

# DYNAMICAL ZETA FUNCTION FOR SEVERAL STRICTLY CONVEX OBSTACLES

VESSELIN PETKOV

ABSTRACT. The behavior of the dynamical zeta function  $Z_D(s)$  related to several strictly convex disjoint obstacles is similar to that of the inverse  $Q(s) = \frac{1}{\zeta(s)}$  of the Riemann zeta function  $\zeta(s)$ . Let  $\Pi(s)$  be the series obtained from  $Z_D(s)$  summing only over primitive periodic rays. In this paper we examine the analytic singularities of  $Z_D(s)$  and  $\Pi(s)$  close to the line  $\Re s = s_2$ , where  $s_2$  is the abscissa of absolute convergence of the series obtained by the second iterations of the primitive periodic rays. We show that at least one of the functions  $Z_D(s)$ ,  $\Pi(s)$  has a singularity at  $s = s_2$ .

KEYWORDS: dynamical zeta function, periodic rays

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## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^n$ ,  $n = 2, 3$ , be an open and connected domain with  $C^\infty$  boundary  $\partial\Omega$  having the form  $\Omega = \mathbb{R}^n \setminus K$ , where

$$K = \bigcup_{j=1}^Q K_j, \quad K_i \cap K_j = \emptyset, \quad \text{for } i \neq j \quad (1.1)$$

and  $K_j$  are strictly convex compact obstacles for  $j = 1, \dots, Q$ ,  $Q \geq 3$ . Throughout this paper we suppose that  $K$  satisfies the following condition introduced by Ikawa ([6]):

(H) The convex hull of every two connected components of  $K$  does not have common points with any other connected component of  $K$ .

Consider the reflecting rays in  $\overline{\Omega}$  (see [6] and Chapter 2 in [19] for a precise definition). Under the condition (H) every periodic ray is ordinary reflecting, that is  $\gamma$  has no tangent segments. Given a *periodic reflecting ray*  $\gamma$  in  $\overline{\Omega}$  with  $m_\gamma$  reflections, we denote by  $T_\gamma$  the primitive period (length) of  $\gamma$ , by  $d_\gamma = lT_\gamma$ ,  $l \in \mathbb{N}$ , the period of  $\gamma$  and by  $P_\gamma$  the linear Poincaré map related to  $\gamma$ . Setting  $|\det(I - P_\gamma)| = |I - P_\gamma|$ , it is easy to prove (see [18], Appendix) that there exist constants  $b_1 > 0$ ,  $b_2 > 0$ ,  $B_0 > 0$  so that

$$B_0 e^{2b_1 d_\gamma} \leq |I - P_\gamma| \leq e^{2b_2 d_\gamma}. \quad (1.2)$$

Denote by  $\Xi$  the set of all reflecting periodic rays in  $\overline{\Omega}$  and set

$$d_0 = \min \operatorname{dist}_{i \neq j} (K_i, K_j), \quad D_0 = \max \operatorname{dist}_{i \neq j} (K_i, K_j).$$

For the counting function of the lengths of periodic rays there exists a constant  $a_0 > 0$  such that

$$\#\{\gamma \in \Xi : d_\gamma \leq q\} \leq e^{a_0 q} \quad (1.3)$$

(see [6], [22] and Chapter 2 in [19]). In this note we examine the dynamical zeta function

$$Z_D(s) = \sum_{\gamma \in \Xi} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-sd_\gamma}, \quad s \in \mathbb{C}, \quad (1.4)$$

where the summation is over all periodic rays  $\gamma \in \Xi$ . This zeta function is related to the trace formula for the unitary group associated to the Dirichlet problem for the wave equation

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 & \text{in } \mathbb{R} \times \Omega, \\ u = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ u(0, x) = f_1(x), \partial_t u(0, x) = f_2(x). \end{cases} \quad (1.5)$$

The form of  $Z_D(s)$  is obtained by the Laplace transformation of the distribution

$$\sum_{\gamma \in \Xi} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} \delta(t - d_\gamma) \quad (1.6)$$

which in turn is the sum of the principal singularities of  $u(t) \in \mathcal{D}'(\mathbb{R}^+)$  given by

$$u(t) = \sum_{\lambda_j} e^{it\lambda_j}, \quad t > 0.$$

Here  $\lambda_j \in \mathbb{C}$  are the poles of the scattering matrix  $S(z)$  related to the problem (1.5) and the summation is over all poles counted with their multiplicities. We refer to [7], [8], [18], [21] for a more detailed description of this link and to [1], [28], [13], [5], [19], [21] for the trace formulas leading to (1.6).

Following a result of Ikawa [7], [8], the existence of an analytic singularity of  $Z_D(s)$  implies the existence of  $\delta > 0$  such that there are infinite number of poles  $\{z_j\}_{j \in \mathbb{N}}$  of the scattering matrix  $S(z)$  satisfying

$$0 < \Im z_j \leq \delta, \quad \forall j \in \mathbb{N}$$

and the last property is known as the modified Lax-Phillips conjecture. Another motivation for the analysis of  $Z_D(s)$  is the folklore conjecture that the singularities of  $Z_D(s)$  should determine approximatively the scattering poles.

By using (1.2) and (1.3), it is easy to see that there exists  $s_1 \in \mathbb{R}$  called abscissa of absolute convergence such that for  $\Re s > s_1$  the series (1.4) is absolutely convergent. Despite many works in physical literature and the numerical analysis results concerning n-disk problems (see for example [3], [29], [12], [28] and the references cited there), to our best knowledge in the general case the problem of the existence of at *least one singularity* of  $Z_D(s)$  is still open. The existence of an analytic non-real singularity has been proved by Ikawa [9] in the case when  $K$  is the union of several balls with radius  $r \leq r_0$ , provided  $r_0 > 0$  sufficiently small. Recently, Stoyanov [25] generalized the result of Ikawa for several obstacles satisfying some geometrical conditions and having diameters less than  $r_0$ . It was proved in [18] that  $Z_D(s)$  has no singularities on the line  $\Re s = s_1$ . In fact we have a stronger result and following the recent works of Stoyanov (see [23], [26]), we know that there exists  $\delta_0 > 0$  such that  $Z_D(s)$  is analytic for  $\Re s > s_1 - \delta_0$  (see also [10] for the special case  $s_1 > 0$ .) This means that  $Z_D(s)$  is analytic in a domain around  $\Re s = s_1$  and this phenomenon of cancellations is typical for dynamical zeta functions (see [4], [16], [23], [24], [26]). On the other hand, since  $Z_D(s)$  is a Dirichlet series with real coefficients changing their signs, the situation is very similar to that for the inverse  $Q(s) = \frac{1}{\zeta(s)}$  of the classical Riemann zeta function  $\zeta(s)$ . It is

well known that  $Q(s)$  is analytic on the line  $\Re s = 1$  and  $Q(s)$  has non-real singularities on the *critical line*  $\Re s = 1/2$ . Moreover, we have the representation

$$\log \zeta(s) = \sum_{m=1}^{\infty} \sum_{p \in \mathbf{P}} \frac{1}{m} \frac{1}{p^{ms}}, \quad \Re s > 1, \quad (1.7)$$

where  $\mathbf{P}$  denotes the set of prime numbers. Consequently, the analytic behavior of  $\log \zeta(s)$  for  $1/2 < \Re s \leq 1$  is characterized by the continuation of the function

$$\pi(s) = \sum_{p \in \mathbf{P}} \frac{1}{p^s}, \quad \Re s > 1$$

and the critical line  $\Re s = 1/2$  is related to  $m = 2$  in the representation (1.7).

Denote by  $\mathcal{P}$  the set of all primitive periodic rays. In this note we examine the analytic singularities of  $Z_D(s)$  close to the line  $\Re s = s_2$ , where  $s_2 < s_1$  is the abscissa of the absolute convergence of the series  $\Pi_2(s)$  obtained from  $Z_D(s)$  when we sum only over the rays  $2\gamma$ ,  $\gamma \in \mathcal{P}$ , that is over the second iteration of primitive rays (see Section 4 for a precise definition). We show that the line  $\Re s = s_2$  plays a role in the investigation of the singularities of  $Z_D(s)$ . Similarly to  $\pi(s)$ , introduce the function

$$\Pi(s) = \sum_{\gamma \in \mathcal{P}} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-sT_\gamma}, \quad \Re s > s_1,$$

where the summation is over the primitive rays  $\gamma \in \mathcal{P}$ . Next let  $h_\Pi < s_1$  be the abscissa of holomorphy of  $\Pi(s)$  given by

$$h_\Pi = \inf\{t \in \mathbb{R} : \Pi(s) \text{ is analytic for } \Re s > t\}.$$

Our main result is the following

**Theorem 1.** *At least one of the functions  $Z_D(s)$ ,  $\Pi(s)$  has a singularity at  $s = s_2$  and the difference  $Z_D(s) - \Pi(s)$  is analytic for*

$$s \in \{z \in \mathbb{C} : \Re z > s_2\}.$$

*Moreover, if  $s_2 \neq h_\Pi$ , then  $Z_D(s)$  has a singularity at  $z$  with  $\Re z > \max\{s_2, h_\Pi\} - \epsilon_1$ , where  $\epsilon_1 > 0$  is sufficiently small.*

Following the same way, we may show that if we consider the series obtained by summing over all iterations of the primitive rays of order  $(2m - 1)$ , the corresponding function will be singular at  $s = s_{2m}$  if  $Z_D(s)$  is analytic at  $s = s_{2m}$ . Here  $s_k$  is the abscissa of absolute convergence of the series obtained by summing over all iterations of order  $k \geq 2$  and we show that  $s_1 - h_t < s_k < s_{k-1}$ ,  $h_t > 0$  being the topological entropy of the billiard flow (see Proposition 2). Thus if  $Z_D(s)$  is analytic for  $\Re s > s_1 - h_t$ , for any fixed  $M \geq 2$  one obtains a singularity of the sum of series related to the iterations  $m \leq M$ . This corollary yields some information for the numerical analysis since in the numerical experiences one treats series with finite number iterations.

The existence of a singularity  $z_0$  of  $\Pi(s)$  such that  $\Re z_0 > s_2 - \epsilon_0$ ,  $\epsilon_0 > 0$ ,  $\Im z_0 \neq 0$ , is an interesting open problem, but it seems that the difficulty of this problem could be compared with that of the existence or the absence of singularities of  $\pi(s)$  for  $1/2 < \Re s < 1$ . In fact, the dynamics of the periodic orbits is chaotic and the random change of signs of the coefficients in (1.4) plays some essential role. We *conjecture* that in general  $Z_D(s)$  is not singular at  $s_2$  and Theorem 1 shows that

in this case  $\Pi(s)$  must be singular at  $s_2$ . It is expected that there exist non-real singularities  $z$  of  $\Pi(s)$  with  $\Re z$  arbitrary close to line of holomorphy  $\Re s = h_\Pi$  of  $\Pi(s)$ . This will lead to singularities of  $Z_D(s)$ . In fact we have two possibilities:

$$(a) \ s_2 \neq h_\Pi, \quad (b) \ s_2 = h_\Pi.$$

Our analysis in Section 4 implies that in the case (a) the function  $Z_D(s)$  must be singular either at  $s = s_2$  ( $s_2 > h_\Pi$ ) or at a point  $z$  close to the line  $\Re s = h_\Pi$  ( $s_2 < h_\Pi$ ) and we obtain a solution of the modified Lax-Phillips conjecture (see [8], [9], [25]). In the case (b) we have a phenomenon similar to the famous Riemann conjecture for  $\zeta(s)$  and the maximal domain  $\Re s > t$ , where  $\Pi(s)$  is analytic, is determined by the line  $\Re s = s_2$ . Finally, it is not clear if the singularities found in [9] and [25] lie in the domain  $\Re s > s_2$  and we will discuss this problem in Section 4.

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## 2. SYMBOLIC DYNAMICS

We will write  $Z_D(s)$  as a Selberg zeta function using the argument of Section 5, [18]. First assume  $n = 3$  and let  $\lambda_{\gamma,i}$ ,  $i = 1, 2$ ,  $|\lambda_{\gamma,i}| > 1$ , be the eigenvalues of the Poincaré map  $P_\gamma$  of the ray  $\gamma \in \mathcal{P}$ . Set

$$\delta_\gamma = -\frac{1}{2} \log(\lambda_{\gamma,1}\lambda_{\gamma,2}), \quad \nu_\gamma = -\log \lambda_{\gamma,1}, \quad \mu_\gamma = -\log \lambda_{\gamma,2}.$$

The product  $\lambda_{\gamma,1}\lambda_{\gamma,2}$  and the sum  $\lambda_{\gamma,1} + \lambda_{\gamma,2}$  are positive and  $\delta_\gamma < 0$ . Given  $\gamma \in \mathcal{P}$ , introduce

$$r_\gamma = \begin{cases} 0 & \text{if } m_\gamma = 2k, \\ 1 & \text{if } m_\gamma = 2k + 1. \end{cases}$$

Then for  $\Re s \gg s_1$  we have

$$Z_D(s) = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{m=1}^{\infty} \sum_{\gamma \in \mathcal{P}} T_\gamma(-1)^{mr_\gamma} e^{m(-sT_\gamma + \delta_\gamma + k\nu_\gamma + p\mu_\gamma)}. \quad (2.1)$$

We refer to [18] for the details of the proof of this representation. For  $n = 2$  we have a simpler formula since there is only one eigenvalue  $\lambda_\gamma > 1$  and we get

$$Z_D(s) = \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \sum_{\gamma \in \mathcal{P}} T_\gamma(-1)^{mr_\gamma} e^{m(-sT_\gamma + \delta_\gamma + k\nu_\gamma)}, \quad (2.2)$$

where  $\delta_\gamma = -\frac{1}{2} \log \lambda_\gamma$ ,  $\nu_\gamma = 2\delta_\gamma$ . Consider the leading term of  $Z_D(s)$  obtained for  $k = p = 0$  (resp.  $k = 0$  for  $n = 2$ ) and having the form

$$Z(s) = -\frac{d}{ds} Z_0(s), \quad Z_0(s) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} (-1)^{mr_\gamma} e^{m(-sT_\gamma + \delta_\gamma)}.$$

We will write  $Z_0(s)$  by using a symbolic model. Let us recall some notations concerning the symbolic dynamics. Given a  $Q \times Q$  matrix  $A(i, j)_{i, j=1, \dots, Q}$  such that

$$A(i, j) = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

introduce the spaces

$$\begin{aligned}\Sigma_A &= \{\xi = \{\xi_i\}_{i=-\infty}^{\infty} : \xi_i \in \{1, \dots, Q\}, A(\xi_i, \xi_{i+1}) = 1\}, \\ \Sigma_A^+ &= \{\xi = (\xi_0, \xi_1, \dots) : A(\xi_i, \xi_{i+1}) = 1, \forall i \geq 0\}.\end{aligned}$$

Let  $\sigma_A$  be the shift on  $\Sigma_A, \Sigma_A^+$  given, respectively, by

$$(\sigma_A \xi)_i = \xi_{i+1}, \forall i \in \mathbb{Z}, \quad (\sigma_A \xi)_i = \xi_{i+1}, \forall i \geq 0.$$

For every  $\xi \in \Sigma_A$  there exists a unique ray  $\gamma(\xi)$  with successive reflection points on

$$\dots, \partial K_{j-1}, \partial K_j, \partial K_{j+1}, \dots$$

(see [6], [19]). Let  $P_j(\xi)$  be the  $j^{\text{th}}$  reflection point of  $\gamma(\xi)$  and let  $f(\xi) = \|P_0(\xi) - P_1(\xi)\|$ . If  $\gamma = \gamma(\xi) \in \mathcal{P}$  has  $m$  reflections and primitive period  $T_\gamma$ , then

$$T_\gamma = f(\xi) + f(\sigma_A \xi) + \dots + f(\sigma_A^{m-1} \xi) = S_m f(\xi).$$

Also (see [8], [9]), there exists a function  $g(\xi)$  such that

$$\delta_\gamma = g(\xi) + g(\sigma_A \xi) + \dots + g(\sigma_A^{m-1} \xi) = S_m g(\xi).$$

For  $\Re s$  large we may write  $Z_0(s)$  as follows

$$Z_0(s) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{\sigma_A^m \xi = \xi} e^{S_m(-sf(\xi) + g(\xi))}.$$

Given a continuous function  $F(\xi) \in C(\Sigma_A)$ , introduce

$$\text{var}_n F = \sup_{\xi, \eta \in \Sigma_A} \{|F(\xi) - F(\eta)| : \xi_i = \eta_i \text{ for } |i| \leq n\}$$

and for  $0 < \theta < 1$  consider the norms

$$|F|_\theta = \sup_n \frac{\text{var}_n F}{\theta^n}, \quad \|F\|_\infty = \sup_{\xi \in \Sigma_A} |F(\xi)|, \quad \|F\|_\theta = \|F\|_\infty + |F|_\theta.$$

Let  $\mathcal{F}_\theta(\Sigma_A) \subset C(\Sigma_A)$ ,  $\mathcal{F}_\theta(\Sigma_A^+) \subset C(\Sigma_A^+)$  be Banach spaces with norm  $\|\cdot\|_\theta$ . It follows from the exponential instability of the billiard ball map that with some constant  $0 < \theta < 1$ , depending on the geometry of  $K$ , we have  $f(\xi), g(\xi) \in \mathcal{F}_\theta(\Sigma_A)$  (see for more details [8], [9], [18], [23], [25]). We introduce the suspended flow  $\sigma^f$  over the space

$$\Sigma_A^f = \{(\xi, t) : \xi \in \Sigma_A, 0 \leq t \leq f(\xi)\}$$

with the identification  $(\xi, f(\xi)) \sim (\sigma_A(\xi), 0)$  (see [17]) and notice that the topological entropy  $h_t > 0$  of the suspended flow  $\sigma^f$  over  $\Sigma_A^f$  is given by

$$h_t = \sup_{\mu \in \mathcal{M}} \frac{h_\mu(\sigma_A)}{\int_{\Sigma_A} f d\mu}.$$

Finally, recall that the pressure  $P(F)$  of a function  $F \in C(\Sigma_A)$  is given by

$$P(F) = \sup_{\mu \in \mathcal{M}} \left( h_\mu(\sigma_A) + \int_{\Sigma_A} F d\mu \right),$$

where  $h_\mu(\sigma_A)$  is the measure entropy of  $\sigma_A$  and the sup is taken over the set  $\mathcal{M}$  of all probabilistic measures on  $\Sigma_A$  invariant with respect to  $\sigma_A$ .

## 3. SUMMATION OVER THE ITERATED PERIODIC RAYS

It is well known (see [17]) that for every function  $\varphi(\xi) \in \mathcal{F}_\theta(\Sigma_A)$  there exists  $h, \psi \in \mathcal{F}_{\theta^{1/2}}(\Sigma_A)$  so that

$$\varphi(\xi) = h(\xi) + \psi(\sigma_A(\xi)) - \psi(\xi)$$

and the function  $h(\xi)$  depends only on the coordinates  $(\xi_0, \xi_1, \dots)$ . In this case we will write  $\varphi \sim h$ . Obviously, if  $F \sim \tilde{F}$ , we have  $P(F) = P(\tilde{F})$ . Passing to functions  $f \sim \tilde{f}$ ,  $g \sim \tilde{g}$ , we get

$$Z_0(s) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{\sigma_A^m \xi = \xi} e^{S_m(-s\tilde{f}(\xi) + \tilde{g}(\xi))}.$$

The function

$$\mathbb{R} \ni s \longrightarrow P(-skf + kg)$$

is strictly decreasing and given an integer  $k \geq 1$  we may introduce the number  $s_k \in \mathbb{R}$  determined uniquely by the equality

$$P(-s_k k f + k g) = 0.$$

It follows easily from the results in [17] that  $s_k$  is the abscissa of absolute convergence of the series

$$P_k(s) = \frac{1}{k} \sum_{\gamma \in \mathcal{P}} (-1)^{km_\gamma} e^{-ksT_\gamma + k\delta_\gamma}.$$

Indeed,  $s_k$  is the abscissa of absolute convergence of the series

$$G_k(s) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\sigma_A^m \xi = \xi} e^{S_m(-skf(\xi) + kg(\xi))}.$$

On the other hand, for  $\Re s > s_k$  we have

$$G_k(s) = \sum_{\gamma \in \mathcal{P}} e^{-skT_\gamma + k\delta_\gamma} + \sum_{m=2}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} e^{m(-skT_\gamma + k\delta_\gamma)}$$

and as in [17], Chapter 6 and [18], Section 4, we deduce that the series

$$\sum_{m=2}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} e^{m(-skT_\gamma + k\delta_\gamma)}$$

is absolutely convergent for  $\Re s \geq s_k - \epsilon$  for some small  $\epsilon > 0$ . Next we will prove the following

**Lemma 1.** *For all  $k \geq 1$  we have  $s_{k+1} < s_k$ .*

**Proof.** The pressure of the function  $-s_k k f + k g$  is zero, so we may find a function  $h \in \mathcal{F}_{\theta^{1/2}}(\Sigma_A^+)$  so that  $h \sim -s_k k f + k g$ ,  $P(h) = 0$  and we may choose  $h$  (see for more details [17]) so that

$$\sum_{\sigma_A \eta = \xi} e^{h(\eta)} = 1, \quad \forall \xi \in \Sigma_A^+.$$

This implies  $h(\eta) \leq \alpha_k < 0$  for all  $\eta \in \Sigma_A^+$  and

$$k \int_{\Sigma_A} (-s_k f + g) d\mu \leq \alpha_k$$

for each  $\mu \in \mathcal{M}$ . It is clear that

$$\begin{aligned} & h_\mu(\sigma) + \int_{\Sigma_A} (-s_k(k+1)f + (k+1)g)d\mu \\ & \leq \sup_{\mu \in \mathcal{M}} \left[ h_\mu(\sigma) + \int_{\Sigma_A} (-s_k k f + k g)d\mu \right] + \frac{\alpha_k}{k} = \frac{\alpha_k}{k} < 0, \forall \mu \in \mathcal{M}. \end{aligned}$$

This implies

$$P\left(-s_k(k+1)f + (k+1)g\right) = \sup_{\mu \in \mathcal{M}} \left[ h_\mu(\sigma) + \int_{\Sigma_A} (-s_k(k+1)f + (k+1)g)d\mu \right] \leq \frac{\alpha_k}{k}.$$

On the other hand,  $P\left(-s_{k+1}(k+1)f + (k+1)g\right) = 0$  and since the function

$$\mathbb{R} \ni s \longrightarrow P\left(-s(k+1)f + (k+1)g\right)$$

is strictly decreasing, we get  $s_{k+1} < s_k$ .

To study the convergence of the series over the iterated rays we need the following

**Proposition 1.** *For every  $k \geq 1$  there exists  $\epsilon_o(k) > 0$ , depending on  $k$ , such that the series*

$$\sum_{m=k+1}^{\infty} P_m(s) = \sum_{m=k+1}^{\infty} \sum_{\gamma \in \mathcal{P}} \frac{(-1)^{mr_\gamma}}{m} e^{m(-sT_\gamma + \delta_\gamma)}$$

*is absolutely convergent for  $\Re s \geq s_k - \epsilon_o(k)$ .*

**Proof.** As in the proof of Lemma 1, we choose  $h$  so that  $h \sim -ks_k f + kg$ ,  $h(\eta) < 0$ , for all  $\eta \in \Sigma_A^+$ . First, assume that  $s_k < 0$ . We choose  $\epsilon = \epsilon(k) > 0$  small enough in order to arrange the inequality

$$\sup_{\eta \in \Sigma_A^+} h(\eta) = \alpha_k \leq (k+1)k\epsilon s_k \|f\|_\infty.$$

Let  $\eta \in \Sigma_A^+$  correspond to a primitive periodic ray  $\gamma \in \mathcal{P}$  with  $m$  reflections as it has been explained in Section 2. We obtain

$$S_m(-ks_k f + kg)(\eta) = -ks_k T_\gamma + k\delta_\gamma.$$

On the other hand, it is clear that  $T_\gamma \leq m\|f\|_\infty$  and we get

$$S_m h(\eta) \leq m(k+1)k\epsilon s_k \|f\|_\infty \leq (k+1)k\epsilon s_k T_\gamma.$$

From the equality  $S_m(-ks_k f + kg)(\eta) = S_m h(\eta)$ , we deduce

$$-s_k T_\gamma + \delta_\gamma \leq (k+1)\epsilon s_k T_\gamma, \forall \gamma \in \mathcal{P}.$$

Now let  $0 \leq u \leq \frac{\epsilon}{k+1}$ . Then

$$\begin{aligned} -s_k(1+u)T_\gamma + \delta_\gamma & \leq (k+1)\epsilon s_k T_\gamma - s_k u T_\gamma \\ & \leq \left( (k+1)\epsilon - \frac{\epsilon}{k+1} \right) s_k T_\gamma \leq \epsilon s_k T_\gamma \end{aligned}$$

and we get the lower bound

$$1 > 1 - e^{-s_k(1+u)T_\gamma + \delta_\gamma} \geq 1 - e^{\epsilon s_k T_\gamma} \geq 1 - e^{2s_k \epsilon d_0} = \frac{1}{C_{\epsilon,k}} > 0.$$

Thus for  $0 \leq u \leq \frac{\epsilon}{k+1}$  the series

$$\sum_{m=k+1}^{\infty} e^{m(-s_k(1+u)T_\gamma + \delta_\gamma)} = \frac{e^{(k+1)(-s_k(1+u)T_\gamma + \delta_\gamma)}}{1 - e^{-s_k(1+u)T_\gamma + \delta_\gamma}} \leq C_{\epsilon,k} e^{(k+1)(-s_k(1+u)T_\gamma + \delta_\gamma)} \quad (3.1)$$

is convergent.

Next we obtain

$$\begin{aligned} & -(k+1)s_k(1+u)T_\gamma + (k+1)\delta_\gamma \\ & \leq -s_k k T_\gamma + k\delta_\gamma + (k+1)\epsilon s_k T_\gamma - (k+1)u s_k T_\gamma \leq -s_k(1-\epsilon)k T_\gamma + k\delta_\gamma. \end{aligned}$$

Since  $s_k$  is the abscissa of absolute convergence of the series of  $k$  iterated rays, we deduce

$$\sum_{\gamma \in \mathcal{P}} e^{-s_k(1-\epsilon)k T_\gamma + k\delta_\gamma} < \infty.$$

Thus we conclude that

$$\sum_{m=k+1}^{\infty} \sum_{\gamma \in \mathcal{P}} e^{m(-s_k(1+u)T_\gamma + \delta_\gamma)} < \infty$$

and the series

$$\sum_{m=k+1}^{\infty} \sum_{\gamma \in \mathcal{P}} \frac{(-1)^{mr_\gamma}}{m} e^{m(-s T_\gamma + \delta_\gamma)}$$

is absolutely convergent for  $\Re s \geq s_k - \frac{\epsilon}{k+1}$ . Setting  $\epsilon_o(k) = \frac{\epsilon}{k+1}$ , we obtain the result in this case.

Passing to the case  $s_k > 0$ , choose  $\epsilon = \epsilon(k) > 0$  to arrange the inequalities

$$\begin{aligned} \sup_{\eta \in \Sigma_A^+} h(\eta) & \leq -(k+1)k\epsilon s_k \|f\|_\infty, \\ -s_k T_\gamma + \delta_\gamma & \leq -(k+1)\epsilon s_k T_\gamma, \quad \forall T_\gamma \in \mathcal{P}. \end{aligned}$$

For  $0 \leq u \leq \frac{\epsilon}{k+1}$  we deduce

$$-s_k(1-u)T_\gamma + \delta_\gamma \leq -(k+1)\epsilon s_k T_\gamma + s_k u T_\gamma \leq -\epsilon s_k T_\gamma$$

which yields

$$\sum_{m=k+1}^{\infty} e^{m(-s_k(1-u)T_\gamma + \delta_\gamma)} \leq C_{\epsilon,k} e^{(k+1)(-s_k(1-u)T_\gamma + \delta_\gamma)}. \quad (3.2)$$

On the other hand,

$$-(k+1)s_k(1-u)T_\gamma + (k+1)\delta_\gamma \leq -s_k(1+\epsilon)k T_\gamma + k\delta_\gamma$$

and this leads to

$$\sum_{m=k+1}^{\infty} \sum_{\gamma \in \mathcal{P}} e^{m(-s_k(1-u)T_\gamma + \delta_\gamma)} < \infty.$$

Finally, in the case  $s_k = 0$ , we arrange

$$\begin{aligned} \sup_{\eta \in \Sigma_A^+} h(\eta) & \leq -(k+1)k\epsilon \|f\|_\infty, \\ \delta_\gamma & \leq -(k+1)\epsilon T_\gamma, \quad \forall T_\gamma \in \mathcal{P}. \end{aligned}$$

Repeating the above argument, we establish for  $0 \leq u \leq \frac{\epsilon}{k+1}$  the convergence of the series

$$\sum_{m=k+1}^{\infty} \sum_{\gamma \in \mathcal{P}} e^{m(uT_\gamma + \delta_\gamma)} < \infty,$$

and this completes the proof.

To compare  $s_k$  and  $s_1$ , consider the measure  $\nu \in \mathcal{M}$  for which we have

$$P(-s_1 f + g) = h_\nu(\sigma_A) + \int_{\Sigma_A} (-s_1 f + g) d\nu = 0.$$

This measure is called equilibrium state of  $-s_1 f + g$  (see [17]). Then we obtain

$$\begin{aligned} P\left(-k\left(s_1 - \frac{k-1}{k}h_t\right)f + kg\right) &\geq h_\nu(\sigma_A) + k \int_{\Sigma_A} (-s_1 f + g) d\nu + (k-1)h_t \int_{\Sigma_A} f d\nu \\ &= (k-1) \left[ h_t \int_{\Sigma_A} f d\nu - h_\nu(\sigma_A) \right] \geq 0. \end{aligned}$$

Comparing this with  $P(-ks_k f + kg) = 0$ , we deduce

$$s_k \geq s_1 - \frac{k-1}{k}h_t. \quad (3.3)$$

Thus we have proved the following

**Proposition 2.** *The sequence  $s_k$  is convergent and  $\lim_{k \rightarrow \infty} s_k \geq s_1 - h_t$ .*

It is interesting to note that the abscissa  $c_0$  of simple convergence of the Dirichlet series  $Z_0(s)$  satisfies the estimate

$$c_0 \geq s_1 - h_t,$$

but it is difficult to compare  $c_0$  with  $s_k$ .

#### 4. SINGULARITIES ON THE LINE $\Re s = s_2$

Consider the Dirichlet series

$$P_2(s) = \frac{1}{2} \sum_{\gamma \in \mathcal{P}} e^{-2sT_\gamma + 2\delta_\gamma}$$

with positive coefficients. According to a classical result, this series has an analytic singularity at  $s = s_2$ . On the other hand, Proposition 1 implies that the sum over all iterated rays  $k\gamma, \gamma \in \mathcal{P}, k \geq 3$ , given by

$$\sum_{k=3}^{\infty} P_k(s),$$

is analytic for  $\Re s \geq s_2 - \epsilon_o(2)$  for some  $\epsilon_o(2) > 0$ . It is clear that the singularities of  $Z_0(s)$  for  $\Re s > s_2$  are related to those of the series obtained by summing only over the primitive rays

$$P_1(s) = \sum_{\gamma \in \mathcal{P}} (-1)^{r_\gamma} e^{-sT_\gamma + \delta_\gamma}.$$

Let  $h_p$  be the abscissa of *holomorphy* of the Dirichlet series  $P_1(s)$ . More precisely,  $h_p$  is the *minimal* real number  $t$  such that  $P_1(s)$  is analytic for  $\Re s > t$ . We have three possibilities:

$$(i) h_p > s_2, \quad (ii) h_p = s_2, \quad (iii) h_p < s_2.$$

In the case (i), the function  $P_1(s)$ , and hence  $Z_0(s)$ , has either a singularity on the line  $\Re s = h_p$  or there exists a sequence of singularities  $z_j$  with  $\Re z_j \rightarrow h_p$ ,  $|\Im z_j| \rightarrow \infty$ . In the case (iii), the function  $P_2(s)$  produces a singularity of  $Z_0(s)$  at  $s = s_2$ . In the case (ii) we must examine the singularities of the sum  $P_1(s) + P_2(s)$ . Of course, if  $P_1(s)$  is analytic at  $s = s_2$ , we have the same situation as in the case (iii). Thus a cancellation of the singularities of  $P_1(s) + P_2(s)$  at the point  $s_2$  is possible only if  $P_1(s)$  is singular at  $s = s_2$ . Thus we have the following

**Theorem 2.** *At least one of the functions  $Z_0(s)$ ,  $P_1(s)$  has a singularity at  $s = s_2$ . Moreover, the difference  $Z_0(s) - P_1(s)$  is analytic for*

$$s \in \{z \in \mathbb{C} : \Re z > s_2\}.$$

We may compare the functions  $Z_0(s)$  and  $Z_D(s)$ . As it was shown in [8], [18], [25] there exists  $\mu_1 > 0$  such that  $Z_D(s) - Z_0(s)$  is analytic for  $\Re s > s_1 - \mu_1$ . The number  $\mu_1$  depends on the geometry of obstacles (see Appendix in [18] and [25]). In some cases we may show that  $s_2 > s_1 - \mu_1$ . For example this is true if  $n = 2$  and  $s_2 < 0$ . Nevertheless, it is more natural to deal with the function  $\Pi(s)$  introduced in Section 1. As above, let  $h_\Pi$  be the abscissa of the holomorphy of the Dirichlet series  $\Pi(s)$  introduced in Section 1. We consider again three cases:

$$(i) h_\Pi > s_2, \quad (ii) h_\Pi = s_2, \quad (iii) h_\Pi < s_2.$$

For  $m \geq 2$  and  $n = 3$  the analysis of the series

$$\Pi_m(s) = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{\gamma \in \mathcal{P}} \frac{1}{m} (-1)^{mr_\gamma} e^{m(-sT_\gamma + \delta_\gamma + k\nu_\gamma + p\mu_\gamma)}, \quad \Re s > s_1$$

is completely similar to that of  $P_m(s)$ . In fact the abscissa of absolute convergence of  $\Pi_m(s)$  coincides with that of  $P_m(s)$  and we may apply Proposition 1 for the series

$$\sum_{m=j+1}^{\infty} \Pi_m(s) = \sum_{m=j+1}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{\gamma \in \mathcal{P}} \frac{1}{m} (-1)^{mr_\gamma} e^{m(-sT_\gamma + \delta_\gamma + k\nu_\gamma + p\mu_\gamma)}$$

assuming  $j \geq 1$ . The case  $n = 2$  is treated in a similar way and repeating the argument of the proof of Theorem 2, we obtain Theorem 1.

Following the same way, we may consider the function

$$\mathbf{\Pi}_3(s) = \Pi(s) + \Pi_2(s) + \Pi_3(s) = \sum_{\gamma \in \Xi_3} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-sd_\gamma}, \quad \Re s > s_1,$$

where the summation is over all rays  $\gamma \in \Xi_3 \subset \Xi$  which are either primitive or they are obtained by two or three iterations of primitive periodic rays. Then at least one of the functions  $Z_D(s)$ ,  $\mathbf{\Pi}_3(s)$  has a singularity at  $s = s_4$  and it is possible to iterate this argument.

Let us mention that from our results it is not clear if the analytic singularity  $z$  of  $\Pi(s)$  or  $Z_D(s)$  given by Theorem 1 is a pole. In fact, it is known that the function  $Z_0(s)$  is meromorphic for

$$\Re s \geq s_1 - \frac{|\log \theta|}{2\|f\|_\infty},$$

$0 < \theta < 1$  being the constant introduced in Section 2. On the other hand, we have  $s_2 \geq h_t/2$  and  $s_2$  lies in the above domain if  $h_t \|f\|_\infty \leq |\log \theta|$ . It is expected that  $Z_0(s)$  and  $Z_D(s)$  are meromorphic in a more larger domain or in the whole complex plan. For  $n = 2$  some results in this direction are obtained by Morita [15].

It is interesting to mention that for all  $k \in \mathbb{N}$  we have

$$s_k > b_0 = \sup_{\gamma \in \mathcal{P}} \frac{\delta_\gamma}{T_\gamma}. \quad (4.1)$$

In [18] it was established that  $b_0 < 0$ , so we need to check (4.1) only for  $s_k < 0$ . In this case the argument of the proof of Proposition 1 shows that

$$-s_k T_\gamma + \delta_\gamma \leq \epsilon_k T_\gamma, \quad \forall \gamma \in \mathcal{P}$$

with some  $\epsilon_k < 0$  and we obtain (4.1). The number  $b_0$  has been introduced in [18] and it is related to the sequence of poles

$$s_{m,\gamma} = \frac{\delta_\gamma}{T_\gamma} + \frac{2m\pi}{T_\gamma} \mathbf{i}, \quad r_{m,\gamma} = \frac{\delta_\gamma}{T_\gamma} + \frac{(2m+1)\pi}{T_\gamma} \mathbf{i}, \quad m \in \mathbb{Z}$$

obtained from the series formed by all iteration of a *fixed* periodic primitive ray  $\gamma$ .

For several strictly convex small obstacles Ikawa [9] and Stoyanov [25] established the existence of a non-real singularity

$$z_0 = \alpha + \mathbf{i} \frac{\pi}{d_1}, \quad \alpha \in \mathbb{R}$$

of  $Z_D(s)$  with  $d_1$  sufficiently close to  $D_0$ . Following the analysis in Section 7, [25], we conclude that  $s_1 - b_K \leq \alpha < s_1$  with

$$b_K \geq \frac{1}{D_0} \ln \left( 1 + \frac{\kappa_{\min}}{\nu_0} D_0 \right).$$

Here  $\kappa_{\min} > 0$  is the minimal normal curvature of  $\partial K$  and  $\nu_0 > 0$  is a constant depending on  $d_0$ , the diameter of  $K$  and

$$\chi_0 = \min \{ \text{dist} \left( K_j, \text{convex hull} (K_i \cup K_l) \right) : j \neq i, i \neq l, l \neq j \} > 0.$$

For obstacles having sufficiently small diameters, we may arrange the inequality  $b_K \geq h_t$ . Indeed, it is sufficient to have

$$h_\mu(\sigma_A) \leq \frac{d_0}{D_0} \ln \left( 1 + \frac{\kappa_{\min}}{\nu_0} D_0 \right) \leq b_K \int_{\Sigma_A} f d\mu$$

for every  $\sigma_A$  invariant measure  $\mu \in \mathcal{M}$ . If the diameters of the obstacles are sufficiently small, then  $\kappa_{\min}$  is large enough, while  $\frac{d_0}{D_0}$  and  $\chi_0$  remain bounded from below. Thus in this case we have

$$\sup_{\mu \in \mathcal{M}} h_\mu(\sigma_A) \leq \frac{d_0}{D_0} \ln \left( 1 + \frac{\kappa_{\min}}{\nu_0} D_0 \right)$$

which implies  $b_K \geq h_t$ . Combining this with (3.3), we obtain immediately

$$s_1 - b_K \leq s_1 - h_t < s_k, \quad \forall k \in \mathbb{N}.$$

Consequently, the line  $\Re s = s_k$  lies in the domain where we have complex singularities and this agrees with the conjecture that we must have complex singularities of  $Z_D(s)$  close to the line  $\Re s = h_{\Pi}$  or close to the line  $\Re s = s_2$ .

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DÉPARTEMENT DE MATHÉMATIQUES APPLIQUÉES, UNIVERSITÉ BORDEAUX I, 351, COURS DE LA LIBÉRATION,  
33405 TALENCE, FRANCE

*E-mail address:* `petkov@math.u-bordeaux1.fr`