Exponential polynomials and $\mathcal{D}$-modules

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Abstract

We introduce $\mathcal{D}$-modules adapted to study ideals generated by exponential polynomials.

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0 Introduction

In this manuscript we introduce a new method to study ideals generated by exponential polynomials, inspired by the theory of $\mathcal{D}$-modules [17, 18, 19]. Let us recall that an exponential polynomial f of n complex variables with frequencies in a finitely generated subgroup Γ of $\mathbf{C}^n$ is a function of the form

$$f(z_1, \ldots, z_n) = f(z) = \sum_{\gamma \in \Gamma} p_\gamma(z) exp(\gamma \cdot z),$$

where the sum is finite, the p_γ are polynomials, and $\gamma \cdot z = \gamma_1 z_1 + \cdots + \gamma_n z_n$. Such a function belongs to the algebra $A_\phi(\mathbf{C}^n)$ of entire functions F satisfying the growth condition:

$$\exists C > 0 \quad |F(z)| \leq C exp(C\phi(z)),$$

where the weight ϕ can be taken as $\mid z \mid$, the Euclidean norm of z, or, more precisely, if we choose a system $\gamma^1, \ldots, \gamma^N$, of $\mathbf{Q}$-linearly independent generators of Γ, as

$$\phi(z) = max(\mid \Re(\gamma^j \cdot z) \mid : j = 1, \cdots, N) + log(1 + \mid z \mid^2),$$

where $\Re z$ denotes the real part of the complex number z.

In the case that $\Gamma \subset i\mathbf{R}^n$, the exponential polynomials are just the Fourier transforms of distributions supported by finitely many points in the lattice $-i\Gamma$, and A_ϕ is a subalgebra of the Paley-Wiener algebra $\widehat{\mathcal{E}'(\mathbf{R}^n)}$ of Fourier transforms of distributions of compact support. It is well-known that the spectral synthesis does not hold for arbitrary systems of convolution equations as soon as $n \geq 2$, equivalently, not all ideals in the Paley-Wiener algebra are localizable [23]. If an ideal is generated by polynomials then, it has been proved by Ehrenpreis and Malgrange, that it is always localizable [21, 27]. The only fairly general criterion to ensure localizability of a finitely generated ideal I is to verify that the generators form a slowly decreasing sequence in the sense of [6]. Among other requirements, the generators must define a complete intersection. The slowly decreasing condition is not too easy to check, especially when the variety V of common zeros of the generators is not discrete. The only general example given in [6] of a slowly decreasing sequence of exponential polynomials is the following. Let $P_1, \ldots, P_n$ be polynomials defining a discrete (hence, finite) variety in $\mathbf{C}^n$ and $k \leq n$, then the sequence of functions

$$f_j(z) = P_j(e^{iz_1}, \ldots, e^{iz_k}, z_{k+1}, \ldots, z_n) \tag{1}$$

is slowly decreasing.

For these reasons, in our previous paper [9] we had considered the case of finitely generated ideals of exponential polynomials with frequencies in a group Γ of rank n and V discrete. Even when $\Gamma = i\mathbf{Z}^n$, we could not find a general criterion for localizability of the ideals generated by such exponential polynomials. Part of the problem was of an arithmetic nature, namely localizability may depend not only on the geometry of V and Γ, but also on the diophantine approximations of the coefficients of the generators of I. For example, the ideal generated by $cos(z_1), cos(z_2), z_2 - \alpha z_1$ is localizable if and only if α is not a Liouville number. As we pointed out in [10], there is a deep relationship between the localizability issue and a conjecture of Ehrenpreis on the zeros of exponential polynomials of a single variable with algebraic coefficients and frequencies.

In this paper we consider a situation that is fairly different from that of [9]. Namely, the group Γ has very low rank, either one or two, and the variety V might not be discrete or complete intersection. We have obtained some results very simple to state. For instance, if $rank(\Gamma) = 1$, any system of exponential polynomials defining a complete intersection generates a localizable ideal in the space A_ϕ. Another example of localizability is that where the generators are of the type (1) and define a non-discrete complete intersection. We have also studied problems related to global versions of the Nullstellensatz and of the Briançon-Skoda theorem, which could be useful when solving the ubiquitous Bezout identity for exponential polynomials without common zeros. The solution of the Ehrenpreis conjecture, as mentioned in [10], is precisely equivalent to solving in general the Bezout identity.

The leitmotiv of our approach is to relate the division problems implicit in the previous questions, to the study of the analytic continuation in $\lambda_1, \ldots, \lambda_m$ of the distribution

$$z \mapsto |f_1(z)|^{\lambda_1} \cdots |f_m(z)|^{\lambda_m},$$

for exponential polynomials f_j and the residues of this distribution-valued meromorphic function. This idea originated in our previous work about residue currents [3] and their applications to the effective solvability of the polynomial membership problem [12, 13]. The theory of $\mathcal{D}$-modules, as introduced by J. Bernstein [17], was precisely formulated to obtain an explicit form of the analytic continuation in λ of the distribution $|P(z)|^\lambda$ when P is a polynomial. Bernstein's results were extended by Björk to the holomorphic setting in [18].

Finally, we should mention that our results can be interpreted in harmonic analysis as providing a representation of all the solutions of certain homogeneous systems of linear partial differential equations with time lags, for instance, in Theorem 3.4 below.

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1 $\mathcal{D}$-modules

The ideas we develop in this section are clearly related to those about the Weyl algebra found in [18, Chapter 1], to which we refer for further developments.

We denote by $\mathbf{N}$ the set of non-negative integers. For an index $\alpha \in \mathbf{N}^n$, its length $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We also let $\mathbf{K}$ be a field of characteristic zero, n and m two positive integers, we define an extension $E_{n,m}(\mathbf{K})$ of the Weyl algebra $A_n(\mathbf{K})$. It is realized as an algebra of operators acting on the algebra of polynomials in $n+m$ variables over $\mathbf{K}$ as follows.

Consider the polynomial algebra $\mathbf{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ and derivations $D_1, \ldots, D_n$ on this algebra such that

$$D_i x_j = \delta_{ij} \qquad (i, j = 1, \ldots, n)$$

$$D_i y_j = \delta_{ij} y_j \qquad (i = 1, \ldots, n;\ j = 1, \ldots, m.)$$

The algebra $E_{n,m}(\mathbf{K})$ is the algebra of operators on $\mathbf{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ generated by $X_1, \ldots, X_n, Y_1, \ldots, Y_m, D_1, \ldots, D_n$, where X_i (resp. Y_j) is the operator of multiplication by x_i (resp. y_j). It is a Lie algebra, with the usual definition of the Lie bracket $[.,.]$ in terms of the composition of operators, i.e.,

$$[P, Q] = P \circ Q - Q \circ P.$$

The Lie bracket satisfies the following commutator relations

$$[X_i, X_j] = [Y_i, Y_j] = [X_i, Y_j] = [D_i, D_j] = 0\ ;$$

$$[X_i, D_j] = -\delta_{ij} \;\; ; \;\; [X_i, D_j] = -\delta_{ij} Y_i.$$

We note that for $m = 0$ our algebra coincides with the Weyl algebra. It is evident that every element P of $E_{n,m}(\mathbf{K})$ can be written in the form of a finite sum

$$P = \sum_{\alpha,\beta,\gamma} c_{\alpha,\beta,\gamma} X^\alpha Y^\beta D^\gamma, \tag{2}$$

$c_{\alpha,\beta,\gamma} \in \mathbf{K}, \alpha, \gamma \in \mathbf{N}^n, \beta \in \mathbf{N}^m$. We want to prove the uniqueness of the representation (2). For that purpose it is convenient to introduce the operators $ad(Q)$ acting on $E_{n,m}(\mathbf{K})$ by $ad(Q)(P) := [P, Q]$. Once the uniqueness is proven, the integer $max(|\alpha| + |\beta| + |\gamma| : c_{\alpha,\beta,\gamma} \neq 0)$ will be denoted $degP$.

Lemma 1.1 *Every element of $E_{n,m}(\mathbf{K})$ can be written in a unique way as in (2).*

Proof. Let us assume we have an expression

$$P = \sum_{\alpha,\beta,\gamma} c_{\alpha,\beta,\gamma} X^\alpha Y^\beta D^\gamma = 0,$$

as an operator on $\mathbf{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. We rewrite P as

$$P = \sum_\gamma P_\gamma D^\gamma, \;\; P_\gamma := \sum_{\alpha,\beta} c_{\alpha,\beta,\gamma} X^\alpha Y^\beta.$$

Observe that if $\gamma = (\gamma_1, \ldots, \gamma_n) = (\gamma_1, \gamma')$ then

$$ad(X_1)(D^\gamma) := [X_1, D^\gamma] = -\gamma_1 D^{\gamma_1 - 1} D^{\gamma'},$$

which vanishes if $\gamma_1 = 0$. Hence

$$ad(X_n)^{\gamma_n} \circ \cdots \circ ad(X_1)^{\gamma_1}(D^\gamma) = (-1)^{|\gamma|}\gamma!.$$

Moreover, for any other index $\overline{\gamma}$ we have

$$ad(X_n)^{\gamma_n} \circ \cdots \circ ad(X_1)^{\gamma_1}(D^{\overline{\gamma}}) = 0,$$

if some $\overline{\gamma_i} < \gamma_i$, in particular, if $|\overline{\gamma}| < |\gamma|$. It follows, using the lexicographical ordering, that for all γ

$$(-1)^{|\gamma|}\gamma! P_\gamma = 0.$$

Since $char(\mathbf{K}) = 0$, we have $P_\gamma = 0$. As P_γ is the operator acting on $\mathbf{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ by multiplication with a polynomial all the coefficients of P_γ are zero.

□

We shall need the following simple calculus rules.

Lemma 1.2 *For any integers $a, b \geq 0$, $1 \leq k \leq n$, we have*

$$[D_k, X_k^a Y_k^b] = a X_k^{a-1} Y_k^b + b X_k^a Y_k^b.$$

Corollary 1.1 *Let $P(X, Y) = \sum_{k=0}^M X_1^k P_k(X', Y) = \sum_{l=0}^N Y_1^l Q_l(X, Y')$, where $X = (X_1, X'), Y = (Y_1, Y')$. Then*

$$[D_1, P] = \sum_{l=0}^N Y_1^l (\frac{\partial Q_l}{\partial X_1} + l Q_l) = X_1^M Y_1 \frac{\partial P_M}{\partial Y_1} + \sum_{k=0}^{M-1} X_1^k \left\{ (k+1) P_{k+1} + Y_1 \frac{\partial P_k}{\partial Y_1} \right\}.$$

Let us define the natural filtration $\mathcal{E}_v$ on $E_{n,m}(\mathbf{K})$ by

$$\mathcal{E}_v := \{ P \in E_{n,m}(\mathbf{K}) : deg P \leq v \}.$$

It is a $\mathbf{K}$-vector space of dimension $\binom{2n+m+v}{v} \approx v^{2n+m}$. We can define the graded algebra $gr(E_{n,m}(\mathbf{K}))$ as

$$gr(E_{n,m}(\mathbf{K})) := \mathcal{E}_0 \bigoplus \mathcal{E}_1 / \mathcal{E}_0 \bigoplus \cdots$$

As always (cf. [18]), it is necessary to show that this is a commutative algebra. The only thing to show is that

$$[\mathcal{E}_u, \mathcal{E}_v] \subseteq \mathcal{E}_{u+v-1}.$$

This is a consequence of the fact that $deg[X_i, D_j] \leq 0$, $deg[Y_i, D_j] \leq 1$.

Finally, we want to show that $gr(E_{n,m}(\mathbf{K}))$ is isomorphic to a polynomial ring in $2n + m$ variables. As in [18] all we need to demonstrate is that if $\overline{X}_i$ (resp., $\overline{Y}_i, \overline{D}_i$) denotes the class of X_i (resp., Y_i, D_i) and

$$\sum c_{\alpha,\beta,\gamma} \overline{X}^\alpha \overline{Y}^\beta \overline{D}^\gamma = 0,$$

in $gr(E_{n,m}(\mathbf{K}))$, then all coefficients $c_{\alpha,\beta,\gamma} = 0$. Assume this is not true and let

$$v := max \{ |\alpha| + |\beta| + |\gamma| : c_{\alpha,\beta,\gamma} \neq 0 \}.$$

We have that

$$\sum_{|\alpha|+|\beta|+|\gamma|=v} c_{\alpha,\beta,\gamma}\overline{X}^{\alpha}\overline{Y}^{\beta}\overline{D}^{\gamma} = 0 \text{ in } \mathcal{E}(v),$$

where $\mathcal{E}(v) := \mathcal{E}_v/\mathcal{E}_{v-1}$. Thus, its representative $\sum_{|\alpha|+|\beta|+|\gamma|=v} c_{\alpha,\beta,\gamma}X^{\alpha}Y^{\beta}D^{\gamma}$ belongs necessarily to $\mathcal{E}_{v-1}$. Since the degree is v, this is clearly a contradiction to the uniqueness of the representation proved earlier, so we are done.

Let M be a (left) $E_{n,m}(\mathbf{K})$-module and Γ_v a filtration of M, i.e., an increasing family of finite dimensional $\mathbf{K}$-vector spaces Γ_v such that

(i) $\bigcup_{v\geq 0}\Gamma_v = M$;

(ii) $X_i\Gamma_v \subseteq \Gamma_{v+1}, Y_i\Gamma_v \subseteq \Gamma_{v+1}$, and $D_i\Gamma_v \subseteq \Gamma_{v+1}$.

Let $\Gamma(v) := \Gamma_v/\Gamma_{v-1}$ and define $gr(M)$ by

$$gr(M) := \Gamma_0 \bigoplus \Gamma_1/\Gamma_0 \bigoplus \cdots = \Gamma(0) \bigoplus \Gamma(1) \bigoplus \cdots$$

Due to property (ii), this graded module is a module over $gr(E_{n,m}(\mathbf{K}))$. One says the filtration is a *good filtration* if $gr(M)$ is of finite type over $gr(E_{n,m}(\mathbf{K}))$. For instance, if M is finitely generated over $E_{n,m}(\mathbf{K})$ by $a_1, \ldots, a_r$ and we choose $\Gamma_v := \mathcal{E}_v a_1 + \cdots + \mathcal{E}_v a_r$, then we have a good filtration.

As in [18, Lemma 3.4], one can prove the following lemma.

Lemma 1.3 *Let $(\Gamma_v)_v, (\Omega_v)_v$ be two filtrations of a $E_{n,m}(\mathbf{K})$ module M, and assume that $(\Gamma_v)_v$ is a good filtration. Then there is an integer w such that $\Gamma_v \subseteq \Omega_{v+w}$ for all $v \geq 0$.*

If $gr(M)$ is of finite type over $gr(E_{n,m}(\mathbf{K}))$, there is a Hilbert polynomial $H \in \mathbf{Q}[t]$ such that for all $v \gg 1$

$$H(v) = dim_{\mathbf{K}}\Gamma_v$$

(see [18, Theorem 3.1]). As a consequence of Lemma 1.3, the degree and the leading coefficient of H do not depend on the choice of the good filtration $(\Gamma_v)_v$. The degree d of H is called the *dimension* $d(M)$ of $gr(M)$ and the *multiplicity* $e(M)$ of $gr(M)$ is the leading term of H times $d!$.

In the case $m = 0$, i.e., for the Weyl algebra $A_n(\mathbf{K})$ one has the fundamental theorem of J. Bernstein that asserts that, for any non-trivial $A_n(\mathbf{K})$-module M so that $gr(M)$ is of finite type,

$$d(M) \geq n.$$

An $A_n(\mathbf{K})$-module M such that $d(M) = n$ is said to be *holonomic*.

One of the applications of the concept of holonomic modules is the existence of the *Bernstein-Sato functional equations* [18, 31, 29], i.e., given polynomials $f_1, \ldots, f_q$ in $\mathbf{K}[x_1, \ldots, x_n]$ there are differential operators Q_j in $A_n(\mathbf{K}[\lambda])$, with $\lambda = (\lambda_1, \cdots, \lambda_q)$, and a non-zero polynomial $b \in \mathbf{K}[\lambda]$ such that the formal relations

$$Q_j(f_1^{\lambda_1} \cdots f_j^{\lambda_j+1} \cdots f^{\lambda_q}) = b(\lambda) f_1^{\lambda_1} \cdots f^{\lambda_q} \quad (j = 1, \ldots, q)$$

hold.

One of the most interesting examples, for us, of $E_{n,m}(\mathbf{K})$-modules, $m \leq n$, is the following. Consider exponential polynomials $P_1, \ldots, P_q$ of n variables with positive integral frequencies and coefficients in a subfield $\mathbf{K}$ of $\mathbf{C}$, that is, finite sums

$$P_j(x) = \sum_{k \in \mathbf{N}^m} c_{j,k}(x) e^{k \cdot x},$$

with $c_{j,k} \in \mathbf{K}[x], j = 1, \ldots, q$. We consider a new field $\mathbf{K}(\lambda) = \mathbf{K}(\lambda_1, \ldots, \lambda_q)$ obtained from $\mathbf{K}$ by adjoining q indeterminates, and define the module M freely generated by a single generator denoted $\mathcal{P}^\lambda = \mathcal{P}_1^{\lambda_1} \cdots \mathcal{P}_q^{\lambda_q}$, namely,

$$M = M(P_1, \ldots, P_q) := \mathbf{K}(\lambda)[x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_m}][\frac{1}{P_1}, \ldots, \frac{1}{P_q}]\mathcal{P}^\lambda, \quad (3)$$

where, to pick up the earlier notation, X_i (resp., Y_j) operates as multiplication by x_i (resp., by e^{x_j}) and D_j acts as the differential operator $\bigtriangledown_j$, defined by

$$\bigtriangledown_j(A\mathcal{P}^\lambda) := \left(\frac{\partial A}{\partial x_j} + A \sum_{k=1}^{q} \frac{\lambda_k}{P_k} \frac{\partial P_k}{\partial x_j} \right) \mathcal{P}^\lambda.$$

The natural filtration of M is

$$\Gamma_v := \left\{ \frac{R(\lambda, x, e^x)}{(P_1 \cdots P_q)^v} \mathcal{P}^\lambda : R \in \mathbf{K}(\lambda)[x, e^x], deg_{x,e^x} R \leq v d_0 \right\},$$

where $d_0 := 1 + deg_{x,e^x}(P_1 \cdots P_q)$. This is a good filtration and

$$dim_{\mathbf{K}(\lambda)}\Gamma_v = \binom{n+m+vd_0}{vd_0}.$$

Hence,

$$d(M) = n + m,\ e(M) = d_0^{n+m}.$$

It is natural to ask whether for every non-trivial $E_{n,m}(\mathbf{K})$-module (or $E_{n,m}(\mathbf{K}(\lambda))$-module) with $m \leq n$, one has $d(M) \geq n + m$. Or, at least, to give conditions that ensure this inequality occurs.

Let us start with the following simple examples where $n = m = 1$. Let $\alpha \in \mathbf{R}$ (or even $\alpha \in \mathbf{C}$) and denote by δ_α the Dirac mass at the point α. Consider $\mathbf{K}$ a subfield of $\mathbf{C}$, and the $E_{1,1}(\mathbf{K})$-module M_α, generated by δ_α. M_α is a family of distributions with support at the point α. When $\alpha = 0$ we have

$$x\delta_0 = 0,\ e^x\delta_0 = \delta_0,\ \frac{d}{dx}\delta_0 = \delta_0',$$

so that

$$M_0 = \{\sum c_k\delta_0^{(k)}\} \cong \mathbf{K}[x],$$

and hence,

$$d(M_0) = 1.$$

On the other hand, when $\alpha \neq 0$, we have

$$x\delta_\alpha = \alpha\delta_\alpha,\ e^x\delta_\alpha = e^\alpha\delta_\alpha,\ \frac{d}{dx}\delta_\alpha = \delta_\alpha',$$

so that this time

$$M_\alpha \cong \mathbf{K}[\alpha, e^\alpha][x].$$

Hence,

$$d(M_\alpha) = 1 + transcdeg(\mathbf{K}[\alpha, e^\alpha]),$$

that is, it depends on the degree of transcendency of the extension of $\mathbf{K}$ by α and e^α. For instance, if $\mathbf{K} = \mathbf{Q}$, $\alpha \neq 0$ is algebraic, then $d(M_\alpha) = 2$. In every case in which $\mathbf{K} = \mathbf{Q}$, $\alpha \neq 0$, $d(M_\alpha) \leq 3$. If $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, then $d(M_\alpha) = 1$.

What this example shows is that the choice of the field may play a crucial role in deciding whether an $E_{n,m}(\mathbf{K})$-module M verifies or $d(M) \geq n + m$ or not. On the other hand, we are mainly interested in modules of the form $M(P_1, \ldots, P_q)$, their submodules, and quotient modules.

Let us now consider the case $m = 1$.

Proposition 1.1 *Let M be a finitely generated $E_{n,1}(\mathbf{K})$-module, then, either $d(M) \geq n+1$ or for every element $m_0 \in M \setminus \{0\}$ there exist two non-zero polynomials $A, B \in \mathbf{K}[s]$, and $t \in \mathbf{N}$ such that*

$$Y_1^t A(X_1) m_0 = B(Y_1) m_0 = 0.$$

Proof. Let us assume that $d(M) \leq n$ and let $m_0 \in M \setminus \{0\}$. We complete m_0 to a system of $E_{n,1}(\mathbf{K})$-generators of M and denote by Γ_0 the $\mathbf{K}$-vector space spanned by this system of generators. We define for $v \in \mathbf{N}$

$$\Gamma_v := \{P(X, Y_1, D)\Gamma_0 \, : \, degP \leq v\}\,.$$

This is a good filtration.

We claim that the map

$$\mathcal{E}_v \longrightarrow Hom_{\mathbf{K}}(\Gamma_v, \Gamma_{2v})$$

$$P \mapsto \{m \in \Gamma_v \mapsto Pm \in \Gamma_{2v}\}$$

cannot be injective for any sufficiently large v. If it were injective, we would have the inequality

$$const.v^{2n+1} \approx dim_{\mathbf{K}} \mathcal{E}_v \leq dim_{\mathbf{K}} Hom_{\mathbf{K}}(\Gamma_v, \Gamma_{2v}) \approx const.v^{2d(M)},$$

which implies $2n+1 \leq 2d(M)$, in other words, $d(M) \geq n + \frac{1}{2}$. This contradicts the fact that we have assumed $d(M) \leq n$.

Hence, for all large v there are differential operators $P_v = P \in \mathcal{E}_v \setminus \{0\}$ such that $P \cdot \Gamma_v = 0$. In other words,

$$Pm = 0 \;\; \forall m \in \Gamma_v \; .$$

Let us write $P = \sum c_{\alpha,\beta,\gamma} X^\alpha Y_1^\beta D^\gamma$, $|\alpha| + \beta + |\gamma| \leq v$. Let γ_0 be the largest power of D, in the lexicographical order, that appears in P. Then, as in Lemma 1.1, we have

$$P_1 := ad(X)^{\gamma_0}(P) = (-1)^{|\gamma_0|}\gamma_0! \sum c_{\alpha,\beta,\gamma_0} X^\alpha Y_1^\beta \neq 0.$$

On the other hand, since $P \cdot \Gamma_v = 0$, we have for any $1 \leq k \leq n$

$$ad(X_k)P \cdot \Gamma_{v-1} = 0$$

because

$$ad(X_k)P \cdot \Gamma_{v-1} = X_k P \cdot \Gamma_{v-1} - P X_k \cdot \Gamma_{v-1}$$

and $X_k \cdot \Gamma_{v-1} \subseteq \Gamma_v$, $\Gamma_{v-1} \subseteq \Gamma_v$. Therefore,

$$P_1 \cdot \Gamma_{v-|\gamma_0|} = ad(X)^{\gamma_0}(P) \cdot \Gamma_{v-|\gamma_0|} = 0,$$

and

$$degP_1 + |\gamma_0| \leq degP \leq v.$$

Let us rewrite P_1 as a polynomial in $X' = (X_2, \ldots, X_n)$,

$$P_1 = \sum b_{\alpha,\beta,\delta} X_1^{\alpha} Y_1^{\beta} (X')^{\delta}.$$

¿From Lemma 1.2, with $D' = (D_2, \ldots, D_n)$, we obtain

$$ad(D')^{\delta}(X')^{\delta} = \delta!$$

and, if for some i, $\overline{\delta_i} < \delta_i$,

$$ad(D')^{\delta}(X')^{\overline{\delta}} = 0.$$

Therefore, if δ_0 is the largest power of X' in the lexicographic order, we have

$$P_2 := ad(D')^{\delta_0} P_1 = \delta_0! \sum b_{\alpha,\beta,\delta_0} X_1^{\alpha} Y_1^{\beta} \neq 0,$$

$$degP_2 + |\delta_0| \leq degP_1,$$

$$P_2 \cdot \Gamma_{v-|\gamma_0|-|\delta_0|} = 0.$$

Clearly, $v - |\gamma_0| - |\delta_0| \geq degP_2 > 0$, if not, P_2 would be a non-zero constant, which contradicts the last identity.

Thus, we have reduced ourselves to the following situation. We have a non-zero polynomial P of the variables X_1, Y_1, $1 \leq degP \leq v$, and $P \cdot \Gamma_v = 0$. Let us write it in the form

$$P(X_1, Y_1) = \sum_{l=0}^{N} Y_1^l Q_l(X_1).$$

Observe that if $P(X_1, Y_1) = Y_1^N Q_N(X_1, Y_1)$ then

$$Y_1^N Q_N(X_1, Y_1) m_0 \in Y_1^N Q_N(X_1, Y_1) \Gamma_v = 0.$$

and we would already have proved the first part of the proposition, so that we can assume that there is more than one index l such that $Q_l \neq 0$. Obviously, we want to reduce ourselves to the case of a single $Q_l \neq 0$. Let us apply Corollary 1.1, then

$$ad(D_1)P = \sum_{l=0}^{N} Y_1^l (Q_l'(X_1) + lQ_l(X_1)),$$

where $Q_l' = \frac{dQ_l}{dX_1}$. We let

$$\tilde{P} := \left[Q_N ad(D_1)P - (Q_N' + NQ_N)P \right],$$

so that we still have

$$\tilde{P} \cdot \Gamma_{v-1} = 0.$$

Let L be the largest index such that $L < N$ and $Q_L \neq 0$, then the leading coefficient of $\tilde{P}$ as a polynomial in Y_1 is

$$Q_N(Q_L' + LQ_L) - Q_L(Q_N' + NQ_N) = (L - N)Q_N Q_L + (Q_N Q_L' - Q_L Q_N'),$$

which is the sum of two polynomials of different degrees. The one of highest degree is $(L - N)Q_N Q_L$, which is evidently different from zero. This shows that $\tilde{P} \neq 0$, $deg_{Y_1}\tilde{P} = L$, and $\tilde{P} \cdot \Gamma_{v-1} = 0$, so that we can repeat the procedure, and in at most $N - 1$ steps arrive to a non-zero polynomial of the form $Y_1^t A(X_1)$, which annihilates Γ_{v-N+1}. This makes sense because $N \leq degP \leq v$. This proves the first part of the proposition.

To prove the second part, we rewrite the original polynomial $P(X_1, Y_1)$ in the form

$$P = \sum_{k=0}^{M} X_1^k P_k(Y_1),$$

and assume $M \geq 1$, otherwise we are done. Hence, by Corollary 1.1 we have

$$ad(D_1)P = X_1^M Y_1 P_M'(Y_1) + \sum_{k=0}^{M-1} {X_1}^k((k + 1)P_{k+1} + Y_1 P_k'),$$

which again kills Γ_{v-1}. We consider

$$\tilde{P} := Y_1 P_M'(Y_1)P - P_M(Y_1)ad(D_1)P. \tag{4}$$

We claim that $deg_{X_1}\tilde{P} = M - 1$. In fact, the leading coefficient of $\tilde{P}$ is

$$Y_1 P'_M(Y_1) P_{M-1}(Y_1) - M(P_M(Y_1))^2 - Y_1 P_M(Y_1) P'_{M-1}(Y_1), \tag{5}$$

which we have to show is not identically zero. For that purpose, we prove the following lemma.

Lemma 1.4 *Let* $R, S \in \mathbf{K}[\xi]$, $R \neq 0$, *and* $a \in \mathbf{K}^*$, *then, the polynomial*

$$aR^2(\xi) - \xi(R'(\xi)S(\xi) - R(\xi)S'(\xi)) \not\equiv 0.$$

Proof. We want to reduce ourselves to the case where the coefficients are complex numbers. For that purpose we consider the collection of a and all the non-zero coefficients of R and S, say $\{\alpha_1, \dots, \alpha_s\}$. Then $\mathbf{Q}(\alpha_1, \dots, \alpha_s)$ is a subfield of $\mathbf{K}$, since $char\mathbf{K} = 0$, and, on the other hand it is a finitely generated extension of $\mathbf{Q}$, which we can decompose as a finite transcendental extension followed by a finite algebraic extension. The first extension can be embedded as a subfield of $\mathbf{R}$, and its algebraic extension as a subfield $\mathbf{k}$ of $\mathbf{C}$.

Therefore, we really have two polynomials $R, S \in \mathbf{C}[\xi]$, $R \neq 0$, and $a \in \mathbf{C}^*$, and we need to show that the identity

$$aR^2 = \xi(R'S - RS')$$

is impossible. Namely, we would have the equation

$$-\frac{a}{\xi} = \frac{RS' - R'S}{S^2} = \frac{d}{d\xi}\left(\frac{S}{R}\right)$$

The function $f(\xi) := \frac{S(\xi)}{R(\xi)}$ is rational, hence it is single valued and holomorphic outside the set of its poles. On the other hand, the differential equation

$$-\frac{a}{\xi} = f'(\xi) \tag{6}$$

has only the solutions $-a \log \xi + c$, $c \in \mathbf{C}$, which are neither single valued nor rational. This concludes the proof of Lemma 1.4.

□

Let us return to the proof of the Proposition 1.1. We have just seen that $deg_{X_1}\tilde{P} = M - 1$, where $\tilde{P}$ is defined by (4). We also have that $\tilde{P} \cdot \Gamma_{v-1} = 0$.

Repeating this procedure a total of M times, we obtain a non-zero polynomial $B(Y_1)$, i.e, a polynomial of degree zero in X_1, such that $B(Y_1) \cdot \Gamma_{v-M} = 0$. This is possible because $M \leq degP \leq v$. This concludes the proof of the second part of the proposition.

□

Let us give an application of Proposition 1.1 to the module

$$M(P_1, \ldots, P_q) = \mathbf{K}(\lambda)[x_1, \ldots, x_n, e^{x_1}][1/P_1, \ldots, 1/P_q]\mathcal{P}^\lambda$$

defined by equation (2), where $P_j \in \mathbf{K}[x_1, \ldots, x_n, e^{x_1}]$, $\mathbf{K}$ a subfield of $\mathbf{C}$.

Proposition 1.2 *There are two non-zero polynomials A_1, A_2 of a single variable s, with coefficients in $\mathbf{K}[\lambda]$, $\lambda = (\lambda_1, \ldots, \lambda_q)$, and $2q$ linear differential operators, $Q_{i,j}$ $(i = 1, 2;\ j = 1, \ldots, q)$, with coefficients belonging $\mathbf{K}[\lambda, x, e^{x_1}, e^{-x_1}]$, such that for every j*

$$A_1(\lambda, x_1)\mathcal{P}^\lambda = Q_{1,j}(\lambda, x, e^{x_1}, e^{-x_1}, \frac{\partial}{\partial x})P_j\mathcal{P}^\lambda, \tag{7}$$

$$A_2(\lambda, e^{x_1})\mathcal{P}^\lambda = Q_{2,j}(\lambda, x, e^{x_1}, e^{-x_1}, \frac{\partial}{\partial x})P_j\mathcal{P}^\lambda. \tag{8}$$

(To simplify the notation we have written $\frac{\partial}{\partial x}$ to denote $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.)

Proof. We follow an idea of Lichtin [29]. The module M,

$$M = M(P_1, \ldots, P_q) = \mathbf{K}(\lambda)[x_1, \ldots, x_n, e^{x_1}][1/P_1, \ldots, 1/P_q]\mathcal{P}^\lambda$$

is an $E_{n,1}(\mathbf{K}(\lambda)$-module of finite type and $d(M) = n + 1$, as stated earlier. Introduce the new $E_{n,1}(\mathbf{K}(\lambda))$-module $\mathcal{N}$ defined by

$$\mathcal{N} := M \bigoplus \cdots \bigoplus M \ (q\,\text{terms}),$$

consider the elements $e_l \in \mathcal{N}$, $l \in \mathbf{N}^*$,

$$e_l = ({P_1}^{l-1}{P_2}^l \cdots {P_q}^l\mathcal{P}^\lambda, \ldots, {P_1}^l \cdots {P_{q-1}}^l {P_q}^{l-1}\mathcal{P}^\lambda),$$

and denote $\mathcal{N}(l)$ the submodule of $\mathcal{N}$ generated by e_l.

We have that

$$d(\mathcal{N}) = n + 1,$$

since it is a direct sum [18]. Hence (cf. [18]),

$$d(\mathcal{N}(l)) \leq n+1.$$

Moreover,

$$\mathcal{N}(l+1) \subseteq \mathcal{N}(l), \quad (l \in \mathbf{N}^*).$$

On the other hand, we can apply Proposition 1.1 to conclude that $d(\mathcal{N}(l)) = n+1$ for every l. If not true, there would be a non-zero polynomial $B \in \mathbf{K}[\lambda, s]$, such that

$$B(\lambda, e^{x_1})e_l = 0.$$

This is impossible, since we are just multiplying exponential polynomials. Furthermore, for every l [18],

$$d(\mathcal{N}(l)/\mathcal{N}(l+1)) \leq n+1.$$

Thus, either for every l we achieve this upper bound or there is a smallest index l_0 such that

$$d(\mathcal{N}(l_0)/\mathcal{N}(l_0+1)) \leq n. \tag{9}$$

Let us show the first case cannot occur. If it did, consider the sequence of modules

$$0 \longrightarrow \mathcal{N}(l+1) \longrightarrow \mathcal{N}(l) \longrightarrow \mathcal{N}(l)/\mathcal{N}(l+1) \longrightarrow 0,$$

which is clearly exact for every l. Since the dimensions of all the terms coincide and it is possible to apply the proof of [18, Proposition 3.6], with $A_n(\mathbf{K})$ replaced by $E_{n,1}(\mathbf{K}(\lambda))$, we conclude that their multiplicities are related by

$$e(\mathcal{N}(l)) = e(\mathcal{N}(l+1)) + e(\mathcal{N}(l)/\mathcal{N}(l+1)),$$

which implies that for all $l \geq 1$

$$1 \leq e(\mathcal{N}(l)) < e(\mathcal{N}(l+1)).$$

This is obviously impossible. Hence, the equation (9) holds for some minimal value of l_0.

It could occur that $\mathcal{N}(l_0)/\mathcal{N}(l_0+1) = 0$, then $e_{l_0} \in \mathcal{N}(l_0+1)$. In this case there is a differential operator $R = R(\lambda, x, e^{x_1}, \frac{\partial}{\partial x}) \in E_{n,1}(\mathbf{K}(\lambda))$ such that

$$e_{l_0} = Re_{l_0+1}.$$

Consider the jth entry. We have

$$\begin{aligned} {P_1}^{l_0} \cdots {P_j}^{l_0-1} \cdots {P_q}^{l_0} \mathcal{P}^\lambda &= R\left({P_1}^{l_0+1} \cdots {P_j}^{l_0} \cdots {P_q}^{l_0+1} \mathcal{P}^\lambda\right) \\ &= R\left(\left(P_1 \cdots \hat{P}_j \cdots P_q\right)\left({P_1}^{l_0} \cdots {P_q}^{l_0} \mathcal{P}^\lambda\right)\right) \\ &= R_j\left({P_1}^{l_0} \cdots {P_q}^{l_0} \mathcal{P}^\lambda\right), \end{aligned}$$

where R_j is another differential operator in $E_{n,1}(\mathbf{K}(\lambda))$ obtained applying Leibniz's rules. Since $\lambda_1, \ldots, \lambda_q$ are transcendental over $\mathbf{K}$, this last formal identity is equivalent to a true identity involving only $P_1, \ldots, P_q$, and their derivatives, instead of $\mathcal{P}^\lambda$. We can therefore change variables $\lambda_1 \mapsto \lambda_1 + l_0, \ldots, \lambda_j \mapsto \lambda_j + l_0 - 1, \ldots, \lambda_q \mapsto \lambda_q + l_0$, and obtain

$$\mathcal{P}^\lambda = R_j(P_j \mathcal{P}^\lambda).$$

Finally, we can clear the denominators from $\mathbf{K}[\lambda]$ in R_j and conclude that there is some $b \in \mathbf{K}[\lambda] \setminus \{0\}$, independent of j, and corresponding differential operators Q_j with coefficients in $\mathbf{K}[\lambda, x, e^{x_1}]$ so that

$$b(\lambda)\mathcal{P}^\lambda = Q_j(P_j \mathcal{P}^\lambda). \tag{10}$$

If $\mathcal{N}(l_0)/\mathcal{N}(l_0+1) \neq 0$, we can apply Proposition 1.1 to this $E_{n,1}(\mathbf{K}(\lambda))$-module and find two non-zero polynomials $A, B \in \mathbf{K}[\lambda, s]$ and an integer $t \in \mathbf{N}$ such that

$$e^{tx_1} A(\lambda, x_1) e_{l_0} \in \mathcal{N}(l_0+1)$$

and

$$B(\lambda, e^{x_1}) e_{l_0} \in \mathcal{N}(l_0+1).$$

We can divide out by e^{tx_1} the first relation and apply the earlier reasoning to conclude there are non-zero polynomials $A_1, A_2 \in \mathbf{K}[\lambda, s]$ and linear differential operators $Q_{i,j}$ with coefficients in $\mathbf{K}[\lambda, x, e^{x_1}, e^{-x_1}]$ so that

$$A_1(\lambda, x_1)\mathcal{P}^\lambda = Q_{1,j} P_j \mathcal{P}^\lambda$$

$$A_2(\lambda, e^{x_1})\mathcal{P}^\lambda = Q_{2,j} P_j \mathcal{P}^\lambda$$

for $j = 1, \ldots, q$.

This concludes the proof of the functional equation in every case.

□

Let us denote by $\mathcal{A}$ the ring of all entire functions f in $\mathbf{C}^{n+q}$ satisfying the growth condition

$$|f(\lambda, x)| \leq \kappa(1 + |\lambda| + |x|)^N e^{D|\Re x_1|} \tag{11}$$

for some $\kappa, N, D > 0$.

If we knew that $A_j(\lambda, s) = b_j(\lambda)B_j(s)$, $j = 1, 2$, then we could simplify the equations (7) and (8) when $\mathbf{K} \subseteq \overline{\mathbf{Q}}$, as follows. The only possible solution $s \in \mathbf{C}$ of the pair of equations

$$B_1(s) = B_2(e^s) = 0$$

is $s = 0$, by the Gelfond-Schneider theorem [2]. Let us denote $m \in \mathbf{N}$ is the multiplicity of this solution. Then, appealing to [10] we know there are two entire functions C_1, C_2 satisfying the growth conditions

$$|C_j(s)| = O((1 + |s|)^N e^{N|\Re s|}) \ \ (s \in \mathbf{C}), \tag{12}$$

for some $N \in \mathbf{N}$, $j = 1, 2$, and

$$C_1(s)B_1(s) + C_2(s)B_2(e^s) = s^m. \tag{13}$$

We could then conclude that there would be a non-zero polynomial $b(\lambda)$ and linear differential operators $\tilde{Q}_j$ with coefficients in $\mathcal{A}$ such that

$$b(\lambda){x_1}^m\mathcal{P}^\lambda = \tilde{Q}_j(P_j\mathcal{P}^\lambda), \ j = 1, \ldots, q.$$

Namely, multiply (7) by $b_2(\lambda)C_2(x_1)$, (8) by $b_1(\lambda)C_1(x_1)$, and add.

In general, we do not have such a factorization of A_1 and A_2. The idea will be to use an approximate factorization. We discuss this point in the following section.

To conclude this introductory section, let us make some remarks about generalizations of the previous results. First, it is convenient to observe that the algebra $\mathcal{A}$ is a subalgebra of the weighted Fréchet algebra usually denoted $A_\rho(\mathbf{C}^n)$, $\rho(x) = \log(1 + |x|) + |\Re x|$, where

$$A_\rho(\mathbf{C}^n) = \{f \text{ entire} : \exists c > 0 \ |f(x)| \leq ce^{c\rho(x)} \ \forall x \in \mathbf{C}^n\}.$$

The spaces $E_{n,m}$ we are considering, are subalgebras of this weighted algebra. In this paper we will essentially consider only this weight ρ.

Let us now see how to apply the previous reasonings to the algebra of polynomials in $e^{z_1}, e^{\alpha z_1}, z_2, \dots, z_n$ with coefficients in $\overline{\mathbf{Q}}$, where we assume $\alpha \in \overline{\mathbf{Q}} \setminus \mathbf{Q}$ and $z_1, \dots, z_n$ are n complex variables. For this purpose, we introduce the algebra $\mathcal{K} = \mathbf{K} < Y_1, Z_1, X_2, \dots, X_n, D_1, \dots, D_n >$ of operators acting on the polynomial algebra $\mathbf{K}[Y_1, Z_1, X_2, \dots, X_n]$, $\mathbf{K}$ a field of characteristic zero. The differential operators D_j obey Leibniz's rule, the X_j, Y_1, Z_1 act by multiplication, and we define

$$D_1 Y_1 = Y_1,\ D_1 Z_1 = \alpha Z_1,\ D_1 X_j = 0,$$

$$D_j Y_1 = D_j Z_1 = 0\ (j \geq 2),\ D_j X_k = \delta_{jk}.$$

As a consequence, we have the conmutation rules

$$[D_j, Y_1] = [D_j, Z_1] = 0\ (j \geq 2),\ [D_j, X_k] = \delta_{jk},$$

$$[Y_1, Z_1] = [Y_1, X_j] = [Z_1, X_j] = [X_k, X_j] = 0,$$

$$[D_1, {Y_1}^k {Z_1}^l] = (k + \alpha l) {Y_1}^k {Z_1}^l \tag{14}$$

$$[Y_1, {D_1}^k] = -Y_1 {D_1}^{k-1} + D_1([Y_1, {D_1}^{k-1}]) = -k Y_1 {D_1}^{k-1} + p(Y_1, D_1), \tag{15}$$

where $p(Y_1, D_1)$ is a polynomial of degree $\leq k - 2$ in D_1.

We remark that the algebra of exponential polynomials in the variables $e^{z_1}, e^{\alpha z_1}, z_2, \dots, z_n$, cannot be isomorphic to the polynomial algebra in Y_1, Z_1, $X_2, \dots, X_n$, unless $\alpha \notin \mathbf{Q}$. Recall that the field $\mathbf{K}$ always contains a copy of $\mathbf{Q}$.

Lemma 1.5 *Every element P of $\mathcal{K}$ can be written in a unique way as*

$$P = \sum c_{ijkl} X^i {Y_1}^j {Z_1}^k D^l.$$

Proof. Denote $degP = |i| + j + k + |l|$. Suppose $P = 0$, as operators, we need to verify that all the coefficients $c_{ijkl} = 0$. As in Lemma 1.1, we can reduce ourselves to the case P is a polynomial in Y_1, Z_1, D_1. We use (13) to diminish the degree of P in D_1, by commutations with Y_1, and conclude we can assume there is no D_1. But, as a multiplication operator, a polynomial in Y_1, Z_1 cannot vanish unless its coefficients are zero.

□

We define the filtration $\mathcal{K}_v$ of $\mathcal{K}$, by degrees, and the corresponding graded ring $gr(\mathcal{K}) = \mathcal{K}_0 \bigoplus \mathcal{K}_1/\mathcal{K}_0 \bigoplus \cdots$ We conclude, as before, that $gr(\mathcal{K})$ is commutative and isomorphic to the polynomial ring in $2n + 1$ variables over $\mathbf{K}$.

The concept of good filtrations is the same as earlier and Lemma 1.4 and its consequences hold.

We need to prove the analogue of Proposition 1.2.

Proposition 1.3 *Let M be a finitely generated $\mathcal{K}$-module. Then, either $d(M) \geq n+1$ or for every element $m_0 \in M \setminus \{0\}$ there are two non-zero polynomials $A, B \in \mathbf{K}[s]$ and two non-negative integers a, b such that*

$$Z_1{}^a A(Y_1) m_0 = Y_1{}^b B(Z_1) m_0 = 0.$$

Proof. We use the same filtration Γ_v as in the proof of the Proposition 1.2, so that $m_0 \in \Gamma_0$. Thus, if $d(M) \leq n$, for all large v there is $P \in \mathcal{K}_v \setminus \{0\}$, such that $P \cdot \Gamma_v = 0$. By the argument we have used in Proposition 1.2, we can assume that P depends only on the variables Y_1, Z_1, D_1, $1 \leq degP \leq v$. Using the relation (13), we can even eliminate D_1. We just observe that $ad(Y_1)P \cdot \Gamma_{v-1} = 0$, and $ad(Y_1)P$ is a non-zero polynomial whose degree in D_1 strictly smaller than that of P. This is verbatim the procedure in Proposition 1.2 to eliminate D_1. So that, from the start we could assume that $1 \leq degP \leq v$, $P \cdot \Gamma_v = 0$, and $P \in \mathbf{K}[Y_1, Z_1]$.

If P were independent of Z_1, then, either $P(Y_1) = cY_1{}^d$, $c \neq 0$, $d = degP$, and we can take $a = 0$, $b = d$, $A(Y_1) = P(Y_1)$, $B(Z_1) = c$, or P has at least two terms. In the latter case, the polynomial $ad(D_1)P(Y_1) - (degP)P(Y_1) \neq 0$, it annihilates Γ_{v-1}, and it has degree $< degP$. Iterating this procedure we would be done. Thus, let us assume that P depends both on Y_1 and on Z_1. Consider

$$P = \sum_{j=0}^{l} Q_j(Y_1) Z_1{}^j,$$

and assume there are at least two non-zero terms in this representation. (Otherwise, we let $A(Y_1) = Q_l(Y_1)$ and $a = l$.) Then,

$$[D_1, P] = \sum_{j=0}^{l} (Y_1 Q_j'(Y_1) + \alpha j Q_j(Y_1)) Z_1{}^j$$

kills Γ_{v-1}, and so does

$$P_1 := Q_l [D_1, P] - (Y_1 Q_l'(Y_1) + \alpha l Q_l(Y_1)) P,$$

which has degree in $Z_1 < l$. The only problem is to show that $P_1 \neq 0$. Since there is an index $j < l$ such that $Q_j \neq 0$, if $P_1 = 0$ we would have that

$$Y_1(Q_l Q_j' - Q_j Q_l') = \alpha(l-j) Q_j Q_l.$$

This leads to the formal differential equation

$$\frac{Q_l'}{Q_l} - \frac{Q_j'}{Q_j} = \frac{\alpha(l-j)}{Y_1} \neq 0,$$

which, by an argument similar to that used in (5), can be shown to be impossible. This concludes the proof of the existence of an $a \in \mathbf{N}$ and an $A \in \mathbf{K}[Y_1] \setminus \{0\}$ with the required properties. The other part of the proposition is proved similarly.

□

As pointed out above, the algebra generated over $\mathbf{K}$ by $e^{x_1}, e^{\alpha x_1}, x_2, \ldots, x_n$, $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, when $\mathbf{K}$ is a subfield of $\mathbf{C}$ and $\alpha \in \mathbf{K} \setminus \mathbf{Q}$, is isomorphic to the algebra $\mathcal{K} = \mathcal{K}_\alpha$ we have considered above ($Y_1 = e^{x_1}$, $Z_1 = e^{\alpha x_1}$). Therefore, we can consider a family $P_1, \ldots, P_q$ of exponential polynomials in $\mathcal{K}$, $\mathbf{K}(\lambda) = \mathbf{K}(\lambda_1, \ldots, \lambda_q)$, and the module

$$M = M(P_1, \ldots, P_q) = \mathbf{K}(\lambda)[e^{x_1}, e^{\alpha x_1}, x_2, \ldots, x_n][1/P_1, \ldots, 1/P_q]\mathcal{P}^\lambda,$$

which is the module generated by the action of $\mathcal{K}$ on the formal generator $\mathcal{P}^\lambda = {P_1}^{\lambda_1} \cdots {P_q}^{\lambda_q}$. So that we obtain the following result, corresponding to Proposition 1.2.

Proposition 1.4 *There are two non-zero polynomials A_1, A_2 of a single variable s, with coefficients in $\mathbf{K}[\lambda]$, $\lambda = (\lambda_1, \ldots, \lambda_q)$, and $2q$ linear differential operators, $Q_{i,j}$ $(1 \leq i \leq 2;\ 1 \leq j \leq q)$, whose coefficients belong to $\mathbf{K}[\lambda, e^{x_1}, e^{-x_1}, e^{\alpha x_1}, e^{-\alpha x_1}, x_2, \ldots, x_n]$, such that for every $j = 1, \ldots, q$ we have*

$$A_1(\lambda, e^{x_1})\mathcal{P}^\lambda = Q_{1,j} P_j \mathcal{P}^\lambda, \tag{16}$$

$$A_2(\lambda, e^{\alpha x_1})\mathcal{P}^\lambda = Q_{2,j} P_j \mathcal{P}^\lambda. \tag{17}$$

2 Functional equations and analytic continuation

In the applications we have in mind, e.g., Bezout identities, division problems, and the like, one needs to determine the principal part of the Laurent development of $|f|^{2\lambda}$ for $\lambda = -k$, $k \in \mathbf{N}$, where f is an exponential polynomial. The reason for this need will become clear later on. Meanwhile, we are going to explain how to obtain sufficient knowledge of the coefficients of the principal parts, even if we do not have the factorization of the polynomials A_1, A_2 mentioned at the end of the last section.

Lemma 2.1 *Let f be an exponential polynomial in $E_{n,1}(\mathbf{K})$, $k \in \mathbf{N}$, there is an integer $q \in \mathbf{N}$ such that for any $N \in \mathbf{N}$ one can find a non-zero polynomial $R_N \in \mathbf{K}[x_1]$ and a functional equation of the form*

$$(\lambda + k)^q R_N |f|^{2\lambda} = Q_{k,N}(|f|^{2\lambda} f^{k+1}) + (\lambda + k)^{q+N} v_N |f|^{2\lambda}, \qquad (18)$$

where $v_N \in \mathbf{K}[\lambda, x_1]$ and $Q_{k,N}$ is a linear differential operator with coefficients in $\mathbf{K}[\lambda, x, e^{x_1}, e^{-x_1}]$.

Proof. From Proposition 1.2 we know the existence of a non-zero polynomial $A \in \mathbf{K}[\lambda, x_1]$ and a differential operator Q with polynomial coefficients such that one has the formal identity

$$A(\lambda, x_1) f^\lambda = Q(\lambda, x, e^{x_1}, e^{-x_1}, \frac{\partial}{\partial x}) f^{\lambda+1} = Q(\lambda) f^{\lambda+1}.$$

We would like to iterate this identity, except that contrary to the usual Bernstein-Sato functional equations, the coefficients of $A(\lambda, x_1)$ depend on x_1. We factor A into two coprime polynomials

$$A(\lambda, x_1) = p_1(\lambda) \mathcal{A}_1(\lambda, x_1).$$

In particular, for every fixed λ the map $x_1 \mapsto \mathcal{A}_1(\lambda, x_1)$ is not identically zero, and hence we deduce the formal identity

$$\begin{aligned} p_1(\lambda+1) A(\lambda, x_1) f^\lambda &= Q(\lambda) \left\{ \frac{A(\lambda+1, x_1) f^{\lambda+1}}{\mathcal{A}_1(\lambda+1, x_1)} \right\} \\ &= Q(\lambda) \left\{ \frac{Q(\lambda+1) f^{\lambda+2}}{\mathcal{A}_1(\lambda+1, x_1)} \right\}. \end{aligned}$$

Applying Leibniz's rules one obtains

$$p_1(\lambda+1)A(\lambda,x_1)f^\lambda = \frac{\tilde{Q}_1(\lambda)f^{\lambda+2}}{\mathcal{A}_1(\lambda+1,x_1)^{m_1}},$$

for some $m_1 \in \mathbf{N}$, and $\tilde{Q}_1$ a new differential operator with coefficients in $\mathbf{K}[\lambda, x, e^{x_1}, e^{-x_1}]$. Thus, we find

$$\tilde{A}_1(\lambda,x_1)f^\lambda := p_1(\lambda+1)\mathcal{A}_1(\lambda+1,x_1)^{m_1}A(\lambda,x_1)f^\lambda = \tilde{Q}_1(\lambda)f^{\lambda+2}.$$

Iterating this procedure, for every $k \in \mathbf{N}$ we find a non-zero polynomial $A_k(\lambda, x_1)$ and a differential operator Q_k with coefficients in the same ring as above, so that

$$A_k f^\lambda = Q_k f^{\lambda+2k+1}. \tag{19}$$

Since multiplication by the formal antiholomorphic function $\overline{f}^\lambda$ commutes with the operators $\frac{\partial}{\partial x_j}$, then

$$A_k|f|^{2\lambda} = Q_k(\lambda)(|f|^{2\lambda}f^{2k+1}).$$

Note that this formal identity can also be interpreted as an identity among distributions. It is convenient to factor A_k into coprime polynomials as follows

$$A_k(\lambda,x_1) = (\lambda+k)^q B_k(\lambda,x_1),$$

which allows us to write

$$(\lambda+k)^q B_k(\lambda,x_1)|f|^{2\lambda} = Q_k(\lambda)(|f|^{2\lambda}f^{2k+1}) \tag{20}$$

Since B_k is coprime with $\lambda+k$, for any $N \in \mathbf{N}^*$ we have a polynomial Bezout identity

$$R_N(x_1) = u_N(\lambda,x_1)B_k(\lambda,x_1) + v_N(\lambda,x_1)(\lambda+k)^N, \tag{21}$$

for some polynomials $u_N, v_N \in \mathbf{K}[\lambda, x_1]$ and $R_N \in \mathbf{K}[x_1]$, $R_N \neq 0$. (It is clear that u_N, v_N, R_N, and q depend also on k. We suppress this index to simplify the notation.) Therefore, we have

$$\begin{aligned}(\lambda+k)^q R_N(x_1)|f|^{2\lambda} &= (\lambda+k)^q u_N B_k|f|^{2\lambda} + (\lambda+k)^{q+N}v_N|f|^{2\lambda} \\ &= u_N Q_k(\lambda)(|f|^{2\lambda}f^{2k+1}) + (\lambda+k)^{q+N}v_N|f|^{2\lambda} \\ &= Q_{k,N}(|f|^{2\lambda}f^{2k+1}) + (\lambda+k)^{q+N}v_N|f|^{2\lambda},\end{aligned}$$

with $Q_{k,N} := u_N Q_k$, which concludes the proof of the lemma.

□

The same proof yields relations of the form

$$(\lambda+k)^{\tilde{q}} S_N(e^{x_1})|f|^{2\lambda} = \tilde{Q}_{k,N}(|f|^{2\lambda} f^{2k+1}) + (\lambda+k)^{\tilde{q}+N} \tilde{v}_N |f|^{2\lambda}, \tag{22}$$

where $\tilde{v}_N \in \mathbf{K}[\lambda, e^{x_1}], S_N(t) = S_{N,k}(t) \in \mathbf{K}[t]$, and $\tilde{Q}_{k,N}$ is a differential operator with the same properties as $Q_{k,N}$.

We know a priori [1] that, in a neighborhood of $\lambda = -k$, the distribution-valued meromorphic function $|f|^{2\lambda}$ has the Laurent expansion

$$|f|^{2\lambda} = \sum_{j=-2n}^{\infty} a_{k,j} (\lambda+k)^j, \tag{23}$$

with $a_{k,j} \in \mathcal{D}'(\mathbf{C}^n)$. The previous lemma allows us to compute explicitly the products $R_N(x_1)a_{k,j}$, $S_N(e^{x_1})a_{k,j}$, for $-2n \le j \le 0$, as soon as we take $N \ge 2n+1$. Namely, the polynomial v_N in (21) can be expanded in powers of $\lambda + k$, i.e.,

$$v_N(\lambda, x_1) = \sum_{l=0}^{m} v_{N,l}(x_1)(\lambda+k)^l. \tag{24}$$

Let $\varphi \in \mathcal{D}'(\mathbf{C}^n)$, then

$$\begin{aligned} (\lambda+k)^q < |f|^{2\lambda}, R_N(x_1)\varphi > &= \sum_{j=-2n}^{\infty} < a_{k,j}, R_N\varphi > (\lambda+k)^{q+j} \\ &= < Q_{k,N}(\lambda)(|f|^{2\lambda} f^{2k+1}), \varphi > \\ &\quad + \sum_{j=-2n}^{\infty} < a_{k,j}, v_N\varphi > (\lambda+k)^{q+N+j} \\ &= < |f|^{2\lambda} f^{2k+1}, Q'_{k,N}(\lambda)\varphi > \\ &\quad + \sum_{j,l} < a_{k,j}, v_{N,l}\varphi > (\lambda+k)^{q+N+j+l}, \end{aligned} \tag{25}$$

where $Q'_{k,N}$ is the adjoint operator of $Q_{k,N}$ (obtained by integration by parts).

The first term of the last sum is holomorphic at $\lambda = -k$, and the series only contains powers of $\lambda + k$ bigger or equal to $q + 1$, due to the choice

$N \geq 2n+1$. Thus, the distribution-valued function $(\lambda+k)^q R_N(x_1)|f|^{2\lambda}$ is holomorphic in a neighboorhood of $\lambda = -k$. Moreover, if we denote

$$(\lambda+k)^q R_N(x_1)|f|^{2\lambda} = \sum_{h=0}^{\infty} b_{k,h}(\lambda+k)^h \tag{26}$$

its Taylor development, then, for $0 \leq h \leq q$, the distributions $b_{k,h}$ are given by

$$< b_{k,h}, \varphi >= \frac{1}{2\pi i} \int_{|\lambda+k|=\varepsilon} \int_{\mathbf{C}^n} |f(x)|^{2\lambda} f(x)^{2k+1} Q'_{k,N}(\lambda)(\varphi(x)) dx \frac{d\lambda}{(\lambda+k)^{h+1}}, \tag{27}$$

where $\varepsilon > 0$ is chosen sufficiently small so that on a neighborhood of $supp(\varphi)$, the function $x \mapsto |f(x)|^{-2\varepsilon}$ is integrable.

We can rewrite the last integral as

$$\frac{1}{2\pi i} \int_{|\lambda+k|=\varepsilon} \int_{\mathbf{C}^n} |f(x)|^{2(\lambda+k)} f(x)(f(x)/\overline{f(x)})^k Q'_{k,N}(\lambda)(\varphi(x)) dx \frac{d\lambda}{(\lambda+k)^{h+1}}$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{2\pi i} \int_{|\lambda+k|=\varepsilon} \int_{\mathbf{C}^n} (\log |f|^2)^j f(f/\overline{f})^k Q'_{k,N}(\lambda)(\varphi(x)) dx \, (\lambda+k)^{j-h-1} d\lambda \, ,$$

which shows that the terms $< b_{k,h}, \varphi >$ are linear combinations of integrals of the form

$$\int_{\mathbf{C}^n} (\log |f|^2)^j f(f/\overline{f})^k Q_\iota(\varphi) dx, \tag{28}$$

where $j \in \mathbf{N}$ (in fact, $0 \leq j \leq h+1$) and the Q_ι are differential operators with coefficients in $\mathbf{K}[x, e^{x_1}, e^{-x_1}]$. Note that the term $(f/\overline{f})^k$ is bounded, and the same holds locally for $f(\log |f|^2)^j$.

Let us take, once for all, $N = 2n+1$ and, since R_N really depends also on k we shall denote it $\mathcal{R}_k$ from now on. Therefore, from (23) and (26) we obtain

$$\mathcal{R}_k(x_1) a_{k,j} = 0 \quad \text{if } q+j < 0 \tag{29}$$

$$\mathcal{R}_k(x_1) a_{k,j} = b_{k,q+j} \quad \text{if } 0 \leq q+j \leq q \tag{30}$$

Moreover, if we introduce the polynomials $\mathcal{S}_k$ in a similar way, the same procedure leads to an explicit computation of $\mathcal{S}_k(e^{x_1}) a_{k,j}$ for the same values of j, $-2n \leq j \leq 0$. We summarize these remarks in the following statement.

Proposition 2.1 *Let $f \in E_{n,1}(\mathbf{K})$ and $k \in \mathbf{N}$, there exist non-zero polynomials $\mathcal{R}_k, \mathcal{S}_k$ of a single variable, with coefficients in $\mathbf{K}$, $N_k \in \mathbf{N}$, and positive constants C_k, D_k such that the distributions $a_{k,j}$, $-2n \leq j \leq 0$, defined by the Laurent development*

$$|f|^{2\lambda} = \sum_{j=-2n}^{\infty} a_{k,j}(\lambda + k)^j$$

satisfy the estimates

$$| < \mathcal{R}_k(x_1)a_{k,j}, \varphi > | + | < \mathcal{S}_k(e^{x_1})a_{k,j}, \varphi > | \leq C_k \|\varphi\|_{N_k} \max_{x \in supp(\varphi)} e^{(D_k \rho(x))}, \tag{31}$$

where $\varphi \in \mathcal{D}(\mathbf{C}^n)$, $\rho(x) = \log(1 + |x|) + |\Re x_1|$.

Corollary 2.1 *If $\mathbf{K} \subseteq \overline{\mathbf{Q}}$, there are integers $m_k \in \mathbf{N}$, and two constants $C'_k, D'_k > 0$ such that the estimate (31) implies*

$$| < x_1^{m_k} a_{k,j}, \varphi > | \leq C'_k \|\varphi\|_{N_k} \max_{x \in supp(\varphi)} e^{(D'_k \rho(x))}$$

Proof. We return to the argument at the end of the preceding section. For each $k \in \mathbf{N}$ we can find two entire functions φ_k, ψ_k in the Paley-Wiener class of functions, i.e., $O((1 + |x_1|)^B e^{A|\Re x_1|})$, and an integer $m = m_k \geq 0$ such that

$$\mathcal{R}_k(x_1)\varphi_k(x_1) + \mathcal{S}_k(e^{x_1})\psi_k(x_1) = x_1^m. \tag{32}$$

Thus, we can get estimates for the distributions $x_1^m a_{k,j}$, using (28), (30), and (32).

□

In the following section, we shall use these estimates for the distributions involved in the analytic continuation of distribution-valued holomorphic functions of the form $|f_1|^{2\lambda_1} \cdots |f_p|^{2\lambda_p}/(|f_1|^2 + \cdots + |f_p|^2)^m$. These functions have already appeared in our previous work [12, 3]. The existence of an analytic continuation as a meromorphic function of $\lambda_1, \ldots, \lambda_p$ follows from Hironaka's resolution of singularities, but since we want to control the distributions that appear as coefficients in the Laurent developments about some pole, that is, we would like to obtain estimates similar to those of Proposition 2.1 and Corollary 2.1, we need to find some kind of functional equation that provides the analytic continuation. Since it is easier to provide functional equations for $|f_1|^{2\lambda_1}\cdots|f_p|^{2\lambda_p}$, we need a technical trick to reduce this kind of quotients of functions to products. It is based on a simple lemma about the inverse Mellin transform. In order to simplify its writing let us introduce the following notation.

For $t_1,\ldots,t_p>0$, $\mu_1,\ldots,\mu_p\in\mathbf{C}$, we let

$$t^\mu := t_1^{\mu_1}\cdots t_p^{\mu_p}\,.$$

Given $s_1,\ldots,s_{p-1},\beta\in\mathbf{C}$, let

$$ds := ds_1\cdots ds_{p-1},\ s_p := \beta - s_1 - \cdots - s_{p-1},\ \tilde{\mu_j} := \mu_j - s_j\ (1\le j\le p).$$

We also let $s := (s_1,\ldots,s_{p-1}), s^* := (s_1,\ldots,s_p)$, with s_p as previously defined. Recall also the somewhat standard notation,

$$\Gamma[a] := \Gamma[a_1,\cdots,a_k] := \Gamma(a_1)\cdots\Gamma(a_k),$$

for complex values a_j such that the Euler Gamma function is defined. Finally, as long as there is no possibility of confusion, we shall use the following abbreviated notation for multiple integrals on lines parallel to the imaginary axes. Let $\gamma=(\gamma_1,\ldots,\gamma_{p-1})$ be a vector of real components, then, for any integrable function F

$$\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)ds := \int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_{p-1}-i\infty}^{\gamma_{p-1}+i\infty} F(s)ds_1\cdots ds_{p-1}$$

Lemma 2.2 *Let $t_1,\ldots,t_p>0$, $\mu_1,\ldots,\mu_p\in\mathbf{C}$, $\Re\beta>1$, $P\in\mathbf{C}[\mu_1,\ldots,\mu_p]$, then, with the previous notation,*

$$P(\mu)\frac{t^\mu}{(t_1+\cdots+t_p)^\beta} = \frac{1}{(2\pi i)^{p-1}\Gamma(\beta)}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma[s^*]P(\tilde{\mu})t^{\tilde{\mu}}ds \qquad (33)$$

for any $\gamma_j>0$ such that $\gamma_1+\cdots+\gamma_{p-1}<\Re\beta-1$.

Proof. We start from a known formula about the inverse Mellin transform [24, 6.422.3,p.657], for $0<\gamma<\Re(\beta-1)$, $t>0$, one has

$$\frac{1}{(1+t)^\beta} = \frac{1}{2\pi i\Gamma(\beta)}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)\Gamma(\beta-s)t^{-s}ds, \qquad (34)$$

This integral is absolutely convergent because of the rapid decrease of $\Gamma(s)$ along verical lines in the right-hand plane. (In fact, (34) follows inmediately from the definition of Euler's Beta function and the Mellin inversion formula.) Thus, when $p \geq 2$ we let $\tau = t_2 + \cdots + t_p$, and then, if $0 < \gamma_1 < \Re(\beta - 1)$, we have

$$\begin{aligned}
\frac{1}{(t_1 + \cdots + t_p)^\beta} &= \frac{1}{\tau^\beta} \frac{1}{(1 + (t_1/\tau))^\beta} \\
&= \frac{1}{2\pi i \Gamma(\beta) \tau^\beta} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \Gamma(s_1)\Gamma(\beta - s_1)(t_1/\tau)^{-s_1} ds_1 \\
&= \frac{1}{2\pi i \Gamma(\beta)} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \Gamma(s_1)\Gamma(\beta - s_1) {t_1}^{-s_1} \tau^{-\beta - s_1} ds_1.
\end{aligned}$$

Since $\Re(\beta - s_1) > 1$, we can use a recurrence argument when $p \geq 3$, which will become clear after we write down the next step. We rewrite $\tau = t_2 + \sigma$, so that

$$\begin{aligned}
\frac{1}{\tau^{\beta - s_1}} &= \frac{1}{\sigma^{\beta - s_1}} \frac{1}{(1 + (t_2/\sigma))^{\beta - s_1}} \\
&= \frac{1}{2\pi i \Gamma(\beta - s_1)} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} \Gamma(s_2)\Gamma(\beta - s_1 - s_2) t^{-s_2} \sigma^{-(\beta - s_1 - s_2)} ds_2,
\end{aligned}$$

as long as $0 < \gamma_2 < \Re(\beta - s_1 - 1)$, i.e., $\gamma_1 + \gamma_2 < \Re\beta - 1$. Therefore, with $T = t_1 + \cdots + t_p$, we have

$$\frac{1}{T^\beta} = \frac{1}{(2\pi i)^2 \Gamma(\beta)} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} \Gamma[s_1, s_2, \beta - s_1 - s_2] t^{-s_1} t^{-s_2} \sigma^{-(\beta - s_1 - s_2)} ds_1 ds_2,$$

which shows, by induction on p, that the formula (33) is correct when $P \equiv 1$, $\mu_j = 1 (j = 1, \ldots, p)$. In other words, with the notation introduced above, we have proved that

$$\frac{1}{T^\beta} = \frac{1}{(2\pi i)^{p-1} \Gamma(\beta)} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma[s^*] t^{-s^*} ds. \tag{35}$$

Multiplying (35) by $t^\mu = t_1^{\mu_1} \cdots t_p^{\mu_p}$, we obtain the formula (33) in the case $P \equiv 1$:

$$\frac{t^\mu}{T^\beta} = \frac{1}{(2\pi i)^{p-1} \Gamma(\beta)} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma[s^*] t^{\tilde{\mu}} ds. \tag{36}$$

To obtain the general case, let us rewrite (36) by choosing new variables $r_1, \ldots, r_p$ defined by

$$r_j := \frac{t_j}{T} = \frac{t_j}{t_1 + \cdots + t_p} \quad (1 \leq j \leq p).$$

It follows that for any $r_j > 0, \mu_j \in \mathbf{C}$,

$$r^{\mu} = \frac{1}{(2\pi i)^{p-1}\Gamma(\beta)} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma[s^*] r^{\tilde{\mu}} ds. \tag{37}$$

If we now apply the differential operator $r_j \frac{\partial}{\partial r_j}$ to both sides of (37) we find

$$\mu_j r^{\mu} = \frac{1}{(2\pi i)^{p-1}\Gamma(\beta)} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma[s^*] \tilde{\mu_j} r^{\tilde{\mu}} ds.$$

It is clear now that for any polynomial P,

$$P(\mu) r^{\mu} = \frac{1}{(2\pi i)^{p-1}\Gamma(\beta)} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma[s^*] P(\tilde{\mu}) r^{\tilde{\mu}} ds.$$

Replacing r_j by their values in terms of the t_k, we obtain the expression (33).

□

Let us now apply this lemma to the study of the coefficients in the Laurent expansion about $\mu = 0$ of the analytic continuation of

$$\mu \mapsto \frac{|f|^{2(\mu t - \underline{k})}}{\|f\|^{2m}}, \tag{38}$$

where $t \in]0, \infty[^p$ is a vector to be chosen below, $\mu \in \mathbf{C}$, $k \in \mathbf{Z}$, $\underline{k}$ is the p-dimensional vector $(k, \ldots, k)$, $m \in \mathbf{N}^*$, $f_j \in E_{n,1}(\mathbf{K})$, $\|f\|^{2m} = (|f_1|^2 + \cdots + |f_p|^2)^m$, and, keeping with the previous notation $|f|^r = |f_1|^{r_1} \cdots |f_p|^{r_p}$ for any vector $r = (r_1, \ldots, r_p)$ (Similar meaning for f^r). From Proposition 1.2 we conclude that there is a polynomial $A(\lambda_1, \ldots, \lambda_p, x_1)$ and differential operators $Q_{1,j}(\lambda, x, e^{x_1}, e^{-x_1}, \frac{\partial}{\partial x})$ such that

$$A(\lambda, x_1) {f_1}^{\lambda_1} \cdots {f_p}^{\lambda_p} Q_{1,j}(\lambda)({f_1}^{\lambda_1} \cdots {f_j}^{\lambda_j + 1} \cdots {f_p}^{\lambda_p}).$$

As we have done in the proof of Lemma 2.1, for any $k \in \mathbf{Z}, l \in \mathbf{N}$ there is a polynomial $A_l \in \mathbf{K}[\lambda, x_1]$ and a functional equation (in which we use the abbreviated notation introduced earlier)

$$A_l f^{\lambda - \underline{k}} = Q_{1;l}(\lambda)(f^{\lambda - \underline{k} + \underline{l}}).$$

The polynomial A_l and the new differential operator $Q_{1;l}$ depend also on k. Multiplying this equation by $\overline{f_1}^{\lambda_1-k}\cdots\overline{f_p}^{\lambda_p-k}$, we obtain a functional equation that has also meaning in the sense of distributions

$$A_l|f|^{2(\lambda-\underline{k})} = Q_{1;l}(\lambda)(|f|^{2(\lambda-\underline{k})}\, f^{\underline{l}}). \tag{39}$$

As a consequence of Lemma 2.2, and using the same notation, for any point x such that $f_1(x)\cdots f_p(x)\neq 0$ we have

$$A_l(\lambda, x_1)\frac{|f(x)|^{2(\lambda-\underline{k})}}{\|f(x)\|^{2m}} = \frac{1}{(2\pi i)^{p-1}\Gamma(m)}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma[s^*]A_l(\tilde{\lambda}, x_1)|f(x)|^{2(\tilde{\lambda}-\underline{k})}ds. \tag{40}$$

Let us fix $l = 2m+2k+1$ and choose a vector $t\in]0,\infty[^p$ such that the one variable polynomial $\mu\mapsto A_l(\mu t, x_1)$ is not identically zero. Almost every choice of t works for all k and m. To emphasize the dependence on k, we now denote $\mathcal{A}_k(\mu, x_1) := A_l(\mu t, x_1)$. Factor $\mathcal{A}_k$ into two coprime terms,

$$\mathcal{A}_k(\mu, x_1) = \mu^q\mathcal{B}_k(\mu, x_1),\ (q = q(k)).$$

Therefore, there are polynomials $\mathcal{R}_k(x_1)\neq 0, u_k(\mu, x_1)$, and $v_k(\mu, x_1)$, with the property that

$$\mathcal{R}_k(x_1) = u_k(\mu, x_1)\mathcal{B}_k(\mu, x_1) + \mu^{2n+1}v_k(\mu, x_1).$$

Consider from now on $\lambda = \mu t$. For $\Re\mu \gg 1$ and $\varphi\in\mathcal{D}(\mathbf{C}^n)$ we can integrate φ against (40) to obtain

$$\begin{aligned}
&\frac{1}{(2\pi i)^{p-1}\Gamma(m)}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_{\mathbf{C}^n}\Gamma[s^*]u_kA_l(\tilde{\lambda}, x_1)|f(x)|^{2(\tilde{\lambda}-\underline{k})}\varphi dx ds\\
&= \int_{\mathbf{C}^n}u_k(\mu, x_1)\mathcal{A}_k(\mu, x_1)\frac{|f(x)|^{2(\lambda-\underline{k})}}{\|f(x)\|^{2m}}\varphi(x)dx\\
&= \mu^q\int_{\mathbf{C}^n}\mathcal{R}_k(x_1)\frac{|f(x)|^{2(\lambda-\underline{k})}}{\|f(x)\|^{2m}}\varphi dx - \mu^{q+2n+1}\int_{\mathbf{C}^n}v_k\frac{|f(x)|^{2(\lambda-\underline{k})}}{\|f(x)\|^{2m}}\varphi dx
\end{aligned}$$

We remind the reader that $A_l(\tilde{\lambda}, x_1)$ is really a polynomial in $\mu, s_1, \ldots, s_{p-1}$, and x_1. For s fixed we apply the functional equation (39) and integration by parts, to conclude that

$$\int_{\mathbf{C}^n}\Gamma[s^*]u_k A_l(\tilde{\lambda},x_1)|f|^{2(\tilde{\lambda}-\underline{k})}\varphi dx \quad = \int_{\mathbf{C}^n} Q_{1;l}(\tilde{\lambda})(|f|^{2(\tilde{\lambda}-\underline{k})}f^{\underline{l}})u_k\varphi dx$$

$$= \int_{\mathbf{C}^n} |f|^{2(\tilde{\lambda}-\underline{k})}f^{\underline{l}}\, Q'_{1;l}(\tilde{\lambda})(u_k\varphi)dx,$$

where $Q'_{1;l}$ represents the adjoint operator. Using Fubini's theorem we get

$$\begin{aligned}
&\mu^q\int_{\mathbf{C}^n}\mathcal{R}_k\frac{|f|^{2(\lambda-\underline{k})}}{\|f\|^{2m}}\varphi dx = \\
&\quad \frac{1}{(2\pi i)^{p-1}\Gamma(m)}\int_{\mathbf{C}^n}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma[s^*]|f|^{2(\tilde{\lambda}-\underline{k})}f^{\underline{l}}\, Q'_{1;l}(\tilde{\lambda})(u_k\varphi)dxds \\
&\quad +\mu^{q+2n+1}\int_{\mathbf{C}^n} v_k\frac{|f|^{2(\lambda-\underline{k})}}{\|f\|^{2m}}\varphi dx \\
&\quad = I_1(\mu)+I_2(\mu)
\end{aligned} \tag{41}$$

Similarly to the case of a single equation considered earlier, we have that in a neighborhood of $\mu = 0$, the distribution-valued function (38) has the Laurent development

$$\frac{|f|^{2(\mu t-\underline{k})}}{\|f\|^{2m}} = \sum_{j=-2n}^{\infty} a_{k,j}\mu^j,\ a_{k,j}\in\mathcal{D}'(\mathbf{C}^n). \tag{42}$$

The choice $l = 2m+2k+1$, ensures that the distribution valued function

$$\mu\mapsto |f(x)|^{2(\tilde{\lambda}-\underline{k})}(f_1(x)\cdots f_p(x))^l$$

is holomorphic in a neighborhood W of $\mu = 0$, uniformly with respect to s , and independent of x as long as x is near $supp(\varphi)$. Thus, the Taylor coefficients of $I_1(\mu)$ about $\mu = 0$ are linear combinations of expressions of the form

$$\int_{\gamma-i\infty}^{\gamma+i\infty}\int_{\mathbf{C}^n}\Gamma[s^*]F(s^*)(\log|f|)^{\alpha}|f|^{-2(s^*+\underline{k})}f^l E(x,e^{x_1},e^{-x_1})\varphi^{(\beta)}dxds,$$

where E, F are polymonials, $\alpha\in\mathbf{N}^p, \beta\in\mathbf{N}^n$, and we have written $(\log|f|)^{\alpha} = (\log|f_1|)^{\alpha_1}\cdots(\log|f_p|)^{\alpha_p}$. Altogether, due to the choice of l and the constraints on the γ_j, there are constants κ_α and $N\in\mathbf{N}$ such that these integrals can be estimated by

$$\kappa_\alpha max\{|f_1(x)\cdots f_p(x)E(x,e^{x_1},e^{-x_1})| : x\in supp(\varphi)\}\|\varphi\|_N,$$

with $\|\varphi\|_N$ denoting a Sobolev norm of φ. It is clear that there are distributions $b_{k,h} \in \mathcal{D}'(\mathbf{C}^n)$ such that

$$I_1(\mu) = \sum_{h=0}^{\infty} < b_{k,h}, \varphi > \mu^h.$$

On the other hand, if $v_k(\mu, x_1) = \sum_{i=0}^{d} v_{k,i}(x_1)\mu^i$, then

$$I_2(\mu) = \sum_{j=-2n}^{\infty} \sum_{i=0}^{d} < a_{k,j}, v_{k,i}(x_1)\varphi > \mu^{i+j+q+2n+1}$$

Summarizing,

$$\begin{aligned} &\sum_{j=-2n}^{\infty} < \mathcal{R}_k(x_1) a_{k,j}, \varphi > \mu^{j+q} \\ &\quad = \sum_{h=0}^{\infty} < b_{k,h}, \varphi > \mu^h + \sum_{i,j} < a_{k,j}, v_{k,i}(x_1)\varphi > \mu^{i+j+q+2n+1}. \end{aligned}$$

The second series on the right hand side does not contain any power of μ smaller than $q+1$. This allows us to identify the coefficients on the left hand side with indices $-2n \le j \le 0$. Namely,

$$\begin{array}{lll} \mathcal{R}_k(x_1) a_{k,j} = & 0 & \text{if } q+j<0 \\ \mathcal{R}_k(x_1) a_{k,j} = & b_{k,q+j} & \text{if } 0 \le q+j \le q \end{array} \tag{43}$$

Note that if the f_j are polynomials (no exponentials) then the polynomial factor can be taken to be $\mathcal{R}_k \equiv 1$ for any k. This follows from the fact that $E_{n,0}(\mathbf{K})$ is holonomic and, hence, there are always functional equations (7) with A_1 independent of x and $Q_{1,j}$ with coefficients in $\mathbf{K}[\lambda, x]$.

The same reasoning holds when we start with the system of formal identities (8), and the only thing to remark is that we can choose the vector $t \in]0,\infty[^p$ so that for every k, m the corresponding exponential polynomials in $\mathbf{K}[\lambda, e^{x_1}]$ are not identically zero on the complex line $\lambda = \mu t$. Correspondingly, we obtain $\mathcal{S}_k \in \mathbf{K}[e^{x_1}]$, $\tilde{q} = \tilde{q}(k) \in \mathbf{N}$, and distributions $c_{k,j}$, with the same properties as the $b_{k,j}$ such that

$$\begin{array}{lll} \mathcal{S}_k(e^{x_1}) b_{k,j} = & 0 & \text{if } \tilde{q}+j<0 \\ \mathcal{S}_k(e^{x_1}) b_{k,j} = & b_{k,\tilde{q}+j} & \text{if } 0 \le \tilde{q}+j \le \tilde{q}. \end{array}$$

In other words, we have proved entirely the following proposition.

Proposition 2.2 *Let* $f_1,\dots,f_p \in E_{n,1}(\mathbf{K})$*, then, for any* $t \in]0,1[^p$ *(outside a countable union of* $\mathbf{K}$*-algebraic hypersurfaces, which depend on the* f_j*) and any* $k \in \mathbf{Z}, m \in \mathbf{N}^*$*, there are polynomials* $\mathcal{R}_k$ *and* $\mathcal{S}_k$ *in* $\mathbf{K}[u]$ *and constants* $C_k, D_k > 0, N_k \in \mathbf{N}$ *such that if* $a_{k,j} \in \mathcal{D}'(\mathbf{C}^n)$ *denote the coefficients of the Laurent expansion*

$$\frac{|f|^{2(\mu t-\underline{k})}}{\|f\|^{2m}} = \sum_{j=-2n}^{\infty} a_{k,j}\mu^j,$$

then, for $-2n \leq j \leq 0$, $\varphi \in \mathcal{D}(\mathbf{C}^n)$*,*

$$|<\mathcal{R}_k(x_1)a_{k,j},\varphi>| + |<\mathcal{S}_k(e^{x_1})a_{k,j},\varphi>| \leq C_k\|\varphi\|_{N_k} \max_{x\in supp(\varphi)} e^{(D_k\rho(x))},$$

where $\rho(x) = \log(1+|x|) + |\Re x_1|$*.*

Note that in this proposition, $\mathcal{R}_k, \mathcal{S}_k, C_k, D_k$ depend also on m and t, while N_k depends on k, m.

Corollary 2.2 *If* $\mathbf{K} \subseteq \overline{\mathbf{Q}}$*, there is an integer* $\nu_k \in \mathbf{N}$*, and positive constants* C'_k, D'_k *such that*

$$|<x_1^{\nu_k}a_{k,j},\varphi>| \leq C'_k\|\varphi\|_{N_k} \max_{x\in supp(\varphi)} e^{(D'_k\rho(x))}$$

Proof. It is the same as that of Corollary 2.1.

□

Let us examine now the situation where $f_1,\dots,f_p$ are polynomials in $e^{x_1}, e^{\alpha x_1}, x_2,\dots,x_n$, with coefficients in $\overline{\mathbf{Q}}$, $\alpha \in \overline{\mathbf{Q}} \setminus \mathbf{Q}$. The same procedure as earlier shows there are polynomials of a single variable $A, B \in \overline{\mathbf{Q}}[s] \setminus \{0\}$ such that if $a_{k,j}$ denote the distributions that appear in (42), then $A(e^{x_1})a_{k,j}$ and $B(e^{\alpha x_1})a_{k,j}$ have good estimates for $-2n \leq j \leq 0$. In this case the two entire functions $A(e^{x_1})$ and $B(e^{\alpha x_1})$ can only have $x_1 = 0$ as a common zero. In fact, if $x_1 = \zeta$ is a common zero, then $\omega = e^{\zeta}$ satisfies the algebraic equation $A(\omega) = 0$, so that $\omega \in \overline{\mathbf{Q}}$. For the same reason $\omega^\alpha \in \overline{\mathbf{Q}}$. Gelfond's theorem [2] implies that $\zeta = 0$. Let us factor

$$A(s) = (s-1)^{\nu_1}\prod_{j=1}^{l_1}(s-\xi_j),$$

$$B(s) = (s-1)^{\nu_2}\prod_{j=1}^{l_2}(s-\eta_j),$$

where $\xi_j, \eta_j \in \overline{\mathbf{Q}} \setminus \{1\}$.

Lemma 2.3 *Let* $A(e^{x_1}) = x_1^\nu A_1(x_1)$, $B(e^{\alpha x_1}) = x_1^\nu B_1(x_1)$, *where* $\nu \in \mathbf{N}$, *and* A_1, B_1 *are entire functions without any common zeros. Then there are constants* $c_1, c_2, \varepsilon, \kappa > 0$ *such that*

$$\varepsilon \exp(-\kappa\rho(x)) \leq |A_1(x_1)| + |B_1(x_1)| \leq c_1 \exp(c_2\rho(x)).$$

Proof. Clearly $\nu = inf(\nu_1, \nu_2)$. For the sake of definiteness, let us assume $\nu = \nu_1$. The proof now follows from the fact that if $|A_1(x_1)| + |B_1(x_1)|$ is small, then, either $|e^{x_1} - \xi_j| + |e^{\alpha x_1} - \eta_l|$ is small for some pair of indices j, l, or $|e^{x_1} - \xi_j| + |\alpha x_1 - 2m\pi i|$ is small for some index j and some integer m. Baker's theorem [2] on lower bounds for linear combinations over $\overline{\mathbf{Q}}$ of logarithms of algebraic numbers yields the lower bound of the lemma. (Otherwise, either $|x_1 - \log \xi_j| + |\alpha x_1 - \log \eta_l|$ is too small or $|x_1 - \log \xi_j| + |\alpha x_1 - 2m\pi i|$ is too small. Since $\alpha \in \overline{\mathbf{Q}} \setminus \mathbf{Q}$ these two simultaneous estimates are impossible [2, 10].) The upper bound is clear.

$\square$

As a consequence of this lemma, we conclude that $x_1^\nu a_{k,j}$ can be estimated as in Corollary 2.2. For future use, we state this in the form of a proposition.

Proposition 2.3 *Let* $f_1, \ldots, f_p$ *are polynomials in* $e^{x_1}, e^{\alpha x_1}$, $x_2, \ldots, x_n$, *with coefficients in* $\overline{\mathbf{Q}}$, $\alpha \in \overline{\mathbf{Q}} \setminus \mathbf{Q}$ *and* $t \in]0, 1[^p$ *(outside a countable union of* $\mathbf{K}$*-algebraic hypersurfaces, which depend only on the* f_j*) and any* $k \in \mathbf{Z}, m \in \mathbf{N}^*$, *there are an integer* $\nu_k \in \mathbf{N}$ *and positive constants* C_k, D_k *such that if* $a_{k,j} \in \mathcal{D}'(\mathbf{C}^n)$ *denote the coefficients of the Laurent expansion*

$$\frac{|f|^{2(\mu t - \underline{k})}}{\|f\|^{2m}} = \sum_{j=-2n}^{\infty} a_{k,j}\mu^j,$$

then, for $-2n \leq j \leq 0$, $\varphi \in \mathcal{D}(\mathbf{C}^n)$, *we have the estimate*

$$| < x_1^{\nu_k} a_{k,j}, \varphi > | \leq C'_k \|\varphi\|_{N_k} \max_{x \in supp(\varphi)} e^{(D'_k\rho(x))},$$

where $\rho(x) = \log(1 + |x|) + |\Re x_1|$.

3 Division formulas and representation theorems

In [9] we gave some sufficient conditions, albeit sometimes hard to verify, so that if $f_1, \ldots, f_n$ are exponential polynomials in n variables with integral frequencies whose variety of common zeros $V = \{z \in \mathbf{C}^n : f_1(z) = \cdots = f_n(z) = 0\}$ is discrete or empty, then the ideal I generated by them in the space $A_\rho(\mathbf{C}^n), \rho(z) = \log(1+|z|) + |\Re z|$ coincides with I_{loc} the ideal of those functions in $A_\rho(\mathbf{C}^n)$ which can locally be obtained as linear combinations of the f_j with holomorphic coefficients. In particular, I is closed and localizable (i.e., $I = \overline{I} = I_{loc}$). In fact, the conditions given in [9] implied that the n-tuple $f_1, \ldots, f_n$ was slowly decreasing in the sense of [6]. This has a certain number of interesting consequences for the harmonic analysis of the solutions of the system of difference-differential equations in $\mathbf{R}^n$ with symbol given by the f_j. In [8] we had proved that in case $n = 2$, the discreteness of V was enough to ensure that the pair f_1, f_2 is slowly decreasing. This led to the conjecture in [9] that if the coefficients of the f_j are algebraic numbers, the discreteness of V should be enough to prove that $f_1, \ldots, f_n$ is slowly decreasing or, at least, that I is closed and localizable. Examples were given showing that this last statement could fail if the algebraicity of the coefficients was not true. On the other hand, we show in this section that if $f_1, \ldots, f_p \in E_{n,1}(\overline{\mathbf{C}})$ define a complete intersection variety, that is $dimV \leq n - p$, then I is closed and, moreover, $I = I_{loc}$. In the case V is not a complete intersection we show that the local algebraic closure $\hat{I}$ and the radical $\sqrt{I}$ are closed. That is, these theorems are valid without any restrictions on the coefficients, whereas to extend them to exponential polynomials with two main frequencies one needs to impose arithmetic restrictions both on the frequencies and the coefficients.

The section ends with some representation theorems for the solutions of systems of difference-differential equations corresponding to exponential polynomials $f_1, \ldots, f_p \in E_{n,1}(\overline{\mathbf{C}})$, which define a complete intersection, as an illustration of the applications of the previous results to harmonic analysis.

Theorem 3.1 *Let $f_1, \ldots, f_p \in E_{n,1}(\mathbf{C})$ define a complete intersection variety V. The ideal I generated by them in $A_\rho(\mathbf{C}^n), \rho(z) = \log(1+|z|) + |\Re z_1|$ is localizable.*

Proof. The first thing to do is to replace $f_1, \ldots, f_p$ by some linear combinations of them, $g_1, \ldots, g_p$, that have the additional property that for any sequence of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq p$,

$$dim\{z \in \mathbf{C}^n : g_{i_1}(z) = \cdots = g_{i_k}(z) = 0\} \leq n - k.$$

We say that the sequence $g_1, \ldots, g_p$ is a *normal sequence.* The existence of such a normal sequence is guaranteed by the following lemma.

Lemma 3.1 *Given any collection of entire functions* $f_1, \dots, f_p$ *such that*

$$dim\{z \in \mathbf{C}^n : \ f_1(z) = \cdots = f_p(z) = 0\} \leq n - p.$$

There exist $\zeta_{ij} \in \mathbf{C}$ *such that the functions defined by*

$$g_i := \sum_{j=1}^{p} \zeta_{ij} f_j \ (1 \leq i \leq p)$$

form a normal sequence. Moreover $det(\zeta_{ij}) \neq 0$.

Proof of Lemma 3.1. Let $g_1 = f_1$ and $V_{1;i}$ denote the irreducible components of $V(g_1) = \{z \in \mathbf{C}^n : \ g_1(z) = 0\}$. Pick a regular point $z'_{1;i}$ in each $V_{1;i}$. Since $dimV \leq n-p$ and we can assume $p \geq 2$, for each $z'_{1;i}$ there is a nearby regular point $z_{1;i} \in V_{1;i}$ and some $2 \leq k \leq p$ such that $f_k(z_{1;i}) \neq 0$. Consider now the system of linear equations

$$\sum_{k=2}^{p} c_k f_k(z_{1;i}) = 0.$$

Since the number of equations is countable, the Baire category theorem ensures there is a complex vector $(c_2, \dots, c_p)$ such that $g_2 := c_2 f_2 + \cdots + c_p f_p$ does not vanish at any of the points $z_{1;i}$. It is clear that the two vectors $\zeta_1 = (1, 0, \dots, 0)$ and $\zeta_2 = (0, c_2, \dots, c_p)$ are linearly independent. We claim that $dimV(g_1, g_2) \leq n-2$. If not, g_2 would be identically zero on a component of $V(g_1)$, which is impossible.

Assume now that $p \geq 3$. By the previous reasoning we can choose regular points $z'_{2;j} \in V(g_2)$ (resp., $z'_{1,2;h} \in V(g_1, g_2)$), one for each component, such that for some index $1 \leq k \leq p$, $f_k(z'_{2;j}) \neq 0$ (resp. $f_k(z'_{1,2;h}) \neq 0$). The index k clearly depends on the point. Let us denote now $\{z_{2;l}\}_l$ the collection of all the points $z_{1;i}, z_{2;j}, z_{1,2;h}$. Then we consider the countable family of linear equations in $\mathbf{C}^p$

$$\sum_{k=1}^{p} \zeta_{3,k} f_k(z_{2;l}) = 0,$$

augmented by the linear equation in $\zeta_3 = (\zeta_{3,1}, \dots, \zeta_{3,p})$

$$rank[\zeta_1, \zeta_2, \zeta_3] = 2.$$

The earlier considerations imply the existence of a point ζ_3 not satisfying any of the equations. We define $g_3 := \sum_{k=1}^{p} \zeta_{3,k} f_k$ for this choice. It is clear now that also $dimV(g_1, g_3) \leq n-2$, $dimV(g_2, g_3) \leq n-2$, and $dimV(g_1, g_2, g_3) \leq n-3$. If $p > 3$ it is easy to continue this process. This way we obtain a normal sequence with the desired properties.

□

Let us return to the proof of Theorem 3.1. We assume henceforth that $f_1, \ldots, f_p$ is a normal sequence. We recall from the proof of Proposition 2.1, applied to the function $f := f_1^{m_1} \cdots f_p^{m_p}$, $m_j \in \mathbf{N}$, the existence of polynomials $\mathcal{R}_{1,m}(x_1)$ such that the coefficients $a_{m;1,j}$, $-2n \leq j \leq 0$, of the Laurent development of $|f|^{2\lambda}$ at $\lambda = -1$, have the property that the distributions $\mathcal{R}_{1,m}(x_1) a_{m;1,j}$ are linear combinations of distributions of the form

$$\varphi \mapsto \int_{\mathbf{C}^n} f(f/\overline{f})(\log|f|^2)^l Q_\kappa(\varphi) dx, \ (\varphi \in \mathcal{D}(\mathbf{C}^n)) \tag{44}$$

where $l \in \mathbf{N}$ and Q_κ are differential operators in $\frac{\partial}{\partial x}$ with coefficients that are polynomials in x, e^{x_1}, e^{-x_1}. (See equation (28), note that $k = 1$ in this case.)

For simplicity, we define $\mathcal{R}$ to be the product of $\mathcal{R}_{1,m}$ for all the choices of indices m with length $|m| \leq p$. This choice allows us to control all the coefficients $a_{m;1,j}$ simultaneously.

Let $\alpha_1, \ldots, \alpha_k$ be the distinct roots of the polynomial $\mathcal{R}(x_1)$ and $\nu_1, \ldots, \nu_k$ their respective multiplicities. Fix one such root α_l. Then each function f_j can be considered as a power series in $x_1 - \alpha_l$, with coefficients that are polynomials in $x' = (x_2, \ldots, x_n)$. It is clear that when we truncate this series at the term $(x_1 - \alpha_l)^{\nu_l}$, we obtain a polynomial $P_{j,l}$. Moreover, if a function is locally in the ideal generated by $f_1, \ldots, f_p, (x_1 - \alpha_l)^{\nu_l}$, then, it is also locally in the ideal generated by $P_{1,l}, \ldots, P_{p,l}, (x_1 - \alpha_l)^{\nu_l}$. Let $F \in A_\rho(\mathbf{C}^n)$ belong to I_{loc}, then, for each l it is locally in the ideal generated by $P_{1,l}, \ldots, P_{p,l}, (x_1 - \alpha_l)^{\nu_l}$. We can apply Ehrenpreis' Fundamental Principle to obtain a representation

$$F = \sum_{j=1}^{p} G_{j,l} P_{j,l} + (x_1 - \alpha_l)^{\nu_l} G_{p+1,l},$$

with functions $G_{j,l} \in A_\rho(\mathbf{C}^n)$ (cf. [21, 18, 26]). If we write

$$f_j = P_{j,l} + (x_1 - \alpha_l)^{\nu_l} Q_{j,l},$$

then $Q_{j,l} \in A_\rho(\mathbf{C}^n)$ and F can be expressed as

$$F = \sum_{j=1}^{p} G_{j,l} f_j + (x_1 - \alpha_l)^{\nu_l} \tilde{G}_{p+1,l}, \tag{45}$$

where $\tilde{G}_{p+1,l} := G_{p+1,l} - \sum_{j=1}^{p} Q_{j,l} P_{j,l}$, so that this function also belongs to $A_\rho(\mathbf{C}^n)$.

We claim that there are functions $G_j \in A_\rho(\mathbf{C}^n)$ such that

$$F = \sum_{j=1}^{p} G_j f_j + \mathcal{R}(x_1) G_{p+1}. \tag{46}$$

In fact, for x' fixed, we apply the Lagrange interpolation formula to the points $\alpha_1, \ldots, \alpha_k$, with multiplicities $\nu_1, \ldots, \nu_k$, so that we construct functions $G_j(x)$ with the property that for each l

$$G_j(x_1, x') - G_{j,l}(x_1, x') = O((x_1 - \alpha_l)^{\nu_l}). \tag{47}$$

The Lagrange interpolation formula guarantees that $G_j \in A_\rho(\mathbf{C}^n)$, and (45),(47) imply that

$$\begin{aligned} F - \sum_{j=1}^{p} G_j f_j &= F - \textstyle\sum_{j=1}^{p} G_{j,l} f_j + \sum_{j=1}^{p} (G_{j,l} - G_j) f_j \\ &= O((x_1 - \alpha_l)^{\nu_l}). \end{aligned}$$

Hence, $F - \sum_{j=1}^{p} G_j f_j$ is divisible by the polynomial $\mathcal{R}$, and the entire function G_{p+1} defined by (46) also belongs to $A_\rho(\mathbf{C}^n)$ by the Pólya-Ehrenpreis-Malgrange division lemma [21, 27].

Note that the remainder term in (46), namely $H := \mathcal{R}(x_1) G_{p+1} \in I_{loc}$, since $F \in I_{loc}$ and $\sum_{j=1}^{p} G_j f_j \in I$. The idea of the rest of the proof of Theorem 3.1 is to show that, thanks to the fact that H is also divisible by $\mathcal{R}$, we have $H \in I$, using the explicit division formulas considered in [3].

Let us recall the construction from [3, 14], except that here we will need three weights as in [14]. Let $\mathcal{N}$ be a sufficiently large integer and $\kappa \gg 1$ (both shall be chosen below.) Let $\theta \in C_0^\infty(\mathbf{R}^{2n})$, non-negative, radial, $\theta(x) = 0$ for $|x| > 1$, $\int \theta \, dx = 1$. The weights we consider are constructed starting with an auxiliary entire function $\Gamma(t)$ of a single variable, $\Gamma(1) = 1$, and a smooth

(1, 0)-differential form Q in $\mathbf{C}^{2n}$. In fact, we take three such pairs, the first one depends on λ and it is

$$
\begin{aligned}
Q_1(x,\xi,\lambda) &:= \frac{1}{p}\sum_{j=1}^{p}|f_j(\xi)|^{2\lambda}\frac{g_j(x,\xi)}{f_j(\xi)},\ \Re\lambda \gg 1.\\
\Gamma_1(t) &:= \frac{1}{p!}\prod_{j=0}^{p-1}(pt-j).
\end{aligned}
\tag{48}
$$

The $g_j(x,\xi)$ are differential forms given by

$$
g_j(x,\xi) = \sum_{k=1}^{n} g_{jk}(x,\xi)\, d\xi_k,
$$

where the entire functions $g_{jk} \in A_{\rho\otimes\rho}(\mathbf{C}^{2n})$, with $(\rho\otimes\rho)(x,\xi) = \rho(x)+\rho(\xi)$, and satisfy the identities

$$
\sum_{k=1}^{n}(x_k-\xi_k)g_{jk}(x,\xi) = f_j(x) - f_j(\xi)\ \ (1\le j\le p).
$$

The existence of such functions is well-known [27, 6]. The second pair is given by

$$
\begin{aligned}
Q_2(x,\xi) &:= \partial\log(1+|\xi|)^2\\
\Gamma_2(t) &:= t^{\mathcal{N}}.
\end{aligned}
\tag{49}
$$

Finally,

$$
\begin{aligned}
Q_3(x,\xi) &:= \kappa\partial(|\Re\xi_1| * \theta)\\
\Gamma_3(t) &:= \exp(t-1).
\end{aligned}
\tag{50}
$$

To every pair we associate a function

$$
\Phi_j(x,\xi) := 1+ < Q_j, x-\xi > := 1 + \sum_{k=1}^{n} Q_{jk}(x,\xi)(x_k-\xi_k),
$$

where $Q_j(x,\xi) := \sum_{k=1}^{n} Q_{jk}(x,\xi)d\xi_k$. A simple computation shows that

$$
\begin{aligned}
\Phi_1(x,\xi,\lambda) &= \frac{1}{p}\sum_{j=1}^{p}|f_j(\xi)|^{2(\lambda-1)}\overline{f_j}(\xi)f_j(x) + \frac{1}{p}\sum_{j=1}^{p}(1-|f_j(\xi)|)^{2\lambda}\\
\Phi_2(x,\xi) &= \frac{1+x\cdot\xi}{1+|\xi|^2}\\
\Phi_3(x,\xi) &= \kappa(|\Re\xi_1| * \partial_\xi\theta)\cdot(x_1-\xi_1)+1.
\end{aligned}
\tag{51}
$$

Here $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$. We also need a few extra auxiliary functions

$$\Gamma_j^{(\alpha)} = \Gamma_j^{(\alpha)}(x,\xi) := \frac{d^\alpha}{dt^\alpha}\Gamma_j(t)|_{t=\Phi_j(x,\xi)},\ \alpha \in \mathbf{N}.$$

$$\Gamma^{(\alpha)} := \Gamma_1^{(\alpha_1)}\Gamma_2^{(\alpha_2)}\Gamma_3^{(\alpha_3)},\ \alpha_j \in \mathbf{N}, \alpha = (\alpha_1, \alpha_2, \alpha_3).$$

Following Henkin's ideas [11, 5, 25, 16], we can represent an arbitrary function u in $C_0^\infty(\mathbf{C}^n)$ by the formula

$$u(x) = \frac{1}{(2\pi i)^n}\int_{\mathbf{C}^n} u(\xi)P(x,\xi) - \int_{\mathbf{C}^n} \overline{\partial}u(\xi) \wedge K(x,\xi), \tag{52}$$

where

$$\begin{aligned}
P(x,\xi) = P(x,\xi,\lambda) &:= \sum_{|\alpha|=n} \frac{1}{\alpha!}\Gamma^{(\alpha)}(\overline{\partial}Q)^\alpha, \\
(\overline{\partial}Q)^\alpha &:= (\overline{\partial}_\xi Q_1(x,\xi))^{\alpha_1} \wedge (\overline{\partial}_\xi Q_2(x,\xi))^{\alpha_2} \wedge (\overline{\partial}_\xi Q_3(x,\xi))^{\alpha_3}, \\
K(x,\xi) = K(x,\xi,\lambda) &:= \sum_{\alpha_0+|\alpha|=n-1} \frac{1}{\alpha!}\Gamma^{(\alpha)}\frac{S \wedge (\overline{\partial}S)^{\alpha_0} \wedge (\overline{\partial}Q)^\alpha}{|x-\xi|^{2\alpha_0+2}}, \\
S = S(x,\xi) &:= \sum_{j=1}^n (\overline{x_j} - \overline{\xi_j})d\xi_j, \\
\overline{\partial}S = \overline{\partial_\xi}S &= \sum_{j=1}^n d\xi_j \wedge d\overline{\xi_j}.
\end{aligned}$$

Let us now apply (52) to prove that $H \in I$. We choose a radial function $\chi \in C_0^\infty(\mathbf{C}^n)$, $\chi \equiv 1$ for $|\xi| \leq 1$, $\chi \equiv 0$ for $|\xi| \geq 2$, $0 \leq \chi \leq 1$. For a fixed $R > 1$, apply the representation formula (52) to the function $u(\xi) := \chi(\xi/R)H(\xi) = \chi(\xi/R)\mathcal{R}(\xi_1)G_{p+1}(\xi)$. Note that (52) is a priori defined only when the parameter λ satisfies $\Re\lambda \gg 1$, and we apply it to a fixed x, $|x| < R/2$. The two integrals in (52) admit an analytic continuation to the whole complex plane as meromorphic functions of λ. We are going to identify the zeroth coefficient of their Laurent development at $\lambda = 0$, which will provide a representation for $H(x)$.

Following the computations in [3, p.42-43], we can conclude that because $H \in I_{loc}$, the first integral in (52) represents an element of the ideal generated by I in $C_0^\infty(\mathbf{C}^n)$. (It is here that one uses the fact that $f_1, \ldots, f_p$ is a normal sequence, a point left implicit in [3].) More precisely, if we consider

$$\Gamma_1^{(\alpha_1)}(x,\xi,\lambda) - \frac{d^{\alpha_1}\Gamma_1}{dt^{\alpha_1}}(t = \frac{1}{p}\sum_{j=1}^{p}(1-|f_j(\xi)|^{2\lambda})) = \sum_{j=1}^{p} f_j(x)\gamma_{\alpha_1,j}(x,\xi,\lambda), \quad (53)$$

it is possible to show (cf. [3]) that to compute the zeroth coefficient of the first integral in (52) at $\lambda = 0$, we can replace everywhere in P, $\Gamma_1^{(\alpha_1)}$ by $\sum_{j=1}^{p} f_j(x)\gamma_{\alpha_1,j}(x,\xi,\lambda)$.

The other important terms where λ appears are $(\overline{\partial}Q_1)^{\alpha_1}, \alpha_1 \in \mathbf{N}$. We have

$$(\overline{\partial}Q_1)^{\alpha_1} = (\frac{\lambda}{p}\sum_{j=1}^{p}|f_j(\xi)|^{2(\lambda-1)}\overline{\partial f_j(\xi)} \wedge g_j(x,\xi))^{\alpha_1}, \quad (54)$$

which is a linear combination of terms of the form

$$\lambda^{\alpha_1}|f_{i_1}(\xi)\cdots f_{i_{\alpha_1}}(\xi)|^{2(\lambda-1)}\bigwedge_{l=1}^{\alpha_1}\overline{\partial f_{i_l}(\xi)} \wedge g_{i_l}(x,\xi). \quad (55)$$

A typical term in $\Gamma_1^{(\alpha_1)}$ is

$$\left\{\sum_{j=1}^{p}|f_j(\xi)|^{2(\lambda-1)}\overline{f_j(\xi)}f_j(x) + \sum_{j=1}^{p}(1-|f_j(\xi)|^{2\lambda})\right\}^q, \ 0 \le q \le p-\alpha_1. \quad (56)$$

So that

$$\int_{\mathbf{C}^n} u(\xi)\Gamma^{(\alpha)}(\overline{\partial}Q)^\alpha = \int_{\mathbf{C}^n} H(\xi)\chi(\xi/R)\Gamma_1^{(\alpha_1)}(\overline{\partial}Q_1)^{\alpha_1} \wedge \Psi_\alpha, \quad (57)$$

for some differential form Ψ_α, independent of λ.

Similarly, for the second integral in (52)

$$\int_{\mathbf{C}^n} \overline{\partial}u(\xi) \wedge \Gamma^{(\alpha)}\frac{S \wedge (\overline{\partial}S)^\alpha \wedge (\overline{\partial}Q)^\alpha}{|x-\xi|^{2\alpha_0+2}}$$
$$= \frac{1}{R}\int_{\mathbf{C}^n} H(\xi)(\overline{\partial}\chi)(\frac{\xi}{R}) \wedge \Gamma_1^{(\alpha_1)}(\overline{\partial}Q_1)^{\alpha_1} \wedge \Theta_\alpha, \quad (58)$$

Θ_α a form independent of λ, smooth on $supp((\overline{\partial}\chi)(\xi/R))$. Finally, (57) and (58) are both linear combinations of expressions of the following type

$$\frac{\lambda^{\alpha_1}}{R^i}\int_{\mathbf{C}^n}|f_1^{m_1}\cdots f_p^{m_p}|^{2(\lambda-1)}|f_1^{n_1}\cdots f_p^{n_p}|^2 H(\xi)\chi^{(i)}(\xi/R)\Omega(x,\xi), \tag{59}$$

where $m_j\in\mathbf{N}, \sum_{j=1}^p m_j \le p, n_j\in\mathbf{N}, \chi^{(i)}$, the ith derivative of χ, $i=0,1$, Ω is a form of degree (n,n), smooth on the support of $\chi^{(i)}(\xi/R)$. The form Ω involves the coefficients of the second and third pairs. It is this formula (59) that will eventually allow us to let $R\to\infty$.

Let us recall that $H(\xi)=\mathcal{R}(\xi_1)G_{p+1}(\xi)$, $G_{p+1}\in A_\rho(\mathbf{C}^n)$. Each expression of the form (59), when analytically continued to $\lambda=0$, contributes one term to the zeroth term of the Laurent expansion of (52), namely that corresponding to the coefficient $a_{m;1,-\alpha_1}$ of the Laurent expansion of $|f_1^{m_1}\cdots f_p^{m_p}|^{2\lambda}$ at $\lambda=-1$. Therefore, the contribution of (59) is given by terms of the form

$$< \mathcal{R}(\xi_1)a_{m;1,-\alpha_1}, \frac{1}{R^i}G_{p+1}|f_1^{n_1}\cdots f_p^{n_p}|^2\chi^{(i)}(\xi/R)\omega(x,\xi) >, \tag{60}$$

where $\omega(x,\xi)$ is one of the coefficients of $\Omega(x,\xi)$. We know from (44) how the distributions $\mathcal{R}(\xi_1)a_{m;1,-\alpha_1}$ act on test functions, which shows that their limit exist when $R\to\infty$ and, in fact, are zero for $i=1$ (i.e., the terms corresponding to the kernel K), while that for $i=0$ (i.e., those corresponding to the kernel P) they are entire functions of x, with the correct growth conditions, that is, they belong to $A_\rho(\mathbf{C}^n)$. All these estimates are achieved thanks to the previous choices of $Q_2,\Gamma_2,Q_3,\Gamma_3$ for sufficiently large constants $\mathcal{N},\kappa$. (We dot really need to use the exact form (44) of the distributions $\mathcal{R}(\xi_1)a_{m;1,-\alpha_1}$, it is enough to apply the estimates of the Proposition 2.1.) This is similar to what we have done elsewhere, [3], in the algebraic case, and [14], in the analytic case. In other words, we have shown that $H\in I$.

□

Let us consider now the case where we do not assume the ideal is either complete intersection or its variety is discrete. We shall study several ideals containing $I=I(f_1,\dots,f_p)$. First, let us recall that $\sqrt{I}$, the radical of I, is the set of all elements $F\in A_\rho(\mathbf{C}^n)$ such that $F^k\in I$ for some $k\in\mathbf{N}$. Second, let $\hat{I}$, the local integral closure of I, be the set of all elements $F\in A_\rho(\mathbf{C}^n)$ such that for every point $x_0\in\mathbf{C}^n$ there is a neighborhood U and a constant

$C_{x_0} > 0$ such that

$$|F(x)| \leq C_{x_0}\|f(x)\| = C_{x_0}(\sum_{j=1}^{p} |f_j(x)|^2)^{1/2}, \ \forall x \in U.$$

For W open in $\mathbf{C}^n$, let I_W denote the ideal generated by $f_1, \ldots, f_p$ in $\mathcal{H}(W)$. It follows from [28] that $F \in \hat{I}$ if and only if for every $x_0 \in \mathbf{C}^n$ there is an open neighborhood W, a positive integer N, and functions $\varphi_1, \ldots, \varphi_N$ such that

$$F^N + \varphi_1 F^{N-1} + \cdots + \varphi_N = 0 \text{ in } W, \text{ and } \varphi_j \in I_W^j.$$

Finally, let $I(V) = \{F \in A_\rho(\mathbf{C}^n) : \ F \mid V = 0\}$. Note that for a function F to belong to I_{loc} means that it vanishes on the points of the variety V with some multiplicity, whereas in $I(V)$ the common multiplicities of $f_1, \ldots, f_p$ are disregarded. It is obvious that $I(V)$ is a closed ideal, and we recall that the same is true for I_{loc}. Some inclusions between these ideals are clear

$$I \subseteq I_{loc} \subseteq \hat{I} \subseteq I(V), \ \sqrt{I} \subseteq I(V).$$

It is also clear that, in general, we do not have $I_{loc} = I(V)$. We are now ready to state two important results.

Theorem 3.2 *Let I be the ideal in $A_\rho(\mathbf{C}^n)$ generated by $f_1, \ldots, f_p \in E_{n,1}(\mathbf{C})$, $V = \{x \in \mathbf{C}^n : \ f_1(x) = \cdots = f_p(x) = 0\}$. Then $\sqrt{I} = I(V)$.*

Theorem 3.3 *Let I be the ideal of the previous theorem and let m be given by $m = inf(p+1, n)$, then $\hat{I}^{2m} \subseteq I$.*

The crucial step in the proof of these two theorems is the following proposition. We state it in a slightly more general form that actually needed for future reference.

Proposition 3.1 *Let φ be a convex, non negative function in $\mathbf{C}^n$, satisfying the inequality*

$$\varphi(x) \leq K_0\varphi(y) + K_1 \quad \text{if } |x-y| \leq 1, \tag{61}$$

for some constants $K_0, K_1 > 0$. Let $\mathcal{A}$ be the space of entire functions given by

$$\mathcal{A} := \{g \in \mathcal{H}(\mathbf{C}^n) : \ \exists A > 0 \ \log|g(x)| \leq A(\log(2+|x|) + \varphi(x))\}.$$

Let $f_1, \ldots, f_p, \mathcal{R} \in \mathcal{A}$, $m = inf(p,n)$, *and assume there are* $t \in]0,\infty[^p$, $B > 0, N \in \mathbf{N}$ *such that the coefficients* a_j, $-2n \leq j \leq 0$, *of the Laurent expansion*

$$\frac{|f|^{2(\mu t - \underline{1})}}{\|f\|^{2m}} = \sum_{j=-2n}^{\infty} a_j \mu^j \tag{62}$$

satisfy for any $\psi \in \mathcal{D}(\mathbf{C}^n)$ *the estimates*

$$| < \mathcal{R} a_j, \psi > | \leq B \exp(B max\{\log(2+|x|) + \varphi(x) : \; x \in supp(\psi)\}) \|\psi\|_N. \tag{63}$$

Then

(i) *If* $F \in \mathcal{A}$ *and* $F(x) = 0$ *whenever* $f_1(x) = \cdots = f_p(x) = 0$, *then*

$$\mathcal{R} F^{N+1} \in f_1 \mathcal{A} + \cdots + f_p \mathcal{A}.$$

(ii) *If* $F \in \mathcal{A}$ *is such that every* $x_0 \in \mathbf{C}^n$ *has a neighborhood* U_{x_0} *in which*

$$|F(x)| \leq C_{x_0} \|f(x)\| \quad \forall x \in U_{x_0}$$

for some constant $C_{x_0} > 0$, *then*

$$\mathcal{R} F^m \in f_1 \mathcal{A} + \cdots + f_p \mathcal{A}.$$

Proof. The proof is based on the representation formula (52) with

$$u(\xi) := \chi(\xi/R) \mathcal{R}(\xi) F(\xi)^k, \tag{64}$$

for some $R > 0, k \in \mathbf{N}$, χ a plateau function as in the proof of Theorem 3.1. We need to make explicit the three pairs Q_j, Γ_j that appear in the kernels P and K. First, for $\Re \mu \gg 1$,

$$\begin{aligned} Q_1(x,\xi;\mu) \; &:= \frac{|f|^{2\mu t}}{\|f\|^2} \sum_{j=1}^{p} \overline{f_j} g_j(x,\xi) \\ \Gamma_1(s) \; &:= s^q, \;\; q = min(p, n+1), \end{aligned} \tag{65}$$

where we have left implicit the variable ξ of $\overline{f_j}$ in the definition of Q_1, as we shall do elsewhere. The differential forms g_j are defined exactly as in (48), for the present growth conditions.

As before, for some $\mathcal{N} \gg 1$ to be chosen later

$$Q_2(x, \xi) := \partial \log(1 + |\xi|^2), \ \Gamma_2(s) := s^{\mathcal{N}}. \tag{66}$$

Finally, for some $\kappa \gg 1$ and $\theta \in C^\infty$, non-negative and radial, $supp(\theta) \subseteq \{\xi : |\xi| \leq 1\}$, $\int \theta d\xi = 1$,

$$Q_3(x, \xi) := \kappa \partial(\varphi * \theta)(\xi), \ \Gamma_3(s) := e^{s-1}. \tag{67}$$

When the corresponding functions Φ_j are defined as before, the function Φ_2 is the same as in (51). The function Φ_3 is given by

$$\Phi(x, \xi) = \kappa \sum_{j=1}^{n} (\varphi * \frac{\partial \theta}{\partial \xi_j}(\xi)(x_j - \xi_j)) + 1.$$

We remark that the function $\varphi * \theta$ and all its partial derivatives of order α can be estimated by

$$|D^\alpha(\varphi * \theta)(\xi)| \leq C'_\alpha(\varphi(\xi) + 1) \leq C_\alpha e^{\varphi(\xi)}$$

and, since the function $\varphi * \theta$ is also convex,

$$\begin{aligned} |\exp \Phi(x, \xi)| &= e \exp\{\kappa \Re(\partial(\varphi * \theta) \cdot (x - \xi)\} \\ &\leq e \exp\{\tfrac{\kappa}{2}(\varphi * \theta(x) - \varphi * \theta(\xi))\}. \end{aligned}$$

On the other hand, not only

$$(\varphi * \theta)(x) \leq K_0 \varphi(x) + K_1,$$

by the hypothesis (61), but moreover,

$$K_0(\varphi*\theta)(\xi) = \int_{|\eta| \leq 1} K_0 \varphi(\xi - \eta)\theta(\eta) d\eta \leq \int_{|\eta| \leq 1} (\varphi(\xi) - K_1)\theta(\eta) d\eta \leq \varphi(\xi) - K_1.$$

It follows that, for some $C > 0$,

$$|\exp \Phi(x, \xi)| \leq C \exp\{\frac{\kappa}{2}(K_0 \varphi(x) - \frac{1}{K_0}\varphi(\xi))\}.$$

The function Φ_1, here, is really different from that in the proof of Theorem 3.1. Namely,

$$\Phi_1(x, \xi; \mu) = (1 - |f|^{2\mu t}) + |f|^{2\mu t} \sum_{j=1}^{p} \frac{\overline{f_j} f_j(x)}{\|f\|^2}. \tag{68}$$

Moreover, it will turn out to be important to make the expression of $(\overline{\partial} Q_1)^{\alpha_1}$ explicit. We have

$$\overline{\partial} Q_1(x,\xi;\mu) = |f|^{2\mu t}\left((\mu \sum_{j=1}^{p} t_j \frac{\overline{\partial f_j}}{\overline{f_j}}) \wedge (\frac{\sum_{i=1}^{p} \overline{f_i} g_i(x,\xi)}{\|f\|^2}) + \sum_{i=1}^{p} \overline{\partial}(\frac{\overline{f_i}}{\|f\|^2}) \wedge g_i(x,\xi) \right). \tag{69}$$

For $\alpha_1 \in \mathbf{N}$, $(\overline{\partial} Q_1)^{\alpha_1}$ is a linear combination of terms of the form

$$\mu^l |f|^{2\mu\alpha_1 t} \left(\frac{\overline{f_{i_1}}}{\|f\|^2} \cdots \frac{\overline{f_{i_l}}}{\|f\|^2} \right) \left(\frac{\overline{\partial f_{j_1}}}{\overline{f_{j_1}}} \wedge \ldots \wedge \frac{\overline{\partial f_{j_l}}}{\overline{f_{j_l}}} \right) \wedge \left(\overline{\partial}(\frac{\overline{f_{h_1}}}{\|f\|^2}) \wedge \ldots \wedge \overline{\partial}(\frac{\overline{f_{h_{\alpha_1 - l}}}}{\|f\|^2}) \right) \wedge \gamma(x,\xi), \tag{70}$$

where $0 \leq l \leq \alpha_1 \leq p, i_1 < \cdots < i_l, j_1 < \cdots < j_l, h_1 < \cdots < h_{\alpha_1 - l}$, and γ is an $(\alpha_1, 0)$-form with holomorphic coefficients, obtained from the wedge product of several g_i. It is clear that, for $\alpha_1 > m$, $(\overline{\partial} Q_1)^{\alpha_1} = 0$, since there are either too many $d\xi_i$ or too many g_i. For $\alpha_1 = p$, the expression of $(\overline{\partial} Q_1)^p$ is particularly simple, namely

$$(\overline{\partial} Q_1)^p = \mu p! (-1)^{p(p-1)/2} (t_1 + \cdots + t_p) \frac{|f|^{2p\mu t}}{\|f\|^{2p}} \overline{\partial f_1} \wedge \ldots \wedge \overline{\partial f_p} \wedge g_1(x,\xi) \wedge \ldots \wedge g_p(x,\xi). \tag{71}$$

In fact, from (69) we see that $\overline{\partial} Q_1$ has the form

$$\overline{\partial} Q_1 = |f|^{2\mu t}(A \wedge B + C),$$

where A and B are 1-forms and C is a 2-form. Since 2-forms commute for the wedge product,

$$(\overline{\partial} Q_1)^p = |f|^{2p\mu t} \sum_{j=0}^{p} \binom{p}{j} (A \wedge B)^j \wedge C^{p-j} = |f|^{2p\mu t}(C^p + p(A \wedge B) \wedge C^{p-1}),$$

since clearly $(A \wedge B)^j = 0$ for $j \geq 2$. In [11, p.61-62] we have shown that $C^p = 0$ (just set $\varepsilon = 0$ in the expression obtained there.) Hence,

$$(\overline{\partial} Q_1)^p = \mu p |f|^{2p\mu t} (\sum_j t_j \overline{\partial f_j}/\overline{f_j}) \wedge (\sum_i (\overline{f_i}/\|f\|^2) g_i) \wedge (\sum_k \overline{\partial}(\overline{f_k}/\|f\|^2) \wedge g_k)^{p-1},$$

which yields the identity (71) after an easy computation.

With these simplifications at hand, let us return to the analysis of the kernel P that appears in (52). The following computations are all made for $\Re\mu \gg 1$, modulo the ideal $\mathcal{I}$ generated by $f_1, \ldots, f_p$ in $C^\infty(\mathbf{C}^n)$. Every term in P contains some $\Gamma^{(\alpha_1)}$ as a factor, $0 \le \alpha_1 \le n$, then it contains $\Phi_1^{q-\alpha_1}$ when $\alpha_1 \le q$ (and vanishes when $\alpha_1 > q$), thus the terms that do not belong to $\mathcal{I}$ are of the form

$$(1 - |f|^{2\mu t})^{q-\alpha_1} (\bar\partial Q_1)^{\alpha_1} \wedge \vartheta_{\alpha_1},$$

$\vartheta_{\alpha_1} = \vartheta_{\alpha_1}(x, \xi)$ is a C^∞ form, which we do not make explicit for the time being. From (70) we conclude that, modulo $\mathcal{I}$, we need to consider the analytic continuation of integrals of the form

$$\mu^l \int_{\mathbf{C}^n} u(1 - |f|^{2\mu t})^{q-\alpha_1} |f|^{2\mu\alpha_1 t} h_I \frac{\overline{\partial f_J}}{\overline{f_J}} \wedge \bar\partial k_M \wedge \Theta(x, \xi), \tag{72}$$

where

$$h_I := \frac{\overline{f_{i_1}} \cdots \overline{f_{i_l}}}{\|f\|^{2l}},$$

$$\frac{\overline{\partial f_J}}{\overline{f_J}} := \frac{\overline{\partial f_{j_1}}}{\overline{f_{j_1}}} \wedge \cdots \wedge \frac{\overline{\partial f_{j_l}}}{\overline{f_{j_l}}},$$

$$\bar\partial k_M := \bar\partial(\overline{f_{m_1}}/\|f\|^2) \wedge \cdots \wedge \bar\partial(\overline{f_{m_{\alpha_1 - l}}}/\|f\|^2),$$

Θ is a C^∞ form, and u is given by (64), so that it has compact support.

Let us distinguish two cases, $p \le n$ and $p > n$. If $p > n$, then $q = n + 1$, and $q - \alpha_1 > 0$ always. If $p \le n$, $(\bar\partial Q_1)^{\alpha_1} = 0$ once $\alpha_1 > p$. On the other hand, when $\alpha_1 = p$, (71) shows that the only possible non-trivial value for l is $l = 1$. Hence, in every case, either $l > 0$ or $q - \alpha_1 > 0$ in (72).

We are now going to consider the case $l = 0$ in (72). Recall that we are only interested in the zeroth term in the Laurent development of the analytic continuation of (72) at $\mu = 0$. As a function of μ, (72) can be written as

$$\sum_{j=0}^{q-\alpha_1} \binom{q-\alpha_1}{j} (-1)^j \int_{\mathbf{C}^n} |f|^{2\mu(\alpha_1 + j)t} u h_I \frac{\overline{\partial f_J}}{\overline{f_J}} \wedge \bar\partial k_M \wedge \Theta(x, \xi), \tag{73}$$

where we have absorbed all other terms into Θ. Even though the powers $|f|^{2\mu(\alpha_1+j)t}$ are different, their contributions to the zeroth term at $\mu = 0$ coincide. (This is evident by considering the variable $\lambda = \mu(\alpha_1 + j)$.) Therefore, the total contribution of (73) is zero.

Consider now the case $l > 0$. As in [3, Proposition 2.3] we can use Hironaka's resolution of singularities to study the current defined by the zeroth term of the analytic continuation at $\mu = 0$ acting on test forms Θ,

$$\mu^l \int_{\mathbf{C}^n} \mathcal{R}|f|^{2\mu\beta t} h_I \frac{\overline{\partial f_J}}{\overline{f_J}} \wedge \overline{\partial} k_M \wedge \Theta, \tag{74}$$

where $\beta = \alpha_1 + j$, for some j, and we have absorbed $\chi(\xi/R)F(\xi)^k$ into Θ. The first thing to observe is that these currents are supported by the variety V of common zeros of $f_1, \ldots, f_p$. Moreover, we shall show that these currents are also annihilated by multiplication by the functions $\overline{f_j}$, as well as multiplication by $f_1^{n_1} \cdots f_p^{n_p}$, whenever $n_1 + \cdots + n_p$ exceeds the order of the current. The order of these currents will be estimated using hypothesis (63). Recall that, after using a partition of unity and resolving the singularities as in [3, p.33-34], we can reduce ourselves to the case where all the f_i are invertible holomorphic functions multiplied by monomials m_j, all the m_j are multiples are m_1, that is, $\pi^* f_i(w) = u_i(w) m_i(w) = u_i(w) m_i'(w) m_1(w)$, $m_1' \equiv 1$, where π is the blowdown of the desingularized variety. Hence,

$$\begin{aligned}
\pi^* k_i &= \pi^*(\overline{f_i}/\|f\|^2) = \frac{\overline{u_i m_i}}{\sum_{j=1}^p |u_j m_j|^2} = \frac{\overline{u_i m_i'} \overline{m_1}}{|m_1|^2 \sum_{j=1}^p |u_j m_j'|^2} \\
&= \frac{\overline{u_i m_i'}}{m_1(|u_1|^2 + \sum_{j=2}^p |u_j m_j'|^2)} = \frac{v_i}{m_1},
\end{aligned}$$

with $v_i \in C^\infty$. Thus,

$$\pi^*(\overline{\partial} k_M) = \overline{\partial} \pi^*(k_M) = \frac{\omega_M}{m_1^{\alpha_1 - l}},$$

for some smooth ω_M. Similarly,

$$\pi^*(h_I) = \frac{v_I}{m_1^l}.$$

Finally,

$$\begin{aligned}
\pi^*\left(\frac{\overline{\partial f_J}}{\overline{f_J}}\right) &= \left(\frac{\overline{\partial u_{j_1}}}{\overline{u_{j_1}}} + \frac{\overline{\partial m_{j_1}}}{\overline{m_{j_1}}}\right) \wedge \cdots \wedge \left(\frac{\overline{\partial u_{j_l}}}{\overline{u_{j_l}}} + \frac{\overline{\partial m_{j_l}}}{\overline{m_{j_l}}}\right) \\
&= \sum_{|\delta| \le l} \Psi_\delta \wedge \frac{d\overline{w_\delta}}{\overline{w_\delta}},
\end{aligned}$$

where, as above, $w_1, \ldots, w_n$ are local coordinates in the desingularized variety, Ψ_δ are smooth forms, $\delta := (\delta_1, \ldots, \delta_r), 1 \leq \delta_1 < \cdots < \delta_r \leq n, |\delta| := r \leq l$, and $\frac{d\overline{w_\delta}}{\overline{w_\delta}} := \frac{d\overline{w_{\delta_1}}}{\overline{w_{\delta_1}}} \wedge \cdots \wedge \frac{d\overline{w_{\delta_r}}}{\overline{w_{\delta_r}}}$. Hence, in the coordinates w the integral in (74) is a linear combination of

$$\mu^l \int |u|^{2\mu\beta t}|m|^{2\mu\beta t} \frac{1}{m_1^{\alpha_1}} \frac{d\overline{w_\delta}}{\overline{w_\delta}} \wedge \eta_\delta \wedge \pi^*(\mathcal{R}\Theta), \tag{75}$$

for some smooth form η_δ. Recall we are assuming that $l > 0$ and note that when $\alpha_1 = 0$ the integrand in (75) is integrable up to $\mu = 0$, thus it contributes nothing to the zeroth term of the Laurent development. We can therefore assume that $\alpha_1 > 0$ in what follows.

Let us assume now that Θ is a smooth multiple of some $\overline{f_j}$, then $\eta_\delta \wedge \pi^*(\mathcal{R}\Theta) = \overline{m_1}\Theta'$, Θ' a smooth form in the w-coordinates (it depends on x also, but that is irrelevant at this moment.) In this case we can integrate by parts (75) and obtain

$$\mu^l \int |u|^{2\mu\beta t}|m|^{2\mu\beta t} \frac{\overline{m_1}}{m_1^{\alpha_1}} \frac{d\overline{w_\delta}}{\overline{w_\delta}} \wedge \Theta'$$

$$= \frac{\mu^l}{P(\mu)} \int |m|^{2\mu\beta t} \frac{\overline{m_1}}{m_1} \frac{d\overline{w_\delta}}{\overline{w_\delta}} \wedge \Theta''.$$

Here $\Theta'' = \Theta''(x, w, \mu)$ such that $\mu \mapsto \Theta''$ is holomorphic at $\mu = 0$ and P is a polynomial which does not vanish at $\mu = 0$ (cf. [3, eqns.(1.20)-(1.22)] for the details.) It is now clear that the integrand is integrable for $\mu = 0$, so that this term cannot contribute to the zeroth term of the Laurent expansion. This is equivalent to say that the currents we are computing are annhilated by $\overline{f_1}C^\infty + \cdots + \overline{f_p}C^\infty$, which implies that their support lies in $\bigcap_j \{\overline{f_j} = 0\} = V$.

Remark that if F satisfies the local estimates in part (ii) of the statement of this proposition and $k \geq m = min(p, n) \geq \alpha_1$, then $(\pi^* F)^k / m_1^{\alpha_1}$ is bounded, so that again the integral (75) will not contribute to the currents we are looking for, because everything is integrable up to $\mu = 0$. We will use this remark in the proof of part (ii) of the proposition.

We are now ready to conclude the proof of statement (i) in the proposition. We observe that the integrand in (74) is a linear combination of terms of the form

$$\mathcal{R} \frac{|f|^{2(\mu\beta t - 1)}}{\|f\|^{2m}} \Theta''',$$

where Θ''' is smooth and has absorbed some factors $|f_j|^2$ and $\|f\|^2$. This expression is obtained by using the definitions of h_I, f_J, k_M. The hypothesis (63) of the proposition ensures that the orders of the currents that appear in (74) are at most N (plus giving some precision in the estimates in terms of x.) As these currents are supported by V, it follows that if the power k of F is $N+1$ or larger, then these integrals do not contribute to the zeroth Laurent coefficient at $\mu = 0$.

We can summarize these statements in the observation that, for $|x| \leq R/2$, we need only to consider the zeroth term of the Laurent expansion of (52) at $\mu = 0$ and obtain

$$\begin{aligned} \mathcal{R}(x)F(x)^{N+1} &= \textstyle\sum_{j=1}^{p} f_j(x) < T_j(x,\xi), \mathcal{R}(\xi)F(\xi)^{N+1}\chi(\xi/R) > \\ &+ \frac{1}{(2\pi i)^n R}\int_{\mathbf{C}^n} \mathcal{R}(\xi)F(\xi)^{N+1}(\overline{\partial}\chi)(\xi/R) \wedge K(x,\xi;\lambda=0) \end{aligned} \tag{76}$$

where the distributions T_j are holomorphic in the variable x and the kernel $K(x,\xi;\lambda=0)$ involves the distributions $a_j, -2n \leq j \leq 0$, from (62), so that, as we did in the proof of Theorem 3.1, we can choose the constants $\mathcal{N}, \kappa$ in (66) and (67) to ensure that all the limits exist when $R \to \infty$ and that the last term of (76) vanishes for $R = \infty$. It follows that the coefficients of the $f_j(x)$ in (76) belong to the space $\mathcal{A}$. This concludes the proof of part (i) of the Proposition 3.5.

Because of the earlier remark, the same representation (76) is valid for $\mathcal{R}(x)F(x)^m, m = inf(p,n)$, and the conclusion of part (ii) follows. This ends the proof of the proposition.

□

Remarks

1. We have pointed out, in the proof of Proposition 3.1, the remarkable fact that the currents involved in the remainder terms of the division formulas we have used, are annihilated by the conjugates $\overline{f_j}$ of the generators of the ideal. In the case of a complete intersection, there is only one remainder term, given by the residue current, and hence the remainder is also annhilated by the generators f_j. The fact that we do not know that the remainder terms are killed by the f_j in the case of not complete intersection, is what prevents us from obtaining holomorphic division theorems.

2. In the algebraic case, that is, all the f_j are polynomials, we know from

the Bernstein-Sato functional equations that hypothesis (63) is valid taking $\mathcal{R} \equiv 1$ and a convenient choice of t.

Proof of Theorem 3.2. Let $F \in I(V)$, we need to show the existence of $k \in \mathbf{N}$ such that $F^k \in I$. We follow the lines of the proof of Theorem 3.1.

¿From Proposition 2.2 of the previous section we conclude there are $t \in]0, \infty[^p$ and a polynomial $\mathcal{R}(x_1)$ so that the Laurent coefficients $a_j := a_{1,j}$, $-2n \leq j \leq 0$, of the expansion

$$\frac{|f|^{2(\mu t - \underline{1})}}{\|f\|^{2n}} = \sum_{j=0}^{\infty} a_j \mu^j \tag{77}$$

have the property

$$| < \mathcal{R}(x_1) a_j, \varphi > | \leq C \exp(Dmax\{\rho(x) : \, x \in supp(\varphi)\}) \|f\|_{N_0} \tag{78}$$

for some positive constants C, D, N_0, and any $\varphi \in \mathcal{D}(\mathbf{C}^n)$.

Let $\alpha_1, \ldots, \alpha_k$ be the zeros of $\mathcal{R}$, with respective multiplicities $\nu_1, \ldots, \nu_k$. Consider I_l, the ideal generated by $f_1, \ldots, f_p, (x_1 - \alpha_l)^{\nu_l}$. This ideal is generated by polynomials $P_{1,l}, \ldots, P_{p,l}, (x_1 - \alpha_l)^{\nu_l}$, as observed in the proof of Theorem 3.1. Since F vanishes on the set V of common zeros of I, it also vanishes on the set V_l of common zeros of I_l. We can therefore apply Proposition 3.1, and obtain a decomposition

$$F^{N_l} = \sum_{j=1}^{p} G_{j,l} P_{j,l} + (x_1 - \alpha_l)^{\nu_l} G_{p+1,l} \tag{79}$$

for some $N_l \in \mathbf{N}, G_{j,l} \in A_\rho(\mathbf{C}^n)$. (There is no factor in front of F^{N_l} since we are considering a polynomial ideal.) Let $N = max(N_l : \, 1 \leq l \leq k)$, then, as done in the proof of Theorem 3.1, we conclude there are functions $G_1, \ldots, G_{p+1} \in A_\rho(\mathbf{C}^n)$ such that

$$F^N = \sum_{j=1}^{p} G_j f_j + G_{p+1} \mathcal{R}. \tag{80}$$

We can apply again Proposition 3.1 to F, this time with $f_1, \ldots, f_p$ as generators and $\mathcal{R} = \mathcal{R}(x_1)$ as in (78), to obtain

$$\mathcal{R} F^{N_0+1} \in I,$$

so that

$$F^{N+N_0+1} \in I,$$

which concludes the proof of Theorem 3.2.

□

Proof of Theorem 3.3. Let $F \in \hat{I}$. We follow the proof of the previous theorem and introduce a polynomial $\mathcal{R}$ as in (78), polynomials $P_{1,l}, \ldots, P_{p,l}$, associated to $f_1, \ldots, f_p$ and a root α_l of $\mathcal{R}$. For any $x_0 \in \mathbf{C}^n$ we have

$$|F(x)| \leq C_{x_0} \|f(x)\| \quad \text{for } x \in U_{x_0},$$

where U_{x_0} is a neighborhood of x_0, which we can assume is bounded, and $C_{x_0} > 0$. Hence, for $x \in U_{x_0}$,

$$|f_j(x)| \leq |P_{j,l}(x)| + C'_{x_0}|(x_1 - \alpha_l)^{\nu_l}|,$$

for some constant $C'_{x_0} > 0$. It follows that for another constant $C''_{x_0} > 0$,

$$|F(x)| \leq C''_{x_0}(\sum_{j=1}^{p} |P_{j,l}(x)|^2 + |(x_1 - \alpha_l)^{\nu_l}|^2)^{1/2}$$

We can apply now Proposition 3.1 to the polynomials $P_{1,l}, \ldots, P_{p,l}, (x_1 - \alpha_l)^{\nu_l}$, and conclude that

$$F^m = \sum_{j=1}^{p} G_{j,l} P_{j,l} + (x_1 - \alpha_l)^{\nu_l} G_{p+1,l},$$

for some $G_{j,l} \in A_\rho(\mathbf{C}^n)$, $m = min(p + 1, n)$. As earlier, we conclude that

$$F^m = \sum_{j=1}^{p} G_j f_j + G_{p+1}\mathcal{R},$$

$G_j \in A_\rho(\mathbf{C}^n)$. Let $m' = min(p, n)$, once more Proposition 3.1 ensures that

$$\mathcal{R}F^{m'} \in I.$$

Hence $F^{m+m'} \in I$. Since $m + m' \leq 2m$, the theorem has been proved.

□

As a corollary of the last two proofs we can obtain a theorem about representation of entire functions in $A_\rho(\mathbf{C}^n)$, modulo an ideal I, which defines a zero-dimensional, complete intersection variety.

Proposition 3.2 *Let* $f_1, \dots, f_n \in E_{n,1}(\overline{\mathbf{Q}})$ *be such that* $dimV = 0$. *Assume further that the algebraic variety in* $\mathbf{C}^{n-1}$ *defined by* $f_1(0, x') = \cdots = f_n(0, x') = 0$, $(x = (x_1, x'))$, *is empty. Then, there are constants* $N \in \mathbf{N}$ *and* $\kappa > 0$ *such that any entire function satisfying the estimates*

$$|F(x)| \leq A(1 + |x|)^B \exp(C|\Re x|),$$

can be represented (modulo the ideal I*) as*

$$F(x) =< \overline{\partial}\frac{1}{f_1} \wedge \cdots \wedge \overline{\partial}\frac{1}{f_n}(\xi), F(\xi)\Psi(x,\xi)g_1(x,\xi) \wedge \cdots \wedge g_n(x,\xi) >, \quad (81)$$

where

$$\Psi(x,\xi) = \left(\frac{1 + x \cdot \overline{\xi}}{1 + |\xi|^2}\right)^{B+N} \exp(2C\partial(|\Re\xi| * \theta_0) \cdot (x - \xi) + \kappa\partial(|\Re\xi_1| * \theta_1)(x_1 - \xi_1)), \quad (82)$$

θ_0 *is a smooth non-negative radial function in* $\mathbf{C}^n$, *supp*(θ_0) *contained in the ball* $\{|\xi| \leq 1\}$, $\int_{\mathbf{C}^n} \theta_0 = 1$, θ_1 *is an even non-negative function in* $\mathbf{C}$, *supp*(θ_1) *is contained in the disk* $\{|\xi_1| \leq 1\}$, $\int_{\mathbf{C}} \theta_1 = 1$, *and* $\overline{\partial}\frac{1}{f_1} \wedge \cdots \wedge \overline{\partial}\frac{1}{f_n}$ *is the residue current associated to* $f_1, \dots, f_n$.

Proof. Before we start the proof we should remark that the residue current $\overline{\partial}\frac{1}{f_1} \wedge \cdots \wedge \overline{\partial}\frac{1}{f_n}$ has been defined in [3] and the proof we give here follows the ideas in [14]. Moreover, the theorem is valid for other growth conditions than $|\Re x| + \log(1 + |x|)$, all we need is to work with a weight $\geq |\Re x_1| + \log(1 + |x|)$. This is done by changing the form Q_3 to incorporate the new weight function, as in Proposition 3.1.

Let us recall from Section 2 that, due to the arithmetic hypothesis on the coefficients of $f_1, \dots, f_n$, there is an integer $m \geq 0$ such that the distributions $x_1^m a_{k,j}$ appearing in the division formula (52) have estimates of the type (63), with $\varphi(x) = |\Re x_1|$. On the other hand, as we already have seen, the ideal generated by $f_1, \dots, f_n, x_1^m$ in $A_\rho(\mathbf{C}^n)$ is also generated by polynomials $P_1, \dots, P_n, x_1^m$. Our extra hypothesis on the zeros of $f_j(0, x')$ translates exactly into the fact that these polynomials have no common zeros. Thus, for any $F \in A_\rho(\mathbf{C}^n)$ we have

$$F = \sum_{j=1}^{n} G'_j P_j + x_1^m G'_{n+1},$$

with G'_j entire functions satisfying

$$|G'_j(x)| \leq A_1(1+|x|)^{B+N_1}\exp(C|\Re x|),$$

for some $A_1, N_1 > 0$. This is clear since, for some $\varepsilon > 0, N_0 > 0$,

$$|P_1(x)|^2 + \cdots + |P_n(x)|^2 + |x_1^m|^2 \geq \varepsilon(1+|x|)^{-N_0}.$$

Writing $P_j = f_j + x_1^m h_j$, we obtain

$$F = \sum_{j=1}^{n} G_j f_j + x_1^m G_{n+1}, \tag{83}$$

with the estimates

$$|G_j(x)| \leq AA_2(1+|x|)^{B+N_2}\exp(C|\Re x| + \kappa_0|\Re x_1|),$$

for some $A_2, N_2, \kappa_0 > 0$.

We apply to $x_1^m G_{n+1}$ the division procedure described in the proof of Theorem 3.1, the only changes are in the more precise choice of the weight Q_3 and the fact that $x_1^m G_{n+1}$ is not in the ideal I_{loc}, hence there is a remainder term coming from the kernel K in (52). We set

$$Q_3 = 2C\partial(|\Re\xi| * \theta_0) + \kappa\partial(|\Re\xi_1| * \theta_1),$$

with $\kappa > 0$ to be chosen conveniently.

In [3, Theorem 3.2] the explicit form of the remainder is given as

$$S(x) =< (\overline{\partial}\frac{1}{f_1} \wedge \cdots \wedge \overline{\partial}\frac{1}{f_n})(\xi), \xi_1^m G_{n+1}(\xi)\Psi(x,\xi) g_1(x,\xi) \wedge \cdots \wedge g_n(x,\xi) >, \tag{84}$$

where Ψ is given by (82), with N, κ are chosen so that all the integrals appearing in the representation (52) for $u(\xi) = \chi(\xi/R)\xi_1^m G_{n+1}(\xi)$ converge. This expression shows that

$$x_1^m G_{n+1}(x) \equiv S(x) \pmod{I}.$$

On the other hand, one of the properties of the residue current in (84) is to kill all the functions in I_{loc}. This shows that, with the help of (83), we can replace in (84), $\xi_1^m G_{n+1}(\xi)$ by $F(\xi)$. This proves the proposition.

□

In the particular case when the set of common zeros of $f_1, \ldots, f_n$ is discrete and the zeros are simple, we can obtain that V is an interpolation variety for the weight $|\Re x_1| + \log(2 + |x|)$, thus also for the space $A_\rho(\mathbf{C}^n)$ for any weight $\rho \geq |\Re x_1| + \log(2 + |x|)$. This follows from [7] and the following proposition.

Proposition 3.3 *Let $f_1, \ldots, f_n \in E_{n,1}(\mathbf{C})$ be such that $dim V = 0$ and $J(x) \neq 0$ for every $x \in V$, where J is the Jacobian determinant of the f_j. Then there is a constant $C > 0$ such that*

$$|J(x)| \geq \exp(-C(|\Re x_1| + \log(2 + |x|))) \ \ \forall x \in V. \tag{85}$$

Proof. We only need to apply Theorem 3.3 to the ideal I_0 generated by $f_1, \ldots, f_n, J$, with weight $|\Re x_1| + \log(2 + |x|)$ instead of ρ. Then $\hat{I_0}^{2m} \subseteq I_0$. Since $V(I_0) = \emptyset$, then $1 \in \hat{I_0}$, so that $1 \in I_0$. It follows that there are $g_1, \ldots, g_{n+1}$, entire functions, satisfying the inequalities

$$|g_j(x)| \leq \exp(C(|\Re x_1| + \log(2 + |x|))),$$

for some $C > 0$, and also the Bezout identity

$$f_1(x)g_1(x) + \cdots + f_n(x)g_n(x) + J(x)g_{n+1}(x) = 1 \ \ \forall x \in \mathbf{C}^n.$$

Considering a point $x \in V$, we obtain the inequality (85) from the earlier estimate of g_{n+1}.

□

Remark. In fact, one has a stronger result. Let $f_1, \ldots, f_p \in E_{n,1}(\mathbf{C})$ be such that $dim V = k$ and assume that, at every point $x \in V$, there is a $k \times k$ minor of the Jacobian matrix Df of $f_1, \ldots, f_p$, which does not vanish. Then, the variety V is an interpolation variety for any weight $\geq |\Re x_1| + \log(2 + |x|)$. Namely, if we let $J_1, \ldots, J_l$ denote all the $k \times k$ minors of Df, then the ideal I_0 generated by $f_1, \ldots, f_p, J_1, \ldots, J_l$ does not have any common zeros, and the previous proof applies, allowing us to conclude that for $x \in V$

$$|J_1(x)| + \cdots + |J_l(x)| \geq \exp(-C(|\Re x_1| + \log(2 + |x|))).$$

¿From [7, Theorem 1], one obtains that V is an interpolating variety.

Let us now observe that essentially all the previous results of this section are valid for exponential-polynomials $f_j(e^{x_1}, e^{\alpha x_1}, x_2, \ldots, x_n)$, $\alpha \in \overline{\mathbf{Q}} \setminus \mathbf{Q}$, $f_j \in$

$\overline{\mathbf{Q}}[y_1, z_1, x_2, \ldots, x_n]$. As before, one could replace the weight ρ by $|\Re x_1| + \log(2 + |x|)$, if necessary for the applications.

Lemma 3.2 *Let $f_1, \ldots, f_p$ be polynomials in $e^{x_1}, e^{\alpha x_1}, x_2, \ldots, x_n$, with coefficients in $\overline{\mathbf{Q}}$, and $\alpha \in \overline{\mathbf{Q}} \setminus \mathbf{Q}$. Assume $dimV \leq n-p$. Then, there are linear combinations $\varphi_1, \ldots, \varphi_p$ of f_j with integral coefficients such that the f_j are also linear combinations of the φ_j, and, moreover, $\varphi_1, \ldots, \varphi_p$ form a normal sequence.*

Proof. We follow the procedure of Lemma 3.2. We can assume that $f_1 \not\equiv 0$, and choose $\varphi_1 = f_1$. Assume that we have already found a normal sequence $\varphi_1, \ldots, \varphi_k, k < p$, such that $\varphi_j = \sum_{i=1}^{p} c_{j,i} f_i$, $c_{j,i} \in \mathbf{Z}, 1 \leq j \leq k$, and $rank(c_{j,i}) = k$. We need to choose φ_{k+1} so that for any subfamily $\varphi_{j_1}, \ldots, \varphi_{j_l}$ of $\{\varphi_1, \ldots, \varphi_k\}$, we have $dimV(\varphi_{j_1}, \ldots, \varphi_{j_l}, \varphi_{k+1}) \leq n - (l+1)$. To simplify the notation consider $\varphi_1, \ldots, \varphi_l$, then $V(\varphi_1, \ldots, \varphi_l)$ is a countable union of irreducible varieties of dimension $n - l$. There are two kinds of components, those contained in some hyperplane $\{x_1 = const\}$, say $\{U_i\}$, and those that are not, say $\{V_h\}$. Let $Q_1, \ldots, Q_p, P_1, \ldots, P_k$ be the polynomials in $\overline{\mathbf{Q}}[y_1, z_1, x_2, \ldots, x_n]$ such that $f_j(x) = Q_j(e^{x_1}, e^{\alpha x_1}, x_2, \ldots, x_n), \varphi_j = P_j(e^{x_1}, e^{\alpha x_1}, x_2, \ldots, x_n)$ and consider the finitely many irreducible components W_r of the algebraic variety $P_1 = \cdots = P_l = 0$ in $\mathbf{C}^{n+1}$. Each of the varieties V_h is contained in some W_r. We have that

$$W_r \bigcap \{y_1 = e^{x_1}, z_1 = e^{\alpha x_1}\} \supseteq V_h.$$

Locally, near a point in V_h, $\{y_1 = e^{x_1}, z_1 = e^{\alpha x_1}\}$ is the analytic variety $z_1 = y_1^{\alpha}$, so that either (locally) $W_r \subseteq \{z_1 = y_1^{\alpha}\}$ or $n + 1 - l \geq dimW_r \geq 1 + dimV_h = n - l + 1$. If y_1 is not constant on W_r, we can fix generic $x_2, \ldots, x_n$, so that near a point in V_h we have that z_1 is an algebraic function of y_1. Considering the Puiseux development of z_1 we see that only rational powers of y_1 can appear in it, which contradicts the fact that $z_1 = y_1^{\alpha}$ (since $\alpha \notin \mathbf{Q}$. On the other hand, if y_1 is locally constant, then x_1 is constant in V_h, which is impossible by the definition of V_h. Hence

$$dimW_r = n - l + 1.$$

Assume all the polynomials $Q_1, \ldots, Q_p$ vanish identically on W_r, then the functions $f_1, \ldots, f_p$ vanish identically on V_h, which contradicts the hypothesis

$dimV \leq n-p$. Thus, for $\lambda \in \mathbf{C}^p$ outside a hyperplane, we have $\lambda_1 Q_1 + \cdots + \lambda_p Q_p \not\equiv 0$ on W_r. We claim that $\sum_{j=1}^p \lambda_j f_j(x) \not\equiv 0$ on V_h. If this were not the case, let

$$\tilde{W_r} = W_r \bigcap \{\sum_{j=1}^p \lambda_j Q_j = 0\}.$$

Then

$$\tilde{W_r} \bigcap \{y_1 = e^{x_1}, z_1 = e^{\alpha x_1}\} \supseteq V_h$$

and $dim\tilde{W_r} \leq n-l$. This implies that $\tilde{W_r} \subseteq \{y_1 = e^{x_1}, z_1 = e^{\alpha x_1}\}$, which, as we have just seen, leads to contradiction. So that for λ outside a finite union of hyperplanes, we have $\sum_{j=1}^p \lambda_j f_j \not\equiv 0$ on any V_h. In particular, one can always choose all the $\lambda_j \in \mathbf{Z}$.

There are also finitely many components U_i contained in the hyperplane $\{x_1 = 0\}$. This is the case for those that are components of the algebraic variety $P_1(1,1,x_2,\ldots,x_n) = \cdots = P_l(1,1,x_2,\ldots,x_n) = 0$. The previous reasoning shows that we can choose integers $\lambda_1,\ldots,\lambda_p$ such that $\sum \lambda_j f_j \not\equiv 0$ in any V_h, that $\sum \lambda_j Q_j \not\equiv 0$ on any W_r, which contains points of $V(\varphi_1,\ldots,\varphi_l)$ and has dimension $dimW_r = n-l+1$, and that $\sum \lambda_j f_j \not\equiv 0$ on those U_i which lie in $\{x_1 = 0\}$. If we run over all possible families $\varphi_{j_1},\ldots,\varphi_{j_l}, 1 \leq l \leq k$, we can obtain the λ_j simultaneously satisfying these conditions, not only for $\varphi_1,\ldots,\varphi_l$, but also for all such families. Moreover, we can also assume that the rank of the matrix of coefficients of $\varphi_1,\ldots,\varphi_k, \sum \lambda_j f_j$ in terms of all f_j is exactly $k+1$. We claim that this is a good choice of λ_j.

Consider now whether there are any U_i not contained in $\{x_1 = 0\}$. For such U_i we would have a unique W_r such that $U_i \subseteq W_r \bigcap \{y_1 = e^{x_1}, z_1 = e^{\alpha x_1}\}$. If $W_r \subseteq \{y_1 = e^{x_1}, z_1 = e^{\alpha x_1}\}$, we have already seen that y_1 and z_1 are constant on W_r. Let us denote these constants by $y_1 = \eta$, $z_1 = \zeta$, and let $x_1 = \xi$ be such that $\eta = e^{\xi}$ and $\zeta = e^{\alpha\xi}$. Now, Noether's Normalization Theorem allows us to choose, near a regular point, $n-l (= dimW_r)$ coordinates, which parametrize W_r by algebraic functions with algebraic coefficients. Choosing a point with algebraic coordinates shows that η and ζ are algebraic numbers. Since they are related by $\eta = e^{\xi}, \zeta = e^{\alpha\xi}$, it follows from Gelfond's theorem that $\xi = 0$ and $\eta = \zeta = 1$. This implies that U_i is contained in $\{x_1 = 0\}$, a contradiction. The only possibility left is that W_r is not contained in $\{y_1 = e^{x_1}, z_1 = e^{\alpha x_1}\}$. In this case $dimW_r = n-l+1$. Then, by the earlier choice of λ_j, $\sum \lambda_j Q_j \not\equiv 0$ on W_r, thus

$$U_i \subseteq W_r \bigcap \{\sum \lambda_j Q_j = 0\}.$$

Since both sides have the same dimension $n - l$, U_i is a component of the algebraic variety $W_r \bigcap \{\sum \lambda_j Q_j = 0\}$. On this component $y_1 = \eta$, $z_1 = \zeta$. For the same reasons as above, the constants η, ζ are algebraic, so that $x_1 = 0$ on U_i. Again a contradiction.

This proves that the choice $\varphi_{k+1} = \sum \lambda_j f_j$, defines a normal system $\varphi_1, \ldots, \varphi_{k+1}$ such that the rank of the integral matrix $(c_{j,i})_{1 \leq j \leq k+1, 1 \leq i \leq p}$ is exactly $k + 1$. Iterating this procedure we conclude the proof of the lemma.

□

With the help of this lemma and Proposition 2.3, we can repeat the proofs of the previous results of this section and obtain the following statements.

Proposition 3.4 *Let $\alpha \in \overline{\mathbf{Q}} \setminus \mathbf{Q}$ and $f_1, \ldots, f_p$ be polynomials in $e^{x_1}, e^{\alpha x_1}$, $x_2, \ldots, x_n$, with coefficients in $\overline{\mathbf{Q}}$. Assume that the exponential polynomials $f_1, \ldots, f_p$ define a complete intersection variety. Let I be the ideal they generate in the space $A_\rho(\mathbf{C}^n)$. Then $I = I_{loc}$.*

Proposition 3.5 *Let $\alpha \in \overline{\mathbf{Q}} \setminus \mathbf{Q}$ and let I be the ideal in $A_\rho(\mathbf{C}^n)$ generated by $f_1, \ldots, f_p$, polynomials in $e^{x_1}, e^{\alpha x_1}, x_2, \ldots, x_n$, with coefficients in $\overline{\mathbf{Q}}$. Denote $V = \{x \in \mathbf{C}^n : f_1(x) = \cdots = f_p(x) = 0\}$. Then $\sqrt{I} = I(V)$ and $\hat{I}^{2m} \subseteq I$, where $m = min(p + 1, n)$.*

Proposition 3.6 *Let $\alpha \in \overline{\mathbf{Q}} \setminus \mathbf{Q}$, let $f_1, \ldots, f_p$ be polynomials in $e^{x_1}, e^{\alpha x_1}$, $x_2, \ldots, x_n$, with coefficients in $\overline{\mathbf{Q}}$, and let the variety of common zeros be $V = \{x \in \mathbf{C}^n : f_1(x) = \cdots = f_p(x) = 0\}$. If V is discrete and all the zeros are simple (or if the f_j define a manifold), then V is an interpolation variety for $A_\rho(\mathbf{C}^n)$.*

We conclude this manuscript with an indication of some simple applications to harmonic analysis that can be obtained from the earlier results and the methods of [6]. For that purpose, let us recall that a linear differential operator $P(D)$ with constant coefficients and commensurable time lags is a finite sum of the form

$$(P(D)\varphi)(t, x) = \sum p_{jk}(D^j \varphi)(t - kT, x), \tag{86}$$

$t \in \mathbf{R}$, $x \in \mathbf{R}^n$, $(n \geq 0)$, $D = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$, $j \in \mathbf{N}^{n+1}$, $k \in \mathbf{Z}$, $T > 0$, and $p_{jk} \in \mathbf{C}$. The symbol of this operator $P(\tau, \xi)$ is the element of $E_{n+1,1}(\mathbf{C})$ given by

$$\begin{aligned} P(\tau, \xi) := & \quad e^{i(t\tau + x\cdot\xi)} P(D) e^{-i(t\tau + x\cdot\xi)} \\ = & \quad \sum p_{jk} (-i\zeta)^j e^{ikT\tau}, \end{aligned} \tag{87}$$

with $\zeta = (\tau, \xi)$. (By the introduction of the new coordinate $\xi_0 = iT\tau$, we are in the case of exponential polynomials considered at the beginning of this section.)

Theorem 3.4 *Let $P_1(D), \dots, P_{n+1}(D)$ be differential operators with time lags as in (86), with the property that the characteristic variety*

$$V := \{\zeta \in \mathbf{C}^{n+1} : \ P_l(\zeta) = 0,\ 1 \leq l \leq n+1\}$$

is discrete and all the points of V are simple. Then, every solution $\varphi \in \mathcal{E}(\mathbf{R}^{n+1})$ (resp., $\varphi \in \mathcal{D}'(\mathbf{R}^{n+1})$) of the overdetermined system

$$P_1(D)\varphi = \cdots = P_{n+1}(D)\varphi = 0, \tag{88}$$

can be represented in a unique way in the form of a series of exponential solutions of the system (88), namely,

$$\varphi(t, x) = \sum_{\zeta \in V} c_\zeta e^{i(t\tau + x\cdot\xi)}\,.$$

This series is convergent in the topology of $\mathcal{E}(\mathbf{R}^{n+1})$ (resp., $\mathcal{D}'(\mathbf{R}^{n+1})$).

Similarly, if we allow two non-commensurable time lags, but we assume that: (i) their ratio is algebraic, (ii) there are no derivatives in the time variable, and (iii) the coefficients of the operators are algebraic, then we can prove the same representation theorem for the solutions of a corresponding system.

We shall present the applications of this type of result to Control Theory elsewhere. Meanwhile, we refer the reader to [20, 22, 30] for some results in that direction, and to [32, 4] for related applications to deconvolution problems.

References

[1] M.F. Atiyah, Resolution of singularities and division of distributions, *Commun. Pure Applied Math.* 23 (1970), 145-150.

[2] A. Baker, "Transcendental Number Theory," Cambridge University Press, London,1979.

[3] C.A. Berenstein, R. Gay, and A. Yger, Analytic continuation of currents and division problems, *Forum Math.* 1 (1989), 15-51.

[4] C.A. Berenstein and E.V. Patrick, Exact deconvolution for multiple operators-an overview plus performance characterizations for imaging sensors, *IEEE Proc. MultidimensionalSignal Processing* 78 (1990), 259-287.

[5] C.A. Berenstein and D.C. Struppa, Complex analysis and convolution equations, in "Contemporary Problems in Mathematics. Fundamental Directions", vol. 54, VINITI, Moscow, 1990, 5-113.

[6] C.A. Berenstein and B.A. Taylor, Interpolation problems in $\mathbf{C}^n$ with applications to harmonic analysis, *J. Analyse Math.* 37 (1980), 188-254.

[7] C.A. Berenstein and B.A. Taylor, On the geometry of interpolating varieties, "Seminaire Lelong-Skoda 1980-81," Springer-Verlag, Heidelberg, 1-25.

[8] C.A. Berenstein, B.A. Taylor and A. Yger, Sur les systèmes d'équations différence-différentelles, *Ann. Inst. Fourier* 33 (1983), 109-130.

[9] C.A. Berenstein and A. Yger, Ideals generated by exponential polynomials, *Advances in Math.* 60 (1986), 1-80.

[10] C.A. Berenstein and A. Yger, On Lojasiewicz type inequalities for exponential polynomials, *J. Math. Anal. Appl.* 129 (1988), 166-195.

[11] C.A. Berenstein and A. Yger, Analytic Bezout identities, *Advances in Applied Math.* 10 (1989), 51-74.

[12] C.A. Berenstein and A. Yger, Effective Bezout estimates in $\mathbf{Q}[z_1, \ldots, z_n]$, *Acta Math.* 66 (1991), 69-120.

[13] C.A. Berenstein and A. Yger, Une formule de Jacobi et ses conséquences, *Ann. scient. Éc. Norm. Sup.* 24 (1991), 363-377.

[14] C.A. Berenstein and A. Yger, About Ehrenpreis' Fundamental Principle, "Geometric and alg. aspects in several complex variables," C. A. Berenstein and D.C. Struppa (ed.), Editel, Rende, 1991, 47-61.

[15] C.A. Berenstein and A. Yger, Formules de représentation intégrale et problèmes de division, "Diophantine Approximations and Transcendental Numbers," P. Phillipon (ed.), Walter de Gruyter & Co., Berlin, 1992, 15-37.

[16] B. Berndtsson and M. Andersson, Henkin-Ramirez formulas with weight factors, Ann. Inst. Fourier, 32 (1982), 91-110.

[17] I.N. Bernstein, The analytic continuation of generalized functions with respect to a parameter, *Funct. Anal. Appl.* 6 (1972), 273-285.

[18] J.-E. Björk, " Rings of Differential Operators," North-Holland Publ. Co., Amsterdam, 1979.

[19] A. Borel, Operations on algebraic $\mathcal{D}$-modules, "Algebraic $\mathcal{D}$-modules", A. Borel (ed.), Academic Press, Boston, 1987, 207-269.

[20] A.G. Butkovskiy, "Green's functions and transfer functions handbook," Ellis Horwood, Chichester, 1982.

[21] L. Ehrenpreis, "Fourier Analysis in Several Complex Variables", John Wiley & Sons, 1970.

[22] B. Frankpitt, Frequency domain methods for the control of distributed parameter systems, M.S. thesis, Univ. of Maryland, 1992.

[23] D.I. Gurevich, Counterexamples to a problem of L. Schwartz, *Funct. Anal. Appl.*, 9 (1975), 116-120.

[24] I.S. Gradshteyn and I.M. Ryzhik, "Table of integrals, series, and products," Academic Press, New York, 1980.

[25] G.M. Henkin and J. Leiterer, "Theory of Functions on Complex Manifolds," Birkhäuser, Basel, 1984.

[26] L. Hörmander, "An Introduction to Complex Analysis in Several Complex Variables", second edition, North-Holland Publ. Co., Amsterdam, 1991.

[27] L. Hörmander, "The Analysis of Linear Partial Differential Operators", II, Springer-Verlag, Berlin, 1983.

[28] M. Lejeune and B. Teissier, Quelques calculs utiles pour la résolution des singularités, Séminaire École Polytechnique, 1972, 130.

[29] B. Lichtin, Generalized Dirichlet series and B-functions, *Compositio Math.* 65 (1988), 81-120.

[30] D.L. Russell, Nonharmonic Fourier series in the control theory of distributed parameter systems, *J. Math. Analysis Appl.* 18 (1967), 542-559.

[31] C. Sabbah, Proximité évanescente II. Equations fonctionnelles pour plusieurs fonctions analytiques, *Compositio Math.* 64 (1987), 213-241.

[32] W. Symes, A trace theorem for the wave equation and remote determination of acoustic sources, preprint, Math. Dept., Michigan State Univ., June 1981.