Bounds for the Degrees in the Division Problem

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0. Introduction

One of the basic questions in computational algebra is to find sharp bounds for the complexity of the possible algorithms solving the following problem. Let $f_1, \dots, f_m \in \mathbf{C}[z]$ $(z = (z_1, \dots, z_n))$, and let I be the ideal they generate. Assuming that $f \in I$, what can be said about the polynomials $q_1, \dots, q_m$ that satisfy the equation

$$(*) \qquad f = q_1 f_1 + \cdots + q_m f_m?$$

To be more concrete, let us assume that $\max\{\deg f, \deg f_j, (1 \le j \le m)\} = D$. Can we find q_j whose degrees are relatively small—for example, bounded by D^n?

In the case where the homogenized polynomials ${}^h f_1, \dots, {}^h f_m$ of $f_1, \dots, f_m$ define a complete intersection variety in $\mathbf{C}^{n+1}$, such bound is correct. In fact, one can find polynomials $q_1, \dots, q_m$ such that

$$\max_{1 \le j \le m} \deg(f_j q_j) \le \deg f + d_1 \dots d_m,$$

where $d_j = \deg f_j$. This is a consequence of a result of Macaulay dating from the beginning of the century. It follows from the fact that for a locally regular sequence the exponent in the local Nullstellensatz is bounded by the product of the degrees (see [14]).

It is surprising to find that such an estimate is false in general. An example of Mayr–Meyer [9] shows that for any $D \ge 5$, $k \ge 1$ and $n = 10k$, there are $n+1$ polynomials $f_1, \dots, f_{n+1} \in \mathbf{C}[z]$ such that $z_1 \in I$, and that if $q_1, \dots, q_{n+1} \in \mathbf{C}[z]$ satisfy $z_1 = q_1 f_1 + \cdots + q_{n+1} f_{n+1}$ then $\max \deg q_j > (D-2)^{2^{k-1}}$. This implies that, in general, the complexity of any algorithm capable of solving this kind of problem must be doubly exponential. This applies in particular to the algorithms that construct the standard (or Groebner) bases. The reason is that, as soon as such a basis is known, the problem of deciding whether $f \in I$ and finding the q_j can be solved immediately.

There are several ways to obtain division formulas with better bounds for the degrees of the q_j. For instance, in comparing $(*)$ with Hilbert's Null-

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stellensatz, one is led to the idea of representing instead an element $f \in I$, a power of f. That is,

$$(**) \qquad f^s = q_1 f_1 + \cdots + q_m f_m.$$

Recently, Brownawell [5], Kollár [8], and Fitchas-Galligo [6] have found good bounds for the exponent s and the degrees of the corresponding q_j. They are

$$s \leq D^n, \qquad \deg q_j \leq (1+\deg f)D^n.$$

Using analytic methods, we found integral representation formulas for a solution $q_1, \ldots, q_m$ of $(*)$ in two cases: when the variety $V = \{z \in \mathbf{C}^n : f_1(z) = \cdots = f_m(z) = 0\}$ was discrete, or when $\dim V = n - m$. In the first case we showed in the original version of [2] that the example of Mayr–Meyer could not hold. In fact, we obtained $\deg q_j \leq n^2 D^{3n}$. In the second case, the case of a complete intersection, we found bounds for $\deg q_j$ in terms of quantities related to the Bernstein–Sato functional equation [3].

In this paper we show that in both cases one has an estimate of the form

$$\deg q_j \leq \deg f + \kappa(n) D^n$$

(see Theorems 2 and 3 for the precise estimates). Except for the constant $\kappa(n)$, these bounds are sharp. The proofs depend on the existence of the Lojasiewicz type inequalities found by Brownawell [4] and on the analytic methods developed in [1] and [8] to study the equations $(*)$ and $(**)$ in spaces of entire functions with growth conditions. If one could find a purely algebraic proof of these two results, one could deal with polynomials in $K[z]$, even if $\operatorname{char} K \neq 0$, and probably sharpen the value of the constant $\kappa(n)$. This has been done by Kollár [10] for the original bounds of Brownawell [5] for $(**)$.

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1. Koszul Complex

The aim of this section is to redo in detail the analysis of the Koszul complex sketched in Theorem 2.6 of [7]. The reason for this is the need to keep track of the constants left implicit in [7], which is not altogether trivial as the reader will see. The knowledge of these constants is crucial for the results of Section 2.

We recall some notation about differential forms. For $r \in \mathbf{N}$ and $\alpha = \sum_j \alpha_j d\bar{z}^j$ a differential form of type $(0, r)$ in n variables, we denote

$$|\alpha| = \Big(\sum_j |\alpha_j|^2\Big)^{1/2},$$

where $j = (j_1, \ldots, j_r)$, $1 \leq j_k \leq n$, is an increasing sequence.

Let p be a plurisubharmonic function in $\mathbf{C}^n$. In what follows p will be referred to as a weight. We denote by L_r^0 the space of $(0,r)$ differential forms α in $\mathbf{C}^n$ (with measurable coefficients) that have finite L^2-norm with respect to this weight. That is,

$$\int_{\mathbf{C}^n} |\alpha(z)|^2 \exp[-Cp(z)]\, d\lambda(z) < \infty$$

for some constant $C = C(\alpha) \geq 0$. The weight p is omitted in the notation. Here $d\lambda$ denotes the Lebesgue measure.

Given N entire functions, $F_1, \ldots, F_N$, $N \geq 2$, let

$$F = (F_1, \ldots, F_N) \quad \text{and} \quad |F(z)|^2 = \sum_{j=1}^{N} |F_j(z)|^2.$$

We will assume, in this section, that they satisfy the following estimates:

(1) $$|F(z)|^2 \leq A_0 \exp[B_0 p(z)] \quad (\forall z \in \mathbf{C}^n),$$

(2) $$\max_{1 \leq j \leq N} |\partial F_j(z)|^2 \leq A_0 \exp[B_0 p(z)] \quad (\forall z \in \mathbf{C}^n),$$

for some positive constants A_0, B_0.

Let $L_r^s = \Lambda^s(\mathbf{C}^N) \otimes_{\mathbf{C}} L_r^0$. It is clear that $L_r^s = 0$ if $r > n$ or $s > N$. Associated with the $\mathbf{C}^N$-valued function F we can consider the Koszul complex given by the maps

$$\bar{\partial}: L_r^s \to L_{r+1}^s,$$

where $\bar{\partial}$ is the usual densely defined Cauchy-Riemann operator, and

$$P: L_r^{s+1} \to L_r^s,$$

where $P\alpha = P_F \alpha$ is the *interior product* of an element $\alpha \in L_r^{s+1}$ with F. Namely, $P\alpha$ is defined by

$$(P\alpha)(i_1, \ldots, i_s) = \sum_{j=1}^{N} F_j \alpha(i_1, \ldots, i_s, j).$$

As usual, we have $\bar{\partial}^2 = P^2 = 0$ and $P\bar{\partial} = \bar{\partial} P$.

We need to define also the exterior product with $\bar{F} = (\bar{F}_1, \ldots, \bar{F}_N)$. Let $\alpha \in L_r^s$; then $\alpha \wedge \bar{F} \in L_r^{s+1}$ is given by

$$(\alpha \wedge \bar{F})(i_1, \ldots, i_{s+1}) = \sum_{k=1}^{s+1} (-1)^{s+1-k} \bar{F}_{i_k} \alpha(i_1, \ldots, \hat{i}_k, \ldots, i_{s+1}),$$

where (as always) the "hat" denotes a deleted entry.

We extend the definition of the norm of a differential form of type $(0,r)$ to the elements of L_r^s. For instance, if $\alpha = \sum_i \alpha_i e_i$, $\{e_i\}$ a basis for $\Lambda^s(\mathbf{C}^N)$, then $|\alpha|^2 = \sum_i |\alpha_i|^2$.

PROPOSITION 1. *Let $\alpha_1 \in L_1^1$ be such that $\bar{\partial}\alpha_1 = 0$, $P\alpha_1 = 0$, and*

(3) $$\int_{\mathbf{C}^n} |F(z)|^{-2k} |\alpha_1(z)|^2 \exp[-Cp(z)]\, d\lambda(z) < \infty,$$

with $C = C(\alpha) \geq 0$, $k = 2\mu - 1$, *and* $\mu = \min\{n, N-1\}$. *Let* $\epsilon > 0$ *be fixed. Then one can find* $\delta_1 \in L_1^2$ *such that*

(4) $$\bar{\partial}\delta_1 = 0, \qquad P\delta_1 = \alpha_1,$$

and

(5) $$\int_{\mathbf{C}^n} |\delta_1(z)|^2 \frac{\exp[-C'p(z)]}{(1+|z|^2)^{(\mu-1)(1+\epsilon)}}\, d\lambda(z) < \infty,$$

with

(6) $$C' = C + 2(\mu - 1)B_0.$$

Proof. We define recursively $\beta_1 \in L_1^2$, $\beta_2 \in L_2^3, \ldots, \beta_\mu \in L_\mu^{\mu+1}$ as follows. Let

$$\beta_1 = |F|^{-2}\alpha_1 \wedge \bar{F} \quad \text{and} \quad \alpha_2 = \bar{\partial}\beta_1.$$

One verifies without difficulty that $P\beta_1 = \alpha_1$. It follows that $\bar{\partial}\alpha_2 = 0$ and $P\alpha_2 = P\bar{\partial}\beta_1 = \bar{\partial}P\beta_1 = \bar{\partial}\alpha_1 = 0$. Moreover, we have the two estimates

$$\int |F|^{2(1-k)}|\beta_1|^2 \exp[-Cp]\, d\lambda < \infty$$

and

$$\int |F|^{2(2-k)}|\alpha_2|^2 \exp[-(C+B_0)p]\, d\lambda < \infty.$$

The first estimate is immediate from (3). The second one follows from a simple computation and (2).

Now we can define

$$\beta_2 = |F|^{-2}\alpha_2 \wedge \bar{F} \quad \text{and} \quad \alpha_3 = \bar{\partial}\beta_2,$$

and so on. We claim that $\alpha_{\mu+1} = 0$.

There are two cases. If $\mu = n$, then $\alpha_{\mu+1} \in L_{n+1}^{\mu+1} = 0$; if $\mu = N - 1 < n$, then we have $P\alpha_{\mu+1} = 0$. Hence we can define $\beta_{\mu+1}$ as above so that $P\beta_{\mu+1} = \alpha_{\mu+1}$. But $\beta_{\mu+1} \in L_{\mu+1}^{N+1} = 0$. Therefore $\alpha_{\mu+1} = 0$ as we wanted to show.

It is clear that we have the estimates

(7) $$\int |F|^{2(2j-1-k)}|\beta_j|^2 \exp[-(C+(j-1)B_0)p]\, d\lambda < \infty.$$

In particular, for $j = \mu$,

$$\int |\beta_\mu|^2 \exp[-(C+(\mu-1)B_0)p]\, d\lambda < \infty.$$

Since $\bar{\partial}\beta_\mu = 0$ we can solve the equation

$$\bar{\partial}\gamma_{\mu-1} = \beta_\mu, \quad \gamma_{\mu-1} \in L_{\mu-1}^{\mu+1}$$

in such a way that the estimate

$$\int |\gamma_{\mu-1}|^2 \frac{\exp[-(C+(\mu-1)B_0)p]}{(1+|z|^2)^{1+\epsilon}}\, d\lambda < \infty$$

holds (see [15]; a slightly weaker estimate can be found in [8]). Consider now the equation

$$\delta_{\mu-1} = \beta_{\mu-1} - P\gamma_{\mu-1} \in L^{\mu}_{\mu-1}.$$

We have

$$P\delta_{\mu-1} = \alpha_{\mu-1} \quad \text{and} \quad \bar{\partial}\delta_{\mu-1} = 0.$$

It is clear from (1) that

$$\int |P\gamma_{\mu-1}|^2 \frac{\exp[-(C+\mu B_0)p]}{(1+|z|^2)^{1+\epsilon}}\, d\lambda < \infty.$$

We need to show that this last estimate holds when $P\gamma_{\mu-1}$ is replaced by $\beta_{\mu-1}$. This is a consequence of the estimate (7) for the case $j = \mu - 1$. We divide the integral in (7) in two, over the regions where $|F| < 1$, respectively, $|F| \geq 1$. In the first one we have:

$$\int_{\{|F|<1\}} |\beta_{\mu-1}|^2 \exp[-(C+(\mu-2)B_0)p]\, d\lambda$$

$$\leq \int_{\{|F|<1\}} |F|^{-4} |\beta_{\mu-1}|^2 \exp[-(C+(\mu-2)B_0)p]\, d\lambda.$$

In the other region, we can use (1) to see that $|F|^4 \leq \exp[2B_0 p]$. Therefore,

$$\int_{\{|F|\geq 1\}} |\beta_{\mu-1}|^2 \exp[-(C+\mu B_0)p]\, d\lambda$$

$$= \int_{\{|F|\geq 1\}} |F|^{-4} |\beta_{\mu-1}|^2 |F|^4 \exp[-(C+\mu B_0)p]\, d\lambda$$

$$\leq \int_{\{|F|\geq 1\}} |F|^{-4} |\beta_{\mu-1}|^2 \exp[-(C+(\mu-2)B_0)p]\, d\lambda.$$

Hence

$$\int |\delta_{\mu-1}|^2 \frac{\exp[-(C+\mu B_0)p]}{(1+|z|^2)^{1+\epsilon}}\, d\lambda < \infty.$$

It is clear we can repeat this procedure, obtaining successively $\delta_{\mu-2}, \ldots, \delta_1$. In particular, $\delta_1 \in L^2_1$ will satisfy

$$P\delta_1 = \alpha_1, \qquad \bar{\partial}\delta_1 = 0,$$

and

$$\int_{\mathbf{C}^n} \frac{|\delta_1(z)|^2}{(1+|z|^2)^{(\mu-1)(1+\epsilon)}} \exp[-(C+2(\mu-1)B_0)p(z)]\, d\lambda < \infty$$

as claimed. □

2. Discrete Varieties

In this section we prove that the Mayr–Meyer example cannot occur when the polynomials $f_1, \ldots, f_m$ define a discrete variety. We need the following notation. If $d_1 \geq \cdots \geq d_m$ is a collection of positive integers and $n \geq 2$, then we denote

$$\mathfrak{N} = N(n, d_1, \ldots, d_m) = \begin{cases} d_1 \ldots d_m & \text{if } m \leq n, \\ d_1 \ldots d_{n-1} d_m & \text{if } m > n. \end{cases}$$

From now on, the weight p mentioned in the previous section will be given by $p(z) = \operatorname{Log}(1+|z|^2)$.

THEOREM 2. *Let $f_1, \ldots, f_m$ be polynomials in* $\mathbf{C}[z]$ *with respective degrees* $d_1 \geq d_2 \geq \cdots \geq d_m$. *Assume that* $V = \{z \in \mathbf{C}^n : f_1(z) = \cdots = f_m(z) = 0\}$ *is discrete. If* $f \in I(f_1, \ldots, f_m)$ *then there are polynomials* $q_j \in \mathbf{C}[z]$ *such that*

$$f = \sum_{j=1}^{m} q_j f_j \tag{8}$$

and

$$\max_j (\deg q_j) \leq \deg f + 3(n+1)\mathfrak{N}. \tag{9}$$

Proof. Let $F = (f_1, \ldots, f_m)$. From [10, Prop. 1] we know there are two constants $\eta, \rho > 0$ such that if $|z| > \rho$ then

$$|F(z)| \geq \eta |z|^{-\mathfrak{N}}, \tag{10}$$

with $\mathfrak{N} = N(n, d_1, \ldots, d_m)$ as defined above.

Let χ be a C^∞ function in $\mathbf{C}^n$, $0 \leq \chi \leq 1$, $\chi = 1$ in a neighborhood of the closed ball $\bar{B}(0, \rho)$ and $\operatorname{supp} \chi \subseteq B(0, \rho + 1)$. Since $f \in I(f_1, \ldots, f_m)$, there are polynomials g_j such that

$$f = \sum_{j=1}^{m} f_j g_j.$$

Therefore

$$f(z) = \sum_{j=1}^{m} \left(\chi(z) g_j(z) + \frac{(1 - \chi(z)) \overline{f_j(z)} f(z)}{|F(z)|^2} \right) f_j(z) = \sum_{j=1}^{m} h_j(z) f_j(z), \tag{11}$$

where $h = (h_1, \ldots, h_m)$ is defined by the identity (11). We define a vector α_1 of (0, 1) differential forms by the formula

$$\alpha_1 = (\bar{\partial} h_1, \ldots, \bar{\partial} h_m). \tag{12}$$

It is obvious that $\bar{\partial} \alpha_1 = 0$ and

$$P\alpha_1 = P_F \alpha_1 = \sum_{j=1}^{m} f_j \bar{\partial} h_j = \bar{\partial} \left(\sum_{j=1}^{m} f_j h_j \right) = \bar{\partial} f = 0.$$

It is clear from the inequality (10) that $\alpha_1 \in L_1^1$. We can apply Proposition 1, with $\mu = \min\{n, m-1\}$, $k = 2\mu - 1$, and the constant C defined by

$$C = \deg f + \mathfrak{N}(k+2) + d_1 + n + \epsilon, \tag{13}$$

for $\epsilon > 0$ fixed. From that proposition we conclude there is $\delta_1 \in L_1^2$ such that

$$\bar{\partial} \delta_1 = 0, \tag{14}$$

$$P\delta_1 = \alpha_1, \tag{15}$$

and

$$\int |\delta_1(z)|^2 \exp[-Dp(z)] \, d\lambda(z) < \infty, \tag{16}$$

where

$$D = C + 2(\mu - 1) d_1 + (\mu - 1)(1 + \epsilon). \tag{17}$$

As in Section 1, since $\bar\partial\delta_1=0$ we can find $\beta_0\in L_0^2$ such that $\bar\partial\beta_0=\delta_1$ and

$$\int|\beta_0|^2\exp[-D'p]\,d\lambda<\infty,$$

with $D'=D+(1+\epsilon)$.

Consider now the vector valued function $q=h-P\beta_0$. We have $\bar\partial q=\bar\partial h-\bar\partial P\beta_0=\alpha_1-P\bar\partial\beta_0=\alpha_1-P\delta_1=0$ and $Pq=Ph-P^2\beta_0=Ph=f$. Hence the components of q are entire functions that satisfy the estimates

$$\int|q|^2\exp[-D''p]\,d\lambda<\infty, \tag{18}$$

$$D''=D'+d_1=\deg f+2\mu d_1+\mathfrak{N}(k+2)+n+\epsilon(\mu+1). \tag{19}$$

Choosing $\epsilon>0$ so that $\epsilon(\mu+1)<1$, we can conclude from (18) and (19) that each q_j is a polynomial of degree at most

$$\begin{aligned}\deg q_j&\le\deg f+(2\mu+1)\mathfrak{N}+2\mu d_1+n\\&\le\deg f+(2n+1)\mathfrak{N}+2nd_1+n\\&\le\deg f+3(n+1)\mathfrak{N},\end{aligned}$$

as we wanted to show. (Here we have assumed $d_m\ge 2$; if not, we can eliminate at least one variable before starting the proof.) □

Slightly sharper bounds for the degrees of the q_j have been obtained by Shiffman in [14, Thm. 2] under the stronger hypothesis that the zero locus of I is zero-dimensional at all finite and infinite points. He obtains in that case the sharp estimates:

$$\max_{1\le j\le m}\deg(q_jf_j)\le\deg f+N(n,d_1,\dots,d_m).$$

Moreover, his estimates remain valid for any algebraically closed field (even when the characteristic $\neq 0$).

3. Complete Intersection Varieties

In this section we consider polynomials $f_1,\dots,f_m\in\mathbf{C}[z]$, $2\le m\le n-1$. We suppose their respective degrees are $d_1\ge\cdots\ge d_m$. Let $V=\{z\in\mathbf{C}^n: f_1(z)=\cdots=f_m(z)=0\}$. We assume throughout this section that

$$\dim V=n-m. \tag{20}$$

Under these conditions we want to show the analogue of Theorem 2 still holds. Note that the case $m=n$ is included in Theorem 2 and the case $m=1$ is trivial.

THEOREM 3. *Let $f_1,\dots,f_m$ be given as above, and $f\in I(f_1,\dots,f_m)$. Then one can find $q_j\in\mathbf{C}[z]$ such that*

$$f=\sum_{j=1}^m f_jq_j$$

and

(21)
$$\max_{1\le j\le m} \deg q_j \le \deg f + \kappa(m,n) d_1 \dots d_m,$$

where
$$\kappa(m,n) = 6(n+1)^2(2m^{m+1}+n).$$

The proof depends entirely on the fact that the family $f_1, \dots, f_m$ is slowly decreasing in the sense of [1]. Here we must follow the bounds in the process of Jacobi interpolation and the Koszul complex associated to a "good" covering of V, which have been described in [1, §5]. The only extra element we have at our disposal is the local Lojasiewicz inequality of [4]. We take here the opportunity to clarify certain obscure points from [1, §5].

Before proceeding with the proof, we will give a few preliminary observations. The first one is a simple consequence of the classical Noether normalization theorem [16]. A more precise version of the following lemma can be found in [2].

LEMMA 4. *There is a linear change of coordinates, $z = Aw$, and constants $\eta_0, K > 0$ such that if $g_j(w) = f_j(Aw)$, then the set*

(22)
$$\mathfrak{S}_{\eta_0} = \{w \in \mathbf{C}^n : \log \max_{1\le j\le m} |g_j(w)| \le \log \eta_0 - (n+1)^2 d_1 \dots d_m \log(1+|w|^2)\}$$

is included in the cone

(23)
$$\mathfrak{C} = \{w = (w', w'') \in \mathbf{C}^m \times \mathbf{C}^{n-m} : |w'| \le K(1+|w''|)\}.$$

Proof. It is a well-known consequence of the Noether normalization theorem that there is a linear change of coordinates, $z = Aw$, and a constant $K' > 0$ such that

(24)
$$W = \{w \in \mathbf{C}^n : g_1(w) = \dots = g_m(w) = 0\} \subseteq \{w : |w'| \le K'(1+|w''|)\}.$$

(This is a result of Sadullaev; for a more general statement see [7].) On the other hand, it follows from the argument of the proof of Theorem A in [4] that the following local Lojasiewicz inequality holds:

(25)
$$\log \max_{1\le j\le m} |g_j(w)| \ge -\gamma - (n+1)^2 d_1 \dots d_m [\log(1+|w|^2) - \log d(w, W)],$$

where $\gamma > 0$ and $d(w, W) = \min\{1, \operatorname{dist}(w, W)\}$. If $\eta_0 > 0$ is chosen sufficiently small then inequality (25) implies that, for any point $w \in \mathfrak{S}_{\eta_0}$, $d(w, W) \le 1/2$. The inclusion (24) now shows that $\mathfrak{S}_{\eta_0} \subseteq \mathfrak{C}$ for a convenient choice of K. □

The second observation is that we can assume that for any k, $1 \le k \le m$, the dimension of the variety of $\{z \in \mathbf{C}^n : f_1(z) = \dots = f_k(z) = 0\}$ is exactly $n-k$. In fact, if necessary, we can use [12, §4, Lemma 2] to find a triangular invertible matrix $M \in \mathbf{C}^{m\times m}$ with the property that the sequence of polynomials

$\tilde{f}_1, \ldots, \tilde{f}_m$ defined by $\tilde{f} = Mf$ is a regular sequence in $\mathbf{C}[z_1, \ldots, z_n]$. (Here $f, \tilde{f}$ are the column vectors of components f_j, resp. $\tilde{f}_j$.) We can further assume that the change of variables from Lemma 4 (applied to $\tilde{f}_j$) has been performed. Henceforth we will revert to the old notation $f_1, \ldots, f_m$ for the polynomials and $z = (z', z'') \in \mathbf{C}^m \times \mathbf{C}^{n-m}$ for the coordinates. The constants $K, \eta_0 > 0$ will have the meaning given in Lemma 4. To simplify the notation we will set

$$\Delta = (n+1)^2 d_1 \ldots d_m. \tag{26}$$

As we have said above, one can show that the polynomials $f_1, \ldots, f_m$ form a slowly decreasing family in the sense of [1, Def. 5.1], with respect to the family $\mathfrak{L}$ of m-dimensional affine subspaces $\{z'' = \text{constant}\}$. Namely, given $0 < \eta < \eta_0$ and $u \in \mathbf{C}^{n-m}$, let

$$\mathfrak{D} = \mathfrak{D}(u, \eta) = \mathfrak{S}_\eta \cap \{z = (z', z''): z'' = u\}. \tag{27}$$

The connected components of this set are bounded since $\mathfrak{S}_\eta \subseteq \mathfrak{C}$. Moreover, if z_1, z_2 are in the same connected component $\mathfrak{G}$ of $\mathfrak{D}$ then

$$\|z_1\| \leq \|z_1'\| + \|u\| \leq (K+1)(1 + \|u\|) \leq (K+1)(1 + \|z_2\|).$$

Since the weight p is given by $p(z) = \log(1 + \|z\|^2)$, we conclude that

$$p(z_1) \leq p(z_2) + K_0 \tag{28}$$

for some $K_0 > 0$.

Following [1, §5], given $0 < \eta \leq \eta_0$ and $\tau > 0$, we consider the family $\mathfrak{C}(\eta, \tau)$ of "good" open sets Ω defined as follows: Fix $u \in \mathbf{C}^{n-m}$, and $\mathfrak{G}$ a component of $\mathfrak{D}(u, \eta)$. Let Ω be the union of balls:

$$\Omega = \bigcup_{\zeta \in \mathfrak{G}} B(\zeta, \tau \exp[-(\Delta + \tfrac{1}{2} d_1) p(\zeta)]). \tag{29}$$

It is clear that $\mathfrak{C}(\eta, \tau)$ is a covering of $\mathfrak{S}_\eta$. Also, if $\eta' < \eta$ and $\tau' < \tau$, then $\mathfrak{C}(\eta', \tau')$ is a refinement of $\mathfrak{C}(\eta, \tau)$. In fact, the refinement map $\rho: \mathfrak{C}(\eta', \tau') \to \mathfrak{C}(\eta, \tau)$ is defined as follows. Let $\Omega' \in \mathfrak{C}(\eta', \tau')$. Its definition depends on a choice of $u \in \mathbf{C}^{n-m}$ and $\mathfrak{G}' \subseteq \mathfrak{D}(u, \eta')$. This connected set $\mathfrak{G}'$ is included in a unique connected component $\mathfrak{G}$ of $\mathfrak{D}(u, \eta)$. The corresponding set Ω defined by (29) clearly contains Ω', so that $\rho(\Omega') = \Omega$ is well defined. This refinement map has the extra property, called *almost parallelism* in [1], that for any $\mathfrak{C}(\eta, \tau)$ there is a refinement $\mathfrak{C}(\eta', \tau')$ such that, if $\Omega_1', \Omega_2' \in \mathfrak{C}(\eta', \tau')$,

$$\Omega_1' \cap \Omega_2' \neq \emptyset \text{ implies that } \overline{\Omega_1'} \cup \overline{\Omega_2'} \subseteq \rho(\Omega_1') \cap \rho(\Omega_2'). \tag{30}$$

This is an immediate consequence of the fact that there is a constant $A > 0$ for which

$$\left| \frac{\partial f_j(\zeta)}{\partial \zeta_k} \right| \leq A \exp\left[\frac{(d_1 - 1) p(\zeta)}{2} \right], \quad j = 1, \ldots, m, \ k = 1, \ldots, n, \ \zeta \in \mathbf{C}.$$

These observations show that $f_1, \ldots, f_m$ is indeed a slowly decreasing family.

The main idea of the proof of Theorem 3 will be to show that there is a set $\mathfrak{S}_\eta$ and holomorphic functions $h_1, \ldots, h_m$ in $\mathfrak{S}_\eta$, with good bounds in $\mathfrak{S}_\eta$, chosen so that

$$f = f_1 h_1 + \cdots + f_m h_m \quad \text{in } \mathfrak{S}_\eta. \tag{31}$$

Once this is done, the remainder of the proof will follow the argument in Theorem 2.

The construction of the h_j involves two double complexes. The first one is a local version of the Koszul complex used in Section 1. Given a covering $\mathcal{C} = \mathfrak{C}(\eta, \tau)$, we can define the group $\mathfrak{U}^r = \mathfrak{U}^r(\mathcal{C})$ of alternating analytic r-cochains. Namely, $\gamma \in \mathfrak{U}^r$ means that for $\Omega_0, \ldots, \Omega_r \in \mathcal{C}$ we have

$$\gamma(\Omega_0, \ldots, \Omega_r) \in H(\Omega_0 \cap \cdots \cap \Omega_r) \cap L^\infty(\Omega_0 \cap \cdots \cap \Omega_r),$$

and γ is alternating on the indices $(\Omega_0, \ldots, \Omega_r) \in \mathcal{C}^{r+1}$. We denote

$$\|\gamma(\Omega_0, \ldots, \Omega_r)\| = \sup\{|\gamma(\Omega_0, \ldots, \Omega_r)(z)| : z \in \Omega_0 \cap \cdots \cap \Omega_r\}; \tag{32}$$

$$\|\gamma(z)\| = \sup\{\|\gamma(\Omega_0, \ldots, \Omega_r)\| : \text{for every } (\Omega_0, \ldots, \Omega_r) \in \mathcal{C}^{r+1} \text{ such that } z \in \Omega_0 \cap \cdots \cap \Omega_r\}. \tag{33}$$

The associated Koszul complex is given by

$$\mathfrak{U}^r_q = \mathfrak{U}^r \otimes_{\mathbf{C}} \Lambda^q \mathbf{C}^m, \quad r \geq 0, \ q \geq 0. \tag{34}$$

Its elements can be represented as families $\{w_I^J\}$ of analytic functions, $J \in \mathcal{C}^{r+1}$, $I \subseteq \{1, \ldots, m\}$, $\#I = q$, which are alternating in both indices. Formulas (32) and (33) define norms for $\{w_I^J\}$, simply by considering the largest entry in the index I. As in Section 1, we consider the commutative diagram of maps:

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \\
\longrightarrow & & \mathfrak{U}^r_q & \xrightarrow{\ \delta\ } & \mathfrak{U}^{r+1}_q & \longrightarrow & \\
 & & \downarrow P & & \downarrow P & & \\
\longrightarrow & & \mathfrak{U}^r_{q-1} & \xrightarrow{\ \delta\ } & \mathfrak{U}^{r+1}_{q-1} & \longrightarrow & \\
 & & \downarrow & & \downarrow & &
\end{array}$$

given by

$$P(w)^J_I = \sum_{i=1}^{m} w^J_{Ii} f_i \tag{35}$$

and

(36) $$(\delta w)_I^J = \sum_{j=0}^{r+1} (-1)^j w_I^{J(j)},$$

where if $I = \{i_1, \ldots, i_q\}$ and $J = \{\Omega_0, \ldots, \Omega_{r+1}\}$, then $Ii = \{i_1, \ldots, i_q, i\}$ and $J(j) = \{\Omega_0, \ldots, \Omega_{j-1}, \Omega_{j+1}, \ldots, \Omega_{r+1}\}$. Clearly $P^2 = 0$ and $\delta^2 = 0$. All the above considerations clearly depend on the covering $\mathfrak{C}$. Moreover, if $\mathfrak{C}' = \mathfrak{C}(\eta', \tau')$ is a refinement of $\mathfrak{C}$ and $\rho: \mathfrak{C}' \to \mathfrak{C}$ is the corresponding refinement map, then ρ induces maps

(37) $$\rho: \mathfrak{U}_q^r(\mathfrak{C}) \to \mathfrak{U}_q^r(\mathfrak{C}')$$

such that $P\rho = \rho P$ and $\rho\delta = \delta\rho$.

The second complex, as in [8, Prop. 7.6.1], relates the Čech cohomology to the de Rham cohomology. Let $L_{s,r}(\mathfrak{C}) = L_{s,r}$ be the spaces of r-cochains for the covering $\mathfrak{C}$ with values in the space of $(0, s)$ differential forms in $\mathbf{C}^n$, which have finite L^2-norms in the following sense. Namely, let

$$\gamma(\Omega_0, \ldots, \Omega_r) = \sum_{|j|=s} \gamma_j d\bar{z}^j;$$

$$\gamma_j: \gamma_j(\Omega_0, \ldots, \Omega_r) \in L^2(\Omega_0 \cap \cdots \cap \Omega_r, d\lambda);$$

$$\|\gamma(\Omega_0, \ldots, \Omega_r)\|_2 = \sum_{|j|=s} \int_{\Omega_0 \cap \cdots \cap \Omega_r} |\gamma_j(z)|^2 \, d\lambda(z) < \infty.$$

For γ an r-cochain, we define a new pointwise norm (compare with (32) and (33)):

(38) $$|||\gamma(z)||| = \sup\{\|\gamma(\Omega_0, \ldots, \Omega_r)\|_2 : \text{for every } (\Omega_0, \ldots, \Omega_r) \in \mathfrak{C}^{r+1} \text{ such that } z \in \Omega_0 \cap \cdots \cap \Omega_r\}.$$

The following three technical lemmas, involving properties of these two complexes, are crucial for the proof of Theorem 3.

Let f be a polynomial in $I(f_1, \ldots, f_m)$, and consider the element $\alpha_0 \in \mathfrak{U}_0^0(\mathfrak{C}(\eta_0, 1))$ defined by

(39) $$\alpha_0(\Omega) = f|_\Omega.$$

For some constant $A_0 > 0$, we have

$$\|\alpha_0(z)\| \leq A_0 \exp[\tfrac{1}{2}(\deg f) p(z)] \quad (z \in \mathbf{C}^n).$$

Moreover, we can show the following.

LEMMA 5. *There is a refinement* $\mathfrak{C}_1 = \mathfrak{C}(\eta_1, \tau_1)$ *of* $\mathfrak{C}(\eta_0, 1)$ *and a* $\beta_0 \in \mathfrak{U}_1^0(\mathfrak{C}_1)$ *such that*

(40) $$\rho(\alpha_0) = P\beta_0,$$

(41) $$\|\beta_0(z)\| \leq A_0' \exp[(\tfrac{1}{2}(\deg f) + N_0) p(z)] \quad (z \in \mathbf{C}^n)$$

for some constant $A_0' > 0$, *and*

(42) $$N_0 = (m+2)\Delta + 2md_1.$$

Proof. Let $\Omega \in \mathcal{C}_0 = \mathfrak{C}(\eta_0, 1)$. Ω is defined in terms of $u \in \mathbf{C}^{n-m}$ and $\mathfrak{G} \subseteq \mathfrak{D}(u, \eta_0)$. Fix a point $\zeta_0 \in \mathfrak{G}$ and define σ by

(43) $$\sigma = \tfrac{1}{4}\eta_0 \exp[-\Delta p(\zeta_0) - \Delta K_0].$$

For $\tau > 0$, let $\tau_0 = \tau \exp[-(\Delta + \frac{1}{2}d_1)p(\zeta_0)]$. We can choose $\tau > 0$ so that, if $z \in \mathfrak{G}$ and $|w - z| \leq \tau_0$, then we have $w \in \Omega$ and $|f_i(z) - f_i(w)| < \sigma$ ($i = 1, \ldots, m$). Therefore, for $|t - u| \leq \tau_0$, we have $\max_{1 \leq i \leq m}|f_i(z)| \geq 3\sigma$ for any $z \in \partial(\Omega \cap \{z'' = t\})$. Let

$$\mathfrak{G}_\sigma = \left\{ z \in \mathfrak{G} \colon \max_{1 \leq i \leq m} |f_i(z)| \leq \frac{\sigma}{2} \right\}.$$

It is clear now that we can find a refinement $\mathcal{C}_1 = \mathfrak{C}(\eta_1, \tau_1)$ of $\mathcal{C}_0$ such that if $\Omega' \in \mathcal{C}_1$ and $\rho(\Omega') = \Omega$ then the following condition holds:

$$\Omega' \subseteq \{z + w \colon z \in \mathfrak{G}_\sigma, |w| \leq \tau_0\} \subseteq \rho(\Omega') = \Omega.$$

Therefore, whenever $|t - u| \leq \tau_0$, one can conclude that

$$\Omega' \cap \{z'' = t\} \subseteq \left\{ z \in \Omega \colon \max_{1 \leq i \leq m} |f_i(z)| \leq \frac{3\sigma}{2} \right\}.$$

Using Sard's theorem we can choose σ', $3\sigma/2 \leq \sigma' \leq 2\sigma$, with the property that for any t, $|t - u| \leq \tau_0$, the analytic polyhedron in $\mathbf{C}^m \times \{t\}$ given by

$$U_t = \{z = (s, t) \in \Omega \colon \max_{1 \leq i \leq m} |f_i(z)| \leq \sigma'\}$$

has a smooth boundary whose $(2m-1)$-dimensional volume L satisfies

$$L \leq \exp[m(\Delta + d_1 + 1)p(\zeta_0)]$$

for some constant $A > 0$ that depends on m, K_0, and η_0 (see [1, Cor. 1.6]).

We can now apply [1, Prop. 1.3 and following Remark] and the fact that $f \in I$, to obtain a $\mathbf{C}^m$ valued holomorphic function $\beta_0(\Omega')$ such that

$$P(\beta_0(\Omega')) = \alpha_0|_{\Omega'}$$

and for which, at any point $z \in \Omega'$, we have

$$\|\beta_0(\Omega')(z)\| \leq A_0' \exp\left[\left(\frac{\deg f}{2} + N_0\right)p(z)\right].$$

This conclude the proof of Lemma 5. □

Lemma 5 can be generalized as follows.

LEMMA 6. *Given a covering $\mathcal{C} = \mathfrak{C}(\eta, \tau)$ $(\eta \leq \eta_0, \tau \leq 1)$ and integers $q \geq 1$ and $r \geq 0$, there is a refinement $\mathcal{C}' = \mathfrak{C}(\eta', \tau')$ such that, if $\alpha \in \mathfrak{U}_q^0(\mathcal{C})$ satisfies $P\alpha = 0$ and the estimate*

(44) $$\|\alpha(z)\| \leq B \exp[Dp(z)],$$

then there is $\beta \in \mathfrak{U}_{q+1}^r(\mathcal{C}')$ with $P\beta = \alpha$ and

(45) $$\|\beta(z)\| \le B' \exp[(D+N_q)p(z)],$$

where $N_q = m^q(N_0 + d_1/2)$.

Proof. Note that Lemma 5 covers the case where $q=0$ and $r=0$. For $r=0$, the proof in [1, Thm. 5.3] reducing the case $q \ge 1$ to the case $q=0$ leads immediately to the existence of the refinement and the above estimates. It is here that one needs to use that the sequence $\{f_1, \ldots, f_m\}$ is regular.

In [1] the reduction of the case $r \ge 1$ to the case $r=0$ is not done, so we furnish a proof here.

Fix $r \ge 1$. Given a covering $\mathcal{C}$, first construct a refinement $\mathcal{C}'$ with the property that if $\Omega_j' \in \mathcal{C}'$ $(j=0, \ldots, r)$ and corresponding $\Omega_j = \rho(\Omega_j')$, then

(46) $$\overline{\Omega_0'} \subseteq \Omega_0 \cap \cdots \cap \Omega_r \quad \text{if } \Omega_0' \cap \cdots \cap \Omega_r' \ne \emptyset.$$

This is a consequence of the almost parallelism. Now we apply the result for $r=0$ to the covering $\mathcal{C}'$ and obtain a covering $\mathcal{C}''$ so that the conclusion of the lemma holds (still for $r=0$). Let $\alpha \in \mathfrak{U}_q^r(\mathcal{C})$, $P\alpha = 0$, and $\Omega_0'', \ldots, \Omega_r'' \in \mathcal{C}''$ with $\Omega_0'' \cap \cdots \cap \Omega_r'' \ne \emptyset$; it follows that (46) holds with $\Omega_j' = \rho(\Omega_j'')$, because $\Omega_0'' \cap \cdots \cap \Omega_r'' \subseteq \Omega_0' \cap \cdots \cap \Omega_r'$. Consider

$$\gamma = \gamma(\Omega_0') = \alpha(\Omega_0, \ldots, \Omega_r)|_{\Omega_0'};$$

then $P\gamma = 0$. Hence, there is $\tilde{\beta} \in \mathfrak{U}_{q+1}^0(\mathcal{C}'')$ such that $P\tilde{\beta} = \rho(\gamma)$, and $\tilde{\beta}$ satisfies the estimate (45) if α satisfies (44) (the constant B' might change). Now we define

$$\beta(\Omega_0'', \ldots, \Omega_r'') = \tilde{\beta}(\Omega_0'')|_{\Omega_0'' \cap \cdots \cap \Omega_r''}.$$

Then $\beta \in \mathfrak{U}_{q+1}^r(\mathcal{C}'')$, $P\beta = \rho(\alpha)$, and (45) holds. □

For the complex $L_{s,r}$ we have an effective version of [1, Thm. 5.2] (see also [8]).

LEMMA 7. *Given a covering* $\mathcal{C} = \mathfrak{C}(\eta, \tau)$ *and* $r \in \mathbf{N}$, *there exists a refinement* $\mathcal{C}' = \mathfrak{C}(\eta', \tau')$ *such that if* $\beta \in L_{s,r+1}(\mathcal{C})$, $\delta\beta = 0$, $\bar{\partial}\beta = 0$, *and the estimate*

(47) $$|||\beta(z)||| \le A \exp[Dp(z)]$$

is satisfied, then there is $\gamma \in L_{s,r}(\mathcal{C}')$ *such that* $\delta\gamma = \rho(\beta)$, $\bar{\partial}\gamma = 0$, *and*

(48) $$|||\gamma(z)||| \le A' \exp[(D+M_r)p(z)],$$

with $M_r = (r+1)(\Delta + d_1/2 + 2) + n/2$.

Proof. We prove this lemma by induction on r starting with $r=0$. Let $\mathcal{C} = \mathfrak{C}(\eta, \tau/2)$. Then we can consider a covering $\mathfrak{Q}$ of $\mathbf{C}^n$ by open cubes Q_j with sides parallel to the axes, centered at points a_j, such that

(i) if $\Omega' \in \mathcal{C}'$ and $\Omega' \cap Q_j \ne \emptyset$ then $Q_j \subseteq \rho(\Omega')$;

(ii) diam $Q_j = c \exp[-(\Delta + d_1/2)p(a_j)]$ for some constant $c > 0$; and

(iii) $\forall z \in \mathbf{C}^n$: $\#\{j : z \in Q_j\} \le A_n$, where A_n is a positive constant depending only on the dimension n.

The existence of such a covering $\mathfrak{Q}$ follows from the well-known Whitney's lemma. Associated with $\mathfrak{Q}$ there is a partition of unity $\varphi_j \in C_0^\infty(Q_j)$, for which $|\bar\partial\varphi_j(z)| \leq c' \exp[(\Delta + d_1/2)p(z)]$ for some $c' > 0$.

Let $J = \{j : Q_j \cap \Omega' \neq \emptyset$ for some $\Omega' \in \mathfrak{C}'\}$. For $j \in J$ let Ω_j be any choice of $\rho(\Omega')$ with $\Omega' \cap Q_j \neq \emptyset$. Now we can define an element $b \in L_{s,0}(\mathfrak{C}')$ by

$$b(\Omega') = \sum_{j \in J} \varphi_j \beta(\rho(\Omega'), \Omega_j)|_{\Omega' \cap Q_j}.$$

Since $\delta\beta = 0$, $\delta b = \rho(\beta)$. We claim now that $\bar\partial b \in L_{s+1,0}(\mathfrak{C}')$. In fact, we have the estimate

$$\|\bar\partial b(z)\| \leq c'' \|\beta(z)\| \exp\left[\left(\Delta + \frac{d_1}{2}\right)p(z)\right]. \tag{49}$$

The constant c'' depends on c' and A_n.

We have $\delta\bar\partial b = \bar\partial\delta b = \bar\partial\beta = 0$. Hence we can extend $\bar\partial b$ as zero outside of $\bigcup_{\Omega' \in \mathfrak{C}'} \Omega'$ and obtain a global differential form in L_{s+1}^0 (see §1). We can appeal to the same theorem of solvability of $\bar\partial$ used in Section 1 and obtain $v \in L_s^0$ with $\bar\partial v = \bar\partial b$ and

$$\int_{\mathbf{C}^n} |v(z)|^2 \exp[-(2D + 2\Delta + d_1 + n + 2)p(z)]\, d\lambda(z) < \infty.$$

In fact, one can also conclude that when we consider v as a 0-cochain

$$\|v(z)\| \leq c''' \exp[(D + \Delta + 1 + \tfrac{1}{2}(d_1 + n))p(z)], \tag{50}$$

for some $c''' > 0$. Consider now $b - v \in L_{s,0}(\mathfrak{C}')$. We have that $\delta(b - v) = \rho(\beta)$, $\bar\partial(b - v) = 0$, and $\|b - v\|$ satisfies the same estimate (50) (with a different constant c'''). If we take $M_0 = \Delta + 2 + \frac{1}{2}(d_1 + n)$, $\gamma = b - v$, we have proved the lemma in the case $r = 0$.

For the inductive step from r to $r+1$, we need to take a bit more care with the refinement. We start as before with a first refinement $\mathfrak{C}' = \mathfrak{C}(\eta_1, \tau/2)$, and $\mathfrak{Q}$ as above. We can define b similarly:

$$b(\Omega_0', \ldots, \Omega_r') = \sum_{j \in J} \varphi_j \beta(\rho(\Omega_0'), \ldots, \rho(\Omega_r'), \Omega_j)|_{\Omega_0' \cap \cdots \cap \Omega_r' \cap Q_j}. \tag{51}$$

We have $\delta b = \rho(\beta)$ and $\bar\partial b \in L_{s+1,r+1}(\mathfrak{C}')$ with the estimate (49) still correct. Since $\delta\bar\partial b = \bar\partial\beta = 0$, we can apply the inductive argument and find a refinement $\mathfrak{C}''$ and $\gamma_0 \in L_{s+1,r}(\mathfrak{C}'')$ such that $\bar\partial\gamma_0 = 0$, $\delta\gamma_0 = \rho(\bar\partial b)$, and

$$\|\gamma_0(z)\| \leq A' \exp\left[\left(D + \Delta + \frac{d_1}{2} + M_r\right)p(z)\right]. \tag{52}$$

As we have done in the proof of Lemma 5, we can construct another refinement $\mathfrak{C}'''$ such that if $\Omega''' \in \mathfrak{C}'''$ then there is an analytic polyhedron $\mathfrak{U}$ such that

$$\overline{\Omega'''} \subseteq \mathfrak{U} \subseteq \bar{\mathfrak{U}} \subseteq \rho(\Omega''').$$

We can apply [8, Thm. 4.4.2] to solve the equation $\bar\partial\gamma_1 = \gamma_0$ in $\mathfrak{U}$. One has

$$\int_{\mathfrak{U}} \frac{|\gamma_1(z)|^2\, d\lambda(z)}{(1+|z|^2)^2} \leq \int_{\mathfrak{U}} |\gamma_0(z)|^2\, d\lambda(z) \leq \|\gamma_0(z_0)\|$$

for z_0 any point of $\mathfrak{U}$. From here we conclude that $\gamma_1 \in L_{s,r}(\mathfrak{C}''')$ and

$$\|\gamma_1(z)\| \leq A'' \exp\left[\left(D+\Delta+\frac{d_1}{2}+M_r+2\right)p(z)\right].$$

Finally, consider $\gamma = \rho(b) - \delta\gamma_1$. Then

$$\delta\gamma = \delta\rho(b) = \rho(\beta), \qquad \bar\partial\gamma = \rho(\bar\partial b) - \delta(\bar\partial\gamma_1) = \rho(\bar\partial b) - \delta\gamma_0 = 0,$$

and γ satisfies the estimate

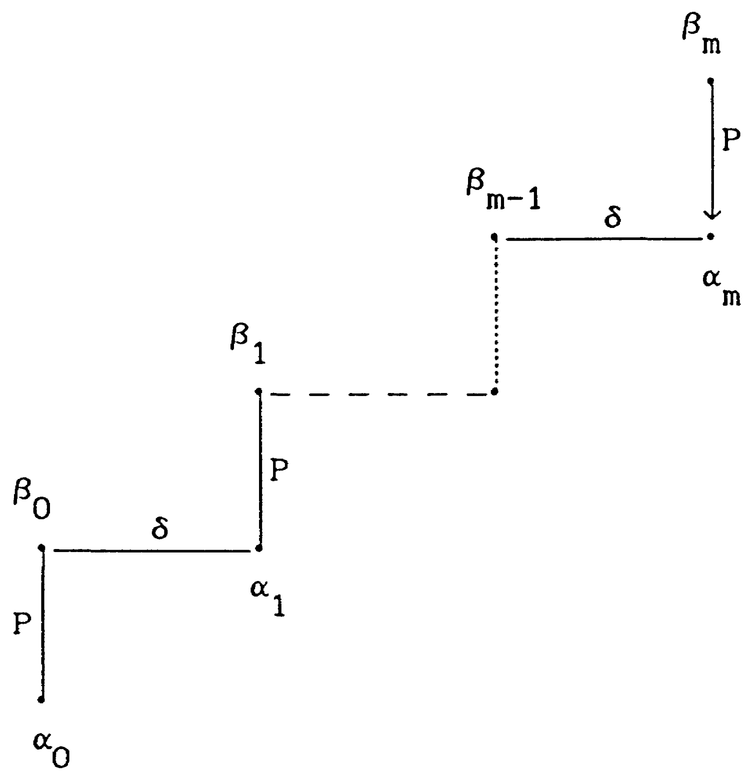
$$\|\gamma(z)\| \leq A''' \exp[(D+M_{r+1})p(z)],$$

with $M_{r+1} = M_r + \Delta + 2 + d_1/2$. □

REMARK. Let $L^q_{s,r} = L_{s,r} \otimes \Lambda^q \mathbf{C}^m$. One can see without difficulty that Lemma 7 holds when we replace $L_{s,r}$ by $L^q_{s,r}$ for any fixed q, $0 \leq q \leq m$; it is simply a question of using the above construction one component at a time.

Proof of Theorem 3. Finding the h_j required in (31) amounts to constructing a covering $\tilde{\mathfrak{C}}$ and $\beta \in \mathfrak{U}^0_1(\tilde{\mathfrak{C}})$ such that $P\beta = \alpha_0$ and $\partial\beta = 0$. We recall that α_0 is defined by (39).

We first construct a sequence $\beta_0, \ldots, \beta_m$, using the local Koszul complex, following the diagram:



What we mean is that, modulo successive refinements, we can define

$$\begin{aligned}
&\beta_0 \in \mathfrak{U}_1^0, \quad P\beta_0 = \alpha_0 \qquad \text{(Lemma 5)}\\
&\alpha_1 = \delta\beta_0 \in \mathfrak{U}_1^1, \quad P\alpha_1 = \delta P\beta_0 = \delta\alpha_0 = 0\\
&\beta_1 \in \mathfrak{U}_2^1, \quad P\beta_1 = \alpha_1 \qquad \text{(Lemma 6 and ff.)}\\
&\vdots\\
&\beta_{m-1} \in \mathfrak{U}_m^{m-1}, \quad P\beta_{m-1} = \alpha_{m-1}\\
&\alpha_m = \delta\beta_{m-1} \in \mathfrak{U}_m^m\\
&\alpha_m = P\beta_m, \quad \beta_m \in \mathfrak{U}_{n+1}^m.
\end{aligned}$$

In the last line we have $\mathfrak{U}_{m+1}^m = 0$, hence $\alpha_m = 0$. Furthermore, we have

$$\|\beta_j(z)\| \le A \exp\left[\left(\frac{\deg f}{2} + N_0 + N_1 + \cdots + N_j\right)p(z)\right] \tag{53}$$

for some $A > 0$.

We know that $\delta\beta_{m-1} = 0$ and $\bar\partial\beta_{m-1} = 0$. Since all the sets in any covering have uniformly bounded volumes (they are very thin by construction), we have that the L^2-norm $|||\beta_{m-1}(z)|||$ has, up to a multiplicative constant, the same bounds as the L^∞-norm $\|\beta_{m-1}(z)\|$. That is, for some constant $A > 0$,

$$|||\beta_{m-1}(z)||| \le A \exp\left[\left(\frac{\deg f}{2} + 2m^{m+1}(\Delta + d_1)\right)p(z)\right]. \tag{54}$$

We have $\beta_{n-1} \in \mathfrak{U}_m^{m-1} \cap L_{0,m-1}^m$. Hence we can apply the remark following Lemma 7 and find, after a refinement of the covering, $\gamma_{m-2} \in L_{0,m-2}^m$ such that $\bar\partial\gamma_{m-2} = 0$, $\delta\gamma_{m-2} = \rho(\beta_{m-1})$, and γ_{m-2} satisfies the estimate

$$|||\gamma_{m-2}(z)||| \le A' |||\beta_{m-1}(z)||| \exp\left[\left((m-1)\left(\Delta + \frac{d_1}{2} + 2\right) + \frac{n}{2}\right)p(z)\right] \tag{55}$$

for some new constant $A' > 0$.

As in Proposition 1, we consider the element

$$\epsilon_{m-2} = \beta_{m-2} - P\gamma_{m-2} \in L_{0,m-2}^{m-1}.$$

We have that $\bar\partial\epsilon_{m-2} = 0$, $P\epsilon_{m-2} = \alpha_{m-2}$,

$$\delta\epsilon_{m-2} = \delta\beta_{m-2} - P\delta(\gamma_{m-2}) = \alpha_{m-1} - P\beta_{m-1} = 0,$$

and the estimate

$$|||\epsilon_{m-2}(z)||| \le A'' \exp\left[\left(\mathfrak{d} + \mathfrak{d}' + \frac{d_1}{2}\right)p(z)\right]$$

holds, with

$$\begin{cases} \mathfrak{d} = (\deg f)/2 + 2m^{m+1}(\Delta + d_1),\\ \mathfrak{d}' = (m-1)(\Delta + d_1/2 + 2) + n/2. \end{cases} \tag{56}$$

We can now iterate the procedure and obtain $\epsilon_0 \in L_{0,0}^1$ with $\bar\partial\epsilon_0 = \delta\epsilon_0 = 0$, $P\epsilon_0 = \alpha_0$, and having similar estimates. This means that ϵ_0 is a holomorphic $\mathbf{C}^m$-valued function defined on the support $\mathfrak{I}$ of a covering $\tilde{\mathfrak{C}}$, $\bigcup_{\tilde\Omega \in \tilde{\mathfrak{C}}} \tilde\Omega = \mathfrak{I}$, such that $P\epsilon_0 = f$ in $\mathfrak{J}$. We need pointwise bounds for ϵ_0; these can be obtained in the support $\mathfrak{J}'$ of a refinement $\tilde{\mathfrak{C}}'$ of $\tilde{\mathfrak{C}}$, by means of Cauchy's formula. Recall that we have

$$\|\epsilon_0(z)\| \le A \exp\left[\left(\mathfrak{d} + (m-2)\left(\mathfrak{d}' + \frac{d_1}{2}\right)\right)p(z)\right]$$
$$\le A \exp\left[\left(\frac{\deg f}{2} + 3m^{m+1}(\Delta + d_1) + m^2 n\right)p(z)\right],$$

which holds for $\tilde{\mathfrak{C}} = \mathfrak{C}(\tilde{\eta}, \tilde{\tau})$. Hence on $\mathfrak{J}'$, the support of $\tilde{\mathfrak{C}}' = \mathfrak{C}(\tilde{\eta}, \tilde{\tau}/2)$, we will have

(57)
$$|\epsilon_0(z)| \le A' \exp\left[\left(\frac{\deg f}{2} + 3m^{m+1}(\Delta + d_1) + n\left(\Delta + \frac{d_1}{2}\right) + m^2 n\right)p(z)\right].$$

The set $\mathfrak{J}'$ contains the set $\mathfrak{S}_{\tilde{\eta}}$. We take a C^∞ function χ with $0 \le \chi \le 1$, $\operatorname{supp} \chi \subseteq \mathfrak{S}_{\tilde{\eta}}$, $\chi = 1$ on $\mathfrak{S}_{(\tilde{\eta}/2)}$, and such that one has

$$|\bar{\partial}\chi(z)| \le \text{const.} \exp\left[\left(\Delta + \frac{d_1}{2}\right)p(z)\right].$$

We are now ready to follow the proof of Theorem 2. We set

$$h_j = \chi\epsilon_{0,j} + \frac{(1-\chi)\bar{f}_j f}{|F|^2},$$

where $\epsilon_0 = (\epsilon_{0,1}, \dots, \epsilon_{0,n})$, $F = (f_1, \dots, f_m)$, and $|F|^2 = \sum_{j=1}^m |f_j|^2$. Therefore $\sum_{j=1}^m h_j f_j = f$. Moreover, $|F|$ has good lower bounds on the support of $\bar{\partial}\chi$. Namely, from (22) we have that

$$\log|F(z)| \ge \log(\tilde{\eta}/2) - (n+1)^2 d_1 \dots d_m \log(1+|z|^2).$$

This allows us to conclude that

$$|\bar{\partial}h_j(z)| \le A'' \exp[\mathfrak{a}p(z)];$$
$$\mathfrak{a} = \frac{\deg f}{2} + 3m^{m+1}(\Delta + d_1) + (n+1)\left(\Delta + \frac{d_1}{2}\right) + m^2 n.$$

Furthermore, if we let

$$C = 2\mathfrak{a} + (2m-3)\Delta + n + 1,$$

then we have

$$\int |F|^{-2(2m-3)} |\bar{\partial}h|^2 \exp[-Cp]\, d\lambda < \infty.$$

As in Theorem 2, we have a $\Lambda^2\mathbf{C}^m$-valued function u such that

$$\int |u|^2 \exp[-C'p]\, d\lambda < \infty,$$

with $C' = C + 2(m-2)d_1 + 2m - 2$ and $P\bar{\partial}u = \bar{\partial}h$.

Let now $g = h - P\dot{u}$; then g is a $\mathbf{C}^m$-valued function, $Pg = f$, $\bar{\partial} g = 0$, and

$$\int_{\mathbf{C}^n} |g|^2 \exp[-(C' + d_1)p]\, d\lambda < \infty. \tag{58}$$

Therefore the components g_j of g are polynomials of degree

$$\deg g_j \leq C' + d_1 \leq \deg f + (6m^{m+1} + 3n)(\Delta + d_1).$$

This conclude the proof of Theorem 3. □

REMARKS. (1) Theorems 2 and 3 remain valid in $K[z]$ for any number field K of characteristic zero, as a consequence of the Lefschetz principle, and in particular for polynomials with rational coefficients.

(2) In case $f, f_1, \dots, f_m \in \mathbf{Q}[z]$ and the situation of Theorems 2 and 3 occurs, it would be interesting to find good estimates for the logarithmic heights of the polynomials $g_j \in \mathbf{Q}[z]$. One would hope for bounds analogous to those found in [2] for the case $1 \in I(f_1, \dots, f_m)$. The existence of good bounds would be an indication that there must be in these cases algorithms capable of finding g_j in polynomial time.

Added in proof. F. Amoroso has recently found an algebraic proof of our results; see *Tests d'appartenance d'après un Théorème de Kollár,* C. R. Acad. Sci. Paris 390 (1989), 691–694.

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