

# Number Theory 2009

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## Topics for the theoretical examination

1. Finite fields
  - (a) Multiplicative group
  - (b) Extensions, Frobenius
  - (c) Theorem of Warning
  - (d) Theorem of Weil-Lang (statement only)
  - (e) Characters
  - (f) Gauss sums
  - (g) Exponential sums
  - (h) Theorem of Weil-Lang for diagonal polynomials
2.  $p$ -adic numbers
  - (a) Definition of  $\mathbb{Z}_p$  and of  $\mathbb{Q}_p$
  - (b) Arithmetic of  $\mathbb{Z}_p$  and of  $\mathbb{Q}_p$
  - (c) Topology of  $\mathbb{Z}_p$  and of  $\mathbb{Q}_p$  (compactness, local compactness)
  - (d)  $p$ -adic absolute value,  $\mathbb{Q}_p$  as the  $p$ -adic completion of  $\mathbb{Q}$
  - (e) Equation over  $\mathbb{Q}_p$  as congruences modulo  $p^n$
  - (f) Hensel
3. Quadratic forms
  - (a) Quadratic forms over  $\mathbb{Q}_p$ 
    - i. Generalities
    - ii. Binary forms
    - iii. Ternary forms, Hilbert's symbol
    - iv. Product formula for Hilbert
    - v. Forms in 4 variables
    - vi. Forms in 5 and more variables
- (b) Hasse -Minkowski theorem
  - i. Proof for binary forms
  - ii. Proof for ternary forms
  - iii. Proof for forms in 4 variables
  - iv. Proof for forms in 5 or more variables
4. Extensions of valuations
  - (a) Extensions of complete fields
    - i. Uniqueness
    - ii. Existence (only in the non-archimedean case)
    - iii. Ramification index, residue field degree, inequality  $ef \leq n$
    - iv. Discrete valuations, equality  $ef = n$
    - v. Unramified extensions
    - vi. Tame ramified extensions
  - (b) Extensions of incomplete fields, equality  $n = \sum e_i f_i$
  - (c) Absolute values of number fields, product formula
5. The theorems of Sprindzhuk
  - (a) Hilbert's Irreducibility theorem (statement only)
  - (b) Sprindzhuk's irreducibility and decomposition theorems (statements)
  - (c) Heights of algebraic numbers, definition and properties
  - (d) Proof of the theorem of Sprindzhuk