xi$ be the *family of all cylinders* $\mathcal{C}$ contained in U'_0 such that a separation by an ξ-plane occurs in $\mathcal{C}$.

(b) Given an open subset V of U'_0 which is a finite union of open cylinders and $\delta > 0$, let $\mathcal{C}_1, \ldots, \mathcal{C}_p$ ($p = p(\delta) \geq 1$) be the family of maximal closed cylinders in $\overline{V}$ with $\mathrm{diam}(\mathcal{C}_j) \leq \delta$. For any unit vector $\xi \in E^u(z_0)$ set $M^{(\delta)}_\xi(V) = \cup\{\mathcal{C}_j : \mathcal{C}_j \in \mathcal{S}_\xi\ ,\ 1 \leq j \leq p\}$.

In what follows we will construct, amongst other things, a sequence of unit vectors $\xi_1, \xi_2, \ldots, \xi_{j_0} \in E^u(z_0)$. For each $\ell = 1, \ldots, j_0$ set $R_\ell = \{\eta \in \mathbf{S}^{n-1} : \langle \eta, \xi_\ell \rangle \geq \theta_0\}$. For $t \in \mathbb{R}$ and $s \in E^u(z_0)$ set $I_{\eta,t}\, g(s) = \frac{g(s + t\,\eta) - g(s)}{t}$, $t \neq 0$ (*increment* of g in the direction of η).

Lemma 3. ([12]) *There exist integers $1 \leq n_1 \leq N_0$ and $\ell_0 \geq 1$, a sequence of unit vectors $\eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^u(z_0)$ and a non-empty open subset U_0 of U'_0 which is a finite union of open cylinders of length n_1 such that setting $\mathcal{U} = \sigma^{n_1}(U_0)$ we have:*

(a) *For any integer $N \geq N_0$ there exist Lipschitz maps $v^{(\ell)}_1, v^{(\ell)}_2 : U \longrightarrow U$ ($\ell = 1, \ldots, \ell_0$) such that $\sigma^N(v^{(\ell)}_i(x)) = x$ for all $x \in \mathcal{U}$ and $v^{(\ell)}_i(\mathcal{U})$ is a finite union of open cylinders of length N ($i = 1, 2$; $\ell = 1, 2, \ldots, \ell_0$).*

(b) *There exists a constant $\hat{\delta} > 0$ such that for all $\ell = 1, \ldots, \ell_0$, $s \in r^{-1}(U_0)$, $0 < |h| \leq \hat{\delta}$ and $\eta \in R_\ell$ with $s + h\,\eta \in r^{-1}(U_0 \cap \Lambda)$ we have*

$$\left[I_{\eta,h} \left(\tau^N(v^{(\ell)}_2(\tilde{r}(\cdot))) - \tau^N(v^{(\ell)}_1(\tilde{r}(\cdot))) \right) \right](s) \geq \frac{\hat{\delta}}{2}\,.$$

(c) *We have $\overline{v^{(\ell)}_i(U)} \bigcap \overline{v^{(\ell')}_{i'}(U)} = \emptyset$ whenever $(i, \ell) \neq (i', \ell')$.*

(d) *For any open cylinder V in U_0 there exists a constant $\delta' = \delta'(V) > 0$ such that*

$$V \subset M^{(\delta)}_{\eta_1}(V) \cup M^{(\delta)}_{\eta_2}(V) \cup \ldots \cup M^{(\delta)}_{\eta_{\ell_0}}(V)$$

for all $\delta \in (0, \delta']$.

Fix U_0 and $\mathcal{U}$ with the properties described in Lemma 3; then $\overline{\mathcal{U}} = U$.

Set $\hat{\delta} = \min_{1 \leq \ell \leq \ell_0} \hat{\delta}_j$, $n_0 = \max_{1 \leq \ell \leq \ell_0} m_\ell$, and fix an arbitrary point $\hat{z}_0 \in U^{(\ell_0)}_0 \cap \widehat{U}$.

Fix integers $1 \leq n_1 \leq N_0$ and $\ell_0 \geq 1$, unit vectors $\eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^u(z_0)$ and a non-empty open subset U_0 of W_0 with the properties described in Lemma 3. By the choice of U_0, $\sigma^{n_1} : U_0 \longrightarrow \mathcal{U}$ is one-to-one and has an inverse map $\psi : \mathcal{U} \longrightarrow U_0$, which is Lipschitz.

Next, assume that $B > 1$**,** $\beta \in (0, \alpha)$ **and** $E \geq \max\left\{4A_0\ ,\ BC_1\ ,\ \frac{2A_0 T}{\gamma - 1}\right\}$ **are fixed constants**, where $A_0 \geq 1$ is the constant from Lemma 2 and C_1 is the constant from the proof of Lemma 2 in the Appendix. **Fix an integer** $N \geq N_0$ such that

$$\hat{\gamma}^N \geq \max\left\{ 6A_0\ ,\ \frac{200\,\gamma_1^{n_1}\,A_0}{c_0^2}\ ,\ \frac{512\,\gamma_1^{n_1}\,E}{c_0\,\hat{\delta}\,\rho} \right\}. \tag{4.6}$$

We will also **assume now that the parameter** $t = t(a_0, N) > 1$ **is fixed** with

$$a_0 \leq \frac{1}{t^{\alpha-\beta}} \leq 2a_0 \quad , \quad t \leq \frac{c_0 \hat{\delta} \rho \hat{\gamma}^N}{500 E \gamma_1^{n_1}}. \tag{4.7}$$

(Part of this condition will be needed for the proof of Theorem 1.) Clearly the above requires to assume that $a_0 = a_0(N)$ satisfies $a_0^{1/(\alpha-\beta)} \leq t$. Some other conditions on the small parameter $a_0 = a_0(N) > 0$ will be imposed later. We will also need to choose

$$b_0 \geq t e^{A_0 t}.$$

Let the parameters $b, w \in \mathbb{R}$ be so that $|w| \leq B\,|b|$ and $|b|, |w| \geq b_0$.

Next, fix maps $v_i^{(\ell)} : U \longrightarrow U$ ($\ell = 1, \ldots, \ell_0$, $i = 1, 2$) with the properties (a), (b), (c) and (d) in Lemma 3. In particular, (c) gives $\overline{v_i^{(\ell)}(U)} \cap \overline{v_{i'}^{(\ell')}(U)} = \emptyset$ for all $(i, \ell) \neq (i', \ell')$.

Since U_0 is a finite union of open cylinders, it follows from Lemma 3(d) that there exist a constant $\delta' = \delta'(U_0) > 0$ such that $M_{\eta_1}^{(\delta)}(U_0) \cup \ldots \cup M_{\eta_{\ell_0}}^{(\delta)}(U_0) \supset U_0$ for all $\delta \in (0, \delta']$. **Fix** δ' **with this property**. Set

$$\epsilon_1 = \min\left\{ \frac{1}{32C_0}\ ,\ c_1\ ,\ \frac{1}{4E}\ ,\ \frac{1}{\hat{\delta}\,\rho^{p_0+2}}\ ,\ \frac{c_0 r_0}{\gamma_1^{n_1}}\ ,\ \frac{c_0^2(\gamma-1)}{16T\gamma_1^{n_1}} \right\}.$$

We will also assume that b_0 is chosen so that $\frac{\epsilon_1}{b_0} \leq \delta'$.

Let $\mathcal{C}_m$ ($1 \leq m \leq p$) be the family of *maximal closed cylinders* contained in $\overline{U_0}$ with $\mathrm{diam}(\mathcal{C}_m) \leq \frac{\epsilon_1}{|b|}$ such that $U_0 \subset \cup_{j=m}^p \mathcal{C}_m$ and $\overline{U_0} = \cup_{m=1}^p \mathcal{C}_m$. As in [13],

$$\rho\,\frac{\epsilon_1}{|b|} \leq \mathrm{diam}(\mathcal{C}_m) \leq \frac{\epsilon_1}{|b|} \quad , \quad 1 \leq m \leq p\,. \tag{4.8}$$

Fix an integer $q_0 \geq 1$ such that $32\rho^{q_0-1} < \theta_1 - \theta_0$, i.e. $\theta_0 < \theta_1 - 32\,\rho^{q_0-1}$. Next, let $\mathcal{D}_1, \ldots, \mathcal{D}_q$ be the list of all closed cylinders contained in $\overline{U_0}$ that are *subcylinders of co-length* $p_0\,q_0$ of some $\mathcal{C}_m$ ($1 \leq m \leq p$). Then $\overline{U_0} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_p = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_q$. Moreover,

$$\rho^{p_0\,q_0+1} \cdot \frac{\epsilon_1}{|b|} \leq \mathrm{diam}(\mathcal{D}_j) \leq \rho^{q_0} \cdot \frac{\epsilon_1}{|b|} \quad , \quad 1 \leq j \leq q.$$

Given $j = 1, \ldots, q$, $\ell = 1, \ldots, \ell_0$ and $i = 1, 2$, set $\widehat{\mathcal{D}}_j = \mathcal{D}_j \cap \widehat{U}$, $Z_j = \overline{\sigma^{n_1}(\widehat{\mathcal{D}}_j)}$, $\widehat{Z}_j = Z_j \cap \widehat{U}$, $X_{i,j}^{(\ell)} = \overline{v_i^{(\ell)}(\widehat{Z}_j)}$, and $\widehat{X}_{i,j}^{(\ell)} = X_{i,j}^{(\ell)} \cap \widehat{U}$. It then follows that $\mathcal{D}_j = \psi(Z_j)$, and $U = \cup_{j=1}^q Z_j$. Moreover, $\sigma^{N-n_1}(v_i^{(\ell)}(x)) = \psi(x)$ for all $x \in \mathcal{U}$, and all $X_{i,j}^{(\ell)}$ are cylinders such that $X_{i,j}^{(\ell)} \cap X_{i',j'}^{(\ell')} = \emptyset$ whenever $(i, j, \ell) \neq (i', j', \ell')$, and $\mathrm{diam}(X_{i,j}^{(\ell)}) \geq \dfrac{c_0\,\rho^{p_0\,q_0+1}}{\gamma_1^N} \cdot \dfrac{\epsilon_1}{|b|}$ for all $i = 1, 2$, $j = 1, \ldots, q$ and

$\ell = 1, \ldots, \ell_0$. The *characteristic function* $\omega_{i,j}^{(\ell)} = \chi_{\widehat{X}_{i,j}^{(\ell)}} : \widehat{U} \longrightarrow [0,1]$ of $\widehat{X}_{i,j}^{(\ell)}$ belongs to $C_D^{\rm Lip}(\widehat{U})$ and $\mathrm{Lip}_D(X_{i,j}^{(\ell)}) \leq 1/\mathrm{diam}(X_{i,j}^{(\ell)})$. Set

$$\mu_0 = \mu_0(N) = \min\left\{ \frac{1}{4} \; , \; \frac{c_0\, \rho^{p_0 q_0 + 2}\, \epsilon_1}{4\, \gamma_1^N} \; , \; \frac{1}{4\, e^{2TN}} \, \sin^2\left(\frac{\hat{\delta}\, \rho\, \epsilon_1}{256} \right) \right\}.$$

Let J be a *subset of the set* $\Xi = \{ \, (i,j,\ell) \; : \; 1 \leq i \leq 2 \; , \; 1 \leq j \leq q \; , \; 1 \leq \ell \leq \ell_0 \, \}$. Define the function $\omega = \omega_J : \widehat{U} \longrightarrow [0,1]$ by

$$\omega = 1 - \mu_0 \sum_{(i,j,\ell) \in J} \omega_{i,j}^{(\ell)}.$$

Clearly $\omega \in C_D^{\rm Lip}(\widehat{U})$ and $1 - \mu_0 \leq \omega(u) \leq 1$ for any $u \in \widehat{U}$. Moreover, $\mathrm{Lip}_D(\omega) \leq \Gamma = \frac{2\mu_0\, \gamma_1^N}{c_0\, \rho^{p_0 q_0 + 2}} \cdot \frac{|b|}{\epsilon_1}$.

Next, define *the contraction operator* $\mathcal{N} = \mathcal{N}_J(a,b,t,c) : C_D^{\rm Lip}(\widehat{U}) \longrightarrow C_D^{\rm Lip}(\widehat{U})$ by

$$(\mathcal{N} h) = \mathcal{M}_{atc}^N (\omega_J \cdot h).$$

Using Lemma 2 above, the proof of the following lemma is very similar to that of Lemma 5.6 in [12] and we omit it.

Lemma 4. *Under the above conditions for N and μ the following hold :*

(a) $\mathcal{N} h \in K_{E|b|}(\widehat{U})$ for any $h \in K_{E|b|}(\widehat{U})$;

(b) If $h \in C_D^{\rm Lip}(\widehat{U})$ and $H \in K_{E|b|}(\widehat{U})$ are such that $|h(v) - h(v')| \leq E\, t\, |b| H(v')\, D(v,v')$ for any $v, v' \in U_j$, $j = 1, \ldots, k$ and $|h| \leq H$ on $\widehat{U}$ and, then for any $i = 1, \ldots, k$ and any $u, u' \in \widehat{U}_i$ we have $|(\mathcal{L}_{abtz}^N h)(u) - (\mathcal{L}_{abtz}^N h)(u')| \leq E\, t\, |b|\, (\mathcal{N} H)(u')\, D(u,u')$.

Definition. A subset J of Ξ will be called *dense* if for any $m = 1, \ldots, p$ there exists $(i,j,\ell) \in J$ such that $\mathcal{D}_j \subset \mathcal{C}_m$.

Denote by $\mathsf{J} = \mathsf{J}(a,b,z)$ the *set of all dense subsets* J of Ξ.

Although the operator $\mathcal{N}$ here is different, the proof of the following lemma is very similar to that of Lemma 5.8 in [12] and we omit it.

Lemma 5. *Given the number N, there exist $\rho_2 = \rho_2(N) \in (0,1)$ and $a_0 = a_0(N) > 0$ such that*

$$\int_{\widehat{U}} (\mathcal{N}_J H)^2 d\nu \leq \rho_2 \int_{\widehat{U}} H^2 d\nu$$

whenever $|a|, |c| \leq a_0$, $t \geq 1/a_0$, J is dense and $H \in K_{E|b|}(\widehat{U})$.

Until the end of this section we will assume that $h, H \in C_D^{\rm Lip}(\widehat{U})$ are **fixed functions such that**

$$H \in K_{E|b|}(\widehat{U}) \quad , \quad |h(u)| \leq H(u) \quad , \quad u \in \widehat{U} \; , \tag{4.9}$$

and

$$|h(u) - h(u')| \leq E\, t\, |b| H(u')\, D(u,u') \quad \text{whenever } \; u, u' \in \widehat{U}_i \; , \; i = 1, \ldots, k \; . \tag{4.10}$$

Let again $z = c + \mathbf{i}w$. Define the functions $\chi_\ell^{(i)} : \widehat{U} \longrightarrow \mathbb{C}$ ($\ell = 1, \ldots, j_0$, $i = 1, 2$) by

$$\chi_\ell^{(1)}(u) = \frac{\left| e^{(f_{atc}^N - \mathbf{i}b\tau^N + \mathbf{i}wg_t^N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(f_{atc}^N - \mathbf{i}b\tau^N + \mathbf{i}wg_t^N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \right|}{(1-\mu) e^{f_{atc}^N(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + e^{f_{atc}^N(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

$$\chi_\ell^{(2)}(u) = \frac{\left| e^{(f_{atc}^N - \mathbf{i}b\tau^N + \mathbf{i}wg_t^N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(f_{atc}^N - \mathbf{i}b\tau^N + \mathbf{i}wg_t^N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \right|}{e^{f_{atc}^N(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + (1-\mu) e^{f_{atc}^N(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

and set

$$\gamma_\ell(u) = |b| \, [\tau^N(v_2^{(\ell)}(u)) - \tau^N(v_1^{(\ell)}(u))].$$

for all $u \in \widehat{U}$.

Definitions ([12]) We will say that the cylinders $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are *adjacent* if they are subcylinders of the same $\mathcal{C}_m$ for some m. If $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are contained in $\mathcal{C}_m$ for some m and for some $\ell = 1, \ldots, \ell_0$ there exist $u \in \mathcal{D}_j$ and $v \in \mathcal{D}_{j'}$ such that $d(u, v) \geq \frac{1}{2} \operatorname{diam}(\mathcal{C}_m)$ and $\left\langle \frac{r^{-1}(v) - r^{-1}(u)}{\|r^{-1}(v) - r^{-1}(u)\|} \, , \, \eta_\ell \right\rangle \geq \theta_1$ we will say that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are *η_ℓ-separable in $\mathcal{C}_m$*.

As a consequence of Lemma 3(b) one gets the following whose proof is almost the same as that of Lemma 5.9 in [12], so we omit it.

Lemma 6. *Let $j, j' \in \{1, 2, \ldots, q\}$ be such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are contained in $\mathcal{C}_m$ and are η_ℓ-separable in $\mathcal{C}_m$ for some $m = 1, \ldots, p$ and $\ell = 1, \ldots, \ell_0$. Then $|\gamma_\ell(u) - \gamma_\ell(u')| \geq A\, c_2\, \epsilon_1$ for all $u \in \widehat{Z}_j$ and $u' \in \widehat{Z}_{j'}$, where $c_2 = \frac{\hat{\delta}\,\rho}{16}$.*

The following lemma is the analogue of Lemma 5.10 in [12] and represents the main step in proving Theorem 1.

Lemma 7. *Assume $|b| \geq b_0$ for some sufficiently large $b_0 > 0$, $|a|, |c| \leq a_0$, and let $|w| \leq B|b|$. Then for any $j = 1, \ldots, q$ there exist $i \in \{1, 2\}$, $j' \in \{1, \ldots, q\}$ and $\ell \in \{1, \ldots, \ell_0\}$ such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are adjacent and $\chi_\ell^{(i)}(u) \leq 1$ for all $u \in \widehat{Z}_{j'}$.*

To prove this we need the following lemma which is the analogue of Lemma 14 in [4] and its proof is very similar, so we omit it.

Lemma 8. *If h and H satisfy* (4.9)-(4.10), *then for any $j = 1, \ldots, q$, $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$ we have:*

(a) $\frac{1}{2} \leq \frac{H(v_i^{(\ell)}(u'))}{H(v_i^{(\ell)}(u''))} \leq 2$ for all $u', u'' \in \widehat{Z}_j$;

(b) Either for all $u \in \widehat{Z}_j$ we have $|h(v_i^{(\ell)}(u))| \leq \frac{3}{4} H(v_i^{(\ell)}(u))$, or $|h(v_i^{(\ell)}(u))| \geq \frac{1}{4} H(v_i^{(\ell)}(u))$ for all $u \in \widehat{Z}_j$.

Sketch of proof of Lemma 7. We use a modification of the proof of Lemma 5.10 in [12].

Given $j = 1, \ldots, q$, let $m = 1, \ldots, p$ be such that $\mathcal{D}_j \subset \mathcal{C}_m$. As in [12] we find $j', j'' = 1, \ldots, q$ such that $\mathcal{D}_{j'}, \mathcal{D}_{j''} \subset \mathcal{C}_m$ and $\mathcal{D}_{j'}$ and $\mathcal{D}_{j''}$ are η_ℓ-separable in $\mathcal{C}_m$.

Fix ℓ, j' and j'' with the above properties, and set $\widehat{Z} = \widehat{Z}_j \cup \widehat{Z}_{j'} \cup \widehat{Z}_{j''}$. If there exist $t \in \{j, j', j''\}$ and $i = 1, 2$ such that the first alternative in Lemma 8(b) holds for $\widehat{Z}_t$, ℓ and i, then $\mu \leq 1/4$ implies $\chi_\ell^{(i)}(u) \leq 1$ for any $u \in \widehat{Z}_t$.

Assume that for every $t \in \{j, j', j''\}$ and every $i = 1, 2$ the second alternative in Lemma 8(b) holds for $\widehat{Z}_t$, ℓ and i, i.e. $|h(v_i^{(\ell)}(u))| \geq \frac{1}{4} H(v_i^{(\ell)}(u))$, $u \in \widehat{Z}$.

Since $\psi(\widehat{Z}) = \widehat{\mathcal{D}}_j \cup \widehat{\mathcal{D}}_{j'} \cup \widehat{\mathcal{D}}_{j''} \subset \mathcal{C}_m$, given $u, u' \in \widehat{Z}$ we have $\sigma^{N-n_1}(v_i^{(\ell)}(u)), \sigma^{N-n_1}(v_i^{(\ell)}(u')) \in \mathcal{C}_m$. Moreover, $\mathcal{C}' = v_i^{(\ell)}(\sigma^{n_1}(\mathcal{C}_m))$ is a cylinder with $\mathrm{diam}(\mathcal{C}') \leq \frac{\epsilon_1}{c_0\, \gamma_0^{N-n_1}\, |b|}$. Now the estimate (6.2) in the Appendix below implies $|g_t^N(v_i^{(\ell)}(u)) - g_t^N(v_i^{(\ell)}(u'))| \leq \frac{C_1 t \epsilon_1}{c_0\, \gamma_0^{N-n_1}\, |b|}$. Assume for example that $e^{c g_t^N(v_i^{(\ell)}(u))} |h(v_i^{(\ell)}(u))| \geq e^{c g_t^N(v_i^{(\ell)}(u))} |h(v_i^{(\ell)}(u'))|$. Then[2]

$$
\begin{aligned}
& \frac{|e^{z g_t^N(v_i^{(\ell)}(u))} h(v_i^{(\ell)}(u)) - e^{z g_t^N(v_i^{(\ell)}(u'))} h(v_i^{(\ell)}(u'))|}{\min\{|e^{z g_t^N(v_i^{(\ell)}(u))} h(v_i^{(\ell)}(u))|, |e^{z g_t^N(v_i^{(\ell)}(u'))} h(v_i^{(\ell)}(u'))|\}} \\
\leq \;& \frac{|e^{z g_t^N(v_i^{(\ell)}(u))} - e^{z g_t^N(v_i^{(\ell)}(u'))}|}{e^{c g_t^N(v_i^{(\ell)}(u'))}} + \frac{e^{c g_t^N(v_i^{(\ell)}(u))} |h(v_i^{(\ell)}(u)) - h(v_i^{(\ell)}(u'))|}{e^{c g_t^N(v_i^{(\ell)}(u'))} |h(v_i^{(\ell)}(u'))|} \\
\leq \;& \frac{|e^{z g_t^N(v_i^{(\ell)}(u))} - e^{z g_t^N(v_i^{(\ell)}(u'))}|}{e^{c g_t^N(v_i^{(\ell)}(u'))}} + \frac{e^{c (g_t^N(v_i^{(\ell)}(u')) - g_t^N(v_i^{(\ell)}(u')))}\, E |b| H(v_i^{(\ell)}(u'))}{|h(v_i^{(\ell)}(u'))|} D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) \\
\leq \;& \frac{|e^{c g_t^N(v_i^{(\ell)}(u))} - e^{c g_t^N(v_i^{(\ell)}(u'))}|}{e^{c g_t^N(v_i^{(\ell)}(u'))}} + |e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u))} - e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u'))}| + 4E|b| e^{2a_0 NT}\, \mathrm{diam}(\mathcal{C}') \\
\leq \;& (e^{C_1 t} C_1 t + |w| C_1 t)\, D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) + 4E|b| e^{2 N a_0 T}\, \frac{\gamma^{n_1} \epsilon_1}{c_0 \gamma^N} \\
\leq \;& \frac{(B + A_0)\gamma^{n_1} \epsilon_1}{c_0 \gamma^N} + \frac{4E \gamma^{n_1} \epsilon_1}{c_0 (e^{-2a_0 T} \gamma_0)^N} < \frac{\pi}{12}
\end{aligned}
$$

assuming $a_0 > 0$ is chosen sufficiently small and N sufficiently large. So, the angle between the complex numbers $e^{z g_t^N(v_i^{(\ell)}(u)} h(v_i^{(\ell)}(u))$ and $e^{z g_t^N(v_i^{(\ell)}(u')} h(v_i^{(\ell)}(u'))$ (regarded as vectors in $\mathbb{R}^2$) is $< \pi/6$. In particular, for each $i = 1, 2$ we can choose a real continuous function $\theta_i(u)$, $u \in \widehat{Z}$, with values in $[0, \pi/6]$ and a constant λ_i such that

$$
e^{z g_t^N(v_i^{(\ell)}(u))} h(v_i^{(\ell)}(u)) = e^{\mathbf{i}(\lambda_i + \theta_i(u))} e^{c g_t^N(v_i^{(\ell)}(u))} |h(v_i^{(\ell)}(u))|
$$

for all $u \in \widehat{Z}$. Fix an arbitrary $u_0 \in \widehat{Z}$ and set $\lambda = \gamma_\ell(u_0)$. Replacing e.g λ_2 by $\lambda_2 + 2m\pi$ for some integer m, we may assume that $|\lambda_2 - \lambda_1 + \lambda| \leq \pi$. Using the above, $\theta \leq 2 \sin\theta$ for $\theta \in [0, \pi/6]$, and some elementary geometry yields $|\theta_i(u) - \theta_i(u')| \leq 2 \sin|\theta_i(u) - \theta_i(u')| < \frac{c_2 \epsilon_1}{8}$.

The difference between the arguments of the complex numbers $e^{\mathbf{i}\, b\, \tau^N(v_1^{(\ell)}(u))} e^{z g_t^N(v_1^{(\ell)}(u)} h(v_1^{(\ell)}(u))$ and $e^{\mathbf{i}\, b\, \tau^N(v_2^{(\ell)}(u))} e^{z g_t^N(v_2^{(\ell)}(u)} h(v_2^{(\ell)}(u))$ is given by the function

$$
\Gamma^{(\ell)}(u) = [b\, \tau^N(v_2^{(\ell)}(u)) + \theta_2(u) + \lambda_2] - [b\, \tau^N(v_1^{(\ell)}(u)) + \theta_1(u) + \lambda_1] = (\lambda_2 - \lambda_1) + \gamma_\ell(u) + (\theta_2(u) - \theta_1(u)) \; .
$$

[2]Using some estimates as in the proof of Lemma 2(b) in the Appendix below and $\|c g_t^N\|_0 \leq a_0 NT$ by (4.5).

Given $u' \in \widehat{Z}_{j'}$ and $u'' \in \widehat{Z}_{j''}$, since $\widehat{\mathcal{D}}_{j'}$ and $\widehat{\mathcal{D}}_{j''}$ are contained in $\mathcal{C}_m$ and are η_ℓ-separable in $\mathcal{C}_m$, it follows from Lemma 6 and the above that

$$|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq |\gamma_\ell(u') - \gamma_\ell(u'')| - |\theta_1(u') - \theta_1(u'')| - |\theta_2(u') - \theta_2(u'')| \geq \frac{c_2\epsilon_1}{2} \ .$$

Thus, $|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq \frac{c_2}{2}\epsilon_1$ for all $u' \in \widehat{Z}_{j'}$ and $u'' \in \widehat{Z}_{j''}$. Hence either $|\Gamma^{(\ell)}(u')| \geq \frac{c_2}{4}\epsilon_1$ for all $u' \in \widehat{Z}_{j'}$ or $|\Gamma^{(\ell)}(u'')| \geq \frac{c_2}{4}\epsilon_1$ for all $u'' \in \widehat{Z}_{j''}$.

Assume for example that $|\Gamma^{(\ell)}(u)| \geq \frac{c_2}{4}\epsilon_1$ for all $u \in \widehat{Z}_{j'}$. Since $\widehat{Z} \subset \sigma^{n_1}(\mathcal{C}_m)$, as in [12] we have for any $u \in \widehat{Z}$ we get $|\Gamma_\ell(u)| < \frac{3\pi}{2}$. Thus, $\frac{c_2}{4}\epsilon_1 \leq |\Gamma^{(\ell)}(u)| < \frac{3\pi}{2}$ for all $u \in \widehat{Z}_{j'}$. Now as in [4] (see also [12]) one shows that $\chi_\ell^{(1)}(u) \leq 1$ and $\chi_\ell^{(2)}(u) \leq 1$ for all $u \in \widehat{Z}_{j'}$. ■

Parts (a) and (b) of the following lemma can be proved in the same way as the corresponding parts of Lemma 5.3 in [12], while part (c) follows from Lemma 4(b).

Lemma 9. *There exist a positive integer N and constants $\hat{\rho} = \hat{\rho}(N) \in (0,1)$, $a_0 = a_0(N) > 0$, $b_0 = b_0(N) > 0$ and $E \geq 1$ such that for every $a, b, c, t \geq 1, w \in \mathbb{R}$ with $|a|, |c| \leq a_0$, $|b| \geq b_0$ such that $|w| \leq B|b|$, there exists a finite family $\{\mathcal{N}_J\}_{J\in\mathsf{J}}$ of operators*

$$\mathcal{N}_J = \mathcal{N}_J(a,b,t,c) : C_D^{\mathrm{Lip}}(\widehat{U}) \longrightarrow C_D^{\mathrm{Lip}}(\widehat{U}),$$

where $\mathsf{J} = \mathsf{J}(a,b,t,c)$, with the following properties:

(a) The operators $\mathcal{N}_J$ preserve the cone $K_{E|b|}(\widehat{U})$;

(b) For all $H \in K_{E|b|}(\widehat{U})$ and $J \in \mathsf{J}$ we have $\int_{\widehat{U}} (\mathcal{N}_J H)^2 \, d\nu_0 \leq \hat{\rho} \int_{\widehat{U}} H^2 \, d\nu_0$.

(c) If $h, H \in C_D^{\mathrm{Lip}}(\widehat{U})$ are such that $H \in K_{E|b|}(\widehat{U})$, $|h(u)| \leq H(u)$ for all $u \in \widehat{U}$ and

$$|h(u) - h(u')| \leq Et|b|H(u')\, D(u,u')$$

whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$, then there exists $J \in \mathsf{J}$ such that $|\mathcal{L}_{abtz}^N h(u)| \leq (\mathcal{N}_J H)(u)$ for all $u \in \widehat{U}$ and for $z = c + \mathbf{i}w$ we have

$$|(\mathcal{L}_{abtz}^N h)(u) - (\mathcal{L}_{abtz}^N h)(u')| \leq E\, t\, |b| (\mathcal{N}_J H)(u')\, D(u,u')$$

whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$.

Proof of Lemma 9. Set $\hat{\rho} = 1 - \epsilon_2$.

Let $a \in \mathbb{R}$ and $b, w \in \mathbb{R}$ be such that $|a| \leq a_0$ and $|w| \leq B|b|$, $|b|, |w| \geq b_0$, and let $J \in \mathsf{J}(a,b)$. Then (a) follows from Lemma 4(a), while (b) follows from Lemma 5.

To check (c), assume that $h, H \in C_D^{\mathrm{Lip}}(\widehat{U})$ satisfy (4.9) and (4.10). Now define the subset J of $\mathsf{J}(a,b)$ in the following way. First, include in J all $(1,j,\ell) \in \Xi$ such that $\chi_\ell^{(1)}(u) \leq 1$ for all $u \in \widehat{Z}_j$. Then for any $j = 1, \ldots, q$ and $\ell = 1, \ldots, \ell_0$ include $(2,j,\ell)$ in J if and only if $(1,j,\ell)$ has not been included in J (that is, $\chi_\ell^{(1)}(u) > 1$ for some $u \in \widehat{Z}_j$) and $\chi_\ell^{(2)}(u) \leq 1$ for all $u \in \widehat{Z}_j$. It follows from Lemma 7 that J is dense.

Consider the operator $\mathcal{N} = \mathcal{N}_J(a,b) : C_D^{\mathrm{Lip}}(\widehat{U}) \longrightarrow C_D^{\mathrm{Lip}}(\widehat{U})$. Then Lemma 4(b) implies

$$|(\mathcal{L}_{abtz}^N\, h)(u) - (\mathcal{L}_{abtz}^N h)(u')| \leq E\, t\, |b| (\mathcal{N}\, H)(u')\, D(u,u')$$

whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$. So, it remains to show that

$$\left|(\mathcal{L}_{abtz}^N h)(u)\right| \leq (\mathcal{N} H)(u) \quad , \quad u \in \widehat{U} \ . \tag{4.11}$$

Let $u \in \widehat{U}$. If $u \notin \widehat{Z}_j$ for any $(i,j,\ell) \in J$, then $\omega(v) = 1$ whenever $\sigma^N v = u$ (since $v \in X_{i,j}^{(\ell)}$ implies $u = \sigma^N v \in Z_j$). Hence

$$\left|(\mathcal{L}_{abtz}^N h)(u)\right| = \left|\sum_{\sigma^N v = u} e^{(f_{atc}^N - \mathbf{i} b\tau^N + \mathbf{i} w g_t^N)(v)} h(v)\right| \le (\mathcal{M}_{atc}^N(\omega H))(u) = (\mathcal{N} H)(u).$$

Assume that $u \in \widehat{Z}_j$ for some $(i,j,\ell) \in J$. We will consider the case $i = 1$; the case $i = 2$ is similar. (Notice that by the definition of J, we cannot have both $(1,j,\ell)$ and $(2,j,\ell)$ in J.) Then $\chi_\ell^{(1)}(u) \le 1$, and therefore

$$\begin{aligned}
\left|(\mathcal{L}_{abtz}^N h)(u)\right| &\le \left|\sum_{\sigma^N v = u,\ v \neq v_1^{(\ell)}(u), v_2^{(\ell)}(u)} e^{(f_{atc}^N - \mathbf{i} b\tau^N + \mathbf{i} w g_t^N)(v)} h(v)\right| \\
&\quad + \left| e^{(f_{atc}^N - \mathbf{i} b\tau^N + \mathbf{i} w g_t^N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(f_{atc}^N - \mathbf{i} b\tau^N + \mathbf{i} w g_t^N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u))\right| \\
&\le \sum_{\sigma^N v = u,\ v \neq v_1^{(\ell)}(u), v_2^{(\ell)}(u)} e^{f_{atc}^N(v)} |h(v)| \\
&\quad + \left[(1-\mu) e^{f_{atc}^N(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + e^{f_{atc}^N(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))\right] .
\end{aligned}$$

Since $(1,j,\ell) \in J$ and $(2,j,\ell) \notin J$, the definition of the function ω gives $\omega(v_1^{(\ell)}(u)) \ge 1 - \mu$ and $\omega(v_2^{(\ell)}(u)) = 1$. This and (4.9) imply

$$\begin{aligned}
\left|(\mathcal{L}_{abtz}^N h)(u)\right| &\le \sum_{\sigma^N v = u,\ v \neq v_1(u), v_2(u)} e^{f_{atc}^N(v)} \omega(v) H(v) \\
&\quad + \left[e^{f_{atc}^N(v_1(u))} \omega(v_1(u)) H(v_1(u)) + e^{f_{atc}^N(v_2(u))} \omega(v_2(u)) H(v_2(u))\right] = (\mathcal{N} H)(u) ,
\end{aligned}$$

which proves (4.11). ■

5. Proofs of Theorems

Proof of Theorem 3. We use an argument from [4].

Let $B > 0$ be a constant. Let N, $\hat{\rho}$, a_0, w_0 and E be as in Lemma 9. Given $a, b, c, w, t \in \mathbb{R}$ with $|a| \le a_0$, $|b| \ge b_0$, $|w| \le B|b|$, let $\{\mathcal{N}_J\}_{J \in \mathsf{J}}$ be a finite family of operators having the properties (a), (b) and (c) in Lemma 9.

Let $h \in C^{\mathrm{Lip}}(U)$ be such that $\|h\|_{\mathrm{Lip},b} \le 1$. Then $|h(u)| \le 1$ for all $u \in U$ and $\mathrm{Lip}(h) \le |b|$. Thus, for any $u, v \in \widehat{U}_i$. $i = 1, \ldots, k$, we have $|h(u) - h(v)| \le |b|\, d(u,v) \le |b|\, D(u,v)$, so $\mathrm{Lip}_D(h) \le |b|$. Set $h^{(m)} = \mathcal{L}_{abtz}^{mN} h$. Define the sequence of functions $\{H^{(m)}\}$ recursively by $H^{(0)} = 1$ and $H^{(m+1)} = \mathcal{N}_{J_m} H^{(m)}$, where $J_m \in \mathsf{J}$ is chosen by induction so that the conclusions of Lemma 9(c) are satisfied with $h = h^{(m)}$, $H = H^{(m)}$ and $J = J_m$.

Since $H^{(0)} \in K_{E|b|}(U)$, it follows that $H^{(m)} \in K_{E|b|}(U)$ for all $m \ge 0$. Moreover, for $h^{(0)} = h$ we clearly have $|h^{(0)}| \le H^{(0)}$ and

$$|h^{(0}(u) - h^{(0)}(u')| \le |b|\, d(u,u') \le Et|b| H^{(0)}(u')\, D(u,u')$$

whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$. Now Lemma 9(c) implies that $h^{(m)}$ and $H^{(m)}$ satisfy similar conditions for all $m \ge 0$. In particular, $|h^{(m)}| \le H^{(m)}$ on $\widehat{U}$ for all m.

Using an induction on m and property (b) in Lemma 9, we get

$$\int_{\widehat{U}} (H^{(m)})^2 \, d\nu \leq \hat{\rho} \int_{\widehat{U}} (H^{(m-1)})^2 \, d\nu \leq \hat{\rho}^m.$$

Hence

$$\int_U |\mathcal{L}_{abtz}^{mN} h|^2 \, d\nu = \int_{\widehat{U}} |\mathcal{L}_{abtz}^{mN} h|^2 \, d\nu = \int_{\widehat{U}} |h^{(m)}|^2 \, d\nu \leq \int_{\widehat{U}} (H^{(m)})^2 \, d\nu \leq \hat{\rho}^m$$

for all $m \geq 1$. This proves the theorem. ■

As in [4] and [12] we need the following lemma whose proof is the same.

Lemma 10. *Let $\beta \in (0, \alpha)$. There exists a constants $A_1 > 0$ such that for all $a, b, c, t, w \in \mathbb{R}$ with $|a|, |c|, 1/|b|, 1/t \leq a_0$ such that $|w| \leq B|b|$, and all positive integers m and all $h \in C^\beta(U)$ we have*

$$|\mathcal{L}_{abtz}^m h(u) - \mathcal{L}_{abtz}^m h(u')| \leq A_1 \left[\frac{|h|_\beta}{\hat{\gamma}^{m\beta}} + |b| \, (\mathcal{M}_{atc}^m |h|)(u') \right] (d(u, u'))^\beta$$

for all $u, u' \in U_i$.

We will derive Theorem 1 from Theorem 3 and Lemma 10 above.

Proof of Theorem 1. We essentially repeat the proofs of Corollaries 2 and 3 in [4] (see also the Appendix in [9]).

Let $\epsilon > 0$, $B > 0$ and $\beta \in (0, \alpha)$. Take $\rho \in (0, 1)$, $a_0 > 0$, $b_0 > 0$, $A_0 > 0$ and N as in Theorem 2. We will assume that $\rho \geq \frac{1}{\gamma_0}$. Let $a, b, c, w \in \mathbb{R}$ be such that $|a|, |c| \leq a_0$ and $|b| \geq b_0$. Let $t > 0$ be such that $1/t^{\alpha - \beta} \leq a_0$. Assume that $|w| \leq B|b|$ and set $z = c + \mathbf{i}w$.

First, as in [4] one derives from Theorem 2 and Lemma 14 (approximating functions $h \in C^\beta(\widehat{U})$ by Lipschitz functions) that there exist constants $C_6 > 0$ and $\rho_3 \in (0, 1)$ such that

$$\|\mathcal{L}_{abtz}^n h\|_{\beta, b} \leq C_6 |b|^\epsilon \rho_3^n \quad , \quad n \geq 0, \tag{5.1}$$

for all $h \in C^\beta(\widehat{U})$.

Next, given $h \in C^\beta(\widehat{U})$, we have $\mathcal{L}_{abtz}^n (h/h_{atc}) = \frac{1}{\lambda_{atc}^n h_{atc}} L_{f_t - s\,\tau + z\,g_t} h$, where again $s = a + \mathbf{i}b$ and $z = c + \mathbf{i}w$, so by (4.3) and (4.4) we get

$$\begin{aligned} \|L_{f_t - s\tau + z g_t}^n h\|_{\beta, b} &\leq \lambda_{atc}^n \|h_{atc} \, \mathcal{L}_{abtz}^n (h/h_{atc})\|_{\beta, b} \\ &\leq \text{Const}\, \lambda_0^n (e^{3C_0 a_0} \rho_3)^n |b|^\epsilon \, \|h/h_{atc}\|_{\beta, b} \leq \text{Const}\, \lambda_0^n \, \rho_4^n \, |b|^\epsilon \, \|h\|_{\beta, b} \,, \end{aligned}$$

where $\lambda_{atc} \leq e^{3C_0 a_0} \lambda_0$ and $\rho_3 e^{3C_0 a_0} = \rho_4 < 1$, provided $a_0 > 0$ is small enough.

We will now approximate $L_{f - s\tau + zg}$ by $L_{f_t - s\tau + c g_t}$ in two steps. First, the above implies

$$\|L_{f - s\tau + cg + \mathbf{i}w g_t}^n h\|_{\beta, w} = \left\| L_{f_t - s\tau + z g_t}^n \left(e^{(f^n - f_t^n) + c(g^n - g_t^n)} h \right) \right\|_{\beta, b} \leq C \, \lambda_0^n \, \rho_4^n \, |b|^\epsilon \, \left\| e^{(f^n - f_t^n) + c(g^n - g_t^n)} h \right\|_{\beta, b} .$$

for some constant $C > 0$. Choosing C appropriately, we have $\|f - f_t\|_0 \leq C\, a_0$ and $|f - f_t|_\beta \leq C/t^{\alpha - \beta} \leq C$, so $\|f^n - f_t^n\|_0 \leq n \, \|f - f_t|_0 \leq C\, n a_0$, and similarly $|f^n - f_t^n|_\beta \leq C\, n a_0$. Similar estimates hold for $g^n - g_t^n$. Thus, $\|e^{(f^n - f_t^n) + c(g^n - g_t^n)} h\|_0 \leq e^{C\, n a_0} \|h\|_0$, and

$$\begin{aligned} |e^{(f^n - f_t^n) + c(g^n - g_t^n)} h|_\beta &\leq \|e^{(f^n - f_t^n) + c(g^n - g_t^n)}\|_0 \, |h|_\beta + |e^{(f^n - f_t^n) + c(g^n - g_t^n)}|_\beta \, \|h\|_\infty \\ &\leq e^{C\, n a_0} |h|_\beta + e^{C\, n a_0} \, |(f^n - f_t^n) + c(g^n - g_t^n)|_\beta \, \|h\|_\infty \leq C\, n \, e^{C\, n a_0} \, \|h\|_\beta, \end{aligned}$$

replacing C by a larger constant where necessary. Combining this with the previous estimate gives $\|e^{(f^n - f_t^n) + c(g^n - g_t^n)} h\|_{\beta,b} \leq C\, n\, e^{C\, n a_0}\, \|h\|_\beta$, so

$$\|L^n_{f - s\tau + cg + \mathbf{i} w g_t} h\|_{\beta,b} \leq C\, \lambda_0^n\, \rho_4^n\, |b|^\epsilon\, n\, e^{C\, n a_0}\, \|h\|_{\beta,b}.$$

Taking $a_0 > 0$ sufficiently small, we may assume that $\rho_4\, e^{C\, a_0} < 1$. Now take an arbitrary ρ_5 with $\rho_4\, e^{C\, a_0} < \rho_5 < 1$. Then we can take the constant $C_7 > 0$ so large that $n\, \rho_4^n\, e^{C n a_0} \leq C_7 \rho_5^n$ for all integers $n \geq 1$. This gives $\|L^n_{f - s\tau + cg + \mathbf{i} w g_t} h\|_{\beta,b} \leq C_7\, \lambda_0^n\, \rho_5^n\, |b|^\epsilon\, \|h\|_{\beta,b}$ for all $n \geq 0$. Using the latter we can write

$$\|L^n_{f - s\tau + zg} h\|_{\beta,b} = \left\| L^n_{f - s\tau + cg + \mathbf{i} w g_t} \left(e^{\mathbf{i} w (g^n - g_t^n)} h \right) \right\|_{\beta,b} \leq C_7\, \lambda_0^n\, \rho_5^n\, |b|^\epsilon\, \left\| e^{\mathbf{i} w (g^n - g_t^n)} h \right\|_{\beta,b}.$$

We have $\|e^{\mathbf{i} w (g^n - g_t^n)} h\|_0 = \|h\|_0$, $|g - g_t|_\beta \leq \text{Const}/t^{\alpha - \beta} \leq 1$ (if $t > 1$ is sufficiently large), so

$$|e^{\mathbf{i} w (g^n - g_t^n)} h|_\beta \leq \|e^{\mathbf{i} w (g^n - g_t^n)}\|_0\, |h|_\beta + |e^{\mathbf{i} w (g^n - g_t^n)}|_\beta\, \|h\|_0 \leq |h|_\beta + |w|\, |g^n - g_t^n|_\beta\, \|h\|_0. \tag{5.2}$$

and therefore $\|e^{\mathbf{i} w (g^n - g_t^n)} h\|_{\beta,b} = \|e^{\mathbf{i} w (g^n - g_t^n)} h\|_0 + \frac{|w|}{|b|} |e^{\mathbf{i} w (g^n - g_t^n)} h|_\beta \leq 2Bn \|h\|_{\beta,b}$. This yields $\|L^n_{f - s\tau + zg} h\|_{\beta,b} \leq C_8\, \lambda_0^n\, \rho_5^n\, |b|^\epsilon\, n\, \|h\|_{\beta,b}$. Choosing ρ_6 with $\rho_5 < \rho_6 < 1$ and taking the constant $C_9 > C_8$ sufficiently large, we get $\|L^n_{f - s\tau + zg} h\|_{\beta,b} \leq C_9\, \lambda_0^n\, \rho_6^n\, |b|^\epsilon\, \|h\|_{\beta,b}$ for all integers $n \geq 0$. ■

6. Appendix: Proof of Lemma 2

(a) Let $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$ and let $m \geq 1$ be an integer. For any $v \in \widehat{U}$ with $\sigma^m(v) = u$, denote by $v' = v'(v)$ the unique $v' \in \widehat{U}$ in the cylinder of length m containing v such that $\sigma^m(v') = u'$. Then

$$|f^m_{atc}(v) - f^m_{atc}(v')| \leq \sum_{j=0}^{m-1} |f_{atc}(\sigma^j(v)) - f_{atc}(\sigma^j(v'))| \leq \frac{Tt}{c_0\, (\gamma_0 - 1)}\, d(u, u') \leq C_1\, t\, D(u, u') \tag{6.1}$$

for some constant $C_1 > 0$. Similarly,

$$|g^m_t(v) - g^m_t(v')| \leq C_1\, t\, D(u, u'). \tag{6.2}$$

If $D(u, u') = \text{diam}(\mathcal{C}')$ for some cylinder $\mathcal{C}' = C[i_{m+1}, \ldots, i_p]$,then $v, v'(v) \in \mathcal{C}'' = C[i_0, i_1, \ldots, i_p]$ for some cylinder $\mathcal{C}''$ with $\sigma^m(\mathcal{C}'') = \mathcal{C}'$, so $D(v, v') \leq \text{diam}(\mathcal{C}'') \leq \frac{1}{c_0\, \gamma_0^m}\, \text{diam}(\mathcal{C}') = \frac{D(u, u')}{c_0\, \gamma_0^m}$.

We have

$$\frac{|(\mathcal{M}^m_{atc} H)(u) - (\mathcal{M}^m_{atc} H)(u')|}{\mathcal{M}^m_{atc} H(u')} = \frac{\left| \sum_{\sigma^m v = u} e^{f^m_{atc}(v) + c g^m_t(v)}\, H(v) - \sum_{\sigma^m v = u} e^{f^m_{atc}(v') + c g^m_t(v')}\, H(v') \right|}{\mathcal{M}^m_{atc} H(u')}$$

$$\leq \frac{\left| \sum_{\sigma^m v = u} e^{f^m_{atc}(v) + c g^m_t(v)}\, (H(v) - H(v')) \right|}{\mathcal{M}^m_{atc} H(u')} + \frac{\sum_{\sigma^m v = u} \left| e^{f^m_{atc}(v) + c g^m_t(v)} - e^{f^m_{atc}(v') + c g^m_t(v')} \right|\, H(v')}{\mathcal{M}^m_{atc} H(u')}$$

$$\leq \frac{\sum_{\sigma^m v = u} e^{f^m_{atc}(v) + c g^m_t(v)}\, Q\, H(v')\, D(v, v')}{\mathcal{M}^m_{atc} H(u')}$$

$$+ \frac{\sum_{\sigma^m v = u} \left| e^{[f^m_{atc}(v) + c g^m_t(v)] - [f^m_{atc}(v') + c g^m_t(v')]} - 1 \right|\, e^{f^m_{atc}(v') + c g^m_t(v')}\, H(v')}{\mathcal{M}^m_{atc} H(u')}.$$

By (6.1) and (6.2), $|f^m_{atc}(v)+cg^m_t(v)]-[f^m_{atc}(v')+cg^m_t(v')|\le 2C_1t\,D(u,u')\le 2C_1t$, which implies $\left|e^{[f^m_{atc}(v)+cg^m_t(v)]-[f^m_{atc}(v')+cg^m_t(v')]}-1\right|\le e^{2C_1t}2C_1t\,D(u,u')$. A more precise estimate follows from (4.4) and (4.5):

$$\begin{aligned}
& |f^m_{atc}(v)+cg^m_t(v)]-[f^m_{atc}(v')+cg^m_t(v')| \\
\le\ & |f^m_t(v)-f^m_t(v)|+|P-a|\,|\tau^m(v)-\tau^m(v')|+|(h_{atc}(v)-h_{atc}(u))-(h_{atc}(v')-h_{atc}(u')| \\
& +a_0|g^m_t(v)-g^m_t(v')| \\
\le\ & 2m\|f_t-f^{(0)}\|_0+|(f^{(0)})^m(v)-(f^{(0)})^m(v')|+\mathrm{Const}\,D(u,u')+4C_0+2ma_0\|g_t-g\|_0 \\
\le\ & \mathrm{Const}\,D(u,u')+C_2ma_0\le C_2+C_2m\,a_0
\end{aligned}$$

for some constant $C_2>0$. Assume $a_0>0$ is chosen so that $e^{C_2a_0}<\gamma_0/\hat\gamma$. Then

$$\begin{aligned}
& \frac{|(\mathcal{M}^m_{atc}H)(u)-(\mathcal{M}^m_{atc}H)(u')|}{\mathcal{M}^m_{atc}H(u')} \\
\le\ & \frac{Q\,D(u,u')}{c_0\gamma^m}\,\frac{\sum_{\sigma^m v=u} e^{[f^m_{atc}(v)+cg^m_t(v)]-[f^m_{atc}(v')+cg^m_t(v')]}e^{f^m_{atc}(v')+cg^m_t(v')}\,H(v')}{\mathcal{M}^m_{atc}H(u')} \\
& +e^{2C_1t}\,\frac{\sum_{\sigma^m v=u} 2C_1t\,e^{f^m_{atc}(v'(v))}\,H(v'(v))}{\mathcal{M}^m_{atc}H(u')} \\
\le\ & e^{C_2}\,e^{C_2ma_0}\frac{Q\,D(u,u')}{c_0\gamma^m}+2C_1te^{2C_1t}\,D(u,u')\le A_0\,\left[\frac{Q}{\hat\gamma^m}+e^{A_0t}\,t\right]\,D(u,u'),
\end{aligned}$$

for some constant $A_0>0$ independent of a, c, t, m and Q.

(b) Let $m\ge 1$ be an integer and $u,u'\in\widehat{U}_i$ for some $i=1,\dots,k$. Using the notation $v'=v'(v)$ and the constant $C_2>0$ from part (a) above, where $\sigma^m v=u$ and $\sigma^m v'=u'$, and some of the estimates from the proof of part (a), we get

$$\begin{aligned}
& |\mathcal{L}^m_{abtz}h(u)-\mathcal{L}^m_{abtz}h(u')| \\
=\ & \left|\sum_{\sigma^m v=u}\left(e^{f^m_{atc}(v)+cg^m_t(v)-\mathbf{i}b\tau^m(v)+\mathbf{i}wg^m_t(v)}\,h(v)-e^{f^m_{atc}(v')+cg^m_t(v')-\mathbf{i}b\tau^m(v')+\mathbf{i}wg^m_t(v')}\,h(v')\right)\right| \\
\le\ & \left|\sum_{\sigma^m v=u} e^{f^m_{atc}(v)+cg^m_t(v)-\mathbf{i}b\tau^m(v)+\mathbf{i}wg^m_t(v)}\,[h(v)-h(v')]\right| \\
& +\sum_{\sigma^m v=u}\left|e^{f^m_{atc}(v)+cg^m_t(v)}-e^{f^m_{at}(v')+cg^m_t(v')}\right|\,|h(v')| \\
& +\sum_{\sigma^m v=u}\left|e^{-\mathbf{i}b\tau^m(v)+\mathbf{i}wg^m_t(v)}-e^{-\mathbf{i}b\tau^m(v')-\mathbf{i}wg^m_t(v')}\right|\,e^{f^m_{atc}(v')+cg^m_t(v')}|h(v')| \\
\le\ & \sum_{\sigma^m v=u} e^{f^m_{atc}(v)+cg^m_t(v)}\,|h(v)-h(v')| \\
+\ & \sum_{\sigma^m v=u}\left|e^{[f^m_{atc}(v)+cg^m_t(v)]-[f^m_{atc}(v')+cg^m_t(v')]}-1\right|\,e^{f^m_{atc}(v')+cg^m_t(v')}\,|h(v')| \\
& +\sum_{\sigma^m v=u}\big(|b|\,|\tau^m(v)-\tau^m(v')|+|w|\,|g^m_t(v)-g^m_t(v')|\big)\,e^{f^m_{atc}(v')+cg^m_t(v')}|h(v')|.
\end{aligned}$$

Using the constants $C_1, C_2 > 0$ from the proof of part (a) and $e^{C_2 a_0} < \gamma_0/\hat{\gamma}$ we get

$$\begin{aligned}\sum_{\sigma^m v = u} e^{f^m_{atc}(v) + c g^m_t(v)} |h(v) - h(v')| &\leq e^{C_2} e^{C_2 m a_0} \frac{t\, Q\, D(u,u')}{c_0 \gamma_0^m} \sum_{\sigma^m v = u} e^{f^m_{atc}(v') + c g^m_t(v')} H(v') \\ &\leq \frac{e^{C_2} t\, Q}{c_0 \hat{\gamma}^m} (\mathcal{M}^m_{atc} H)(u')\, D(u,u').\end{aligned}$$

This, implies

$$\begin{aligned}|\mathcal{L}^m_{abtz} h(u) - \mathcal{L}^m_{abtz} h(u')| &\leq \frac{e^{C_2} t\, Q}{c_0 \hat{\gamma}^m} (\mathcal{M}^m_{atc} H)(u')\, D(u,u') + e^{2C_1 t} 2C_1 t\, D(u,u')\, (\mathcal{M}^m_{atc}|h|)(u') \\ &\quad + (\mathrm{Const}\, |b| + |w| C_1\, t)\, D(u,u')\end{aligned}$$

Thus, taking the constant $A_0 > 0$ sufficiently large we get

$$|(\mathcal{L}^N_{abtz} h)(u) - (\mathcal{L}^N_{abtz} h)(u')| \leq A_0 \left(\frac{t\, Q}{\hat{\gamma}^m} (\mathcal{M}^m_{atc} H)(u') + (|b| + e^{A_0 t} t + t|w|)(\mathcal{M}^m_{atc}|h|)(u') \right) D(u,u'),$$

which proves the assertion. ■

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