RESIDUE CURRENTS

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1 Introduction

Integral representations and residues provide a very powerful tool to investigate functions and compute integrals. For instance one can represent by residue integrals the number of roots of a system of algebraic equations, the roots themselves, the solutions of differential equations, and many special functions of mathematical physics. Construction of residues is naturally connected with analytic sets ; therefore residue theory interfers very deeply with algebraic geometry. For this reason residues have been playing in the last years an important role in computational effectivity problems within the frame of computer algebra.

There exist two different approaches towards the concept of multidimensional residue theory. The first one, so called *classical*, is connected with integration of closed differential forms over cycles ; the other one, so called *current-approach*, has to do with integration of smooth differential forms which are not necesseraly closed. The approach towards the concept in its classical variant appeared in 1887 with H. Poincaré, but took form only in the fifties when G. de Rham invented the notion of iterated indefinite integral. At the same time the notion of the *current* came out, and, thanks to the investigations of P. Dolbeault, who pointed out the important role of the $\overline{\partial}$-operator, one began to formalize the current approach towards multidimensional residue theory. From the point of view of general current (or distribution) theory, residue currents and principal values of integrals (which are very closely related) are just examples of currents, but very important and constructive ones indeed. Their role in distribution theory could be compared to the role of numbers such as e and π in real analytic number theory.

The class of residue currents contains the classical Dirac's delta-function as well as its generalization, the integration current over some analytic set. For analytic sets of codimension 1, the so-called integration current $[Z_f]$ along the divisor $Z_f = \{f = 0\}$ of the holomorphic function f can be described as follows thanks to Poincaré-Lelong formula :

$$\frac{1}{2\pi i}\overline{\partial}\partial \log |f|^2 = [Z_f] \,. \tag{1.1}$$

This means that in order to define the action of the delta-function with support the set of solutions of some equation $f = 0$ in n variables on some differential test form with bidegree $(n-1, n-1)$, one has to multiply $\overline{\partial}\partial\varphi/2\pi i$ with the locally integrable function $\log |f|^2$.

Taking first the action of the ∂ operator on the logarithmic function, one can rewrite the Poincaré-Lelong formula as

$$\frac{1}{2\pi i}\overline{\partial}[\frac{1}{f}] \wedge df = [Z_f]\,. \tag{1.2}$$

In fact the relation between (1.1) and (1.2) lies in Stokes's formula and may be expressed (in terms of the action on some $(n-1,n-1)$ test form φ) as

$$\frac{1}{2\pi i}\langle\overline{\partial}[\frac{1}{f}], df \wedge \varphi\rangle = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int\limits_{|f|=\varepsilon} \frac{1}{f}\cdot df \wedge \varphi = \int\limits_{Z_f} \varphi. \tag{1.3}$$

Here comes an important question :

does the current $\overline{\partial}[1/f]$ *exist* ?

One has to remark that (1.3) guarantees its existence in presence of the weight factor df. In 1971, M. Herrera and D. Lieberman [49] proved the existence of $\overline{\partial}[1/f]$ using Hironaka's desingularisation theorem. Curiously, it remains unknown whether there exists some simple proof of such a fact without using desingularization.

The next step in the theory of residue currents was achieved by Coleff & Herrera [26]. They introduced the residue current associated to several functions $f_1, ..., f_m$, which is denoted as

$$\overline{\partial}\Big[\frac{1}{f_1}\Big] \wedge \ldots \wedge \overline{\partial}\Big[\frac{1}{f_m}\Big]\,.$$

Nevertheless, such a current inherit good properties only in the case $f_1, ..., f_m$ define a complete intersection, i.e the codimension of the set of common zeroes $f^{-1}(0)$ in the ambient manifold equals the number of functions, that is m. In order to turn around the inherent disadvantage of the Coleff-Herrera current (which construction is based on the Cauchy kernel), one introduced in the last ten years various alternative approaches (more inspired by the Bochner-Martinelli kernels, generally speaking the Cauchy-Fantappié kernels). In fact, the cycle on which one integrates the Cauchy kernel, that is $\{|z_1| = \epsilon_1\,, \ldots\,, |z_n| = \epsilon_n\}$, depends on n free parameters, namely $\epsilon_1, ..., \epsilon_n$, while the Bochner-Martinelli kernel is integrated on spheres such as $\{|z_1|^2 + \ldots + |z_n|^2 = \epsilon\}$, ϵ being in this case the only parameter. Such circumstances allow the application of one parameter asymptotic approaches, which are usually deduced from multivariate ones by averaging.

The theoretical part of the following survey is inspired by the philosophy we just mentionned above ; since one knows how some very important algebraic or geometric notions (as the multiplicity of intersection) are much more difficult to handle in the non complete intersection case than in the complete one, it seemed important to articulate this survey about the recent developments, together with some of their applications, of residue theory from the current point of view, focusing particularly on these asymptotic various aspects, in the complete as well as non complete intersection case. Some of the applications we took to support our objectives came from the somehow unexpected role complex analysis may play respect to the study of very classical problems, such as membership or nullstellensatz in polynomial algebra ; which is not so much a surprize since one knows how concepts such as trace or duality remain central. In fact, analysis brought also quite powerful tools, also quite inherent to physics, such as Mellin transforms, or even the quite new concept of amoeba. The study of meromorphy or holomorphy on singular analytic sets appeals also deeply to the current approach of residue theory, to the trace formula, to Lagrange interpolation, and it looked to us as another illuminating example to illustrate our presentation.

Of course, the field was so large and there were so many developments in the past ten years (in such various domains as toric geometry, sparse polynomial algebra, study of special functions, especially the hypergeometric ones) that we had to make drastic choices in the presentation. Nevertheless, we hope this survey will play the role of a modest invitation for the reader to enter more deeply in such an interesting area (for itself as well as for its applications).

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2 Residue integral and Coleff-Herrera residue current

2.1 Residue integral ; advantages and disadvantages

One way to express the classical residue of a one variable meromorphic $(1,0)$ form $h(z)\,dz/f$ at a polar point $a \in \mathbb{C}$ is to represent it as the integral

$$\int_{|z-a|=\epsilon} \frac{h(z)\,dz}{f(z)}\,,$$

where ϵ is sufficiently small. The circle of integration $|z-a|=\epsilon$ may of course be replaced by some other path of integration, as for instance the path $\{z \in \mathcal{U}_a : |f(z)| = \epsilon\}$, where $\mathcal{U}_a$ is a small neighborhood of the point a not containing any other zero of the function f. With such a choice for the path of integration the notion of residue has been extended to the n-dimensional case : one can consider instead of the meromorphic integrand $h\,dz/f$ a $2n-1$-semimeromorphic differential form (i.e the quotient φ/f of some smooth $2n-1$-differential form φ by some holomorphic function f) and take formally the limit of the integral

$$\int_{|f|=\epsilon} \frac{\varphi}{f}$$

when ε tends to zero. But when $n>1$, the path of integration is nomore compact, therfore, one has to consider in the numerator compactly supported differential forms. So, if $f\colon \mathcal{X} \to \mathbb{C}$ is a holomorphic function defined on a n-dimensional complex manifold $\mathcal{X}$, then, for any smooth compactly supported test form with degree $2n-1$ in $\mathcal{D}^{2n-1}(\mathcal{X})$, the limit

$$R_f(\varphi) = \lim_{\epsilon \to 0} \int_{|f|=\epsilon} \frac{\varphi}{f}$$

should be considered as the action on φ by the current $\overline{\partial}[1/f]$ we mentionned in the introduction. The first proof of the existence of such a residue current was given by Herrera & Lieberman [49] in 1971.

Theorem 2.1 [49] *For any $\varphi \in \mathcal{D}^{2n-1}(\mathcal{X})$, the limit $R_f(\varphi)$ exists and this action defines a $(0,1)$-current, i.e a continuous linear functional on $\mathcal{D}^{2n-1}(\mathcal{X})$, which takes nonzero values only on $(n,n-1)$-forms $\varphi \in \mathcal{D}^{n,n-1}(\mathcal{X})$.*

In order to generalize this result to some holomorphic mapping

$$f = (f_1, \dots, f_m) : \mathcal{X} \to \mathbb{C}^m ,$$

Coleff & Herrera considered ([26], 1978) the integral

$$I(\epsilon) = \int_{T_\epsilon(f)} \frac{\varphi}{f_1 \cdots f_m}, \qquad \varphi \in \mathcal{D}^{2n-m}(\mathcal{X}), \tag{2.1}$$

over the tube $T_\epsilon(f) = \{|f_1|^2 = \epsilon_1, ..., |f_m|^2 = \epsilon_m\}$. We shall use the following :

Definition 2.1 *The integral $I(\epsilon) = I_f^\varphi(\epsilon_1, ..., \epsilon_p)$ is called the residue integral or residue function.*

As we will see later, the behaviour of the residue integral near the origin $\epsilon = 0$ carries some significant information about the zero set $f^{-1}(0)$ of the mapping f as well as about the structure of this mapping near $f^{-1}(0)$. At the same time it is easy to write down, and these are the advantages of the defined residue integral. But the point is that the residue function will not in general have a well defined limit at the origin $\epsilon = 0$. Let us show this by the simple example of the mapping $f \colon \mathbb{C}^2 \to \mathbb{C}^2$ with $f_1 = z_1$, $f_2 = z_1 z_2$. If we take $\varphi = \tilde{\varphi}(z)\, dz_1 \wedge dz_2$, we obtain the residue function

$$I_f^\varphi(\epsilon) = \frac{1}{(2\pi i)^2} \int_{\substack{|\zeta_1| = \epsilon_1 \\ |\zeta_2| = \epsilon_2/\epsilon_1}} \frac{\tilde{\varphi}(\zeta)\, d\zeta_1 \wedge d\zeta_2}{\zeta_1^2\, \zeta_2},$$

and we see that if one approaches the origin following a path $\delta \mapsto \epsilon(\delta)$ along which $\epsilon_2/\epsilon_1 \to \infty$, then the domain of integration will be disjoint from the compact support of φ for ϵ_2/ϵ_1 sufficiently large, so that the limit of I_f^φ will be zero ; on the other hand, choosing a path along which $\epsilon_2/\epsilon_1 \to 0$ will yield the limit $\partial_{z_1} \tilde{\varphi}(0,0)$.

So, the above example emphasizes the fact that residue integral has an essential disadvantage which is usually inherent to functions in several variables when studied from the asymptotical point of view. This is related to the fact that for non complete intersection mappings $f \colon \mathbb{C}^n \to \mathbb{C}^m$ (when the zero set $f^{-1}(0)$ has dimension bigger then $n - m$) there are various approaches towards the study of $f^{-1}(0)$ (considered as an intersection of supports of divisors) from the point of view of intersection theory. But the curious situation revealing the noncontinuous property of the residue function at $\epsilon = 0$ may also appear in the complete intersection case, which is a case where all intersection theories provide the same notion of multiplicity. We will discuss this question in the next subsection.

Collef & Herrera suggested to consider the limit of the residue function along a special, so called *"admissible"*, path $\epsilon = \epsilon(\delta)$ such that

$$\lim_{\delta\to 0}\epsilon_m(\delta) = 0, \quad \text{and} \quad \lim_{\delta\to 0}\frac{\varepsilon_j(\delta)}{(\varepsilon_{j+1}(\delta))^q} = 0, \quad j = 1, \ldots, m-1,$$

for any positive integer q. They proved the following theorem :

Theorem 2.2 [26] *For any "admissible" path $\delta \mapsto \epsilon(\delta)$, the limit*

$$R_f(\varphi) = \lim_{\delta\to 0}\int_{T_{\epsilon(\delta)}(f)}\frac{\varphi}{f_1 \cdots f_m}, \quad \varphi \in \mathcal{D}^{2n-m}(\mathcal{X}). \tag{2.2}$$

exists and defines a $(0, m)$-current which is independent of the particular choice of "admissible" path.

We will use also for the current defined in (2.2) the notation

$$R_f = \overline{\partial}\Big[\frac{1}{f_1}\Big] \wedge \ldots \overline{\partial}\Big[\frac{1}{f_m}\Big]\,.$$

Up to now, all existing proofs of this theorem are based on Hironaka's desingularisation theorem (see [26, 71, 88]). In the complete intersection case ($m = n$), we will give a new proof without using Hironaka's theorem.

2.2 Complete intersection case $f : \mathbb{C}^n \to \mathbb{C}^n$

Holomorphic mappings $f : \mathbb{C}^n \to \mathbb{C}^n$ defining a complete intersection are locally proper mappings ; we will use in an essential way this property. In this case the residue function looks like the integral

$$I(\epsilon) = \int_{T_\epsilon(f)}\frac{h\, dz_1 \wedge \ldots \wedge dz_n}{f_1 \cdots f_n} \tag{2.3}$$

over the tube $T_\epsilon(f) = \{|f_1|^2 = \epsilon_1, ..., |f_n|^2 = \epsilon_n\}$, where $h = h(z, \bar{z})$ is a smooth compactly supported function. The problem of defining residue currents has a local nature, due to the existence of partitions of unity, hence we can restrict the tube $T_\varepsilon(f)$ to some small neighborhood $\mathcal{U}_a$ of an isolated zero $a \in f^{-1}(0)$. Now we do not have to worry anymore about the compactness property for the support of the fuction h. In the case when h is holomorphic near a our integral does not depend of ϵ and coinsides with the Grothendieck residue [44, 88].

In fact, for complete intersection mappings $f : \mathbb{C}^n \to \mathbb{C}^n$, we prove here a stronger result than Coleff & Herrera's theorem 2.2. Let us consider our "reservoir" of pathes $\delta \mapsto \epsilon(\delta)$ for the definition of the residue integral (2.2) as the set of "parabolic" pathes

$$\epsilon(\delta) = \epsilon_t(\delta) = (\delta^{t_1}, ..., \delta^{t_n})$$

with positive fixed numbers $t_1, ..., t_n$. The admissible pathes which lie in this collection and fit with the Coleff & Herrera approach are those for which $t_1 \gg ... \gg t_n$.

Theorem 2.3 *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a complete intersection. There exist finitely many hyperplanes dividing the positive octant $\mathbb{R}^n_+$ into the finite set $\{C_\nu\}$ of n-dimensional cones such that for any $t \in \overset{\circ}{C_\nu}$ (here $\overset{\circ}{C_\nu}$ denotes the interior of C_ν) the limits*

$$\lim_{\delta \to 0} \int_{T_{\epsilon_t}(f)} \frac{h\, dz_1 \wedge ... \wedge dz_n}{f_1 ... f_n} \tag{2.4}$$

exist and is independent both on $t \in \overset{\circ}{C_\nu}$ and of ν.

Remark. Using the desingularisation theorem in [71], it was proved that even in the non complete intersection case, the "reservoir" of parabolic pathes ensures the partition of the positive octant into a finite number of cones such that the limits (2.4) are the same for t remaining in some fixed $\overset{\circ}{C_\nu}$, but the value then depends on ν.

We will give the proof of this theorem in the next subsection introducing for that the notion of germ of amoeba. Here we consider an example for which the unrestricted limit

$$\lim_{\epsilon \to 0} I(\epsilon)$$

does not exist, in order to emphasize the possible troubles the residue integral is responsible for. Because of theorem 2.3, the unpleasant behaviour of the residue integral $I(\epsilon)$ may appear only when the path $\delta \mapsto \log \epsilon(\delta)$ crosses the boundary of some cone C_ν. The example we propose below was originally constructed in [75] ; then, in [17], another example disproving continuity of the residue integral was given.

Let us consider the mapping $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ given by the polynomials

$$f_1(z) = z_1^4, \;\; f_2(z) = z_1^2 + z_2^2 + z_1^3$$

Let further φ be a smooth compactly supported $(2,0)$-form which in a neighborhood of the origin is equal to

$$\varphi = \bar{z}_2 \, f_2(z) \, dz_1 \wedge dz_2.$$

With these choices of f and $\varphi = h \, dz_1 \wedge dz_2$ the residue function looks like

$$I(\epsilon) = \frac{1}{(2\pi i)^2} \int_{\substack{|f_1|=\epsilon_1 \\ |f_2|=\epsilon_2}} \frac{\varphi}{f_1 \, f_2} = \frac{1}{(2\pi i)^2} \int_{\substack{|z_1^4|=\epsilon_1 \\ |z_1^2+z_2^2+z_1^3|=\epsilon_2}} \frac{\bar{z}_2}{z_1^4} \, dz_1 \wedge dz_2 \, .$$

After the birational coordinate change $z_1 = u$, $z_2 = uv$ we can write

$$I(\varepsilon) = \frac{\sqrt{\varepsilon_1}}{(2\pi i)^2} \int_{\substack{|u^4|=\varepsilon_1 \\ |u^2(v^2+1+u)|=\varepsilon_2}} \frac{\bar{v}}{u^4} \, du \wedge dv. \tag{2.5}$$

Proposition 2.1 *For any fixed positive number $c \neq 1$ one has*

$$\lim_{\delta \to 0} I(\delta^4, c\delta^2) = 0 \, .$$

Proof. In view of (2.5) we are led to the following iterated integral:

$$I(\delta^4, c\delta^2) = \frac{\delta^2}{2\pi i} \int_{|u|=\delta} \left(\frac{1}{2\pi i} \int_{|v^2+1+u|=c} \bar{v} \, dv \right) \frac{du}{u^4}. \tag{2.6}$$

We denote the inner integral by $J(u)$ and apply to it the following version of the trace formula :

$$\int_{|g(v)|=c} \psi(v) \, dv = \int_{|w|=c} Tr[\frac{\psi}{g'}](w) \, dw \, ,$$

where $g(v)$ is a holomorphic function whose level set $|g(v)| = c$ is a μ-fold branched covering over the circle $|w| = c$, and

$$Tr[\psi/g'](w) = \sum_{j=1}^{\mu}[\psi/g'](v_j(w))$$

with $v_j(w)$ denoting the different pre-images in $g^{-1}(\{w\})$. In our case we take g among the family of functions $v \mapsto g_u(v) = v^2 + 1 + u$, depending on the parameter u, and we have for such g_u $v_j(w) = \pm\sqrt{w - (1+u)}$. We get

$$J(u) = \frac{1}{2\pi i}\int_{|v^2+1+u|=c} \bar{v}\, dv = \frac{1}{2\pi i}\int_{|w|=c} \mathrm{Tr}\,[\frac{\bar{v}}{2v}](w)\, dw$$

and hence

$$J(u) = \frac{1}{2\pi i}\int_{|w|=c} \frac{\overline{\sqrt{w-(1+u)}}}{\sqrt{w-(1+u)}}\, dw, \tag{2.7}$$

where the integrand should be understood as $|w - (1+u)|/(w - (1+u))$, and is hence independent of the choice of branch of the square root. From (2.7) it follows that, if $c \neq 1$, the function $u \mapsto J(u)$ is real-analytic in a neighborhood of the origin, and therefore the limit

$$\lim_{\delta\to 0}\int_{|u|=\delta} \frac{J(u)}{u^4}\, du$$

is a finite complex number. Now, in view of the factor δ^2 in formula (2.6) we reach the desired conclusion and Proposition 1 is proved. $\diamondsuit$

Proposition 2.2

$$\lim_{\delta\to 0} I(\delta^4, \delta^2) \neq 0\,.$$

Proof. Comparing to the equations (2.6) and (2.7), we have

$$I(\delta^4, \delta^2) = \frac{\delta^2}{2\pi i}\int_{|u|=\delta} \frac{J(u)}{u^4}\, du, \tag{2.8}$$

where

$$J(u) = \frac{1}{2\pi i}\int_{|w|=1} \frac{\overline{\sqrt{w-(1+u)}}}{\sqrt{w-(1+u)}}\, dw. \tag{2.9}$$

Here again the integrand is equal to $|w-(1+u)|/(w-(1+u))$ and from this it is straight forward to check that the integral (2.9) is actually a function only of the modulus $|1+u|$. So if we denote $|1+u|$ by t we have reduced ourselves to the study of the integral

$$I(t) = \frac{1}{2\pi i} \int_{|w|=1} \frac{\overline{\sqrt{w-t}}}{\sqrt{w-t}}\, dw \tag{2.10}$$

for real parameters $t \geq 0$, and it is related to the previous integral $J(u)$ via the simple formula $I(|1+u|) = J(u)$. We need two lemmas ; for their proof we refer to [75].

Lemma 2.1 *The integral* (2.10) *is a piecewise real-analytic continuous function for $t \geq 0$. It is explicitly given by*

$$I(t) = \begin{cases} F[-\frac{1}{2}, \frac{1}{2}; 1; t^2], & 0 \leq t < 1, \\ \\ \frac{1}{2t} F[\frac{1}{2}, \frac{1}{2}; 2; \frac{1}{t^2}], & t \geq 1, \end{cases}$$

where F denotes the hypergeometric series

$$F[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{n!},$$

with $(a)_0 = 1$, $(a)_n = a\,(a+1)\cdots(a+n-1)$ and similary for b and c.

Using lemma 2.1, we shall next describe the asymptotic behaviour of our function $I(t)$ at the point $t=1$ (we will denote the two expressions $I_{\mp}(t)$, depending on the fact that $t<1$ or $t \geq 1$).

Lemma 2.2 *In a (real) neighborhood of the point $t=1$ the functions $I_{\mp}(t)$ admit representations*

$$I_{\mp}(t) = A_{\mp}(t)\, \log|t^2-1| \; + \; B_{\mp}(t), \tag{2.11}$$

the functions $A_{\mp}$, $B_{\mp}$ being analytic with the properties

(i) $A_{\mp}(t) = (t^2-1)/2\pi + O[(t^2-1)^2]$ as $t \to 1$,

(ii) $B_{-}(1) = B_{+}(1) = I(1)$.

Proof of proposition 2.2, continued. We recall that we have to find the limit of the function (2.8) as $\delta \to 0$, with the integrated function $J(u)$ being equal to the function $I(|1+u|)$, described in Lemma 2.1. According to the lemma we can represent the function $I(t)$ as a series

$$I(t) = b_0(t) + \sum_{n=1}^{\infty} \{a_n(t) \cdot (t^2-1)^n \cdot \log|t^2-1| \;+\; b_n(t) \cdot (t^2-1)^n\},$$

where the coefficients a_n, b_n are piecewise constant functions taking only two values :

$$a_n(t) = \begin{cases} a_n^-, & t < 1, \\ \\ a_n^+, & t \geq 1, \end{cases} \qquad b_n(t) = \begin{cases} b_n^-, & t < 1, \\ \\ b_n^+, & t \geq 1. \end{cases}$$

Moreover, properties (i) and (ii) imply that the first two coefficients are truly constant :

$$b_0(t) \equiv b_0 = I(1), \quad a_1(t) \equiv 1/2\pi.$$

We can therefore write

$$\begin{aligned} J(u) &= I(|1+u|) \\ &= b_0 + \frac{1}{2\pi}\,(|1+u|^2-1)\,\log\big||1+u|^2-1\big| \;+\; b_1(|1+u|)\,(|1+u|^2-1) \\ &\quad + \mathbf{O}\Big(\big||1+u|^2-1\big|^{3/2}\Big), \end{aligned}$$

where

$$b_1(|1+u|) = \begin{cases} b_1^-, & |1+u| < 1, \\ \\ b_1^+, & |1+u| \geq 1 \end{cases}$$

(actually, the exponent 3/2 above can be replaced by any number < 2).

We have thus written $J(u)$ as a sum of four terms. Let us first show that the first, third and fourth terms all give null contribution to the limit of $I(\delta^4, \delta^2)$. This is obvious for the first term, which is just the constant b_0. The fourth term is also easy to handle. Indeed, on the circle of integration $u = \delta\, e^{i\phi}$, $0 \leq \phi \leq 2\pi$, we have $|1+u|^2 - 1 = \delta\,(2\cos\phi + \delta)$, and hence

$$\delta^2 \int_{|u|=\delta} \mathrm{O}\,\big(\big||1+u|^2-1\big|^{3/2}\big)\,\frac{du}{u^4} = \delta^2 \int_0^{2\pi} \frac{\mathbf{O}\,(\delta^{3/2})\,d\phi}{\delta^3\, e^{3i\phi}} \to 0, \quad \text{as } \delta \to 0\,.$$

Let us next consider the contribution of the third term :

$$
\begin{aligned}
\lim_{\delta\to 0}\ \delta^2 \int_{\substack{|u|\delta\\ |1+u|<1}} & \frac{b_1^-\,(|1+u|^2-1)}{u^4}\,du \\
&= \lim_{\delta\to 0}\ b_1^-\,\delta^2\, i \int_{\substack{\phi\in[0,2\pi]\\ 2\cos\phi+\delta<0}} \frac{\delta\,(2\cos\phi+\delta)}{\delta^3\, e^{3i\phi}}\,d\phi \\
&= 2b_1^-\, i \int_{\pi/2}^{3\pi/2} \cos\phi(\cos 3\phi - i\sin 3\phi)\,d\phi = 0\,.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\lim_{\delta\to 0}\ \delta^2 \int_{\substack{|u|=\delta\\ |1+u|>1}} & \frac{b_1^+\,(|1+u|^2-1)}{u^4}\,du \\
&= 2b_1^+\, i \int_{-\pi/2}^{\pi/2} \cos\phi(\cos 3\phi - i\sin 3\phi)\,d\phi = 0\,.
\end{aligned}
$$

This takes care of the third term.

What remains to be shown is that the second term, which contains the logarithm, gives a non-zero contribution to the limit. We have

$$
\begin{aligned}
\lim_{\delta\to 0} I(\delta^4,\delta^2) &= \lim_{\delta\to 0} \frac{\delta^2}{2\pi i}\int_{|u|=\delta} \frac{(1/2\pi)(|1+u|^2-1)\ \log\big||1+u|^2-1\big|}{u^3}\,\frac{du}{u} \\
&= \lim_{\delta\to 0} \frac{1}{4\pi^2}\int_0^{2\pi} \frac{(2\cos\phi+\delta)\ \log(\delta\,|2\cos\phi+\delta|)}{e^{3i\phi}}\,d\phi\,.
\end{aligned}
$$

Observe now that the limit of the last integral does not change if we remove from it the factor δ inside the logarithm. This is because the integral of $(2\cos\phi+\delta)/e^{3i\phi}$ is equal to zero. After the removal of this factor δ the integrand will be a uniformly bounded family of continuous functions, and so by Lebesgue's theorem we may perform the limit procedure inside the integral and obtain :

$$
\lim_{\delta\to 0} I(\delta^4,\delta^2) = \frac{1}{2\pi^2}\int_0^{2\pi} \frac{\cos\phi\ \log|2\cos\phi|}{e^{3i\phi}}\,d\phi\,.
$$

Expanding the function $\log|\cos\phi|$ as a Fourier series we get

$$
\log|\cos\phi| = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\ \cos 2n\phi\,,
$$

containing even frequencies only. Therefore the product $\cos\phi\,\log|2\cos\phi|$ is equal to the uniformly convergent series

$$\cos\phi\,\log|2\cos\phi| = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n}\left(\cos(2n+1)\phi + \cos(2n-1)\phi\right),$$

in which $\cos 3\phi$ appears with the coefficient $1/4$. All the other odd harmonics $\cos(2n-1)\phi$ are orthogonal to $e^{-3i\phi}$, and we get

$$\lim_{\delta\to 0} I(\delta^4,\delta^2) = \frac{1}{2\pi^2}\int_0^{2\pi}\frac{1}{4}\cos 3\phi\, e^{-3i\phi}\,d\phi = \frac{1}{8\pi^2}\int_0^{2\pi}\cos^2 3\phi\,d\phi = \frac{1}{8\pi}\,.$$

Proposition 2.2 follows. $\diamondsuit$

2.3 Proof of theorem 2.3

In order to prove theorem 2.3, one can assume without loss of generality one is in the local situation when the mapping $f\colon \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic at the origin 0 in $\mathbb{C}^n$ with isolated zero the origin. Letting $f(z) = w$, we apply for the residue integral $I(\epsilon_t)$ the trace formula

$$I(\varepsilon_t) = \int\limits_{T_\epsilon(w)} \operatorname{Tr}\,[\frac{h}{J_f}]\,\frac{dw_1}{w_1}\wedge \dots \wedge \frac{dw_n}{w_n}\,, \tag{2.12}$$

where J_f is the Jacobian of f, the integration set being the skeleton

$$T_{\epsilon_t}(w) = \{|w_1| = \varepsilon_{t,1}(\delta), \dots, |w_n| = \varepsilon_{t,n}(\delta)\}\,,$$

and

$$\operatorname{Tr}\,[\frac{h}{J_f}] = \sum_{\nu=1}^{\mu}[\frac{h}{J_f}](z^\nu(w))$$

being the trace function with $\{z^\nu(w)\,,\, \nu = 1,\dots,\mu\} = f^{-1}(\{w\})$ where μ denotes the multiplicity of f. It is well known that in case h is holomorphic, the trace function $w \mapsto \operatorname{Tr}\,[h/J_f](w)$ is also holomorphic [4], [88]. By Cauchy formula one has in this case

$$I(\epsilon) \equiv \operatorname{Tr}\,[\frac{h}{J_f}](0)\,.$$

We shall see below that in order to prove theorem 2.3 (that is considering the set of pathes $\delta \mapsto \epsilon_t(\delta)$), it is enough to take $h(z,\overline{z})$ as a polynomial in z

and $\overline{z}$. Remark that the singular points of the trace function belong to the discriminant set $\sigma(w)=0$ of the mapping f, where

$$\sigma(w) := \prod_{\nu=1}^{\mu} J_f(z^\nu(w))\,.$$

Since the integration set in the trace formula is the skeleton defined by the radius-vector $(\epsilon_1, ..., \epsilon_n)$, it is important to know how this vector is located respect to the image of discriminant set on the Reinhardt diagram in coordinates $|w_1|, ..., |w_n|$. One can see better how this image looks like after taking the one-to-one logarithmic transformation. In such a way will arise the notion of *germ of amoeba*. Recall first the classical notion of amoeba : the *amoeba* $\mathcal{A}_P$ of a Laurent polynomial $P(z)$ (or of the algebraic hypersurface $P^{-1}(0)$) is defined as the image of the hypersurface $P^{-1}(0)$ under the map

$$\mathrm{Log} : (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|).$$

The notion of amoeba was introduced in [39] and studied in details in [37]. We need to generalize the following two statements about the geometry of amoebas.

Statement 1. *The complement $\mathbb{R}^n \setminus \mathcal{A}_P$ consists of finitely many connected components which are open and convex ; these components are in bijective correspondence with the different Laurent series expansions centered at the origin for the rational function $1/P$.*

Let N_P be the Newton polytope of P.

Statement 2. *Let $\{E\}$ be the family of all connected components of $\mathbb{R}^n \setminus \mathcal{A}_P$. There exists an injective function $\nu : \{E\} \to \mathbb{Z}^n \cap N_P$ such that the cone C_ν which is dual to N_P at the point $\nu = \nu(E)$ coincides with the recession cone of E ; that is, for any $u \in E$ one has $u + C_\nu \subset E$ and no strictly larger cone is contained in E.*

Now we consider instead of polynomial function a germ of holomorphic function σ in $\mathcal{O}_{\mathbb{C}^n,0}$ with $\sigma(0)=0$. In this case we define the *germ of amoeba* as the image $\mathrm{Log}\,(\sigma^{-1}(0) \cap B_r)$, where B_r is a ball of (arbitrary) small radius r centered at 0. Denote by R_r the image Log (B_r). The same arguments which were used in [39] and [37] imply the following properties for a germ of amoeba.

Statement 1'. *There exists $r_0 > 0$ such that for the germ $\mathcal{A}_\sigma$ in B_r with $r < r_0$ the complement $R_r \setminus \mathcal{A}_\sigma$ consists of finitely many components which are* *open and convex ; moreover such components are in bijective correspondence with the different Laurent series expansions centered at the origin for the meromorphic function* $1/\sigma$.

Denote as $\operatorname{supp}\sigma$ the set of exponents of monomials which are effetively present in the Taylor series of σ (about the origin) ; let Γ_σ be the Newton diagramm of σ, that is the union of all bounded faces of the polyhedron

$$\widetilde{N}_\sigma = \text{convex hull}\Big[\bigcup_{\alpha\in\operatorname{supp}\sigma}\{\alpha+\mathbb{R}^n_+\}\Big].$$

Statement 2'. *Let* $\mathcal{A}_\sigma$ *be the germ of amoeba of* σ *in* B_r, *with* $r \ll 1$, *and let* $\{E\}$ *be the family of all connected components of* $R_r \setminus \mathcal{A}_\sigma$. *There exists an injective function* $\nu : \{E\} \to \mathbb{Z}^n \cap \Gamma_\sigma$ *such that the cone* C_ν *which is dual to* $\widetilde{N}_\sigma$ *at the point* $\nu = \nu(E)$ *coincides with the recession cone of* E.

Remark that the map Log is the composition of two maps :

$$\begin{aligned} m \;:\; (z_1,\dots,z_n)\in(\mathbb{C}\setminus 0)^n &\;\to\; (|z_1|,\dots,|z_n|)\in\mathbb{R}^n_+ \\ l=\log \;:\; \mathbb{R}^n_+ &\;\to\; \mathbb{R}^n\,, \end{aligned}$$

Denote as A_σ the image $m(f^{-1}(0)\cap B_r)$ and call it the *preamoeba*. We have a one-to-one correspondence $\{\tilde{E}\}\to\{E\}$ between the connected components corresponding to the preamoeba A_σ and those corresponding to the amoeba $\mathcal{A}_\sigma$.

Now we can prove theorem 2.3 as follows : we prove that the n-dimensional cones C_ν which appear in statement $2'$ (in connection with the germ of amoeba of the discriminant σ at the origin in $\mathbb{C}^n_w$) as the dual cones at the vertices of $\widetilde{N}_\sigma$ are in fact the cones we need to formulate our assertion. In order to do so, notice first that the union of such cones gives a decomposition of the positive octant $\mathbb{R}^n_+$. Furthermore, any "parabolic" path $t\mapsto\epsilon_t(\delta)$ with $t\in\overset{\circ}{C}_\nu$ can be tangent at most with some finite order to the coordinate hyperplanes of $\mathbb{R}^n_+$ and the preamoeba A_σ. It follows thus that for the such pathes the denominator $\sigma(w)\,w_1\dots w_n$ in the trace formula (2.12) has finite order respect to the parameter δ. But this means it is enough (in order to prove the theorem) to consider for h some monomial function $h(z,\overline{z})=z^\alpha\overline{z}^\beta$. Since the inverse images $z^j(w)$ are algebroidal functions which are holomorphic outside the discriminant set, they admit a Puiseux expansion in any logarithmic convex domain such as $\operatorname{Log}^{-1}(E_\nu)$, $\nu=1,\dots,\mu$. Consequently the trace $\mathrm{Tr}\,[z^\alpha \overline{z}^\beta / J_f]$ can be expanded as the sum of a series (in w) of monomials of the form $w^k \overline{w}^l |w|^q$, where $k \in \mathbb{Z}^n$, $l \in \mathbb{N}^n$ and $q \in \mathbb{Q}_+^n$. Finally, thanks to the trace formula (2.12), we get that the nontrivial contribution in the limit of the residue integral $I(\epsilon_t(\delta))$ comes from the holomorphic part of h. This explains why the limits (2.4) exist for any "parabolic" path $\delta \mapsto \epsilon_t(\delta)$ with $t \in \overset{\circ}{C}_\nu$ for some ν and why these limits are the same when t remains in the same $\overset{\circ}{C}_\nu$; moreover the value is independent of ν. $\diamondsuit$

2.4 The Mellin transform of a residue function

Let $\mathcal{X}$ be a n-dimensional complex manifold. Recall that given some holomorphic mapping $f\colon \mathcal{X} \to \mathbb{C}^m$ $(m \leq n)$ and some test form $\varphi \in \mathcal{D}^{n,n-m}(\mathcal{X})$, one can associate to these data a residue function $\epsilon \mapsto I(\epsilon) = I_f^\varphi(\epsilon)$ from $\mathbb{R}_+^m$ to $\mathbb{C}$ defined as the integral

$$I(\epsilon) := \frac{1}{(2\pi i)^m} \int_{T_\epsilon} \frac{\varphi}{f_1 \cdots f_m}$$

over the tube $T_\varepsilon = \{z \in \mathcal{X}\,;\, |f_1(z)|^2 = \varepsilon_1, \ldots, |f_m(z)|^2 = \varepsilon_m\}$;note that for the sake of convenience we take here $|f_j|^2 = \varepsilon_j$ in the definition of the tubes, while before there was no square.

The Mellin transform of the function I_f^φ is given as the function

$$\lambda \mapsto \Gamma_f^\varphi(\lambda) = \int_{\mathbb{R}_+^m} I_f^\varphi(\epsilon)\, \epsilon^{\lambda - I}\, d\epsilon\,, \tag{2.13}$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$ is a complex vector and

$$\epsilon^{\lambda - I}\, d\epsilon := \epsilon_1^{\lambda_1 - 1} \cdots \epsilon_m^{\lambda_m - 1}\; d\epsilon_1 \wedge \ldots \wedge d\epsilon_m\,.$$

The Mellin transform is a m-dimensional integral of an integrand which itself is given by some $(2n-m)$-dimensional integral. It may therefore be in a natural way represented as a $2n$-dimensional integral.

Proposition 2.3 [75] *The Mellin transform of the residue function associated to a holomorphic mapping $f\colon \mathcal{X} \to \mathbb{C}^m$ may be expressed as an integral over $\mathcal{X}$ as follows :*

$$\Gamma_f^\varphi(\lambda) = \frac{1}{(2\pi i)^m} \int_{\mathcal{X}} |f|^{2(\lambda - I)}\, \overline{df} \wedge \varphi, \tag{2.14}$$

with the vector notations

$$|f|^{2(\lambda-I)} = |f_1|^{2(\lambda_1-1)}\cdots|f_m|^{2(\lambda_p-1)} \quad , \quad \overline{df} := \overline{df_1}\wedge\ldots\wedge\overline{df_m}\,.$$

Some information about the Mellin transform of the residue integral implies the following theorem (which can be proved using desingularization theorem) :

Theorem 2.4 [74] *The Mellin transform* Γ_f^{φ} *defined by* (2.13) *is holomorphic for* $\mathrm{Re}\,\lambda$ *in* $\mathbb{R}_+^m$ *and has a meromorphic continuation to all of* $\mathbb{C}^m$*. There is a finite collection of non-zero vectors* a^k *in* $\mathbb{N}^m$*, depending only on* f *and on the support of* φ*, such that the poles of* Γ_f^{φ}*, which are all simple, are contained in the hyperplanes* $(a^k,\lambda) = -m$*,* $m \in \mathbb{N}$ *(here* (a^k,λ) *denotes the usual scalar product). In particular, near the origin one has*

$$\Gamma_f^{\varphi}(\lambda) = \sum_{|K|=m}\frac{c_K}{(a^{k_1},\lambda)\cdots(a^{k_m},\lambda)} + Q(\lambda)\,,$$

where the c_K *are constants and* Q *is a finite sum of functions with simple poles along fewer than* m *hyperplanes.*

In case the mapping f defines a complete intersection, one can say a lot more about the structure of the polar set of the function Γ_f^{φ}.

Theorem 2.5 [8, 75] *If* $f : \mathcal{X} \to \mathbb{C}^m$ *defines a complete intersection in* $\mathcal{X}$*, i.e.* $\dim f^{-1}(0) = n-m$*, then in a neighborhood of the origin* $\lambda = 0$ *the function* Γ_f^{φ} *can have (simple) poles only along the coordinate hyperplanes* $\lambda_j = 0$*. In other words, for a complete intersection the function* $\lambda \mapsto \lambda_1\cdots\lambda_m\,\Gamma_f^{\varphi}(\lambda)$ *is holomorphic near the origin.*

In the case $m = n$, the last theorem follows from theorem 2.3 : indeed by theorem 2.3, the residue function is (at the infinitesimal level) a continuous function, since along almost all approaches towards the origin $\epsilon = 0$ in $\mathbb{C}^n$, it has the same limits. But obviously for such function the Mellin transform may be singular near $\lambda = 0$ only on the coordinate lines. We will mention in section 3.1 an alternative way to get this result in the non absolute case $m < n$ (f defining a complete intersection).

In the noncomplete intersection case $\epsilon \mapsto I(\epsilon)$ is (still at the infinitesimal level) piecewice constant (see the remark following theorem 2.3) and its Mellin transform has a finite number of polar hyperplanes near the origin.

One has the following

Corollary 2.1 *In the complete intersection case, the residue current is related with the Mellin transform of the residue integral by the formula*

$$R_f(\varphi) = \lambda_1 \cdots \lambda_m \, \Gamma_f^{\varphi}(\lambda)\Big|_{\lambda=0} .$$

2.5 Principal value of the residue current

There exists another current which is intimely related to the residue current, namely the principal value current. In the simplest case, it is defined for the meromorphic function $1/f$ on the manifold $\mathcal{X}$ and can be introduced as the limit

$$\Big\langle [\frac{1}{f}], \varphi \Big\rangle = \lim_{\epsilon \to 0} \int_{\{|f|>\epsilon\}} \frac{\varphi}{f}$$

for any test $2n$ test form φ (as before, here $n = \dim(\mathcal{X})$). The existence of the last limit was proved in [49]. The fundamental relation between this principal value current and the residue current R_f consists in the formula

$$\overline{\partial}[\frac{1}{f}] = R_f \tag{2.15}$$

which is an avatar of Stokes's formula :

$$\Big\langle \overline{\partial}[\frac{1}{f}], \varphi \Big\rangle = -\lim_{\epsilon \to 0} \int_{\{|f|>\epsilon\}} \frac{\overline{\partial}\varphi}{f} = \lim_{\epsilon \to 0} \int_{\{|f|=\epsilon\}} \frac{\varphi}{f} = \langle R_f, \varphi \rangle$$

for any $(n, n-1)$ test form φ. In particular relation (2.15) shows that the residue current R_f is $\overline{\partial}$-closed.

One can combine the two procedures of taking the residues and principal values. There are two ways to do this. Let

$$f = (f_1, ..., f_m) \ : \ \mathcal{X} \to \mathbb{C}^m, \ g = (g_1, ..., g_k) \ : \ \mathcal{X} \to \mathbb{C}^k$$

two holomorphic mappings on $\mathcal{X}$. One can introduce the notion of the *principal value* of the residue current R_f with respect to g (acting on a $(n, n-m)$-test form) as the limit

$$R_f\, P_g(\varphi) = \lim_{\delta \to 0_+} \int_{T_{\epsilon(\delta)\,;\,\tau(\delta)}(f,g)} \frac{\varphi}{(f_1 ... f_m)(g_1 ... g_k)} \tag{2.16}$$

where

$$T_{\epsilon(\delta)\,;\,\tau(\delta)}(f,g) = \{|f_i| = \epsilon_i(\delta),\, i = 1, ..., m\,;\, |g_j| > \tau_j(\delta),\, j = 1, ..., k\}\ ;$$

here $\delta \mapsto \epsilon(\delta)$ and $\delta \mapsto \tau(\delta)$ are pathes towards zero.

Theorem 2.6 [26, 71] *One has the three following properties :*

- *i) For any "admissible" path $\delta \to (\epsilon(\delta), \tau(\delta))$ in $\mathbf{R}_+^{m+k}$ the limit* (2.16) *exists and defines a $(0, m)$-current independently of the particular choice of the "admissible" paths.*
- *ii) If $k = 1$ then*

$$\overline{\partial}(P_g R_f) = R_{f,g}\ , \tag{2.17}$$

  from which it follows that the Coleff-Herrera current R_f is $\overline{\partial}$-closed ;
- *iii) If $k = 1$ then the principal value current $P_g R_f$ is represented as an iterated limit*

$$P_g R_f = \lim_{\tau \to 0} \lim_{\epsilon(\delta) \to 0} \int_{T_{\epsilon;\tau}(f,g)} \frac{\varphi}{(f_1 ... f_m)(g_1 ... g_k)}$$

Property *iii)* is the basic one when one deals with residues currents. One gets a similar property when considering another notion of principal value. To do this remark that in (2.16), the integration set is the intersection of the "residual" tube $T_\epsilon(f)$ with the set

$$\widetilde{D}_\tau(g) = \{|g_j| > \tau_j\,;\, j = 1, ..., k\}\,.$$

The last tube is the intersection of the sets $|g_j| > \tau_j$, and when $k > 1$ it is not an exterior domain for the zero set $g^{-1}(0)$. In order to get some definition of some principal value on the analytic set $f^{-1}(0)$ respect to the analytic subset $f^{-1}(0) \cap g^{-1}(0)$, we consider instead of the tube $\tilde{D}_\tau(g)$ the following one :

$$D_\tau(g) = \bigcup_{j=1}^{k} \{|g_j| > \tau_j\}\,.$$

Now we introduce the following special limit (let us keep for it the same notation) :

$$P_g R_f = \lim_{\delta \to 0} \int_{T_\epsilon(f) \cap D_\tau(g)} \frac{\varphi}{(f_1 ... f_m)}. \tag{2.18}$$

Theorem 2.7 [86] *One has the three following properties :*

- *i) The limit* (2.16) *exists for any "admissible" path and defines the current of bidegree* $(n, n-m)$ *;*
- *ii) In case* f *defines a complete intersection, with*

  $$\dim(f^{-1}(0) \cap g^{-1}(0)) < \dim(f^{-1}(0),$$

  one has $R_f = P_g R_f$ *;*
- *iii) The two above currents can be represented, acting on a test form* φ*, as the iterated limit*

  $$R_f(\varphi) = P_g R_f(\varphi) = \lim_{\delta_1 \to 0} \lim_{\delta_2 \to 0} \int_{T_{\epsilon(\delta_1)}(f) \cap D_{\tau(\delta_2)}(g)} \frac{\varphi}{f_1 ... f_m}\,,$$

  where $\epsilon(\delta_1), \tau(\delta_2)$ *are admissible pathes.*

This theorem allows to describe the structure of the residue current R_f in the case f defines a reduced complete intesection, i.e $df = df_1 \wedge ... \wedge df_m \neq 0$ almost everywhere on $A = f^{-1}(0)$. In this case the singular part of A coincides with the set (see proposition 5.1)

$$\{z \in A\,;\, df(z) = 0\} = \{z \in A\,;\, J_I = 0, |I| = m\},$$

where $J_I = \partial f / \partial z_I$ is the Jacobian of f with respect to the variables $z_I = (z_{i_1}, ..., z_{i_m})$. For a semimeromorphic form $\varphi / f_1 ... f_m$, $\varphi \in \mathcal{D}^{n,n-m}(\mathcal{X})$, one can define the residue form in the sense of Leray $res\, \omega = \varphi / df|_A$, which can be written on the subset $U_I = \{z \in A\,;\, J_I \neq 0\}$ as

$$\frac{\varphi}{df} = \pm \frac{\varphi / dz_I}{J_I}|_{U_I}.$$

Theorem 2.8 [86] *If* $f : \mathcal{X} \to \mathbb{C}^m$ *defines a reduced complete intersection* $A = f^{-1}(0)$, *then the residue current associated with* f *can be reduced to the principal value on* A *by the formula*

$$R_f(\varphi) = (2\pi i)^m P_{A,J}(\frac{\varphi}{df}),$$

where $P_{A,J}$ *is the principal value on* A *respect to the map* $J = (J_I : |I| = m)$, *i.e.*

$$P_{A,J}(\frac{\varphi}{df}) = \lim_{\delta\to 0} \int\limits_{A\cap D_{\tau(\delta)}(J)} \frac{\varphi}{df}$$

In the case $m = 1$ there is a more precise description of R_f due to P. Dolbeault (see [32] and subsection 3.2 below). Remark also that the last theorem will be used essentially in the last section when studying holomorphic forms on analytic varieties.

3 Residue currents of the Bochner-Martinelli type

3.1 An alternative construction of residue currents

Let $f_1, ..., f_m$ be m holomorphic functions in some open set $V \subset \mathbb{C}^n$, $\|f\|^2$ be the real analytic function in V defined as

$$\|f\|^2 := |f_1|^2 + \cdots + |f_m|^2$$

and

$$C(f) := \{\epsilon > 0\,;\, \epsilon^2 \text{ is a critical value of } \|f\|^2\}\,;$$

let also, for any ordered subset $\mathcal{I} = \{i_1, ..., i_r\} \subset \{1 ..., m\}$ with cardinal $r \leq \min(m, n)$, $\Omega(f\,;\mathcal{I})$ be the $(0, r)$-differential form defined as

$$\Omega(f\,;\mathcal{I}) := \sum_{j=1}^{r} (-1)^{j-1} \overline{f_{i_j}} \bigwedge_{\substack{l=1 \\ l\neq j}}^{r} \overline{df_{i_l}}$$

(note that such a form depends in some alternate way on the ordering of indices in $\mathcal{I}$). For any $m \in \{1, ..., n\}$ and any $\epsilon > 0$, we will denote as $\Sigma_m(\epsilon^2)$ the $m-1$-dimensional open simplex

$$\Sigma_m(\epsilon^2) := \{(t_1, ..., t_m) \in]0, +\infty[^m\,;\, t_1 + \cdots + t_m = \epsilon^2\}$$

and $d\sigma_{m,\epsilon^2}$ the normalized $m-1$-dimensional Lebesgue measure on $\overline{\Sigma_m(\epsilon^2)}$. When $m \leq n$ and $f_1, ..., f_m$ define a complete intersection in V, then one can show that for any $(n, n-m)$-test form φ which is closed in a neighborhood of $f^{-1}(0) := \{\zeta \in V \,;\, f_1(\zeta) = \ldots = f_m(\zeta) = 0\}$, the function defined for any $\epsilon > 0$, $\epsilon \notin C(f)$ sufficiently small (depending on φ) by

$$J_{f\,;\{1,\ldots,m\}}(\epsilon\,;\,\varphi) := \frac{(-1)^{\frac{m(m-1)}{2}}(m-1)!}{(2i\pi)^m}\,\frac{1}{\epsilon^{2m}}\int_{\|f\|^2=\epsilon^2} \Omega(f\,;\,\{1,...,m\}) \wedge \varphi$$

$$= \int_{(\eta_1^2,\ldots,\eta_m^2)\in\Delta_m(\epsilon^2)} \left[\frac{1}{(2i\pi)^m} \int\limits_{\substack{|f_1|=\eta_1\\ \cdots\\ |f_m|=\eta_m}} \frac{\varphi}{f_1\cdots f_m}\right] d\sigma_{m,\epsilon^2}(\eta_1^2, ..., \eta_m^2)$$

is almost everywhere constant and equal to

$$\Big\langle \bigwedge_{j=1}^{m} \overline{\partial}\frac{1}{f_j}\,,\,\varphi\Big\rangle\,.$$

Such a remark suggests a possible definition of residue currents in the non-complete intersection situation ; in fact, ideas arising again from toric geometry allow the following theorem [76] :

Theorem 3.1 *Let $f_1, ..., f_m$ be m holomorphic functions in some open subset $V \subset \mathbb{C}^n$, $r \leq \min(n,m)$, and $\mathcal{I} := \{i_1, ..., i_r\}$ be some ordered subset of $\{1, ..., m\}$; for any element $\varphi \in \mathcal{D}^{n,n-r}(V)$, the limit when ϵ tends to zero outside $C(f)$ of*

$$J_{f\,;\mathcal{I}}(\epsilon\,;\,\varphi) := \frac{(-1)^{\frac{r(r-1)}{2}}(r-1)!}{(2i\pi)^r}\,\frac{1}{\epsilon^{2r}}\int_{\|f\|^2=\epsilon^2} \Omega(f\,;\,\mathcal{I}) \wedge \varphi$$

exists ; moreover, one defines this way the action of a $(0,r)$-current $T_{f\,;\mathcal{I}}$ on V, with support on $V(f)$; one has $T_{f\,;\mathcal{I}} = 0$ when $r < m_f := \operatorname{codim} f^{-1}(0)$; moreover, when $m \leq n$,

$$\overline{\partial}\Big[T_{f\,;\{1,\ldots,m\}}\Big] = 0\,;$$

finally, $\overline{h}T_{f\,;\mathcal{I}} \equiv 0$ for any subset $\mathcal{I}$ with cardinal between m_f and $\min(m,n)$ and any function h holomorphic in some neighborhood of $f^{-1}(0)$ in V, such that $h \equiv 0$ on $f^{-1}(0)$.

The basic idea supporting this proposition can be understood if one looks at the normal crossing case, when V is a neighborhood of the origin in $\mathbb{C}^n$ and each $f_1, ..., f_m$ is of the form

$$f_j(z) = u_j(z)\, z_1^{\alpha_{j1}} \cdots z_n^{\alpha_{jn}}\ ,$$

where u_j is an invertible element in $H(V)$ and $\alpha_j = (\alpha_{j1}, ..., \alpha_{jn})$ is in $\mathbb{N}^n$; let Δ be the closed convex hull of

$$\bigcup_{j=1}^{m} \{\alpha_j + \mathbb{R}_+^n\}$$

and $\overset{\Delta}{\approx}$ the corresponding equivalence relation between elements in $\mathbb{R}_+^n$: $\xi \overset{\Delta}{\approx} \xi'$ if and only if $\mathrm{Tr}_\Delta(\xi) = \mathrm{Tr}_\Delta(\xi')$, where

$$\mathrm{Tr}_\Delta(\xi) := \Big\{\delta \in \Delta,\ <\xi, \delta> = \min_{x\in\Delta} <\xi, x> \Big\}\, ;$$

the set of all closures of the equivalence classes for this relation is a fan $\Sigma(\Delta)$ (see for example [5]), which can be refined in order that all cones are simple ones, so that the corresponding toric variety $\mathcal{X}$ is a n- dimensional complex manifold ; local charts correspond to different copies of $\mathbb{C}^n$ which are glued together via invertible monoidal transformations from the n-dimensional torus $\mathbb{T}^n$ into itself. Since the union of the cones in this fan is $\mathbb{R}_+^n$, the projection map $\Pi\ :\ \mathcal{X} \to \mathbb{C}^n$ (which is monoidal when expressed in local coordinates in each chart) is a proper map ; the key fact here is that in such a local chart (with local coordinates $t_1, ..., t_n$), one can write $\|f\|^2$ as

$$\Pi^* \, \|f\|^2(t) = |\mu(t)|^2 v(t)$$

where μ is a monomial and v some real analytic strictly positive function (in the local chart).

Combining this idea with standard technics involving resolution of singularities for the hypersurface $\{f_1 \dots f_m = 0\}$, one can prove that the Mellin transform of

$$\epsilon \mapsto J_{f\,;\mathcal{I}}(\epsilon\,;\, \varphi)\, ,$$

that is

$$\lambda \mapsto \int_0^{+\infty} \epsilon^{\lambda-1}\, J_{f\,;\mathcal{I}}(\epsilon\,;\, \varphi)\, d\epsilon\, ,$$

is a meromorphic function in $\mathbb{C}$, with no pole at $\lambda = 0$, and uniform rapid decrease in vertical strips, which implies theorem 3.1, thanks to the Mellin inversion formula.

One can notice that the action of such a residue current $T_{f\,;\mathcal{I}}$, when $r = r(\mathcal{I}) \leq \min(m,n)$ on a $(n, n-r)$-test form in V can be also expressed as

$$\langle T_{f\,;\mathcal{I}}\,,\,\varphi\rangle = \frac{(-1)^{\frac{r(r-1)}{2}} r!}{(2i\pi)^r} \lim_{\tau\to 0^+} \int_V \frac{\tau\overline{\partial}\|f\|^2 \wedge \Omega(f\,;\mathcal{I}) \wedge \varphi}{\|f\|^2\,(\|f\|^2+\tau)^{r+1}}\,;$$

in particular, when $m \leq n$,

$$\langle T_{f\,;\{1,\dots,m\}}\,,\,\varphi\rangle = \frac{(-1)^{\frac{m(m-1)}{2}} m!}{(2i\pi)^m} \lim_{\tau\to 0^+} \int_V \frac{\tau \bigwedge\limits_{j=1}^{m} \overline{df_j} \wedge \varphi}{(\|f\|^2+\tau)^{m+1}}\,;$$

therefore, still in the particular case where $m \leq n$ and $\mathcal{I} = \{1,\dots,m\}$, the Γ-type meromorphic function

$$(\lambda_1,\dots,\lambda_m) \in \mathbb{C}^m \mapsto \Gamma_f^{\varphi}(\lambda) := \frac{(-1)^{\frac{m(m-1)}{2}}}{(2i\pi)^m} \int_V \prod_{j=1}^{m} |f_j|^{2(s_j-1)} \bigwedge_{j=1}^{m} \overline{df_j} \wedge \varphi$$

(note that such a function an be uniformly estimated in $C(L)(1+\|\mathrm{Im}\,\lambda\|)^{N(L)}$ when $\|\mathrm{Im}\,\lambda\|$ goes to $+\infty$ in any strip $L = \{\|\mathrm{Re}\,\lambda\| \leq T\}$) is involved in the expression of the action of $\langle T_{f\,;\{1,\dots,m\}}\,,\,\varphi\rangle$ through the multivariate Mellin-Barnes transformation ([76], theorem 2.2) :

Theorem 3.2 *Let $f_1,\dots,f_m$ be m functions holomorphic in some open subset $V \subset \mathbb{C}^n$ $(n \geq m)$ and $(\gamma_1,\dots,\gamma_m)$ in $]0,1[^m$; then, for any test-form φ in $\mathcal{D}^{n,n-m}(V)$, one has*

$$\langle T_{f,\{1,\dots,m\}}\,,\,\varphi\rangle = \lim_{\tau\to 0^+} \frac{1}{(2i\pi)^m} \int_{\gamma+i\mathbb{R}^m} \tau^{-|s|}\Gamma(|s|+1)\Big[\prod_{j=1}^{m}\Gamma(1-s_k)\Big]\,\Gamma_f^{\varphi}(s)ds\,,$$

where $|s| := s_1+\cdots+s_m$ and $ds := ds_1\cdots ds_m$.

An example ([76], proposition 3.1). As an example, we may point here the monomial case $(m=n)$, where

$$f_j(z) = z_1^{\alpha_{j1}}\cdots z_n^{\alpha_{jn}}\,,\quad j=1,\dots,n\,;$$

in this case, if d denotes the codimension of the real closed cone

$$\{x \in \mathbb{R}^n\,;\, \langle \alpha^j\,,\,x\rangle \geq 0\,,\, j=1,...,n\,;\, x_1+\cdots+x_n \leq 0\}\,,$$

where $\alpha^1,...,\alpha^n$ denote the column vectors of the matrix $A=[\alpha_{ij}]$, one has $T_{f\,;\{1,...,n\}}=0$ when $d=0$; otherwise, if $1\leq d\leq n$,

$$T_{f\,;\{1,...,n\}} = \bigwedge_{j=1}^{d} \bar\partial \left[\frac{1}{\zeta_j^{|\alpha^j|}}\right] \wedge \bigwedge_{j=d+1}^{n} \Big(\frac{1}{\zeta_j^{|\alpha^j|}}\cdot \frac{d\overline{\zeta_j}}{\overline{\zeta_j}}\Big)\cdot F(|\zeta_{d+1}|^2,\dots,|\zeta_n|^2)\,,$$

with $|\alpha^j| := \alpha_{1j}+\cdots+\alpha_{nj}$, $j=1,...,n$, F being the hypergeometric function

$$F(|\zeta_{d+1}|^2,\dots,|\zeta_n|^2) := \frac{1}{(2\pi i)^{n-d}}\int_{i\,\mathbb{R}^{n-d}} \prod_{j=1}^{n}\Gamma\left(1-\ell_j(\lambda'')\right)\ \prod_{j=d+1}^{n}|\zeta_j|^{2\lambda_j}\,d\lambda''\,,$$

where, for $\lambda\in\mathbb{C}^n$ with $\lambda' := (\lambda_1,...,\lambda_d)$ and $\lambda''=(\lambda_{d+1},\dots,\lambda_n)$, $\ell_j(\lambda'')$ is the j'th component of the vector $\ell(\lambda'')=A^*(0',\lambda'')$; in the extreme case $d=n$,

$$T_{f\,;\{1,...,n\}} = \bigwedge_{j=1}^{n}\bar\partial\left[\frac{1}{\zeta_j^{|\alpha^j|}}\right]\,.$$

When $m\leq n$ and $(f_1,...,f_m)$ define a complete intersection, we have :

Theorem 3.3 ([76]) *In the complete intersection case ($m=n=m_f$), all Bochner-Martinelli currents $T_{f\,;\mathcal{I}}$ are zero, except the current $T_{f\,;\{1,...,m\}}$ which coincides with the Coleff-Herrera current.*

The second assertion of this proposition is not immediate and its actual proof involves arguments inspired by the proof of the fibered residue formula [26]. It would be essntial to find a proof inspired by the arguments developped for exampe in the proof of theorem 2.3 (section 2.3).

Therefore, when $m\leq n$, though the unconditional limit

$$\lim_{\eta_1,...,\eta_m\to 0_+}\frac{1}{(2i\pi)^m}\int\limits_{\substack{|f_1|=\eta_1\\ \cdots\\ |f_m|=\eta_m}}\frac{\varphi}{f_1\cdots f_m}$$

does not exist in general (see section 2.2) even when $f_1, \dots, f_m$ define a complete intersection, the limit when ϵ tends to zero of the average

$$\int_{(\eta_1^2,\dots,\eta_m^2)\in\Delta_m(\epsilon^2)} \left[\frac{1}{(2i\pi)^m} \int\limits_{\substack{|f_1|=\eta_1\\ \dots\\ |f_m|=\eta_m}} \frac{\varphi}{f_1\cdots f_m}\right] d\sigma_{m,\epsilon^2}(\eta_1^2,\dots,\eta_m^2)$$

always exists ; one has already seen the reason for this in the absolute case $m=n$ in section 2 (see the statement in theorem 2.3 and its proof in section 2.3). One of the advantages of this alternative approach to the Coleff-Herrera current (apart from the fact that it extends to the non-complete intersection case) is that it provides a one parameter approximation (namely ϵ instead of $(\eta_1, \dots, \eta_m)$), which allows to turn around the difficulties arising from the Passare-Tsikh counterexample [75]. In the non-complete intersection case (still with $m \leq n$), it provides a $\overline{\partial}$-closed $(0,m)$-current (namely $T_{f\,;\{1,\dots,m\}}$) with support $f^{-1}(0)$ (instead of the essential intersection in the Coleff-Herrera construction [26]), depending in an alternate way on the ordering of $f_1, \dots, f_m$ (which is not the case for the Coleff-Herrera currents) ; the averaging method which is proposed here needs to be compared to the averaging method proposed by M. Passare in [72, 73].

A key property of the Coleff-Herrera current in the complete intersection case [71] is the fact that one has, for any $h \in H(V)$,

$$h \bigwedge_{j=1}^{m} \overline{\partial}\Big[\frac{1}{f_j}\Big] \equiv 0 \Longleftrightarrow h \in (f_1, \dots, f_m) \ ;$$

on the other hand, as a consequence of the transformation law, one has also in this case that for any $(q_1, \dots, q_m) \in \mathbb{N}^m$,

$$f_1^{q_1} \dots f_m^{q_m} \bigwedge_{j=1}^{m} \overline{\partial}\Big[\frac{1}{f_j^{q_j+1}}\Big] = \bigwedge_{j=1}^{m} \overline{\partial}\Big[\frac{1}{f_j}\Big] .$$

These two properties suggest in the non-complete intersection case the idea to extend the construction proposed in theorem 3.1, replacing the Bochner-Martinelli section $(\overline{f}_1/\|f\|^2, \dots, \overline{f_m}/\|f\|^2)$ that has been used by the section

$$\Big(\frac{\overline{f_1}\,|f_1|^{2q_1}}{\|f\|_q^2}, \dots, \frac{\overline{f_m}\,|f_m|^{2q_m}}{\|f\|_q^2}\Big) , \tag{3.1}$$

where $q=(q_1,...,q_m)\in\mathbb{N}^m$ and

$$\|f\|_q^2 := \sum_{j=1}^m |f_j|^{2(q_j+1)}\,.$$

This has been done in [12] ; it amounts essentially to replace $\|f\|^2$ by $\|f\|_q^2$ and the Bochner-Martinelli forms $\Omega(f\,;\mathcal{I})$, where $\mathcal{I}$ denotes an ordered subset of $\{1,...,m\}$ with cardinal r, by the forms :

$$\Omega(f\,;\mathcal{I}\,;\,q) := \Big(\prod_{l=1}^r f_{i_l}^{q_{i_l}}\Big)\sum_{j=1}^r (-1)^{j-1}\overline{f_{i_j}}^{q_{i_j}+1} \bigwedge_{\substack{l=1\\ l\neq j}}^r \overline{d[f_{i_l}{}^{q_{i_l}+1}]}\,.$$

The corresponding current $T_{f\,;\mathcal{I}\,;\,q}$ is defined as follows [12] : for any $(n,n-r)$ test-form φ with compact support in V,

$$\langle T_{f\,;\mathcal{I}\,;\,q}\,,\varphi\rangle := \frac{(-1)^{\frac{r(r-1)}{2}}(r-1)!}{(2i\pi)^r}\lim_{\epsilon\to 0}\frac{1}{\epsilon^{2r}}\int_{\|f\|_q^2=\epsilon^2}\Omega(f\,;\mathcal{I}\,;\,q)\wedge\varphi\,;$$

when $m\leq n$ and $(f_1,...,f_m)$ define a complete intersection, one has

$$T_{f\,;\{1,...,m\}\,;\,q} = T_{f\,;\{1,...,m\}\,;\,0} = \bigwedge_{j=1}^m \overline{\partial}\Big[\frac{1}{f_j}\Big]\,;$$

this can be obtained repeating the argument developped in the case $q=0$ in [76] (theorem 4.1) ; it means that in the complete intersection case, the action of the Coleff-Herrera current on a $(n,n-m)$-test form φ can be expressed (independently of $q\in\mathbb{N}$) as

$$\begin{aligned}&\Big\langle \bigwedge_{j=1}^m \overline{\partial}\Big[\frac{1}{f_j}\Big]\,,\,\varphi\Big\rangle\\ &\quad= \lim_{\epsilon\to 0}\int_{(\eta_1^2,...,\eta_m^2)\in\Delta_{m,q}(\epsilon^2)}\Bigg[\frac{1}{(2i\pi)^m}\int\limits_{\substack{|f_1|=\eta_1\\ \cdots\\ |f_m|=\eta_m}}\frac{\varphi}{f_1\cdots f_m}\Bigg]\,d\sigma_{m,q,\epsilon^2}(\eta_1^2,...,\eta_m^2)\,,\end{aligned}$$

where $\Sigma_{m,q}(\epsilon^2)$ denotes the $m-1$-dimensional open simplex

$$\Sigma_{m,q}(\epsilon^2)\ := \{(t_1,...,t_m)\in]0,+\infty[^m\,;\,t_1^{q_1+1}+\cdots+t_m^{q_m+1}=\epsilon^2\}$$

and $d\sigma_{m,q,\epsilon^2}$ the normalized $m-1$-dimensional Lebesgue measure on it ; moreover, in the particular case $m=n$, one has

$$\langle T_{f\,;\{1,\dots,m\}\,;q}\,,\,\varphi\rangle$$
$$= \lim_{\tau\to 0^+} \frac{1}{(2i\pi)^m} \int_{\gamma+i\mathbf{R}^m} \tau^{-|s|}\Gamma(|s|+1)\Big[\prod_{j=1}^m \Gamma(1-s_k)\Big]\,\Gamma^{\varphi}_{f,q}(s)\,ds\,,$$

where

$$\Gamma^{\varphi}_{f,q}(s) := \Big[\prod_{j=1}^m (q_j+1)\Big]\,\Gamma^{\varphi}_f\Big((q_1+1)s_1,\dots,(q_m+1)s_m\Big)\,;$$

so the fact that $T_{f\,;\{1,\dots,m\}\,;q}$ is independent of q when $f_1,\dots,f_m$ define a complete intersection implies that in this case, for any $(n,n-m)$-test form in V, the function

$$s=(\lambda_1,\dots,\lambda_m)\mapsto \Gamma^{\varphi}_f(\lambda)$$

has only simple poles along the divisors $\{\lambda_1=0\}$,...,$\{\lambda_m=0\}$ and that the action on φ of the Coleff-Herrera current equals then the Leray iterated residue at the origin in $\mathbb{C}^m$ (respect to these divisors) of the $(m,0)$-meromorphic form

$$\Gamma^{\varphi}_f(s)\,ds_1\wedge\ldots\wedge ds_m\,;$$

this gives an alternative approach to the statement in theorem 2.5.

Besides the fact that such currents $T_{f\,;\mathcal{I}\,;q}$ with $q\in\mathbb{N}^m$,

$$\mathcal{I}\subset\{1,\dots,m\}\,,\ m_f\le r(\mathcal{I})\le \min(m,n)\,,$$

inherit the same properties than in theorem 3.1, one can add that

$$hT_{f\,;\mathcal{I}\,;q}\equiv 0$$

for any $h\in H(V)$ such that for any point z in V, one has that

$$\Big(\prod_{l=1}^r f_{i_l}^{q_{i_l}}\Big)\,h_z$$

lies in the integral closure (in $\mathcal{O}_{\mathbb{C}^n,z}$) of the r-th power of the ideal generated by the $f_j^{q_j+1}$, $j=1,\dots,m$.

Exactly as for the Coleff-Herrera construction, which can be carried on any reduced analytic set, Bochner-Martinelli currents such as the $T_{f\,;\mathcal{I}\,;q}$, where

$f = (f_1, ..., f_m)$ is a collection of holomorphic functions in V and $\mathcal{I}$ is an ordered subset of $\{1, ..., m\}$ with cardinal $r \leq \min(m, n)$, may be multiplied in some natural way with integration currents on analytic sets ; namely, if A is a closed analytic subset in V (with pure dimension $n - m_A$, $1 \leq m_A \leq n-1$), and δ_A denotes the geometric integration current on A,

$$\delta_A = \sum_{\substack{J \subset \{1,\dots,n\} \\ \sharp J = k}} \delta_{A,J} dz_J \wedge \overline{dz_J}\,,$$

all distribution-coefficients $\delta_{A,J}$ are regular holonomic [17], so that, given $f_1, ..., f_m$ holomorphic in V (such that $\dim(A \cap V(f)) < n - m_A = \dim A$) and z_0 in $A \cap V(f)$, there exists (see [16], chapter 3), for each $J \subset \{1, ..., n\}$ with cardinal m_A, a Bernstein-Sato functional equation

$$\mathcal{Q}_{J,z_0}\Big(\zeta, \frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \overline{\zeta}}, \lambda\Big)\Big[\|f\|^{2(\lambda+1)} \otimes \delta_{A,J}\Big] = b_{J,z_0}(\lambda)\Big[\|f\|^{2\lambda} \otimes \delta_{A,J}\Big]\,, \quad b_{J,z_0} \in \mathbb{C}[X]\,;$$

this allow the possibility [13] to define, for any subset $\mathcal{I}$ with cardinal r, $r \leq \min(m, n - m_A)$, restricted residual currents

$$\begin{aligned} & T_{f\,;\mathcal{I}\,;q} \wedge \delta_A \ : \\ &= \frac{(-1)^{\frac{r(r-1)}{2}}(r-1)!}{(2i\pi)^r}\Big[\|f\|_q^{2(\lambda-r-1)}\overline{\partial}\|f\|_q^2 \wedge \Omega(f\,;\,\mathcal{I}\,;\,q) \wedge \delta_A\Big]_{\lambda=0} \\ &= \frac{(-1)^{\frac{r(r-1)}{2}}(r-1)!}{(2i\pi)^r} \lim_{\epsilon \to 0} \frac{1}{\epsilon^{2r}} \int_{A \cap \{\|f\|_q^2 = \epsilon^2\}} \Omega(f\,;\,\mathcal{I}\,;\,q) \wedge (\cdot) \end{aligned}$$

(taking the meromorphic continuation, then the value at $\lambda = 0$, in the first definition, the second being naturally connected to it via the Mellin transform, the limit here been taken when ϵ tends to 0 outside a negligible set in $]0, +\infty[$).

3.2 Division formulas and Bochner-Martinelli residue currents

Let $m, n \in \mathbb{N}$, $m \geq n$; let $f_1, ..., f_m$ be m holomorphic functions in some pseudoconvex domain V in $\mathbb{C}^n$ and $U_1, ..., U_m$ be m planar domains with piecewise smooth boundaries such that $U_j \subset\subset f_j(V)$; let W be a connected composent of the analytic polyedron

$$\{\zeta \in V\,;\, f_j(\zeta) \in U_j\,,\, j = 1, ..., m\}$$

such as the faces of W intersect in general position (that is, for any subset $\mathcal{I}$ of $\{1, ..., m\}$ with cardinal r, the intersection of the r faces

$$\gamma_{i_l} := \{\zeta \in W \, ; \, f_{i_l}(\zeta) \in \partial U_{i_l} \, , \, f_{i_{l'}}(\zeta) \in U_{i_{l'}} \text{ for } l' \neq l\} \, , \quad l = 1, ..., r \, ,$$

defines a piecewise smooth $2n - r$-dimensional real-analytic cycle in V) ; the natural orientation on V induces an orientation on each of the n-dimensional edges $\gamma_{i_1,...,i_n} := \gamma_{i_1} \cap \ldots \cap \gamma_{i_n}$, $1 \leq i_1 < \ldots < i_n \leq m$, which form the skeleton of W ; let $a_{j,i}$, $1 \leq i \leq m$, $1 \leq j \leq n$ be nm holomorphic functions in $V \times V$ such that

$$f_i(\xi) - f_i(\zeta) = \sum_{i=1}^{n} a_{j,i}(\xi, \zeta) \, (\xi_j - z_j) \, , \quad (\xi, \zeta) \in V \times V \, ;$$

then, one can reproduce (in W) any holomorphic function $h \in H(W)$ which extends continuously to $\overline{W}$ with the well known Cauchy-Weil integral representation formula [92] :

$$h(\zeta) = \frac{1}{(2i\pi)^n} \sum_{1 \leq i_1 < \ldots < i_n \leq m} \int_{\gamma_{i_1,\ldots,i_n}} \frac{h(\xi) \bigwedge\limits_{l=1}^{n} \Big(\sum\limits_{j=1}^{n} a_{j,i_l}(\xi, \zeta) d\xi_j \Big)}{\prod\limits_{l=1}^{n} (f_{i_l}(\xi) - f_{i_l}(\zeta))} \, . \tag{3.2}$$

In the particular case where U_j is a disc $D(0, r_j)$, one can derive from (3.2) the Weil expansion theorem :

Theorem 3.4 ([92]) *Any function h holomorphic in W and continuous on $\overline{W}$ admits in W the series expansion*

$$\begin{aligned} h(\zeta) \;\; &= \;\; \frac{1}{(2i\pi)^n} \sum_{1 \leq i_1 < \ldots < i_n \leq m} \sum_{k \in \mathbb{N}^n} \int_{\gamma_{i_1,\ldots,i_n}} \frac{h(\xi) \bigwedge\limits_{l=1}^{n} \Big(\sum\limits_{j=1}^{n} a_{j,i_l}(\xi, \zeta) d\xi_j \Big)}{\prod\limits_{l=1}^{n} f_{i_l}^{k_l+1}(\xi)} \\ & \qquad \times f_{i_1}^{k_1}(\zeta) \cdots f_{i_n}^{k_n}(\zeta) \, , \end{aligned} \tag{3.3}$$

such a development being normally convergent on any compact subset of W.

The key difficulty which is inherent to Cauchy-Weil's formula (3.2) lies in the fact that for any subset $\{i_1, ..., i_n\} \subset \{1, ..., m\}$, $\gamma_{i_1,...,i_n}$ represents a portion of an analytic cycle. Connecting integral formulas via Dolbeault isomorphism (see for example [46]) leads to some infinitesimal versions of the Cauchy-Weil formula in its expanded form (3.3), where Bochner-Martinelli residual currents are involved ; subfamilies $\{f_{i_1}, ..., f_{i_n}\}$ extracted from the family $\{f_1, ..., f_m\}$ will now be considered instead of the pieces of cycles $\gamma_{i_1,...,i_n}$.

In the algebraic context ($f_1, ..., f_n$ being a quasi-regular sequence in some commutative $\mathbf{A}$-algebra $\mathbf{R}$ such that $\mathbf{R}/f\mathbf{R}$ is finitely generated projective module, see [62], section 3, theorem 3.6), all residue symbols

$$\operatorname{Res}\begin{bmatrix} r\, dX_1 \wedge \cdots \wedge dX_n \\ f_1^{k_1+1}, \ldots, f_n^{k_n+1} \end{bmatrix}, \quad k \in \mathbb{N}^n,$$

are simultaneously defined via an algebraic trace formula. In our analytic situation ($f_1, ..., f_m$ being m holomorphic functions in some open subset $V \subset \mathbb{C}^n$), such that $m_f = \operatorname{codim} f^{-1}(0)$, one can attach to any choice of $q \in \mathbb{N}^m$ (that is any choice of a Bochner-Martinelli section such as (3.1)) a collection of residual currents indexed by $\mathcal{I}$, $\mathcal{I} \subset \{1, ..., n\}$ and $k \in \mathbb{N}^m$ defined as follows : if $\mathcal{I} := \{i_1, ..., i_r\}$, $m_f \le r \le \min(n, m)$, $T^k_{f;\mathcal{I};q}$ is the $(n, n-r)$ current in V defined as

$$\begin{aligned}\langle T^k_{f;\mathcal{I};q}, \varphi\rangle &= \frac{(-1)^{\frac{r(r-1)}{2}} (|k| + r - 1)!}{(2i\pi)^r\, k_1! \cdots k_m!} \\ &\quad \times \lim_{\epsilon \to 0} \frac{1}{\epsilon^{2(r+|k|)}} \int_{\|f\|_q^2 = \epsilon^2} \Big(\prod_{j=1}^m |f_j|^{2q_j k_j} \overline{f_j}^{k_j} \Big)\, \Omega(f\,;\, \mathcal{I};\, q) \wedge \varphi\,,\end{aligned}$$

where $|k| := k_1 + \cdots + k_m$, for any test-form $\varphi \in \mathcal{D}^{n,n-m}(V)$. When $m \le n$ and $f_1, ..., f_m$ define a complete intersection one has

$$T^k_{f\,;\{1,...,m\}\,;q} = \bigwedge_{j=1}^m \overline{\partial} \Big[\frac{1}{f_j^{k_j+1}}\Big], \quad \forall q, k \in \mathbb{N}^m.$$

Such objects were already introduced (acting on $\overline{\partial}$-forms) in [70].

It is interesting to note that when $m_f = n$ (which corresponds to the frame of Cauchy-Weil's formulas (3.2) or (3.3)), then, for any subset $\mathcal{I} = \{i_1, ..., i_n\}$ of $\{1, ..., m\}$ with cardinal n, for any $q, k \in \mathbb{N}^m$, one has

$$T^k_{f\,;\mathcal{I}\,;q} = \Big(\sum_{\alpha \in V(f)} \mathcal{Q}_{\alpha,\mathcal{I},k,q}\big(\frac{\partial}{\partial \zeta}\big)\, [\delta_\alpha] \Big) d\zeta_1 \wedge \ldots \wedge d\zeta_n\,,$$

which confers an algebraic character to these currents in case $f^{-1}(0)$ is discrete.

Let now $r, R, \epsilon > 0$, $n > m$ and $\{f_1, ..., f_m\}$ be m holomorphic functions in n variables defining a zero set $f^{-1}(0)$ with codimension m_f in

$$U_{r+\epsilon,R+\epsilon} := \{\zeta = (z, w) \in \mathbb{C}^{n-m_f} \times \mathbb{C}^{m_f}\,;\, \|z\| < r + \epsilon\,,\, \|w\| < R + \epsilon\}$$

such that the projection $\pi\,:\zeta \mapsto z$ from $f^{-1}(0) \cap U_{r+\epsilon,R+\epsilon}$ to $\mathbb{C}^{n-m_f}$ is proper and that

$$\{\zeta \in f^{-1}(0)\,;\, \|z\| < r + \epsilon\,,\, R \leq \|w\| < R + \epsilon\} = \emptyset\,;$$

let

$$\Sigma_k(z) := \frac{1}{(2i\pi)^k} \partial\|z\|^2 \wedge \overline{\partial}\partial\|z\|^2\,, \quad k = 1, ..., n - m_f\,.$$

Let also a_{ji}, $j = 1, ..., n$, $i = 1, ..., m$ be a matrix of $n \times m$ holomorphic functions in $2n$ variables $(\xi, \zeta) = (u, v\,;\, z, w)$ in $U_{r+\epsilon,R+\epsilon} \times U_{r+\epsilon,R+\epsilon}$ such that

$$f_i(\xi) - f_i(\zeta) = \sum_{j=1}^{n} a_{ji}(\xi, \zeta)\,(\xi_j - \zeta_j)\,, \quad i = 1, ..., m \tag{3.4}$$

and

$$A_i(\xi, \zeta) := \sum_{j=1}^{n} a_{ji}(\xi, \zeta)\, d\xi_j\,, \quad i = 1, ..., m\,.$$

Let $\varphi \in \mathcal{D}_z(\{\|z\| < r + \epsilon\})$, such that $\varphi \equiv 1$ in a neighborhood of $\{\|z\| \leq r\}$ and $\psi \in \mathcal{D}_w(\{\|w\| < R + \epsilon\})$, such that $\psi \equiv 1$ in a neighborhood of $\{\|w\| \leq R\}$; then, one can derive from division formulas based on the use of weighted Bochner-Martinelli kernels (in the spirit of [14, 71, 7]) the following generalization of Cauchy-Weil's formula (in its expanded form (3.2)) :

Theorem 3.5 [12, 94] *For any h holomorphic in $U_{r+\epsilon,R+\epsilon}$, for any $q \in \mathbb{N}^m$, one has the following version of Cauchy-Weil's expansion in $U_{r,R}$:*

$$h(\zeta) \equiv$$
$$- \sum_{\substack{k\in\mathbb{N}^m \\ |k|<N}} \sum_{r=m_f}^{m} \sum_{\substack{\mathcal{I}=\{i_1,...,i_r\} \\ 1\leq i_1<...<i_r\leq m}} \Big\langle T^k_{f\,;\mathcal{I}\,;q}\,,\, h\,\psi\overline{\partial}\varphi \wedge \frac{\Sigma_{n-r}(u) \wedge \Big(\bigwedge_{l=1}^{r} A_{i_l}(\xi, \zeta)\Big)}{\langle \overline{u}\,,\, u - z\rangle^{n-r}} \Big\rangle\, f^k(\zeta) \tag{3.5}$$

modulo the ideal $(f_1, ..., f_m)^N$ *in* $H(U_{r,R})$ *for any* $N \in \mathbb{N}^*$ *; moreover, if* $f(0) = 0$*, the series above is normally convergent in some* $U_{\eta,\eta}$ *with* η *sufficiently small.*

In the particular case $m_f = 0$, one can formulate this result as follows :

Theorem 3.6 [94] *If* $f_1, ..., f_m$ *are* m *holomorphic functions with a finite set of common zeroes* $f^{-1}(0)$ *in some bounded pseudoconvex open subset* $V \subset \mathbb{C}^n$ *and* a_{ji}, $i = 1, ..., m$, $j = 1, ..., n$, *are* $n \times m$ *holomorphic functions such that* (3.4) *holds in* $V \times V$*, then for any* $h \in H(V)$*, for any* $N \in \mathbb{N}^*$*, one has*

$$h(\zeta) \equiv \sum_{\substack{k \in \mathbb{N}^m \\ |k| < N}} \sum_{1 \leq i_1 < \ldots < i_n \leq m} \Big\langle T^k_{f;\mathcal{I};q}\, ,\, h \Big(\bigwedge_{l=1}^{n} A_{i_l}(\xi, \zeta) \Big) \Big\rangle\, f^k(\zeta) \qquad (3.6)$$

modulo the ideal $(f_1, ..., f_m)^N$ *in* $H(V)$.

An example of application. As an interesting example where one would expect such division formulas to be involved, we take here the opportunity to mention the following result, which does not happen to be so well known :

Theorem 3.7 *Let* $f_1, ..., f_n$ *be* n *germs of holomorphic functions in* n *variables at the origin of* $\mathbb{C}^n$*, such that* $f_1(0) = \ldots = f_n(0) = 0$ *and* J *be the Jacobian determinant of* $f_1, ..., f_n$ *; then* $J \in (f_1, ..., f_n)$ *if and only if* $\dim V(f) > 0$.

The fact that J cannot lie in $(f_1, ..., f_n)$ when the origin is an isolated zero is a consequence of the formula which was established in section 2,

$$\bigwedge_{j=1}^{n} \overline{\partial}\Big[\frac{1}{f_j}\Big] \wedge dz_1 \wedge \ldots \wedge dz_n = \mathrm{mult}_0(f)\, \delta_0 \bigwedge_{j=1}^{n} \frac{d\overline{z}_j \wedge dz_j}{2i}$$

and of the local duality theorem. The reciprocal assertion is more unexpected ; in fact, what is easy to deduce from theorem 3.6 is that one has always

$$\mathcal{M} J \subset (f_1, ..., f_n)\,,$$

where $\mathcal{M}$ denotes the maximal ideal in $\mathcal{O}_{\mathbb{C}^n,0}$; this happens to be a classical fact in residue theory, namely that the Jacobian determinant of a regular sequence $(f_1, ..., f_n) \in \mathcal{O}_{\mathbb{C}^n,0}$ lies in the socle of the ideal $(f_1, ..., f_n)$; the fact that when $\dim V(f) > 0$, one has in fact $J \in (f_1, ..., f_n)$ follows from this remark, combined with the use of Krull's theorem (this was noticed by M. Hickel) ; the corresponding global result amounts to [90]. It remains an interesting problem to extract from theorem 3.6 an explicit division formula for J in $(f_1, ..., f_n)$ when $\dim f^{-1}(0) > 0$; such a formula can be derived from Euler's identity in the homogeneous case [68, 83] ; in the general situation, it could be done only under some additional hypothesis [12] ; it seems likeky that the proof of such a division formula would provide a better understanding of the algebraic properties of Bochner-Martinelli residue currents.

Given any closed ideal in the space of analytic functions on a n-dimensional Stein manifold $\mathcal{X}$, it is known that any analytic functional $T \in H'(V)$ such that $hT \equiv 0$ for any $h \in I$ can be represented by some element $\widetilde{T} \in \mathcal{D}'(V)$ which satisfies $h\widetilde{T} \equiv 0$ (this identity being understood now from the distribution point of view) ; the question remains whether such $\widetilde{T}$ can be described in terms of residual currents attached to the ideal I. Since the initial developments of the theory, structure problems related to such currents objects, together with their eventual relation with the noetherian operators of Ehrenpreis-Palamodov have been extensively studied [31, 32] ; complete answers have been given in the discrete case and the normal crossings case ; less precise answers have also been proposed to describe the structure of the Coleff-Herrera current in the complete intersection case (see [29]). For example, if $m = 1$ and $f \in H(V)$, one can describe the action of the residual current $\overline{\partial}(1/f)$ on the $(n, n-1)$ test-form

$$\varphi = \sum_{j=1}^{n} \varphi_j d\zeta \wedge \bigwedge_{l \neq j} d\overline{\zeta_j} = \sum_{j=1}^{n} \varphi_j d\zeta \wedge d\overline{\zeta}_{[j]} \, .$$

as

$$\left\langle \overline{\partial} \left[\frac{1}{f} \right], \, \varphi \right\rangle := \sum_{j=1}^{n} \sum_{k} D_k \mathrm{VP}_{\mathrm{k,j}} \left(\varphi_j d\zeta_{[j]} \wedge d\overline{\zeta}_{[j]} \right),$$

where the D_k are differential operators with holomorphic coefficients, the $\mathrm{VP}_{k,j}$ are the images through the canonical injection of some principal value distributions on the irreducible branches of $f^{-1}(0)$ (see [32], theorem 6.4.3).

When I is an ideal of $H(\mathcal{X})$ which is defined as a complete intersection (let say by $f_1, ..., f_m$ which are holomorphic functions in V), then, for any analytic functional $T \in H'(V)$ such that $hT \equiv 0$, for any Stein neighborhood U of a carrier K of T, one can find [30, 86] an element $\varphi_T \in \mathcal{D}^{n,n-m}(U)$ such that :

- $\forall h \in H(V)\,, T(h) = \Big\langle \bigwedge\limits_{j=1}^{m} \overline{\partial}\big[\frac{1}{f_j}\big] \wedge \varphi_T\,,\, h\Big\rangle$
- $\overline{\partial}\varphi_T = \sum\limits_{j=1}^{m} f_j\,\mathcal{D}^{n,n-m+1}(V)\,;$

this shows the crucial role of residue currents in the complete intersection case and the necessity to clarify their relations with noetherian operators attached to the ideal. In the non complete intersection case, the situation is more involved ; what can be proved is the following result, valid in the locally Cohen-Macaulay situation :

Theorem 3.8 [30] *Let $\mathcal{X}$ be a n dimensional Stein manifold and I a closed ideal in $H(V)$ such that $\mathcal{O}_{\mathcal{X},z}/\mathcal{I}_{\mathcal{X},z}$ is a Cohen-Macaulay ring with fixed codimension m for any z in the zero set of I ; let $T \in H'(V)$ such that $I\,T \equiv 0$ (in the sense of analytic functionals). Then one can find a compactly supported residual current $\widetilde{T}$ such that $I\,\widetilde{T} \equiv 0$ (in the sense of currents) and $T(h) = \widetilde{T}(h)$ for any $h \in H(V)$; moreover, given any complete intersection ideal $(f_1, ..., f_m)$ containing I, one can find $\widetilde{T}$ with the form*

$$\widetilde{T} = \bigwedge_{j=1}^{m} \overline{\partial}\Big[\frac{1}{f_j}\Big] \wedge \varphi_{T,f}\,,$$

where

- $\varphi_{T,f} \in [(f_1, ..., f_m)\,:\,I]\,\mathcal{D}^{n,n-m}(V)$ *;*
- $\overline{\partial}\varphi_T = \sum\limits_{j=1}^{m} f_j\,\mathcal{D}^{n,n-m+1}(V)\,.$

Note that in the discrete case $m = n$, one can also realize $\widetilde{T}$ in terms of the currents $T^k_{f;\mathcal{I};q}$, with $\mathcal{I}$ of cardinal n, which have been constructed previously in this section. Nevertheless, we would like to point that the approach developped for the proof of theorem 3.8 (and which is inspired from the notion of *linkage* in algebraic geometry) is different from the direct division approach where Bochner-Martinelli kernels are involved. Let us quote the following result obtained by M. Méo :

Theorem 3.9 [67] *Let $f_1, ..., f_m$ be m holomorphic functions in a neighborhhood of the origin in $\mathbb{C}^n$ such that $V(f_1, ..., f_m)$ is an irreducible analytic set with codimension m_f ; if $m_f < n$, let μ_f be the multiplicity of $(f_1, ..., f_m)$ at any generic (smooth) point of $V(f_1, ..., f_m)$; if $m_f = n$ (that* *is $V(f) = \{0\}$), let μ_f be the multiplicity in the sense of Serre of the primary ideal $(f_1, ..., f_m)$, that is also the multiplicity of $\tilde{f} = (\tilde{f}_1, ..., \tilde{f}_n)$, where $\tilde{f}_1, ..., \tilde{f}_n$ are n linear generic combinations of the f_j's ; then one has the following factorization formula for the integration current on $[V(f_1, ..., f_m)]$:*

$$\begin{aligned}
[V(f_1, ..., f_m)] : &= \mu_f \, \delta_f \bigwedge_{j=1}^{n} \frac{d\overline{z_j} \wedge dz_j}{2i} \quad \textit{when } m_f = n \\
&= \sum_{1 \leq i_1 < ... < i_{m_f} \leq m} T_{f\,;\mathcal{I}} \wedge \bigwedge_{l=1}^{m_f} df_{i_l} \quad \textit{when } m_f < n \, ,
\end{aligned}$$

which can be seen as a generalization to the non-complete intersection case of Poincaré-Lelong equation (here δ_f denotes the Dirac distribution on $f^{-1}(0)$).

As we will see in subsection 6.3, a similar formula exists in case $(f_1, ..., f_m)$ define a purely m_f-codimensional cycle which is not irreducible anymore ; note that the multiplicities involved in the right-hand side of such a generalized Poincaré-Lelong formula are the Hilbert-Samuel multiplicities : for example, in the discrete case $f^{-1}(0) = \{0\}$, the integration current we may represent with such a formula does not change if $(f_1, ..., f_m)$ is replaced by any ideal $(\tilde{f}_1, ..., \tilde{f}_p)$ with the same integral closure, for example any ideal $(\tilde{f}_1, ..., \tilde{f}_n)$, where $\tilde{f}_1, ..., \tilde{f}_n$ are n generic linear combinations of $f_1, ..., f_m$. On the other hand, in [59], M. Lejeune-Jalabert gave an explicit formula, given some primary ideal $(f_1, ..., f_m)$ in $\mathcal{O}_{\mathbb{C}^n,0}$, to express the dimension of the quotient space $\mathcal{O}_{\mathbb{C}^n,0}/(f_1, ..., f_m)$ as

$$\dim \Big(\mathcal{O}_{\mathbb{C}^n,0}/(f_1, ..., f_m)\Big) = \Big\langle \bigwedge_{j=1}^{n} \overline{\partial}\Big[\frac{1}{\tilde{f}_j}\Big] \, , \, \omega_n \Big\rangle$$

where $(\tilde{f}_1, ..., \tilde{f}_n)$ denotes an arbitrary regular sequence in $(f_1, ..., f_m)$ and ω_n (depending of the $\tilde{f}_j$'s) is computed from a free resolution $\mathcal{R}$ (with length n) of $\mathcal{O}_{\mathbb{C}^n,0}/(f_1, ..., f_m)$ using explicitely some morphism $\alpha_\bullet$ of complexes from the Koszul resolution $\Lambda^\bullet \mathcal{O}^n_{\mathbb{C}^n,0}$ to $\mathcal{R}_\bullet$ deduced from the inclusion $(\tilde{f}_1, ..., \tilde{f}_n) \subset (f_1, ..., f_m)$. From this, the following question arises, given a primary ideal $(f_1, ..., f_m)$ in $\mathcal{O}_{\mathbb{C}^n,0}$: can one choose some convenient q in formula (3.6) so that :

- such a formula could reproduce the membership of h to $(f_1, ..., f_m)$?

- for any notion of multiplicity at the origin μ_f (dynamical, algebraic such as the dimension of the quotient space, or more analytic such as the multiplicity in the sense of Serre, which would be the more plausible), one has, for any $h \in \mathcal{O}_{\mathbb{C}^n,0}$,

$$\mu_f\, h(0) = \sum_{1 \leq i_1 < \ldots < i_n \leq m} \Big\langle T^k_{f\,;\mathcal{I}\,;q}\,,\, h\,\Big(\bigwedge_{l=1}^{n} df_{i_l}(\zeta)\Big)\Big\rangle\,?$$

Note that a good test example would be the ideal

$$I := (z_1^2 - z_2^2, z_2^2 - z_3^2, z_1 z_2, z_1 z_3, z_1 z_3)$$

in $\mathcal{O}_{\mathbb{C}^3,0}$, where one can check easily that

$$h \in I \Longleftrightarrow \Big[\sum_{j=1}^{3} \frac{\partial^2}{\partial z_j^2}\Big]\,[hg] = 0 \quad \forall g \in \mathcal{O}_{\mathbb{C}^3,0}\,.$$

Also the Cauchy-Weil formula (3.2), besides its ambivalence in the case $m > n$, needs certainly to be better understood. Note that our construction of Bochner-Martinelli residue currents with various sections correspond to various averaging inside the process that lead to the construction of residual currents of the Coleff-Herrera type (as developped in section 2). One could think also about averaging over the different choices of the sections or even consider for example the parameters $(q_1, ..., q_m)$ that quantify in our examples the choice of sections as complex parameters, then use at this new level analytic continuation techniques in order to realize new residual objects thanks to residue calculus (dealing now with meromorphic functions of q). Here again, one should insist on the new lightening of such ideas that could bring the theory of amoebas (in the spirit of section 2.3).

4 Applications to the effective Nullstellensatz

4.1 About the effectivity of the geometric Hilbert's Nullstellensatz

Let us state here the classical Hilbert's zeroes theorem, also known as the Hilbert's Nullstellensatz :

Theorem 4.1 *Let $\mathbb{K}$ be a commutative field of arbitrary characteristic and $P_1, ..., P_m, Q$, be $m+1$ elements in the polynomial algebra $\mathbb{K}[X_1, ..., X_n]$ such that $Q \equiv 0$ on the set of common zeroes of $P_1, ..., P_m$ in $\overline{\mathbb{K}}^n$, where $\overline{\mathbb{K}}$ is an algebraically closed extension of $\mathbb{K}$; then one can find an integer $M \in \mathbb{N}$ and polynomials $Q_1, ..., Q_m$ in $\mathbb{K}[X_1, ..., X_n]$ such that*

$$Q^M(X) = Q_1(X)P_1(X) + \cdots + Q_m(X)P_m(X) . \tag{4.1}$$

Note that proving such an assertion (when the number n of variables is arbitrary) amounts to prove it when $Q \equiv 1$ and $P_1, ..., P_m$ have no common zeroes in some algebraically closed extension $\overline{\mathbb{K}}$ of $\mathbb{K}$. This is based on what is known as the "Rabinowitz trick" : let us assume Q vanishes on the set of common zeroes of $P_1, ..., P_m$ in $\overline{\mathbb{K}}^n$; this implies that the polynomials

$$1 - X_0\, Q(X_1, ..., X_n)\, ,\ P_1(X_1, ..., X_n)\, ,\ ...\, ,\ P_m(X_1, ..., X_m)$$

(considered as elements in $\mathbb{K}[X_0, ..., X_n]$) have no common zeroes in $\overline{\mathbb{K}}^{n+1}$; assuming that Hilbert's Nullstellensatz holds in this particular situation, one can find polynomials $\widetilde{Q}_0, \ldots, \widetilde{Q}_m$ in $\mathbb{K}[X_0, ..., X_n]$ such that

$$1 \equiv \widetilde{Q}_0(X_0, ..., X_n)\,(1 - X_0\, Q(X_1, ..., X_n)) + \sum_{j=1}^{m} \widetilde{Q}_j(X_0, ..., X_n)\, P_j(X_1, ..., X_n)\ ;$$

substituting $1/Q$ to X_0 and then raising denominators leads to an algebraic identity of the form (4.1). Given m polynomials $P_1, ..., P_m$ in $\mathbb{K}[X_1, ..., X_n]$ which define a nonproper ideal in $\mathbb{K}[X_1, ..., X_n]$ (that is $1 \in (P_1, ..., P_m)$), any polynomial identity of the form

$$1 \equiv \sum_{j=1}^{m} Q_j(X_1, ..., X_n)\, P_j(X_1, ..., X_m)\, , \quad Q_j \in \mathbf{K}[X_1, ..., X_n] \tag{4.2}$$

is known as a Bézout identity ; so what the trick of Rabinowitz says is that solving Hilbert's Nullstellensatz explicitely in $\mathbb{K}[X_1, ..., X_n]$ amounts to solve a Bézout identity in $\mathbb{K}[X_0, ..., X_n]$.

A crucial problem related to Hilbert's Nullstellensatz (or solving Bézout identity) is to give an explicit bound (in terms of the entries $P_1, ..., P_m$, namely of their degrees or the affine geometric degree of the polynomial map $P = (P_1, ..., P_m)$ from $\overline{\mathbb{K}}^n$ to $\overline{\mathbb{K}}^m$) on M and on the degrees of polynomials

$Q_1, ..., Q_m$ involved in (4.1) or (4.2). The first explicit bound respect to the effectivity for Bézout identity was proposed by G. Hermann [48] following a method based on elimination theory ; Hermann's bound (for a particular solution of (4.2) starting from entries $(P_1, ..., P_m)$ such that $1 \in (P_1, ..., P_m)$) is

$$\deg Q_j \leq 2(2D)^{2^{n-1}}, \quad j = 1, ..., m$$

where $D = \max \deg P_j$. The first fundamental decisive progress towards effectivity in Hilbert's Nullstellensatz (or Bézout identity) was made by D. W. Brownawell in 1988 in the case $\mathbb{K} = \mathbb{C}$ (that is essentially the characteristic zero case). For an exhaustive list of references about the beginning of the story and the pionneer work of D. W. Brownawell and J. Kollár, we would like to point out the survey paper [84]). Brownawell's original method combines the theory developped by Y. Nesterenko around Chow forms with the search for inequalities instead of identitities, then division with L^2-estimates in multivariate complex analysis (namely Briançon-Skoda theorem [21], which several years later will be transposed to a algebraic context as Lipman-Sathaye-Teissier theorem [63, 64]) in order to obtain estimates of the form

$$\max \deg Q_j \leq 3 \min(n, m)\, n D^{\min(n,m)}, \quad j = 1, ..., m,$$

for candidates Q_j involved in a Bézout identity (4.2) when D is the maximal degrees of polynomial entries $(P_1, ..., P_m)$ such that $1 \in (P_1, ..., P_m)$; note that analytic tools (namely Briançon-Skoda's theorem or explicit weighted Bochner-Martinelli formulas) where already present in this original proof. A few monthes later, J. Kollár proposed in [54] a geometric method (based on the use of local cohomology, but one could also re-interpret it in terms of Koszul complexes) that lead to a prime power version of Hilbert's Nullstellensatz :

Theorem 4.2 (J. Kollár [54], D. W. Brownawell [23]) *Let $\mathbb{K}$ be a commutative field and $P_1, ..., P_m$ be m elements in $\mathbb{K}[X_1, ..., X_n]$, such that*

$$\deg P_2 \geq \deg P_3 \geq \cdots \geq \deg P_m \geq \deg P_1$$

and $\mathcal{U}(P)$ is the homogeneous ideal in $\mathbb{K}[X_0, ..., X_n]$ generated by the homogeneous polynomials

$$\mathcal{P}_j(X_0, ..., X_n) := X_0^{\deg P_j}\, P_j\Big(\frac{X_1}{X_0}, ..., \frac{X_n}{X_0}\Big), \quad j = 1, ..., m.$$

Suppose that $D_{\mu-\rho+1} \geq 3$, *where* $\mu := \min(n,m)$, $\rho :=$ *height* $(\mathcal{U}(P)) > 1$ *; then, one can find elements* $\mathcal{U}_1, ..., \mathcal{U}_r$ *in* $\mathrm{Ass}(\mathcal{U}(P))$, *together with positive integers* $e_0, ..., e_r$, *such that*

$$(X_0, ..., X_n)^{e_0} \prod_{j=1}^{r} \mathcal{U}_j^{e_j} \subset \mathcal{U}(P)$$

with

$$\sum_{j=0}^{r} e_j \leq D_1 \cdots D_\mu \, ;$$

in particular, if $Q \in \mathrm{rad}\,(P_1, ..., P_m)$, *one can find* $Q_1, ..., Q_m \in \mathbb{K}[X_1, ..., X_n]$ *such that*

$$Q^{\sum\limits_{j=0}^{r} e_j} = \sum_{j=1}^{m} Q_j \, P_j$$

and

$$\deg P_j Q_j \leq e_0 + \cdots + e_r \leq D_1 \cdots D_\mu \, , \quad j = 1, ..., m \, .$$

Under the same hypothesis on the entries P_j's and their degrees, S. Ji, J. Kollár, B. Shiffman derived from such a result Lojasiewicz inegalities, assuming $\mathbb{K}$ was algebraically closed and equipped with some absolute value ; namely, if $P^{-1}(0)$ denotes the set of common zeroes of the P_j's in $\mathbb{K}^n$, one has

$$\max_{1 \leq j \leq m} |P_j(z)| \geq \left(\frac{\min\Big(1, \mathrm{distance}\,[z, P^{-1}(0)]\Big)}{(1 + \|z\|)} \right)^{D_1 \cdots D_\mu} . \qquad (4.3)$$

Since our aim here is to focuse on the role of multidimensional residue calculus, we will insist on the original approach of D. W. Brownawell : get first precise Lojasiewicz inequalities, then go from inequalities to identities using several variables complex analysis tools.

Let us note here that in the very particular case where $m = n$ and $P_1, ..., P_n$ define a discrete (hence finite) variety in $\mathbb{K}^n$, there is a simple way (closer to the ideas we developped previously in this survey) to show that there exist constants $c > 0$ and $K \geq 0$ such that for any $z \in \mathbb{K}$, $|z| \geq K$, one has

$$\max_{1 \leq j \leq n} \frac{|P_j(z)|}{|z|^{D_j}} \geq c|z|^{-D_1 \cdots D_n} \, ; \qquad (4.4)$$

this was already noticed in [54] : consider the homogeneous polynomials $\mathcal{P}_j$ (defined as the homogenizations of the P_j's, $j = 1, ..., n$) and the functions $f_1, \dots, f_n$ which correspond to the functions $\mathcal{P}_j(\zeta)$ expressed in affine coordinates near some point in the hyperplane $z_0 = 0$. Let $\pi : Z \mapsto \mathbb{C}^n$ be the normalized blow-up of $\mathbb{P}^n(\mathbb{C})$ with center the ideal generated by the f_j. It follows from Bézout's theorem that at least one $f_j \circ \pi$ vanishes to an order less or equal to $D_1 \cdots D_n$ along each component of the exceptional divisor of the normalized blow-up, which implies (4.4).

Geometric intersection theory methods in the complex case $\mathbb{K} = \mathbb{C}$ [89, 27, 55] lead to a direct geometric proof of precise local Lojasiewicz type inequalities in the more involved situation where $P_1, ..., P_m$ do not define a zero-dimensional variety : more generally, if $I_1, ..., I_m$ denote m unmixed ideals in $\mathbb{C}[X_1, ..., X_n]$, and $(P_{ij})_j$ is a set of generators for I_j, then, for any bounded set $B \in \mathbb{C}^n$, there is a constant $c(B)$ such that for any $z \in B$,

$$\max_{i,j} |P_{ij}(z)| \geq c\, d(z, V(I_1, ..., I_m))^{\prod_{j=1}^{m} \deg I_j},$$

where $V(I_1, ..., I_m)$ denote the algebraic variety defined by $(I_1, ..., I_m)$ and the degree of an unmixed ideal I is defined as the degree of the corresponding cycle (multiplicities being taken into account). We refer to the paper of J. Kollár [55] for a survey of this geometric approach. Note that Kollár's approach in [55] has been extended in [33] to the case when $\mathbb{P}^n(\mathbb{C})$ is replaced by a smooth projective variety.

Of course, obtaining local Lojasiewicz type inequalities is a problem of different nature than obtaining global Lojasiewicz type inequalities, such as (4.3) (these are consequences from the prime power version of Hilbert's Nullstellensatz) or (4.4) (here a direct geometric global argument in used). Nevertheless, in the spirit of (4.4), global Lojasiewicz inequalities may be established independently of a prime power version of Hilbert's Nullstellensatz ; one has to combine intersection theory technics [38] with methods adapted to the study of singularities [85]. We refer to [27] and to [51] for such a direct approach. Moreover, there is path between Lojasiewicz inequalities (obtained as above through geometric arguments issued from intersection theory combined with methods issued from the field of singularities) and effectivity for the Hilbert's Nullstellensatz. Let us illustrate this with some illuminating example (which will be fundamental for us later on), where $P_0, P_1, ..., P_n$ are $n + 1$ elements in $\mathbb{C}[X_1, ..., X_n]$ without common zeroes in $\mathbb{C}^n$ and such that (4.4) holds for some convenient constants c, K. If one expresses affine coordinates in terms of homogeneous ones, it follows from (4.4) that $h := \zeta_0^{D_1 \cdots D_n}$, considered as a germ in $\mathcal{O}_{\mathbb{C}^{n+1},0}$, is in the integral closure of the ideal $\mathcal{I}(P)$ generated in $\mathcal{O}_{\mathbb{C}^{n+1},0}$ by the $\mathcal{P}_j$, $j = 0, ..., n$, that is satisfies a relation of integral dependency

$$h^M + \Phi_1 h^{M-1} + \cdots + \Phi_{M-1}\, h + \Phi_M = 0\,, \quad \Phi_j \in \mathcal{I}(P)^j\,,\ j = 1, ..., M\,;$$

one knows then from Lipman-Sathaye theorem [63, 64] (which extends to any regular local ring Briançon-Skoda's theorem [21]) that

$$\overline{\mathcal{I}(P)}^l \subseteq \overline{\mathcal{I}(P)^l} \subset \mathcal{I}(P),$$

where $l \leq n$ is the minimal number of elements in $\mathcal{I}$ which generate a reduction of this homogeneous ideal. Therefore, we have $\zeta_0^{lD_1 \cdots D_n} \in \mathcal{I}(P)$; if we now go back to the affine coordinates, we deduce a Bézout identity

$$1 = \sum_{k=1}^{n} Q_k P_k\,, \quad \text{with} \quad \max \deg Q_j P_j \leq l\, D_1 \cdots D_n\,. \tag{4.5}$$

To summarize here this approach towards effectivity for the Bézout identity via intersection theory geometric methods combined with Briançon-Skoda's (or Lipman-Sathaye's) theorem, we will quote a recent result of M. Hickel ([51], theorem 5.1) :

Theorem 4.3 *Let $\mathbb{K}$ an algebraically closed field and $P_1, ..., P_m$ be m elements in $\mathbb{K}[X_1, ..., X_n]$ with degrees less than D such that $1 \in (P_1, ..., P_m)$; let l be the minimal number of generators for a reduction of the ideal $\mathcal{U} = \mathcal{U}(P)$ generated by the homogenizations of the P_j's in $\mathbb{K}[[X_0, ..., X_n]]$; one can define, in terms of the normalized blowing-up with center the ideal sheaf in $\mathcal{O}_{\mathbb{P}^n(\mathbb{K})}$ generated by the homogenizations of the P_j's and the invariants constructed in [85], a positive rational number $\nu_\infty(\mathcal{U})$ bounded from above by $D_1 \cdots D_\mu$, where $\mu := \min(n, m)$, such that :*

- *There exist polynomials $Q_1, ..., Q_m$ in $\mathbb{K}[X_1, ..., X_n]$ such that*

$$1 = \sum_{j=1}^{m} Q_j P_j\,, \quad \max_{1 \leq j \leq m} \deg(P_j Q_j) \leq l \max(\nu_\infty(\mathcal{U}), D)\ ; \tag{4.6}$$

- *for any set of polynomials* $\{Q_1, ..., Q_m\}$ *involved in a Bézout identity of the form* (4.6)*, one has*

$$\max_{1 \le j \le n} \deg P_j Q_j \ge \max(\nu_\infty(\mathcal{U}), D) .$$

Note that there are situations where $\nu_\infty(\mathcal{U})$ and D^μ are comparable : such is the case in the example (see for example [68], volume II, section 411)

$$\begin{aligned} P_1(X) = X_1^D\,,\ P_2(X) &= X_1 - X_2^D\,,\ \ldots, \\ P_{n-1}(X) = X_{n-2} - X_{n-1}^D\,,\ P_n(X) &= 1 - X_{n-1}X_n^{D-1}\,, \end{aligned}$$

where $\max(\deg(Q_j P_j)) \ge D^n - D^{n-1}$ for any Bézout identity (4.6).

One should also mention that J. Kollár in [55] extended (based on the approach via local Lojasiecicz inequalities) the effectivity results respect to the Bézout identity to the case when $P_1, ..., P_m$ were replaced by unmixed ideals $I_1, ..., I_m$ in $\mathbb{K}[X_1, ..., X_n]$, $\mathbb{K}$ being an arbitrary algebraically closed commutative field : for example, if $I_1, ..., I_m$ have no common zero in $\mathbb{K}^n$, then, one can find $F_j \in I_j$, $j = 1, ..., m$, such that $1 = F_1 + \ldots + F_m$ and

$$\deg F_j \le (n+1) \prod_{j=1}^{m} \deg I_j .$$

All the results we quoted above provide an effective solution to the geometric version of Hilbert's Nullstellensatz : "geometric" since one is interested only into degree estimates for M or the degrees of the Q_j's in (4.1) in terms of the geometric degree of $(P_1, ..., P_m)$ (following Kollár's approach in [55]) or multiplicities attached to the minimal primes in the decomposition of the ideal $\mathcal{U}(P)$ (in the spirit of [54, 23]), or even geometric invariants attached to the normalized blowing-up at infinity [51]. Since the Hilbert's Nullstellensatz over $\mathbb{C}$ is known to be an NP-complete decision problem over $\mathbb{C}$ [82], such results could be interesting steps towards the solution of the conjecture $P =$ (?) NP ; of course they need to be more explicit from the algorithmic point of view and here comes the reason why multivariate residue calculus interfers and brings some complementary insight on such questions.

As far as analytic ideas (inspired either by multivariate residue calculus or developments around integral kernels of Bochner-Martinelli type) are concerned, it appears that Briançon-Skoda's theorem [21] or its algebraic companion Lipman-Sathaye-Teissier theorem [63, 64] play a fundamental role (see for example [35, 51]) ; we will just point here three results which take their significance from the fact that we know from Mayr-Meyer's classical example [66] that the membership problem (test explicitely whether a given polynomial Q is in some ideal $(P_1, ..., P_m)$ or not and write it explicitely in this ideal if it is) cannot be solved in polynomial time (degree estimates are are in D^{2^n} where D is the maximal degree of the P_j's) ; analytic ideas inspired the first of these results (relate to Briançon-Skoda's theorem division formulas developped in section 3.2), while residue currents lie behind the two other ones :

Theorem 4.4 [3, 51] *Let $P_1, ..., P_m$ be m polynomials in n variables with respective degrees $D_1 \leq D_2 \leq \cdots \leq D_m = D$ and Q in the integral closure of $(P_1, ..., P_m)$ in $\mathbb{C}[X_1, ..., X_n]$; then, one can find polynomials $Q_1, ..., Q_m$ such that*

$$Q^{\min(n+1,m)} = \sum_{j=1}^{m} Q_j P_j \tag{4.7}$$

and

$$\deg P_j Q_j \leq \min(n+1, m) \left[\deg Q + \begin{cases} D_1 \cdots D_m \ if \ m \leq n \\ \min(D^n, D_1 \cdots D_m / D_1^{m-n}) \ if \ m > n \end{cases} \right]$$

Division interpolation formulas obtained from expanded versions of Cauchy-Weil's formula (see section 3) lead to explicit division formulas of the form (4.7) ; this is the reason to think such formulaes could be used in effectivity questions to make explicit the "black box" which represents Briançon-Skoda's theorem.

Our second example covers a very peculiar situation respect to the membership problem, the situation where $\mathbb{K} = \mathbb{C}$, $m = n$, and $P_1, ..., P_n$ are n elements in $\mathbb{C}[X_1, ..., X_n]$ such that the zero set of $\mathcal{U}(P)$ lies in $\mathbb{C}^n$ (there are no common zeroes at infinity) ; then, one has the following result, due to Max Nöther when $n = 2$ (he called it the $Af + Bg$ theorem [69]), which inspired a lot of further geometric developments (see for example [81, 44]) :

Theorem 4.5 *Let $P_1, ..., P_n$ be n elements in $\mathbb{C}[X_1, ..., X_n]$ as above and $Q \in (P_1, ..., P_n)$; then there are polynomials $Q_1, ..., Q_n$ such that*

$$Q = \sum_{j=1}^{n} Q_j P_j$$

and $\deg Q_j + \deg P_j = \deg Q$ *for any* $j = 1, ..., n$.

A geometric proof for Noether's theorem involving multidimensional residue theory can be found in [87], section 7.20 ; one can also find a somehow more algebraic proof in [91], corollary 4.1 ; both proofs use technics based on multidimensional residue calculus ; here are the different steps :

- first use a deformation argument (using for example as deformation parameter the homogeneous additional variable z_0 when going from the affine setting to the projective one) to show that, for any $\alpha \in \mathbb{N}^n$ such that $\alpha_1 + \cdots + \alpha_n < D_1 + \cdots + D_n - n$, one has

  $$\Big\langle \bigwedge_{j=1}^{n} \overline{\partial}\Big[\frac{1}{P_j}\Big] \, , \, \zeta^\alpha \, d\zeta_1 \wedge \ldots \wedge d\zeta_n \Big\rangle = 0 \, ;$$

  note that this deformation argument was extended to the case where the homogeneous parts of higher degree of the P_j's (after some weighted process of homogenization) define the origin as an isolated zero [25] ; polynomials could be replaced by Laurent polynomial and the process of weighted homogenization by the process of toric homogenization ;
- then represent Q thanks to the Cauchy-Weil's formula (in its expanded form (3.2)) inside some connected polyedron

  $$W := \{|P_1| < R_1 \, , \ldots \, , \, |P_n| < R_n\}$$

  which contains all common zeroes of $P_1, ..., P_n$;
- finally remark that the expansion one obtains does not contain terms in $P_1^0(z) \ldots P_n^0(z)$ (since Q lies in $(P_1, ..., P_n)$ and $\overline{\partial}[1/P]$ is annihilated by $(P_1, ..., P_n)$, so by Q) and truncates automatically where it should do in order to provide the algebraic identity in the theorem with the right estimates $\deg P_j Q_j \leq \deg Q$, $j = 1, ..., n$;
- alternatively, one could also re-interpret the conclusion of the first step just saying that, if $\mathcal{Q}$, $\mathcal{P}_1$,...,$\mathcal{P}_n$ denote the homogenizations of $Q, P_1, ..., P_n$, one has

  $$\mathcal{Q} \, \overline{\partial}\Big[\frac{1}{\zeta_0^{\deg Q + 1}}\Big] \wedge \bigwedge_{j=1}^{n} \overline{\partial}\Big[\frac{1}{\mathcal{P}_j}\Big] = 0 \, ;$$

  the duality theorem implies that $\mathcal{Q}$ lies in the ideal $(z_0^{\deg Q+1}, \mathcal{P}_1, ..., \mathcal{P}_n)$, from which one can conclude (for degree reasons) that $Q \in (P_1, ..., P_n)$.

Our last example is related to the effectiveness of the membership problem in case the entries $P_1, ..., P_m \in \mathbb{C}[X_1, ..., X_n]$ define a complete intersection in $\mathbb{C}^n$; in this case, it follows easily from the prime power version of Hilbert's Nullstellensatz that if $Q \in (P_1, ..., P_m)$ and $\mathcal{Q}$ denotes its homogenization, then, if $D_1 = \deg P_1 \geq \cdots \geq \deg P_m$ and $\iota = \sharp\{j < m-1\,;\, D_j = 2\}$,

$$\mathcal{Q}\, X_0^{(3/2)^\iota D_1 \cdots D_m} \in \mathcal{U}(P)\,,$$

which implies that one can find a division formula

$$Q = \sum_{j=1}^{m} Q_j P_j\,, \quad \deg P_j Q_j \leq \deg Q + (3/2)^\iota D_1 \cdots D_m\,.$$

On the other hand, as it was noticed by A. Dickenstein et C. Sessa in [28], the membership test to decide whether some given polynomial $Q \in \mathbb{C}[X_1, ..., X_n]$ belongs to $(P_1, ..., P_m)$ or not can be performed via residue currents, since we know it is equivalent to decide whether the current

$$Q \bigwedge_{j=1}^{m} \overline{\partial}\Big[\frac{1}{P_j}\Big] \equiv 0$$

or not ; it is natural to think that the membership problem can be explicitely solved (as we will see in the next section) thanks to residue formulas (via the Cauchy-Weil's representation formula).

4.2 How multidimensional residue theory fits in the picture

Let $\mathbb{K} = \mathbb{C}$ and A be an affine algebraic subvariety of $\mathbb{C}^n$ with pure dimension $m \in \{0 ..., n\}$; given m polynomials $p_1, ..., p_m$ (with $p^{-1}(0)$ as set of common zeroes in $\mathbb{C}^n$) such that

$$\dim \Big(A \cap p^{-1}(0)\Big) = 0\,,$$

one can define the Coleff-Herrera current

$$\bigwedge_{j=1}^{m} \overline{\partial}\Big(\frac{1}{p_j}\Big) \wedge [A]\,,$$

where $[A]$ denotes the geometric integration current [60] on A (which means multiplicities are not taken into account) ; such a current can be expressed (via the Bochner-Martinelli approach, see section 3) as

$$\frac{(-1)^{\frac{m(m-1)}{2}}(m-1)!}{(2i\pi)^m}\Big[\lambda\|p\|^{2(\lambda-m)}\Big(\bigwedge_{j=1}^{m}\overline{dp_j}\Big)\wedge[A]\Big]_{\lambda=0},$$

where $\|p\|^2 := |p_1|^2+\ldots+|p_m|^2$, and the notation $[\,\cdot\,]_{\lambda=0}$ means that one considers the current-valued expression for $\operatorname{Re}\lambda >> 1$, follows the meromorphic continuation (as a current valued function of λ) and takes the value at $\lambda = 0$ (which happens not to be a pole) ; our choice to present here the Bochner-Martinelli approach to the Coleff-Herrera current is deliberate since such an approach will be quite helpful towards the solution of some effectivity questions ; in order to fit with the notations used in algebraic multidimensional theory (we will come back to the correspondences in the next subsection), we will denote the action of this current as follows :

$$\varphi \in \mathcal{D}^{m,0}(\mathbb{C}^n) \mapsto \text{Res}\begin{bmatrix} \varphi \\ p_1,\ldots,p_m \end{bmatrix}_A ;$$

note that such a current can be written as

$$\sum_{\alpha\in p^{-1}(0)\cap A}\ \sum_{1\le i_1<\ldots<i_{n-m}\le n}\mathcal{Q}_{\alpha,I}\Big(\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}\Big)\,dz_I\wedge d\overline{z}\,,$$

where the $\mathcal{Q}_{\alpha,I}(\partial/\partial z)$ are differential operators with constant coefficients, which confers to its action an algebraic character. For any $(k_1,\ldots,k_m)\in\mathbb{N}^m$, one can also define the $(n-m,n)$-current in $\mathbb{C}^n$

$$\mathcal{R}^k_{A,p}\ :\ \varphi\in\mathcal{D}^{m,0}(\mathbb{C}^n)\mapsto \text{Res}\begin{bmatrix} \varphi \\ p_1^{k_1+1},\ldots,p_m^{k_m+1} \end{bmatrix}_A$$

and the $C^\infty(\mathbb{C}^n)$-left-module

$$\Sigma_{A,p}\ :=\sum_{k\in\mathbb{N}^m}C^\infty(\mathbb{C}^n)\cdot\mathcal{R}^k_{A,p}\,.$$

A fundamental tool associated to the role of multidimensional residue calculus in commutative algebra happens to be the classical transformation law which appears to be in the algebraic context (see [44], chapter 7) a formulation in a particular setting of H. Wiebe's theorem ; the extension to the current setting can be found for example in [29] ; the more general version we propose here (and which plays an important role in effectivity questions) is due to A. M. Kytmanov [58] ; the presentation we give amounts to [19] and [11], remark 2.3.

Theorem 4.6 *Let* $(p_1, ..., p_m)$ *and* $(\tilde{p}_1, ...\tilde{p}_m)$ *be two collections of elements in* $\mathbb{C}[X_1, ..., X_n]$ *such that the algebraic varieties* $A \cap p^{-1}(0)\,, A \cap \tilde{p}^{-1}(0)$ *are discrete and*

$$\tilde{p}_i(z) = \sum_{j=1}^{m} h_{ji}(z) p_j(z) \quad \forall\, z \in A\,,\ i = 1, \ldots, m\,, \tag{4.8}$$

where the h_{ji}, $1 \leq i, j \leq m$ *are elements in* $\mathbb{C}[X_1, ..., X_n]$. *Let*

$$\Sigma_{A,p,\tilde{p}} = \Sigma_{A,p} + \Sigma_{A,\tilde{p}}$$

and $\sigma_{W,p}$, $\sigma_{W,\tilde{p}}$, *the homomorphisms of* $C^\infty(\mathbb{C}^n)$*-modules from the polynomial ring* $C^\infty(\mathbb{C}^n)[Y_1, ..., Y_m]$ *to* $\Sigma_{A,p,q}$ *such that*

$$\begin{aligned} \sigma_{A,p}\,[Y_1^{k_1} \ldots Y_m^{k_m}] &= k_1! \ldots k_m!\, \mathcal{R}^k_{A,p} \\ \sigma_{A,\tilde{p}}\,[Y_1^{k_1} \ldots Y_m^{k_m}] &= k_1! \ldots k_m!\, \mathcal{R}^k_{A,\tilde{p}} \end{aligned}$$

for any $k = (k_1, ..., k_m) \in \mathbb{N}^m$ *; then, for any element* $\Phi \in C^\infty(\mathbb{C}^n)[Y_1, ..., Y_m]$, *one has*

$$\sigma_{A,p}\,[\Phi(Y)] = \det H \cdot \sigma_{A,\tilde{p}}\left[\Phi({}^tH \cdot Y)\right], \tag{4.9}$$

where $H := [h_{ji}]_{1 \leq i,j \leq m}$.

Remark 1. It is important to notice here that the expression of $\mathcal{R}^k_{A,p}$ in terms of the currents $\mathcal{R}^l_{A,\tilde{p}}$, with $l \in \mathbb{N}^m$ and $l_1 + \ldots + l_m = k_1 + \ldots + k_m$, involves only fictive divisions by $k! := k_1! \ldots k_m!$; this remark will be crucial to extend multidimensional calculus based on such a tool to the positive characteristic setting.

Remark 2. One can state the same result in the analytic context, that is when the p_j's and the $\tilde{p}_j$'s are holomorphic functions in some open set $U \subset \mathbb{C}^n$, A a closed purely dimensional analytic subset of U such that $\dim(p^{-1}(0) \cap A) = \dim(\tilde{p}^{-1}(0) \cap A) \leq 0$ and there exist holomorphic functions h_{ji} in U such that (4.8) holds.

Given A (an affine or projective m-dimensional algebraic variety embedded in some algebraic ambient manifold) and $p_1, ..., p_m$ regular in some neighborhood of A and defining (as zero set in A) a finite subset of A, we will denote with the symbol

$$\mathrm{Res}\left[\begin{array}{c} \\ p_1, ..., p_m \end{array}\right]_A$$

the total sum of residues on A respect to the polynomial map p.

The second key observation that supports the interest of multidimensional residue calculus towards applications in effectivity problems is that some of the formulas it involves (such for example as the transformation law) may be expressed in the affine context, not (as it is usually the case) in a projective (or complete) context (which is more familiar to algebraist geometers).

Let us take an illuminating example : as a consequence of residue theorem on a smooth compact algebraic variety (here $\mathbb{P}^n(\mathbb{C})$), a famous result which amounts to C.G.J. Jacobi [52] asserts that whenever $p_1, ..., p_n, q$ are $n+1$ polynomials such that the supports of the Cartier divisors $\mathcal{D}_1$,...,$\mathcal{D}_n$ induced by the p_j's in $\mathbb{P}^n(\mathbb{C})$ do not intersect on the hyperplane at infinity and

$$\deg q < \sum_{j=1}^{n} \deg p_j - n$$

(which means that the polar set of the $(n, 0)$-rational form

$$q(X_1, ..., X_n)\frac{dX_1 \wedge \ldots \wedge dX_n}{p_1 \cdots p_n}$$

lies entirely in the union of the supports of the divisors $\mathcal{D}_j$), then one has

$$\mathrm{Res}\left[\begin{array}{c} q(X)\, dX_1 \wedge \ldots \wedge dX_n \\ p_1, ..., p_n \end{array}\right]_{\mathbb{C}^n} = 0$$

(note that $p_1, ..., p_n$ define automatically a discrete, hence finite, variety in the affine space $\mathbb{C}^n$). The same holds if $\mathbb{C}^n$ is replaced by the n-dimensional torus $\mathbb{T} := (\mathbb{C}^*)^n$, $p_1, ..., p_n$ by Laurent polynomials with respective Newton polyedra $\Delta_1, ..., \Delta_n$, and $\mathbb{P}^n(\mathbb{C})$ by a smooth projective toric variety $\mathcal{X}(\Delta_1, ..., \Delta_n)$ compatible with the fan associated to the polyedron $\Delta_1 + \ldots + \Delta_n$: when $p_1, ..., p_n$ satisfy the D. Bernstein conditions [15] (that is the supports of the Cartier divisors $\mathcal{D}_j$ induced by the P_j's on $\mathcal{X}$ intersect only in $\mathbb{T}$), then for any Laurent polynomial Q in n-variables which support lies in the (relative) interior of $\Delta_1 + \ldots + \Delta_n$, it was proved by A. Khovanskii [53] when the supports of the $\mathcal{D}_j$ intersect transversally (in fact, such an hypothesis may be omitted, see [24]) that

$$\mathrm{Res}\begin{bmatrix} q(X)\,\frac{dX_1}{X_1}\wedge\ldots\wedge\frac{dX_n}{X_n} \\ p_1, \ldots, p_n \end{bmatrix}_{\mathbb{T}} = 0\,.$$

In the same vein, if A denotes a purely m-codimensional algebraic variety in $\mathbb{C}^n$ (with Zariski closure $\overline{A}$ in $\mathbb{P}^n(\mathbb{C})$) and $p_1, \ldots, p_m$ are m polynomials in n variables such that the supports of the Cartier divisors $\mathcal{D}_j$ that are induced by the p_j's in $\mathbb{P}^n(\mathbb{C})$ are such that

$$|\mathcal{D}_1| \cap \ldots \cap |\mathcal{D}_m| \cap \overline{A} \subset \mathbb{C}^n\,,$$

then, it follows from Stokes's formula that for any polynomial q such that

$$\deg q < \sum_{j=1}^{m} \deg p_j - m\,,$$

one has

$$\mathrm{Res}\begin{bmatrix} q(X)\,dX_1\wedge\ldots\wedge dX_n \\ p_1, \ldots, p_m \end{bmatrix}_m = 0\,;$$

this result, which appears to be the analytic companion to the algebraic residue theorem has been extensively used towards very interesting geometric applications (in the spirit of Cayley-Bacharach theorem) such as in [57, 43, 44].

The strength of residue calculus is that it allows to transpose the results we mentionned to an affine setting, namely to weaken the drastic condition related to the geometric configuration of the supports of the Cartier divisors $\mathcal{D}_j$ respect to the divisor at infinity (on $\mathbb{P}^n(\mathbb{C})$, on the toric variety $\mathcal{X}(\Delta_1, \ldots, \Delta_n)$ or on $\overline{A}$) ; the weaker alternative hypothesis will usually be a properness assumption for the polynomial map p ; note that such an hypothesis can be checked by an observator looking at infinity from the affine space ; here are two results corresponding to the three situations mentionned above.

Theorem 4.7 ([91, 13]) *Let A be a m-purely dimensional algebraic subvariety in $\mathbb{C}^n$; let $p_1, \ldots, p_m$ be m elements in $\mathbb{C}[X_1, \ldots, X_n]$ such that there* *exists m strictly positive rational numbers $\delta_1, ..., \delta_m$ (with $\delta_j \leq \deg p_j$ for $j = 1, ..., m$) and $c > 0$ such that*

$$\max_{1 \leq i \leq m} \frac{|p_j(z)|}{\|z\|^{\delta_j}} \geq c \,, \quad z \in W \,, \; \|z\| >> 1 \;;$$

then, for any $q \in \mathbb{C}[X_1, ..., X_n]$ such that

$$\deg q < \sum_{j=1}^{m} \delta_j - m \,,$$

one has

$$\mathrm{Res} \begin{bmatrix} q(X)\, dX_1 \wedge \ldots \wedge dX_n \\ p_1, ..., p_m \end{bmatrix}_A = 0 \,.$$

Theorem 4.8 ([91]) *Let $p_1, ..., p_n$ be n Laurent polynomials in n variables with respective Newton polyedra $\Delta_1, ..., \Delta_n$; assume that there are compact convex sets $\delta_1, ..., \delta_n$ (with $\mathrm{vec}(\delta_1 + \ldots + \delta_n) = \mathrm{vec}(\Delta_1 + \ldots + \Delta_n)$ and $\delta_j \subset \Delta_j$ for $j = 1, ..., n$) and $c > 0$ such that*

$$\max_{1 \leq j \leq n} \Big[\frac{|p_j(e^{\zeta})|}{\exp\big(\max_{\xi \in \delta_j} \langle \xi \,,\, \mathrm{Re}\, \zeta \rangle \big)} \Big] \geq c \,, \quad \|\mathrm{Re}\, \zeta\| >> 1 \,;$$

then, for any Laurent polynomial q such that the support of q lies in the (relative) interior of $\delta_1 + \ldots + \delta_n$,

$$\mathrm{Res} \begin{bmatrix} q(X)\, \frac{dX_1}{X_1} \wedge \ldots \wedge \frac{dX_n}{X_n} \\ p_1, ..., p_n \end{bmatrix}_{\mathbb{T}} = 0 \,.$$

The transformation law, together with such vanishing results in the affine context for total sums of residues, combine with Cauchy's formula in order to provide immediately an explicit solution for the Hilbert's Nullstellensatz ; this is not really a surprize since Cauchy's formula can be seen as the analytic transcription of Kronecker's duality formula : suppose for example that $p_1, ..., p_n$ are n polynomials defining a proper polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ such that the hypothesis in theorem 4.7 are fulfilled (here $m = n$ and $A = \mathbb{C}^n$), with $\deg p_j = D_j > 0$, $j = 1, ..., n$. Let h_{ji}, $1 \leq j, k \leq n$ be n^2 polynomials in $2n$ variables (X, Y) such that

$$p_i(X) - p_i(Y) = \sum_{i=1}^{n} h_{ji}(X, Y)\, (X_j - Y_j) \,, \quad i = 1, ..., n$$

(one can find such h_{ji} using either divided differences, keeping track of the subfield in which the coefficients of the p_j's lie, either Taylor's formula) ; let also

$$H_0(X,Y) := \det \Big[h_{ji}(X,Y)\Big]_{1\leq i,j\leq n} = \sum_{\substack{\alpha,\beta\in\mathbf{N}^n \\ |\alpha|+|\beta|\leq D_1+\cdots+D_n-n}} \gamma_{\alpha,\beta} X^\alpha Y^\beta \,;$$

then one can write, combining Cauchy's formula, the transformation law (for $m=n$, $A=\mathbb{C}^n$ and $\Phi\equiv 1$) and theorem 4.7 (for $m=n$, $A=\mathbb{C}^n$) :

$$\begin{aligned}
1 \;&\equiv\; \mathrm{Res}\left[\begin{matrix} dX \\ X_1-Y_1,\dots,X_n-Y_n\end{matrix}\right]_{\mathbb{C}^n} \\
&\equiv\; \mathrm{Res}\left[\begin{matrix} H_0(X,Y)\,dX \\ p_1(X)-p_1(Y),\dots,p_n(X)-p_n(Y)\end{matrix}\right]_{\mathbb{C}^n} \\
&\equiv\; \mathrm{Res}\left[\begin{matrix} H_0(\cdot,Y)\,dX \\ p_1,\dots,p_n\end{matrix}\right]_{\mathbb{C}^n} + \\
&+ \sum_{\substack{\alpha,\beta\in\mathbf{N}^n \\ |\alpha|+|\beta|\leq D_1+\cdots+D_n-n}} \; \sum_{\substack{\mu\in\mathbf{N}^n,\,\mu\neq 0 \\ \langle\mu+\underline{1},\delta\rangle\leq|\alpha|+D+n}} \gamma_{\alpha,\beta}\, \mathrm{Res}\left[\begin{matrix} X^\alpha dX \\ p_1^{\mu_1+1},\dots,p_n^{\mu_n+1}\end{matrix}\right]_{\mathbb{C}^n} Y^\beta p(Y)^\mu \,,
\end{aligned} \tag{4.10}$$

where we used the abridged notations

$$dX = dX_1\wedge\cdots\wedge dX_n\,,\; X^\alpha := X_1^{\alpha_1}\cdots X_n^{\alpha_n}\,,\; p(Y)^\mu = p_1(Y)^{\mu_1}\cdots p_n(Y)^{\mu_n}\,.$$

What is essential here (and which will appear again in our section about Abel's theorem) is that what should be *a priori* an analytic formula (with nothing but a formal meaning) truncates automatically in order to become an algebraic identity (combining Cauchy's theorem, the transformation law and theorem 4.8, one can state an analog formula for n Laurent polynomials $p_1,\dots,p_n$ that fulfill the hypothesis of theorem 4.8, as soon as all convex polyedra δ_j, $j=1,\dots,n$, contain the origin as an interior point, see [91], section 4). If now $p_0\in\mathbb{C}[X_1,\dots,X_n]$ is such that $p_0,\dots,p_n$ have no common zeros in $\mathbb{C}^n$ and

$$p_0(X)-p_0(Y) = \sum_{j=1}^n a_{j0}(X,Y)\,(X_j-Y_j)\,,$$

one can transform formula (4.10) into a Bézout identity $1 = p_0 q_0 + \cdots + p_n q_n$ for $(p_0, ..., p_n)$ just noticing that

$$\operatorname{Res}\left[\begin{matrix} H_0(\cdot, Y)\, dX \\ p_1, \ldots, p_n \end{matrix}\right]_{\mathbb{C}^n} = \sum_{j=0}^{n} (-1)^j \operatorname{Res}\left[\begin{matrix} \frac{H_j(\cdot, Y)}{p_0(X)}\, dX \\ p_1, \ldots, p_n \end{matrix}\right]_{\mathbb{C}^n} p_j(Y)\,, \qquad (4.11)$$

where H_j, $j = 1, ..., n$ is deduced from H_0 after replacing the column vector with index j by the column vector $(a_{10}, ..., a_{n0})$. Such a formula appeared in [10] (inspired by analytic technics in connection with the search for explicit deconvolution formulaes in signal analysis) and was transposed later to an algebraic frame [11].

Of course, such a Bézout identity has been obtained under drastic conditions on the entries (namely $p_1, ..., p_n$ define a proper map in $\mathbb{C}^n$) ; what is interesting here is that it does not cost too much to construct, given a collection of n polynomials $P_1, ..., P_n \in \mathbb{C}[X_1, ..., X_n]$ defining a discrete variety in $\mathbb{C}^n$, a proper map $(p_1, ..., p_n)$ with explicit (and quite easily constructible in terms of complexity) polynomials $p_j \in (P_1, ..., P_n)$; in fact, we have the following lemma :

Lemma 4.1 ([11], *section* 4) *Let $P_1, ..., P_n$ be n polynomials with degree D defining a normal family in $\mathbb{C}[X_1, ..., X_n]$ (any subfamily defines a complete intersection in $\mathbb{C}^n$) such that there exists $N \in \mathbb{N}$, $N \geq D$, so that for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that*

$$\max_{1 \leq j \leq n} \frac{|P_j(z)|}{\|z\|^D} \geq c\|z\|^{-N-\epsilon} \quad for \quad \|z\| >> 1\,; \qquad (4.12)$$

then, if $\xi^* = (\xi_1^*, \ldots, \xi_n^*)$ is a generic element in $(\mathbb{P}^n(\mathbb{C})^*)^n$, the polynomial map $(p_{\xi^*,1}, ..., p_{\xi^*,n})$, where

$$p_{\xi^*,j}(z) = \langle \xi_j^*\,,\, z \rangle^{N-D+1}\, P_j$$

is such that for any $\epsilon > 0$, there exists $c_\epsilon(\xi^*) > 0$ such that

$$\max_{1 \leq j \leq n} |p_{\xi^*,j}(z)| \geq c_\epsilon(\xi^*)\, \|z\|^{1-\epsilon} \quad for\ \|z\| >> 1\,; \qquad (4.13)$$

conversely, as soon as $(p_{\xi^*,1}, ..., p_{\xi^*,n})$ satisfies the above assertion (4.13) *for some $\xi^* \in (\mathbb{P}^n(\mathbb{C})^*)^n$, then, for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that* (4.12) *holds.*

The same combination as above of Cauchy's formula (as a realisation of Kronecker's duality formula), the transformation law and theorem 4.7 (in the case $m = n$, $A = \mathbb{C}^n$) gives the following companion lemma, relating multidimensional residue calculus and Lojasiewicz inequalities (which were so important as we pointed in section 4.1 in the second approach we developped towards geometric Nullstellensatz) :

Lemma 4.2 *Let $p = (p_1, ..., p_n)$ be a polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$, defining a non empty 0-dimensional algebraic subvariety in $\mathbb{C}^n$; the two following assertions are equivalent :*

- *for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that*

$$\max_{1 \leq j \leq n} |p_j(z)| \geq c_\epsilon \, \|z\|^{1-\epsilon} \quad \textit{for } \|z\| >> 1\,;$$

- *for any $\alpha, \beta \in \mathbb{N}^n$ such that $\alpha_1 + \ldots + \alpha_n < \beta_1 + \ldots + \beta_n$,*

$$\mathrm{Res}\left[\begin{matrix} X^\alpha \, dX \\ p_1^{\beta_1+1}, ..., p_n^{\beta_n+1} \end{matrix}\right]_{\mathbb{C}^n} = 0\,.$$

These two lemmas (combined together) show how multidimensional residue calculus fits into the general frame we presented in section 4.1 : suppose one wants to solve the Bézout identity

$$1 = \sum_{j=1}^{m} Q_j P_j \tag{4.14}$$

starting with a collection of entries $(P_1, ..., P_m)$ $(m \geq n)$ such that $(P_1, ..., P_n)$ fit with the hypothesis in lemma 4.1 (with some $N \in \mathbb{N}$, $N \geq \deg P_1 = \ldots = \deg P_n$). Once Lojasiewicz inequalities (4.12) have been translated (thanks to lemma 4.1 and lemma 4.2 combined together) in terms of the vanishing of a list of residue symbols, one has essentially two possibilities to get $Q_1, ..., Q_m$ solving (4.14) :

- either use the "black box" which is Lipman-Sathaye's theorem (here in fact Briançon-Skoda's theorem since we are in the complex setting) and conclude as in section 4.1 ;
- either take $\xi^* \in (\mathbb{P}^n(\mathbb{C})^*)^n$ generic, form $(p_1, ..., p_n) = (p_{\xi^*,1}, ..., p_{\xi^*,n})$, construct a linear combination $p_0 = p_{\xi,0}$ of $P_1, ..., P_m$ which does not

vanish on $V(p_{\xi^*,1},...,p_{\xi^*,n})$ and then recover (in terms of residue symbols, as explained from (4.10) and (4.11)) an explicit Bézout identity for $(p_{\xi^*,0},p_{\xi^*,1},...,p_{\xi^*,n})$, therefore for $(P_1,...,P_m)$.

The second approach keeps track of some arithmetic information on the entries, if there is one (for example if $P_1,...,P_m$ are assumed to have all their coefficients in $\mathbb{Z}$ or in the ring of integers of some number field), while the first one of course does not. The fact that this approach was the first one to lead to an effective arithmetic Nullstellensatz with (not quite, but almost) sharp bounds ([10, 11]) as we will explain in the next section does not come as a surprize, since we know today that any elimination procedure leading to optimality in the algorithmic approach appears to be based on the algorithmic elimination "*à la Kronecker*" (or multivariate residue calculus, which amounts basically to the same thing), even though it may use intensively other duality ideas (such as polarization for example).

The important role of multidimensional residue calculus respect to explicit solutions for the Bézout identity (therefore Hilbert's Nullstellensatz) in the complex case is connected with a crucial result for which we will propose a global and a local version. The global version of this statement (which previously existed using a geometric approach, see [1, 88]) amounts to M. Hickel and J. Y. Boyer ([20], proposition 5.1), who pointed out the algebraic aspect of this result and noticed its connection with the Cauchy-Weil formula :

Theorem 4.9 *Let $P=(P_1,...,P_n)$ be a dominant polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$; then, for $u=(u_1,...,u_n)$ generic, $(P_1-u_1,...,P_n-u_n)$ define a complete intersection in $\mathbb{C}^n$ and, for any $Q\in\mathbb{C}[X_1,...,X_n]$, the almost everywhere defined map*

$$(u_1,...,u_n)\mapsto \mathrm{Res}\left[\begin{matrix} Q(X)\,dX \\ P_1-u_1,...,P_n-u_n\end{matrix}\right]_{\mathbb{C}^n}$$

extends to a rational map $F_{P,Q}$ in $(u_1,...,u_n)$; the polynomial map P is proper if and only if for any $j=1,...,n$, $F_{P,X_j^D}\in\mathbb{C}[u_1,...,u_n]$, where $D:=\deg P_1+\cdots+\deg P_n-n+1$.

Here is the local companion of this result :

Theorem 4.10 *Let $h,f_1,...,f_n$ be $n+1$ germs of holomorphic functions in n variables at the origin in $\mathbb{C}^n$, such that $f_1,...,f_n$ define a regular sequence in $\mathcal{O}_{\mathbb{C}^n,0}$; let, for any $r\in\mathcal{O}_{\mathbb{C}^n,0}$, $F_{f,h\,;\,r}$ be the formal power series in*

$\mathbb{C}[[u_1,...,u_n]]$

$$F_{f,h\,;\,r}(u) := \sum_{k\in\mathbb{N}^n} \mathrm{Res}_0 \Big[\frac{h^{|k|}\, r\, d\zeta_1\wedge\ldots\wedge d\zeta_n}{f_1^{k_1+1}\cdots f_n^{k_n+1}}\Big]\, u_1^{k_1}\cdots u_n^{k_n}\,;$$

then, the two following assertions are equivalent :

- *h belongs to the integral closure of $(f_1,...,f_n)$ in the local ring $\mathcal{O}_{\mathbb{C}^n,0}$;*
- *there exists $N\in\mathbb{N}$ such that for any $r\in\mathcal{O}_{\mathbb{C}^n,0}$, $F_{f,h\,;\,r}$ corresponds to the development at the origin of a rational function $F_1(u)/F_2(u)$ of $(u_1,...,u_n)$ with no pole at $u=0$, F_2 independent of r, and*

$$\max(\deg F_1, \deg F_2) \le N\,.$$

Since we could not find a precise reference for this result, we will sketch here its proof. In order to prove that the first assertion implies the second one, one just uses the fact since h is in the integral closure of $(f_1,...,f_n)$, it satisfies a relation of integral dependency

$$h^M + \sum_{k=1}^{M}\Big(\sum_{\substack{q\in\mathbb{N}^n\\|q|=k}} a_{k,q} f_1^{q_1}\ldots f_n^{q_n}\Big) h^{M-k} = 0\,;$$

one can easily deduce from that that the coefficients of the formal power series $F_{f,h\,;\,r}$ obey a difference equation, which shows this formal power series corresponds to the development at the origin of a rational function ; the fact that this rational function has no pole at 0 follows from the fact that $|h|\le C\|f\|$ in a neighborhood of 0, which shows that, for some convenient choice of $\eta_1>0,...,\eta_n>0$ and for $\|u\|$ sufficiently small (depending on η), one has in fact :

$$F_{f,h\,;\,r}(u) := \frac{1}{(2i\pi)^n} \int_{\substack{|f_1|=\eta_1\\ \cdots\\ |f_m|=\eta_m}} \frac{h(\zeta)\, r(\zeta)\, d\zeta_1\wedge\ldots\wedge d\zeta_n}{\prod_{j=1}^{n}(f_j(\zeta)-h(\zeta)u_j)}\,;$$

clearly, the degree of the numerator and denominator of the rational function $F_{f,h\,;\,r}$ are bounded by $N=2M$ (independently of r) ; also the denominator of $F_{f,h\,;\,r}$ is independent on r. Converserly, if

$$F_2(u) = 1 + \sum_{\substack{q\in(\mathbb{N}^n)^*\\|q|\le N}} \xi_q u_1^{q_1}\ldots u_n^{q_n}\,,$$

one can check that, for any $r \in \mathcal{O}_{\mathbb{C}^n,0}$, for any $l \in \mathbb{N}^n$ such that $|l| = 2N$,

$$\mathrm{Res}_0 \left[\frac{r \left(h^{2N} + \sum_{k=1}^{N} \Big(\sum_{\substack{q \in \mathbb{N}^n \\ |q|=k}} \xi_q f_1^{q_1} \dots f_n^{q_n} \Big) h^{2N-k} \right) d\zeta}{f_1^{l_1+1} \cdots f_n^{l_n+1}} \right] = 0\,;$$

this implies (thanks to the local duality theorem) that, for any such $l \in \mathbb{N}^n$ with $|l| = 2N$, one has

$$h^{2N} + \sum_{k=1}^{N} \Big(\sum_{\substack{q \in \mathbb{N}^n \\ |q|=k}} \xi_q f_1^{q_1} \dots f_n^{q_n} \Big) h^{2N-k} \in (f_1^{l_1+1}, ..., f_n^{l_n+1})\,;$$

since $(f_1, ..., f_n)$ is a regular sequence, we have (see for example an argument by M. Hochster in [64])

$$(f_1, ..., f_n)^{2N+1} = \bigcap_{\substack{l \in \mathbb{N}^n \\ |l|=2N+n}} (f_1^{l_1}, ..., f_n^{l_n})\,,$$

so that

$$h^{2N} + \sum_{k=1}^{N} (\sum_{\substack{q \in \mathbb{N}^n \\ |q|=k}} \xi_q f_1^{q_1} \dots f_n^{q_n}) h^{2N-k} \in (f_1, ..., f_n)^{2N+1}\,,$$

which gives a relation of integral dependency for h over $(f_1, ..., f_n)$.

Remark. If $(f_1, ..., f_n)$ and $(g_1, ..., g_n)$ are two regular sequences in the local ring $\mathcal{O}_{\mathbb{C}^n,0}$, one can see easily (from theorem 4.10) that for any $r \in \mathcal{O}_{\mathbb{C}^n,0}$, the two formal power series :

$$\begin{aligned}
\mathrm{Res}_0 \begin{bmatrix} r d\zeta \\ f - ug \end{bmatrix} &= \sum_{k \in \mathbb{N}^n} \mathrm{Res}_0 \Big[\frac{r\, g_1^{k_1} \dots g_n^{k_n}\, d\zeta_1 \wedge \dots \wedge d\zeta_n}{f_1^{k_1+1} \cdots f_n^{k_n+1}} \Big] u_1^{k_1} \dots u_n^{k_n} \\
\mathrm{Res}_0 \begin{bmatrix} r d\zeta \\ g - uf \end{bmatrix} &= \sum_{k \in \mathbb{N}^n} \mathrm{Res}_0 \Big[\frac{r\, f_1^{k_1} \dots f_n^{k_n}\, d\zeta_1 \wedge \dots \wedge d\zeta_n}{g_1^{k_1+1} \cdots g_n^{k_n+1}} \Big] u_1^{k_1} \dots u_n^{k_n}
\end{aligned}$$

correspond to the developments at the origin of rational functions with no pole at $(0, ..., 0)$; note that a formal application of the transformation law would lead (taking $r = g_1 \dots g_n$) to

$$\begin{aligned}
\mathrm{Res}_0 \begin{bmatrix} g_1 \dots g_n\, d\zeta \\ f - ug \end{bmatrix} &= \frac{1}{u_1 \cdots u_n} \mathrm{Res}_0 \begin{bmatrix} f_1 \dots f_n\, d\zeta \\ f - ug \end{bmatrix} \\
&= \frac{(-1)^n}{(u_1 \dots u_n)^2} \mathrm{Res}_0 \begin{bmatrix} f_1 \dots f_n\, d\zeta \\ g - \frac{f}{u} \end{bmatrix}\ ;
\end{aligned}$$

it remains an open interesting question to deduce (using some combinatorial argument) from the fact that both

$$u \mapsto \mathrm{Res}_0 \begin{bmatrix} g_1 \dots g_n \, d\zeta \\ f - ug \end{bmatrix} \quad \text{and} \quad u \mapsto \mathrm{Res}_0 \begin{bmatrix} f_1 \dots f_n \, d\zeta \\ g - uf \end{bmatrix}$$

correspond (as formal power series) to the developments at the origin of rational functions with no pole at $(0, ..., 0)$, together with the formal identity above, that in fact (as we know it is indeed true)

$$\mathrm{Res}_0 \begin{bmatrix} f_1 \dots f_n \, d\zeta \\ f - ug \end{bmatrix} \equiv \mathrm{Res}_0 \begin{bmatrix} g_1 \dots g_n \, d\zeta \\ g - uf \end{bmatrix} \equiv 0$$

(as formal power series) ; this would provide a combinatorial proof of Lipman-Sathaye's theorem [63] (namely in that case $f_1 \dots f_n \in (g_1, ..., g_n)$ as well as $g_1 \dots g_n \in (f_1, ..., f_n)$) which would be free of analytic estimates (compared to the approaches in [21] or [64]). Since the role of such a theorem is crucial respect to the way multidimensional calculus interfers with interpolation or division problems in commutative algebra, it would clarify the intrinsic nature of such a result, together with its deep relation with residue calculus.

As a conclusion for this subsection, one should add that the method which was proposed in this survey to solve the Bézout identity can be transposed to the different examples we proposed at the end of section 3.1 to illustrate the membership problem, namely the $Af + Bg$ theorem of M. Noether and the membership problem in $\mathbb{C}[X_1, ..., X_n]$ when the polynomial entries $(P_1, ..., P_m)$ define a regular sequence. For the first example (extending M. Noether's theorem to the case where $P = (P_1, ..., P_n)$ is a proper polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$), we refer to [36, 91], section 4 ; respect to the second example (entries defining a complete intersection), we refer to [9, 28, 34].

4.3 Multivariate residue calculus and arithmetic division theory

The first reason why multidimensional residue calculus plays a role in arithmetic intersection theory (such as it has been developped in the last decade following the pionner paper of S. J. Arakelov through the active work of many people [41, 18]) lies in the fact it interfers with the "factorisation" of the integration current associated to a cycle in $\mathbb{P}^n(\mathbb{C})$ (note we mentionned this key fact in our introduction since the guideline of this survey is the Poincaré-Lelong formula (1.1).

Given m homogeneous polynomials $\mathcal{P}_1, ..., \mathcal{P}_m$ in $\mathbb{C}[X_0, ..., X_n]$ such that the ideal $\mathcal{I} = (\mathcal{P}_1, ..., \mathcal{P}_m)$ defines a purely m_P-codimensional projective algebraic cycle $A(\mathcal{I})$ in $\mathbb{P}^n(\mathbb{C})$ (here $m - m_P < n$, note also that only the isolated primary components of $\mathcal{I}$ are involved in this definition), one introduces the integration current $[A(\mathcal{I})]$ as follows : if $\Gamma_1, ..., \Gamma_t$ denote the connected components of $|A(\mathcal{I})| \setminus |A(\mathcal{I})|_{\rm sing}$,

$$\langle [A(\mathcal{I})]\,,\, \varphi\rangle := \sum_{l=1}^{t} \mu(\mathcal{I}, x_{\tau_l}) \int_{\Gamma_l} \varphi\,, \quad \varphi \in \mathcal{D}^{m-m_P, m-m_P}(\mathbb{P}^n(\mathbb{C}))\,,$$

where, for each $l = 1, ..., t$, $\mu(\mathcal{I}, x_{\tau_l})$ denotes the Hilbert-Samuel multiplicity of $\mathcal{I}$ at a generic point x_{τ_l} of the branch Γ_l. A normalized Green current G_A attached to the cycle $A = A(\mathcal{I})$ is a $(m_P - 1, m_P - 1)$-current in $\mathbb{P}^n(\mathbb{C})$ such that :

- $\mathrm{Sing\,Supp}(G_A) \subset |A|$;
- $dd^c G_A + [A] = (\deg A)\, \omega^{m_P}$, where ω is the Kähler form attached to the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$;
- $H(G_A) = 0$, where H denotes the harmonic projection.

As it already appears in one variable through Jensen's formula (written for a polynomial P with integer coefficients), normalized Green currents play a crucial role in arithmetic intersection theory since they are explicitely involved in the analytic contribution to the logarithmic height of arithmetic cycles (the product formula shows that indeed, as in Jensen's formula, a balance between some arithmetic contribution and some analytic one is necessary in order to get an intrinsic notion) ; namely, if $\mathcal{P}_1, ..., \mathcal{P}_m$ are homogeneous polynomials in $\mathbb{Z}[X_0, ..., X_n]$ (that is define an arithmetic cycle $\mathcal{Z}$ in $\mathrm{Proj}\,\mathbb{Z}[X_0, ..., X_n]$) and if u^* is a generic point in $(\mathbb{P}^n(\mathbb{C}))^*$ with integer coordinates such that

$$\{[z_0 \,:\, \ldots \,:\, z_n] \in |\mathcal{Z}(\mathbb{C})|\,;\, \langle u^*\,,\, z\rangle = 0\} = \emptyset\,,$$

so that $(\mathcal{P}_1, ..., \mathcal{P}_m, \langle u\,,\,\cdot\rangle)$ define a zero-dimensional arithmetic cycle

$$\sum_{\tau\ \mathrm{prime}} \nu_\tau\, \{\tau\}\,,$$

then

$$h(\mathcal{Z}) := \sum_{\tau \text{ prime}} \nu_\tau \, \log \tau + \frac{\deg \mathcal{Z}(\mathbb{C})}{2} \sum_{l=n-k}^{n} \sum_{j=1}^{l} \frac{1}{j} + \frac{1}{2} \int_{u^*} G_{\mathcal{Z}(\mathbb{C})} \, . \qquad (4.15)$$

The arithmetic Bézout theorem (see [18])

$$h(\mathcal{Z}_1 \bullet \mathcal{Z}_2) \leq h(\mathcal{Z}_1) \deg Z_2 + h(\mathcal{Z}_2) \deg Z_1 + \kappa(\dim Z_1 \, , \, \dim Z_2) \deg Z_1 \deg Z_2$$

(of course, the intersection product between arithmetic cycles $\mathcal{Z}_1$ and $\mathcal{Z}_2$ needs to be correctly defined, which has been done in [41]) controls arithmetic intersection theory.

"Explicit" expressions for (or at least procedures to recover) the integration current $[Z] = [A(\mathcal{I}]$ in terms of generators $\mathcal{P}_1, ..., \mathcal{P}_m$ for the ideal $\mathcal{I}$ lead to the following : through a standard argument developped in [18], lemma 1.2.2 and section 5.1, one can multiply in $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ the integration current on $\mathcal{V}(\mathcal{I}) \times \mathbb{P}^n(\mathbb{C})$ (where $\mathcal{V}(\mathcal{I})$ denotes the support of the cycle $A(\mathcal{I})$) with the Levine form [45] for the diagonal submanifold in $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ and therefore obtain that way explicit constructions (in terms of $\mathcal{P}_1, ..., \mathcal{P}_m$) for normalized Green currents attached to the cycle Z ; this provides closed formulas in terms of $\mathcal{P}_1, ..., \mathcal{P}_m$ for the analytic contribution in the expression of the logarithmic height of the arithmetic cycle $\mathcal{Z}(\mathcal{P}_1, ..., \mathcal{P}_m)$ attached to the ideal generated $\mathcal{P}_j' s$ when these polynomials have integer coefficients. Let us formulate a procedure to express the integration current $[Z]$ attached to some homogeneous ideal which is given in terms of its generators.

Theorem 4.11 ([12], Theorem 3.1) *Let $r \in \{1, ..., n\}$ and $\mathcal{P}_1, ..., \mathcal{P}_m$ be m homogeneous polynomials (with respective degrees D_j, $j = 1, ..., s$, $D = \max D_j$) defining a purely codimension m_P cycle $Z = A(\mathcal{I})$ in $\mathbb{P}^n(\mathbb{C})$; let ξ^* be a generic element in $(\mathbb{P}^n(\mathbb{C}))^*$ and $\widetilde{\mathcal{P}}_1, ..., \widetilde{\mathcal{P}}_{m_P}$ be m_P linear generic combinations of the polynomials $\langle \xi^* \, , \, \cdot \rangle^{D-D_j} \mathcal{P}_j$, $j = 1, ..., m$; let $M \in \mathbb{N}$, $M > m_P D^{m_P}$; let $F_1, ..., F_{m_P+m}$ the functions on $\mathbb{P}^n(\mathbb{C})$ defined in homogeneous coordinates as :*

$$\begin{aligned} F_j([z_0 \, : \, \ldots \, : \, z_n]) &= \frac{\widetilde{\mathcal{P}}_j(z)}{\|z\|^D} \, , \quad j = 1, ..., m_P \\ F_j([z_0 \, : \, \ldots \, : \, z_n]) &= \frac{\mathcal{P}^M_{j-m_P}(z)}{\|z\|^{M D_{j-m_P}}} \, , \quad j = m_P + 1, ..., m_P + m \, ; \end{aligned}$$

then, one has

$$[Z] = \frac{(m_P-1)!}{(2i\pi)^{m_P}} \times$$
$$\times \Big[\lambda \|F\|^{2(\lambda - m_P - 1)} \overline{\partial} \|F\|^2 \wedge \partial \|F\|^2 \wedge \sum_{\substack{i_1 < \dots < i_{m_P - 1} \\ 1 \le i_l \le m_P + m}} \bigwedge_{l=1}^{m_P - 1} \overline{\partial} F_{i_l} \wedge \partial F_{i_l} \Big]_{\lambda = 0}$$

where $\|F\|^2 := |F_1|^2 + \cdots + |F_{m_P+m}|^2$.

When we express it in some local chart on $\mathbb{P}^n(\mathbb{C})$, for example the affine space $\mathbb{C}^n = \{z_0 \neq 0\}$, such a formula involves the action of Bochner-Martinelli type residual currents (as defined in section 3) ; so the construction of the normalized Green current also does ; when $\mathcal{I} = (\mathcal{P}_1, ..., \mathcal{P}_m)$ is a primary ideal (that is $|Z|$ is irreducible), it follows from M. Méo's result [67] (as a consequence of Poincaré-Lelong formula)

$$\begin{aligned} [Z] &= \frac{1}{2} \Big[\lambda \Theta_{\mathcal{P}}^{\lambda - r - 1} \, d\Theta_{\mathcal{P}} \wedge d^c \Theta_{\mathcal{P}} \wedge \Big(\frac{1}{2} dd^c \Theta_{\mathcal{P}} \Big)^{r-1} \Big]_{\lambda = 0} \\ &= \frac{1}{r} \Big[\lambda \Theta_{\mathcal{P}}^{\lambda - r} \Big(\frac{1}{2} dd^c \Theta_{\mathcal{P}} \Big)^{r} \Big]_{\lambda = 0}, \end{aligned}$$

where
$$\Theta_{\mathcal{P}}([z_0 : \ldots : z_n]) := \sum_{j=1}^{m} \frac{|\mathcal{P}_j(z)|^2}{\|z\|^{2D_j}},$$

which provides in this particular case a simpler formula.

Besides its role in arithmetic intersection theory, one could expect from multivariate residue calculus that it controls "arithmetic division theory". As we have already seen in section 3.2, residue currents of the Bochner-Martinelli type are explicitely involved in division-interpolation formulas of the Lagrange-Kronecker's type (which can be seen as expanded forms of the Cauchy-Weil formula). In fact, once again, an intelligent combination of the transformation law (may be with some variants) and theorem 4.7 allows the possibility to get some good control on the logarithmic size of a residue symbol whose entries are polynomials with coefficients in the ring of integers of some number field ; one should emphasize again the affine aspect of the situation, compared to the projective setting in which arithmetic intersection theory is usually settled [41, 18].

To convince the reader, we propose here to describe a sequence of transformations that give the possibility to get good estimates for a residue symbol with entries in $\mathbb{Z}[X_1, ..., X_n]$. Let $p_1, ..., p_n$ be n polynomials in $\mathbb{Z}[X_1, ..., X_n]$ defining a 0-dimensional variety $V(p_1, ..., p_n)$ in $\mathbb{C}^n$, with $\deg p_j = d_j$, $j = 1, ..., n$, and $q \in \mathbb{Z}[X_1, ..., X_n]$; suppose that all $\log|\gamma|$, where γ denotes a non zero coefficient of any of the p_j's are bounded by h ; then the residue symbol

$$\mathrm{Res}\begin{bmatrix} r(X)dX \\ p_1, ..., p_n \end{bmatrix}$$

is a rational number (this is again an easy consequence of the transformation law) μ/ν, $\mu \in \mathbb{Z}$, $\nu \in \mathbb{Z}^*$ (after reduction) ; our goal here is to estimate $\max(\log|\mu|, \log|\nu|)$ in terms of $d_1, ..., d_n, h, d, h(q)$, where $d = \deg q$, $h(q)$ being the maximum of all $\log|\gamma|$, γ being a coefficient of q. First, one introduces a deformation method (inspired for example by [4]) : let λ be a complex generic parametrer and

$$p_{j,\lambda}(X) := \lambda X_j^{d_j+1} + p_j(X)\,, \quad j = 1, ..., n\,;$$

let $\Delta(\cdot, \lambda)$ be the jacobian determinant of the polynomial map $(p_{1,\lambda}, ..., p_{n,\lambda})$ and $M = (d_1 + 1) \cdots (d_n + 1)$; if $V(p_{1,\lambda}, ..., p_{n,\lambda}) = \{\alpha_\lambda^{(1)}, ..., \alpha_\lambda^{(M)}\}$, then, for any $j = 1, ..., n$, the Newton sums $S_{j1}, ..., S_{jM}$ of the finite set $\{\alpha_{\lambda,j}^{(1)}, ..., \alpha_{\lambda,j}^{(M)}\}$ can be expressed immediately as

$$S_{jk}(\lambda) = \mathcal{R}_0\Bigg[z_j^k \Delta(z, \lambda) \frac{1}{\lambda^n \ z_1^{d_1} \cdots z_n^{d_n}} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \le k}} \prod_{l=1}^n \Big(\frac{-p_l}{\lambda z_l^{d_l+1}}\Big)^{\alpha_l}\Bigg]\,, \quad k = 0, ..., M\,,$$

where $\mathcal{R}_0$ is the functional which assigns to any Laurent series in the variables $z = (z_1, ..., z_n)$ the coefficient of its free term [1]. We deduce from these expressions (and from the Newton relations connecting Newton sums and elementary symetric functions) explicit relations of integral dependency for $X_1, ..., X_n$ over $\mathbb{C}(\lambda)[p_{1,\lambda}, ..., p_{n,\lambda}]$, namely polynomial identities of the form :

$$\sum_{k=0}^M a_{jk}(\lambda, \lambda X_1^{d_1+1} + p_1(X), \ldots, \lambda X_n^{d_n+1} + p_n(X))\, X_j^{M-k} \equiv 0\,, \qquad (4.16)$$

$j = 1, ..., n$, where $a_{jk} \in \mathbb{Z}[\lambda, Y_1, ..., Y_n]$ (in this argument we use the fact that we work in characteristic 0) ; introducing n additional complex parameters $u_1, ..., u_n$, relations (4.16) can be rewritten as polynomial identities

$$Q_j(X_j, u, \lambda) \equiv \sum_{l=1}^n Q_{jl}(X, p(X), u, \lambda)(p_l - u + \lambda X_l^{d_l+1})\,, \quad j = 1, ..., n\,,$$

where $Q_j \in \mathbb{Z}[T,u,\lambda]$ and $Q_{jl} \in \mathbb{Z}[X,Y,u,\lambda]$; let ν_j, $j = 1,...,n$ be the valuation of Q_j as a polynomial in (u,λ) and $(t,w_0,...,w_n)$ be $n+2$ generic complex parameters so that

$$Q_j(X_j, tw_1, ..., tw_n, tw_0) = t^{\nu_j}[U_j(w,X_j) - tV_j(t,w,X_j)]\,,\ j = 1,...,n\,,$$

where $U_j \in \mathbb{Z}[s,T]$ and $V_j \in \mathbb{Z}[t,w,X_j]$; note that up to here, all these constructions are easy to handle from the computional point of view ; here comes the crucial point, which is an avatar of the transformation law : since we have the polynomial identities

$$t^{\nu_j}[U_j(w,X_j) - tV_j(t,w,X_j)] = \sum_{l=1}^{n} Q_{jl}(X,p,tw)(p_l - t(w_l - w_0 X_l^{d_l+1}))\,,$$

$j = 1,...,n$, we deduce from a variant of the transformation law ([11], proposition 2.5) the key identities between residual symbols (for generic w) :

$$\begin{aligned}
&\operatorname{Res}\begin{bmatrix} q(X)dX \\ \\ p_1,...,p_n \end{bmatrix} \\
&= \operatorname{Res}\begin{bmatrix} q(X)\,dt\wedge dX \\ \\ t, p_1 - t(w_1 - w_0X_1^{d_1+1}),...,p_n - t(w_n - w_0X_n^{d_n+1}) \end{bmatrix} \\
&= \operatorname{Res}\begin{bmatrix} \det[Q_{jl}(X,p(X),tw)]\, q(X)\,dt\wedge dX \\ \\ t^{1+|\nu|}, U_1(w,X_1) - tV_1(t,w,X_1),...,U_n(w,X_n) - tV_n(t,w,X_n) \end{bmatrix} \\
&= \sum_{\substack{k\in\mathbb{N}^n \\ |k|\le|\nu|}} \operatorname{Res}\begin{bmatrix} \det[Q_{jl}(X,p(X),tw)]\left(\prod_{j=1}^{n} V_j(t,w,X_j)^{k_j}\right) q(X)\,dt\wedge dX \\ t^{1+|\nu|-|k|}, U_1(w,X_1)^{k_1+1},...,U_n(w,X_n)^{k_n+1} \end{bmatrix}
\end{aligned} \tag{4.17}$$

The right-hand side of (4.17) can be computed using either Euclidean division, either explicit formulas as in [1] ; these computations provide a rational function in w, with integer coefficients γ such that any $\log|\gamma|$ can be roughly estimated in $\kappa(n)M(M+d)h$; from (4.17), we know that this rational function in $\Phi(w)/\Psi(w)$, where $\Phi, \Psi \in \mathbb{Z}[w]$ equals in fact a rational number, which is precisely the residue symbol we are looking for.

What we sketched above reveals the interesting role of multivariate residue calculus in the formulation of an effective arithmetic Nullstellensatz : in fact, the first arithmetic Nullstellensatz followed D. W. Brownawell [22] and J. Kollár's [54] results [10] ; analytic technics, centered around the extensive use of Bochner-Martinelli kernels, made somehow difficult the transposition of this result to a more algebraic setting (compared to the somehow less effective alternative approach developped almost simultaneously with, as a fundamental tool, Cauchy's formula used as an algebraic trace formula) ; nethertheless, this could be done some years later in [11] and lead to the following theorem :

Theorem 4.12 ([11], section 6) *Let $P_1, ..., P_m$ be m elements in $\mathbb{Z}[X_1, ..., X_n]$, $m \geq n$, with degrees in decreasing order and such that any $\log|\gamma|$, where γ is a coefficient of some P_j, is bounded by h ; assume that $P_1, ..., P_m$ have no common zeroes in $\mathbb{C}^n$; then, there exists $\gamma_0 \in \mathbb{Z}^*$ and polynomials $Q_1, ..., Q_m \in \mathbb{Z}[X_1, ..., X_n]$ such that*

$$\gamma_0 = \sum_{j=1}^{m} Q_j P_j \, ,$$

and

$$\begin{aligned} \max \deg P_j Q_j &\leq n(n+1)^3 B + n(D-1) \\ \max(\log|\gamma_0|, h(Q_j)) &\leq \kappa(n) B^4 D^2 (h + n \log D + D \log n + \log m) \, , \end{aligned} \tag{4.18}$$

where

$$B := (3/2)^\iota \deg P_1 \cdots \deg P_n \, , \quad \iota = \#\{j < n-1 \, ; \, \deg P_j = 2\} \, ,$$

$D := \deg P_1$ and $\kappa(n)$ is an explicit constant.

Analytic ideas may be transposed to an algebraic context (even working in positive characteristic) so that theorem 4.12 remains essentially true when one works with polynomials in an integral domain with infinite quotient field which can be equipped with a logarithmic size (for example $(\mathbb{Z}/p\mathbb{Z})[\tau_1, ..., \tau_L]$, where $\tau_1, ..., \tau_L$ are transcendental independent parameters), for the general statement, see theorem 6.1 in [11]). It remains a challenge to settle an "ideal" version of this arithmetic Nullstellensatz : let $I_1, ..., I_m$ m ideals in $\mathbb{Z}[X_1, ..., X_n]$ such that their logarithmic arithmetic sizes (in the sense of arithmetic intersection theory) are bounded by h and their degrees by D ; suppose that the zero sets $V(I_j)$, $j = 1, ..., m$, of $I_1, ..., I_m$ in $\mathbb{C}^n$ do not intersect ; can one find $F_1, ..., F_m$, $F_j \in I_j$ and $\gamma_0 \in \mathbb{Z}^*$ such that

$$1 = \sum_{j=1}^{m} F_j$$

with estimates of the form (4.18) for $\log |\gamma_0|$, the logarithmic sizes (and the degrees) of the F_j's ? In order to do so, one should be able to state an explicit restricted version of Hilbert's Nullstellensatz, $\mathbb{C}^n$ being replaced by some algebraic subvariety W (theorem 4.7 would play again an essential role). This would extend to the arithmetic context the results of J. Kollár in [55]. Though residue technics do not lead to optimality respect to the bounds (still due to their imperfection), sharp bounds were obtained very recently by T. Krick, L. M. Pardo, M. Sombra in [56] for the arithmetic Nullstellensatz over $\mathbb{Z}$ using a somehow different approach (where residue calculus methods are hidden, but Kronecker's ideas are present) ; pursuing the results obtained previously towards a better algorithmic understanding of the path towards effectivity (note a formula such as the formula obtained in [10] is very different in nature from an algorithm) they proved the fundamental theorem we quote here :

Theorem 4.13 *Let $P_1, ..., P_m$ be $m \geq n$ elements in $\mathbb{Z}[X_1, ..., X_n]$, with degrees less than D and such that any $\log |\gamma|$, where γ is a coefficient of some P_j, is bounded by h ; assume that $P_1, ..., P_m$ have no common zeroes in $\mathbb{C}^n$; then, there exists $\gamma_0 \in \mathbb{Z}^*$ and polynomials $Q_1, ..., Q_m \in \mathbb{Z}[X_1, ..., X_n]$ such that*

$$\gamma_0 = \sum_{j=1}^{m} Q_j P_j \, ,$$

and

$$\max \deg P_j Q_j \leq 4nD^n$$
$$\max(\log |\gamma_0|, h(Q_j)) \leq 4n(n+1)D^n(h + \log m + (n+7)\log(n+1)\ D) \, .$$

The connection between the methods developped in this survey (involving explicit tools in multidimensional residue calculus) and optimal results such as theorem 4.13 remains to be understood, so that one could unify intersection theory (where Poincaré-Lelong formula plays a crucial role) and division theory (where the Lagrange-Jacobi division-interpolation formula appears as a fundamental tool) ; the role of results such as Lipman-Sathaye theorem and their interpretation in terms of multidimensional residue calculus (from the operational point of view) remains still unclear ; for example, a direct proof of Briançon-Skoda theorem in the local ring $\mathcal{O}_{\mathbb{C}^n,0}$ as a consequence of theorems 4.6 and 4.10 would help to a better understanding of this role (in the geometric, then arithmetic context).

5 Residue currents and holomorphy on analytic varieties

5.1 Universal denominator and discriminant

There are many ways to define holomorphic objects on non smooth analytic varieties. A usual one suggests to define them as holomorphic objects on the subset of regular points and ensure the holomorphicity property remains preserved after normalization of the variety.

Let us start with the notion of weakly holomorphic function on an analytic set A : a function h is called *weakly holomorphic* if it is defined and holomorphic at regular points of A and is locally bounded on A (that is, for any point z_0 in A, there exists $C(z_0) > 0$ such that $|h(z)| \leq C(z_0)$ for any regular point z inside some neighborhood of z_0). One can say such a definition is intrinsic, since it does not use the fact that A is embedded in some ambient manifold. On the other hand, extrinsically defined functions on some analytic set A are those which are restrictions to A of holomorphic functions in some ambient manifold ; we shall call them *strongly holomorphic* functions. Thus, given A analytic set embedded in some ambient manifold, a weakly holomorphic function on A is strongly holomorphic if it admits an holomorphic extension to the ambient manifold.

Let us give a couple of simple examples. As analytic set A, let us take first the union of the complex coordinate lines in the space $\mathbb{C}^2$, that is, $A = \{(z_1, z_2) \in \mathbb{C}^2 \,;\, z_1 \cdot z_2 = 0\}$. The function h that is equal to one on the line

$\{z_1 = 0\}$ and minus one on the line $\{z_2 = 0\}$ is weakly holomorphic on A and does not admit an holomorphic (even continuous) extension to $\mathbb{C}^2$; however, it is the restriction of the meromorphic function

$$(z_1, z_2) \mapsto \frac{z_2 - z_1}{z_2 + z_1}.$$

Here is another example : let $A = \{(z_1, z_2) \in \mathbb{C}^2\,;\, z_1^2 = z_2^3\}$ be a semicubic parabola in $\mathbb{C}^2$, which admits the parametrisation $z = \varphi(t) = (t^3, t^2)$. The mapping φ maps the complex plane (with complex coordinate t) one-to-one onto the parabola A, and the inverse $h = t = \varphi^{-1}(z)$ is a weakly holomorphic function on A. Clearly, there is no holomorphic function H in a neighborhood of the origin in $\mathbb{C}^2$ such that $H(t^3, t^2) \equiv t$ (that is $H_{|A} = h$) ; at the same time the function $h \equiv t$ is the restriction to A of the meromorphic function $(z_1, z_2) \mapsto z_1/z_2$.

In these examples, it did not occur as an accident that weakly holomorphic functions on the analytic set A could be represented as restrictions to A of some meromorphic functions in the ambient space (here $\mathbb{C}^2$). A well-known theorem of Oka (see for example [50]) asserts that at least locally such a fact holds for any purely dimensional analytic set. Moreover Oka's theorem claims that, given any point $a \in A$, one can find a *universal denominator* for weakly holomorphic functions, that is some function g which is holomorphic in a neighborhood $\mathcal{U}$ of a in the ambient manifold and satisfies :

- g does not vanish identically on any irreducible component of the analytic set A at the point a ;
- for any weakly holomorphic h on A in a neighborhood of a, the product $\tilde{h} = h \cdot g$ can be extended holomorphically to a neighborhood of a in the ambient manifold, that is, weakly holomorphic functions can be represented locally as meromorphic functions with a unique (universal) local denominator.

Let us describe the construction of the universal denominator. It is known (see [42], p.72) that any analytic set A of pure codimension m (in some ambient complex manifold $\mathcal{X}$) can be realized locally (in a convenient neighborhood $\mathcal{U}_{a_0}$ of some arbitrary point a_0) as the union of some irreducible components of the complete intersection

$$\tilde{A} = \{\zeta \in \mathcal{U}_{a_0}\,;\, f_1(\zeta) = \ldots = f_m(\zeta) = 0\},$$

where $df_1 \wedge ... \wedge df_m$ does not vanish identically on any component of A in $\mathcal{U}_{a_0}$. Given the above notations, one has the following :

Theorem 5.1 *For any set of indices $\mathcal{I} = (i_1, ..., i_m) \subset (1, ..., n)$, the jacobian determinant*

$$J_{\mathcal{I}} := \frac{\partial f}{d\zeta_{\mathcal{I}}} = \frac{\partial(f_1, ..., f_m)}{\partial(\zeta_{i_1}, ..., \zeta_{i_m})}$$

has the property that, for any weakly holomorphic function h on A, the product $h \cdot J_{\mathcal{I}}$ locally at each point $a \in \mathcal{U}_{a_0}$ can be holomorphically extended to some neighborhood of a in the ambient manifold. Consequently, if the Jacobian $J_{\mathcal{I}}$ is not identically zero on each irreducible component of A in $\mathcal{U}_{a_0}$, then it is a universal denominator at each point $a \in \mathcal{U}_{a_0} \cap A$; in the case when $\mathcal{U}_{a_0}$ is a domain of holomorphy, $J_{\mathcal{I}}$ is a global universal denominator.

In case A is a complete intersection, the proof of this theorem is given in [88]. But in fact this proof remains valid in the general case. The proof follows from the next lemma, which we will need also later. Since we are interested in local properties of some analytic set $A \subset \mathbb{C}^n$ (together with the functions one wants to define on it), we may choose the coordinates

$$\zeta = (z, w) \in \mathbb{C}_z^{n-m} \times \mathbb{C}_w^m$$

in the ambient space $\mathbb{C}^n$ and the polycylinder $U = U_z \times U_w$ such that $\pi : \tilde{A} \cap U \to U_z$ is a proper projection. Assume then that

$$\widetilde{A} = \{\zeta \in U_z \times U_w \,;\, f_1(z, w) = \cdots = f_m(z, w) = 0\}$$

and let $D = \{z \in U_z \,;\, \sigma(z) = 0\}$ be the discriminant set of the projection $\pi|_{\widetilde{A}}$, that is the image of the zero set $\{J = 0\}$ under this projection, where J is the Jacobian $\partial(f)/\partial(w)$.

Lemma 5.1 *For any holomorphic function $h \in \mathcal{O}(\operatorname{reg} A)$, there exists a holomorphic continuation to $U \setminus \{(z, w) \,:\, \sigma(z) = 0\}$. In the case when h is locally bounded (i.e. weakly holomorphic) there exists a meromorphic continuation to U.*

Proof. It is based on the the usual Lagrange interpolation. Let

$$\pi^{-1}(\{z\}) = \{\zeta^\nu(z) = (z, w^{(\nu)}(z)) \,,\, \nu = 1, \ldots, N\} \,.$$

For the functions f_i defining $\widetilde{A}$ we may write the Hefer expansions

$$f_i(z,u) - f_i(z,w) = \sum_{j=1}^{m} h_{ji}(z,w,u)(u_j - w_j),\ i = 1,\dots,m.$$

Let $H(z,w,u)$ be the determinant of the matrix whose coefficients are the coefficients h_{ji} involved in such expansions. Assuming that the function h is defined on all $\operatorname{reg}\widetilde{A}$ (taking zero values on the components of $\widetilde{A}$ which do not belong to A), consider the function

$$(z,w) \mapsto \tilde{h}(z,w) = \sum_{\nu=1}^{N} h(z, w^{(\nu)}(z))\, H(z,w,w^{(\nu)}(z)),$$

which is holomorphic in $(U_z \setminus \{\sigma = 0\}) \times U_w$. In fact, holomorphicity in $w \in U_w$ follows from the holomorphy of the coefficients h_{ji} while holomorphicity in $z \in U_z \setminus \{\sigma = 0\}$ follows from the facts that on $A \setminus \{\sigma = 0\} \subset \operatorname{reg} A$ the variables z play the role of local coordinates (hence h depends from z holomorphically), and that $\tilde{h}$ is a symmetric function with respect to the (multivalued) holomorphic functions $z \mapsto \zeta^{\nu}(z)$ in $U \setminus \{\sigma = 0\}$. The restriction $\tilde{h}$ to A coincides with that of $h \cdot J$, where $J = \partial(f)/\partial(w)$, since the determinant $H(z, w^{(j)}(z), w^{(\nu)}(z))$ equals $J(\zeta^{(j)}(z))$ if $j = \nu$ and zero if $j \neq \nu$. Thus $h = \frac{\tilde{h}}{J}|_A$, where $\tilde{h}$ is holomorphic in $U \setminus \{\sigma = 0\}$ since $\pi(\{J = 0\}) = D$. When h is locally bounded, $\tilde{h}$ is holomorphic in U (because of Riemann's theorem) and this concludes the proof of the lemma and of theorem 5.1. $\diamondsuit$

Let us point also here how can be described the singular set of any reduced complete intersections :

Proposition 5.1 [88] *Let $\widetilde{A}$ be a complete intersection in some open set U of $\mathbb{C}^n$, defined as*

$$\widetilde{A} = \{\zeta \in U\,;\, f_1(\zeta) = \cdots = f_m(\zeta) = 0\}\,.$$

If the set

$$\mathcal{J} = \{\zeta \in \widetilde{A}\,;\, \partial f_1 \wedge \dots \wedge \partial f_m(z) = 0\}$$

is nowhere dense in $\widetilde{A}$, it coincides with the singular locus $\operatorname{sing}\widetilde{A}$.

5.2 Holomorphic differential forms on analytic varieties

Consider in some open set $U \subset \mathbb{C}^n$ a closed analytic variety A with pure dimension r. We want to decide which holomorphic differential forms ψ on $\operatorname{reg} A$ should be considered as holomorphic on A. Of course, holomorphy on $\operatorname{reg} A$ is a necessity for such forms ; but what kind of additional condition can one suggest, for instance, instead of the condition of being locally bounded that appears in the definition of weakly holomorphic functions.

Taking the distribution point of view, one can remark that, for $q \leq n-r$, any $(q,0)$ holomorphic differential form ψ on $\operatorname{reg} A$ defines a $(q+n-r, n-r)$ $\bar{\partial}$-closed integration current on $U \setminus \operatorname{sing} A$, namely

$$\varphi \to \int_{\operatorname{reg} A} \psi \wedge \varphi, \tag{5.1}$$

where φ runs over the subspace of $(r-q,r)$-test forms with support *disjoint* from $\operatorname{sing} A$.

Definition 5.1 *A $(q,0)$ holomorphic differential form $\psi \in \Omega^q(\operatorname{reg} A)$ on the regular part of some closed analytic set A in an open subset $U \subset \mathbb{C}^n$ is called holomorphic on A if the integration current* (5.1) *admits a continuation to the whole ambient manifold space $\mathcal{D}^{r-q,r}(U)$ as a $\bar{\partial}$-closed current. The family of such forms is denoted by $\omega^q(A)$.*

The family $\omega^q(A)$ coincides with the space of sections (on A) of the Barlet sheaf [6] (see also [47], where definition 5.1 is proposed for the notion of holomorphicity on A and compared to other possible definitions, together with interesting applications). Let us give one among the definitions of this sheaf. For this, recall that the Grothendieck dualizing module of the germ (A,a) is defined as

$$\omega^r_{A,a} = \operatorname{Ext}^m_{\mathcal{O}_{\mathbb{C}^n,a}}(\mathcal{O}_{A,a},\ \Omega^n_{\mathbb{C}^n,a})$$

(here $r = \dim A$, and $m = n-r$). In the case A is a reduced complete intersection defined by functions $f_1, ..., f_m$, the module $\omega^r_{A,a}$ is the following free module of rank one :

$$\omega^r_{A,a} = \mathcal{O}_{A,a}\left(\frac{dz_1 \wedge \ldots \wedge dz_n}{df_1 \wedge \ldots \wedge df_m}\right).$$

For $0 \leq q \leq r$ one has then the following

Definition 5.2 *The sheaf ω_A^q is locally given by the fiber modules*

$$\omega_{A,a}^q = \mathrm{Hom}_{\mathcal{O}_{A,a}}(\Omega_{A,a}^{r-q},\ \omega_{A,a}^r)\ ;$$

equivalently, $\omega_{A,a}^q$ consists of all germs on A at a of $(q,0)$ meromorphic forms ψ such that $\psi \wedge \eta \in \omega_{A,a}^r$ for any $\eta \in \Omega_{A,a}^{r-q}$.

Note that this definition implies that the sections of ω_A^q are meromorphic differential forms, which means that for the class $\omega^q(A)$ of q-holomorphic forms on A, one can generalize Oka's theorem about the existence of a local universal denominator (for the proof, see [6] and [47]). In the next subsection we will give a more direct proof without using the desingularization.

So now, given $\psi \in \omega^q(A)$ (locally near any point a) one has $\psi = \xi/g$ with some holomorphic $(q,0)$ form ξ in the ambient manifold and some holomorphic function g vanishing on $\operatorname{sing} A$ (ξ and g being defined and holomorphic in some neighborhood U of a). By the Herrera & Liberman theorem [49], there is a natural continuation of the current (5.1) through the principal value integral

$$\langle [\psi], \varphi \rangle = \lim_{\epsilon \to 0} \int_{A \cap \{|g| > \epsilon\}} \psi \wedge \varphi, \qquad \varphi \in \mathcal{D}^{r-q,r}(U) \tag{5.2}$$

Let us clarify what means that such a current is $\overline{\partial}$-closed, or equivalently, that ψ is holomorphic on A. We know that the action of the $\overline{\partial}$ operator on $[\psi]$ gives the residue current

$$\langle \overline{\partial}[\psi], \varphi \rangle = \lim_{\epsilon \to 0} \int_{A \cap \{|g| = \epsilon\}} \psi \wedge \varphi$$

with $\varphi \in \mathcal{D}^{r-q,r-1}(U)$. For instance, when A is a curve ($r = 1$), this current is the usual residue at the singular points $a \in \operatorname{sing} A$. Consequently, roughly speaking, what we could have done is the following :

"*define the $(q,0)$ holomorphic forms on A as forms $\psi \in \Omega^q(\operatorname{reg} A)$ which have no residues with respective to singular locus* $\operatorname{sing} A$"

Remark that such circumstances emphasize the difference between the notions of weakly holomorphic functions and holomorphic forms even when $q = 0$: the holomorphy property depends on the embedding $A \subset \mathbb{C}^n$; in case this embedding is non proper, it is more sensible to it than the property of being weakly holomorphic. Let us give a simple example : let $A = \{(z_1, z_2) \in \mathbb{C}^2 \,;\, z_1^2 = z_2^3\}$ be again the semicubic parabola in $\mathbb{C}^2$ which admits the parametrization $z = \varphi(t) = (t^3, t^2)$; consider the meromorphic function $\psi = (z_2/z_1)|_A = 1/t$. Since on A one has $dz_1 = 3t^2 dt$, $dz_2 = 2tdt$, we get for any test form $\varphi = a_1(z)dz_1 + a_2(z)dz_2$

$$\langle \overline{\partial}[\psi], \varphi \rangle = \int_{|t|=\epsilon} \frac{\varphi(z(t))}{t} = 0.$$

Therefore we have to admit that $1/t$ is holomorphic on A but not weakly holomorphic.

The fact we just mentionned that holomorphicity on A for differential forms is a notion depending on the embedding $A \subset \mathbb{C}^n$ will allow us to describe later holomorphic forms on A as residues of meromorphic forms in the ambient space (see subsection 5.4). The point which ultimately depends (in the most crucial way) on the embedding $A \subset \mathbb{C}^n$ is to check whether (or not) an holomorphic object on A admits an holomorphic continuation to the ambient space. Residue currents provide some nice set of tools to solve such problem. The criterion for continuation of weakly holomorphic functions was given in [88] in the case $A = \{f_1 = ... = f_m = 0\}$ is a reduced complete intersection. One can transpose the proof of this criterion to the setting of $(q, 0)$ holomorphic forms on A just replacing the symbol h (for some weakly holomorphic function on A) by the symbol ψ (for some $(q, 0)$ holomorphic form on A), so that one gets the following :

Theorem 5.2 *Let A be a reduced complete intersection defined as*

$$A := \{\zeta \in U \,;\, f_1(\zeta) = \ldots = f_m(\zeta) = 0\}$$

in some open subset $U \subset \mathbb{C}^n$. For any $\psi \in \omega^q(A)$, one has

$$\psi \in \Omega^q_A \Leftrightarrow [\psi/df] \textit{ is } \overline{\partial}-\textit{closed},$$

where $df = df_1 \wedge ... \wedge df_m$ and the current $\left[\frac{\psi}{df}\right]$ is defined in the ambient manifold U as by

$$\langle [\frac{\psi}{df}], \varphi \rangle = \lim_{\epsilon \to 0} \int_{A \cap \{|df| > \epsilon\}} \frac{\psi}{df} \wedge \varphi \,, \quad \varphi \in \mathcal{D}^{r-q,r}(U)\,.$$

As it was pointed in [47], the same criterion holds for meromorphic forms in U which admit a holomorphic restriction to $\operatorname{reg} A$. The main points in the proof of the theorem are Coleff & Herrera theorem [26] and Passare's result [71]. In subsection 5.4 we will show how to relate theorem 5.2 to other questions.

5.3 Holomorphic forms on A are meromorphic in the ambient space

We give in this section a proof of the generalization of Oka's theorem we mentionned in section 5.2.

Theorem 5.3 *Let A be a purely $n-m$-dimensional closed analytic variety in some open subset of $\mathbb{C}^n$. Then, at least locally, there exists a universal denominator for elements $\psi \in \omega^q(A)$, $0 \leq q \leq n-m$; for example, if A can be realized as the family of some irreducible components of the complete intersection*

$$\widetilde{A} = \{\zeta = (z, w) \in U = U_z \times U_w \subset \mathbb{C}_z^{n-m} \times \mathbb{C}_w^m \,;\, f_1(\zeta) = \ldots = f_m(\zeta) = 0\}$$

such that the projection $\pi \ : \widetilde{A} \cap U \to U_z$ is proper, the discriminant $\sigma(z)$ of $\pi_{|\widetilde{A}}$ (as introduced in section 5.1) plays the role of universal denominator (in the sense that any ψ in $\omega^q(A)$ admits as a denominator some power of σ).

In order to prove this theorem, we need some preliminary steps :

Let $f(z,w) = (f_1(z,w), \ldots, f_m(z,w))$ be m holomorphic functions defining a complete intersection in some neighborhood of the origin in $\mathbb{C}_z^{n-m} \times \mathbb{C}_w^m$ such that $w \mapsto f(0,w)$ has an isolated zero at $w = 0$. Choose in the local algebra $\mathcal{O}_w/\langle f(0,w)\rangle$ some arbitrary monomial basis $w^{\alpha_1}, \ldots, w^{\alpha_N}$.

Lemma 5.2 *There exist polycylinders $U_z \subset \mathbb{C}_z^{n-m}$, $U_w \subset \mathbb{C}_w^m$ centered respectively at $z = 0$ and $w = 0$ such that for each $z^{(0)} \in U_z$, the monomials $\{w^{\alpha_k}\}_{k=\overline{1,N}}$ form a basis of $\mathcal{O}(U_w)/\langle f(z^{(0)}, w)\rangle$. In fact, for any h in $\mathcal{O}(U_z \times U_w)$, there exists a unique representation*

$$h(z,w) = \sum_k c_k(z) w^{\alpha_k} + \langle f(z,w)\rangle, \ c_k \in \mathcal{O}(U_z), \tag{5.3}$$

where $\langle f(z,w)\rangle$ is the ideal generated by the system f in $\mathcal{O}(U_z \times U_w)$

Proof. Let us recall that given a regular sequence $F = (F_1, \ldots, F_n)$ in the ring $\mathcal{O}_t$ of germs of holomorphic functions in t at the origin in $\mathbb{C}^n$, the Weierstrass satz claims (see [4], 5, or [87], 16.4) that if $e_1, \ldots, e_N$ is a monomial basis of $\mathcal{O}_t/\langle F\rangle$, then, for any $g \in \mathcal{O}_t$, there exist $g_i \in \mathcal{O}_s$, $j = 1, \ldots, N$, such that

$$g(t) = g_1(s)e_1(t) + \ldots + g_N(s)e_N(t), \text{ where } s = F(t).$$

This means that

$$g(t) - \sum_k g_k(s)e_k(t) \in \langle s - F(t)\rangle.$$

In particular, for $s = F(\zeta)$, one has

$$g(t) - \sum_k g_k(F(\zeta))e_k(t) \in \langle F(s) - F(t)\rangle. \tag{5.4}$$

We now propose to apply Cauchy-Weil's formula in the analytic polyhedron

$$W \ := \{|F_j(z, w)| < \epsilon, \ j = 1, \ldots, n\}$$

where

$$F(z, w) = (f_1(z, w), \ldots, f_m(z, w), z_1, \ldots, z_{n-m}) = (f(z, w), z).$$

Using notations $s = (u, v)$, $t = (z, w) \in \mathbb{C}^{n-m} \times \mathbb{C}^m$, it follows from this formula that for any $h \in \mathcal{O}(W)$

$$h(z, w) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma(W)} \frac{h(u, v)\, H(u, v; z, w)\, du \wedge dv}{[f(u, v) - f(z, w)]\, [u - z]},$$

where H is the determinant of the matrix of some Hefer divisors respectively associated to the F_j's and $\Gamma(W)$ denotes the skeleton of W, with the notations

$$[f(u, v) - f(z, w)] := \prod_{j=1}^{m} (f_j(u, v) - f_j(z, w)), \quad [u - z] := \prod_{j=1}^{n-m} (u_j - z_j).$$

It follows from (5.4) that one can write

$$H(u, v; z, w) = \sum_k g_k(u, v)\, w^{\alpha_k} \mod \quad \langle F(u, v) - F(z, w)\rangle,$$

since, as it is easy to see, $\mathcal{O}_{z,w}/F(z,w) \simeq \mathcal{O}_w/f(0,w)$. Now we have

$$\begin{aligned} h(z,w) &= \frac{1}{(2\pi i)^n} \int\limits_{\Gamma(W)} \frac{h(u,v)\left(\sum\limits_1^N g_k(u,v) w^{\alpha_k}\right) du_wedgedv}{[f(u,v)-f(z,w)]\,[u-z]} \\ &= \frac{1}{(2\pi i)^n} \int\limits_{\Gamma(W)} \frac{h(u,v)\left(\sum\limits_1^N g_k(u,v) w^{\alpha_k}\right) du \wedge dv}{[f(u,v)]\,[u-z]} + \langle f(z,w) \rangle. \end{aligned}$$

This leads to (5.3) with

$$c_k(z) := \frac{1}{(2\pi i)^n} \int\limits_{\Gamma(W)} \frac{h(u,v)\, g_k(u,v)\, du \wedge dv}{[f(u,v)]\,[u-z]}. \qquad \diamondsuit$$

Proof of theorem 5.3. We recall that on the manifold $A \backslash \{\sigma = 0\}$ variables z can be used as local coordinates, so one can write in these coordinates

$$\psi = \sum_{|I|=q} a_I(z,w) dz_I\,, \tag{5.5}$$

where $a_I \in \mathcal{O}((U_z \setminus \{\sigma = 0\}) \times U_w)$; this follows from the first part of the proof of Lemma 5.1. Let, as before, for $z \in U_z \setminus \{\sigma = 0\}$,

$$\pi^{-1}(\{z\}) = \{(z, w^{(\nu)}(z))\,,\, \nu = 1, ..., N\}\ ;$$

we will also extend the form ψ to $\operatorname{reg} \widetilde{A}$, setting $\psi \equiv 0$ on thr components of $\widetilde{A}$ that do not belong to A. We need then the following lemma :

Lemma 5.3 *The differential form ψ is meromorphic if and only if the trace*

$$\mathrm{Tr}_\pi\,[g\psi] := \sum_{|I|=q} \Big(\sum_{j=1}^N (g a_I)(z, w^{(i)}(z)) \Big) dz_I$$

is meromorphic for any holomorphic $g \in \mathcal{O}(U)$.

Proof of Lemma 5.3. The trace of any meromorphic function is obviously meromorphic. Therefore it is enough to prove that if for any $g \in \mathcal{O}(U)$, the traces

$$\mathrm{Tr}_\pi\,[g a_I](z) := \sum_{j=1}^N (g a_I)(z, w^{(i)}(z))$$

are meromorphic, then each a_I is meromorphic.

Let us choose the monomial basis w^{α_i}, $i = 1, \ldots, N$ of the quotient space

$$\mathcal{O}_w / \langle f_1(0,w), \ldots, f_m(0,w) \rangle.$$

We remark that the dimension of this quotient space equals N denotes the cardinal of $\pi^{-1}(\{z\}z))$ for z generic. Consider the following analog of Vandermonde's determinant

$$\mathcal{W}(z) = \det \left[[w^{\alpha_i}]_j \right]_{i,j},$$

where $[w^{\alpha_i}]_j = (w^{(j)}(z))^{\alpha_i}$ denotes the value of the monomial w^{α_i} (from the basis) evaluated at the point $w^{(j)}(z)$. We remark that $\mathcal{W}(z)$ vanishes only on the discriminant set $D = \{\sigma = 0\}$: indeed, under the assumption $\mathrm{W}(z) = 0$ for $z \notin D$, the system of linear equations

$$\sum_i b_i \, [w^{\alpha_i}]_j(z) = 0, \;\; j = 1, \ldots, N$$

does have a nontrivial solution $b = (b_1(z), \ldots, b_N(z))$; hence the function $\sum b_i(z) w^{\alpha_i}$ vanishes at any root $w^{(j)}(z)$ of the system $f(z,w) = 0$, and in view of condition $df(z, w^{(j)}(z)) \neq 0$, this function has to belong to the ideal $\mathcal{I} = \langle f_1(z,w), \ldots, f_m(z,w) \rangle$; but this is impossible, since $\{w^{\alpha_i}\}$ form the basis of $\mathcal{O}_{z,w}/\mathcal{I}$.

Denote now as s_α the Newton sum

$$s_\alpha(z) = \sum_{j=1}^{N} [w^{(j)}(z)]^\alpha, \quad z \in U_z \setminus \{\sigma = 0\}, \; \alpha \in \mathbb{N}^m.$$

Let C^t be the transposed of the matrix $C = \left[[w^{\alpha_i}]_j \right]_{i,j}$ and note that

$$C \cdot C^t = \left[s_{\alpha_i + \alpha_j} \right]_{i,j},$$

so that

$$\mathcal{W}^2 = \det \left[s_{\alpha_i + \alpha_j} \right]_{i,j}.$$

It means that the determinant $\det \left[s_{\alpha_i + \alpha_j} \right]_{i,j}$ vanishes on the discriminant set D.

Let us now conclude the proof of lemma 5.3 as follows. Since the function $a_I \in \mathcal{O}((U_z \setminus D) \times U_w)$ for any multi-index I in $\{1, ..., n-m\}$, one can represent (in $U_z \setminus D$) by lemma 5.2 each a_I as

$$a_I(z,w) = \sum_{i=1}^{N} b_i(z)\, w^{\alpha_i} \pmod{\mathcal{I}},$$

with $b_i \in \mathcal{O}(U_z \setminus D)$, $i = 1, ..., N$.

Consider the traces of $g\psi$ for the finite family of functions $g = w^{\alpha_j}$. Our assumption asserts that for any multi-index I in $\{1, ..., n-m\}$, the traces

$$\mathrm{Tr}\,[w^{\alpha_j} a_I](z) := \sum_{i=1}^{N} b_i(z) s_{\alpha_i+\alpha_j}(z),\ \ j = 1, \ldots, N$$

are all meromorphic. Since $\det \left[s_{\alpha_i+\alpha_j}\right]_{i,j} \not\equiv 0$ is holomorphic, we conclude by Kramer's rule that the $b_i(z)$ corresponding to each a_I are meromorphic, and Lemma 5.3 is therefore proved. $\diamondsuit$

Let us go back to the proof of theorem 5.3. Since ψ is in $\omega^q(A)$, we know that the $(m+q, m)$ current in $U \setminus \operatorname{sing} A$ which action on a test form φ in $\mathcal{D}^{n-m-q,n-m}(U \setminus \operatorname{sing} A)$ is given as

$$\varphi \mapsto \int_{\operatorname{reg} A} \psi \wedge \varphi$$

can be extended as a $(m+q, m)$ current T in U (with support on A) ; since ψ is holomorphic on $\operatorname{reg} A$, this extension T is such that $\overline{\partial} T$ is supported by $A \cap \pi^{-1}(D)$.

Suppose ψ is not meromorphic. We now construct a sequence $\{\varphi_k\}_{k=1}^{\infty}$ of test forms such that φ_k vanishes on $\pi^{-1}(D)$ at least with order k, and such that

$$\langle \bar{\partial} T, \varphi_k \rangle \neq 0,\ k = 1\,2, \ldots \tag{5.6}$$

Since $\overline{\partial} T$ has a finite order and is supported by $\pi^{-1}(D)$, this will lead to a contradiction. For the construction of φ_k, let us consider some multi-index I such that the coefficient a_I in (5.5) is not meromorphic. By lemma 5.3, there exists a holomorphic function g for which the trace $\mathrm{Tr}[g \cdot a_I]$ is not meromorphic. It is known ([1], 27) that, since this trace is holomorphic in $(U_z \setminus D) \times U_w$, it can be represented as a series

$$\mathrm{T}r[g \cdot a_I](z) = \sum_{k=-\infty}^{\infty} c_k(z) \sigma^k(z),$$

where c_k are pseudopolynomials with respective to z_{n-m-1} with degree less or equal than $p-1$ if we assume that coordinates $= (z_1, \ldots, z_{n-m})$ have been chosen in such a way that $\sigma(z)$, up to some invertible function, is a Weierstrass polynomial with degree p in z_{n-m}.) Moreover, under our assumption, $c_k(z) \not\equiv 0$ for an infinite number of negative indices k. Now we take

$$\varphi_k(\zeta) = g(\zeta)\, z_{n-m}^{p-q(k)-1}\, \sigma^k(z)\, dz_{\widehat{I}} \wedge \Phi_k(z')\, d\bar{z}',$$

where

$$\begin{aligned} q(k) : &= \deg_{z_{n-m}} c_{-k-1}(z) \\ \widehat{I} : &= \{1, \ldots, n-m\} \setminus I \\ z' : &= (z_1, \ldots, z_{n-m-1}) \end{aligned}$$

and $\Phi_k(z')$ is a compactly supported function which will be defined below. Remark that we do not need to worry about the compactnes property of the support respect to the coordinate z_{n-m} and w since the intersection of $A \cap \pi^{-1}(D)$ (which supports $\overline{\partial} T$) with $\{\|z'\| \leq \delta_0\}$ for δ_0 small enough is compact. One has :

$$\langle \bar{\partial}[\psi], \varphi_k \rangle = \lim_{\epsilon \to 0} \int\limits_{|\sigma|=\epsilon} \mathrm{Tr}[g \cdot a_I](z) z_{n-m}^{p-q(k)-1} \sigma(z)^k \wedge dz_{n-m} \wedge \Phi_k(z') dz' \wedge d\bar{z}'$$

If we take into account the theorem on total residues with respect to variable z_{n-m}, we get

$$\begin{aligned} \langle \bar{\partial}[\psi], \varphi_k \rangle &= \lim_{\epsilon \to 0} \int_{|\sigma|=\epsilon} \frac{c_{-k-1}(z) \cdot z_{n-m}^{p-q(k)-1}}{\sigma(z)}\, dz_{n-m} \wedge \Phi_k(z')\, dz' \wedge d\bar{z}' \\ &= 2\pi i \int_{\mathbb{C}^{n-m-1}} A_k(z')\, \Phi_k(z')\, dz' \wedge d\bar{z}', \end{aligned} \tag{5.7}$$

where $A_k(z')$ is, in the pseudopolynomial c_{-k-1}, the leading coefficient (that is the coefficient of $z_{n-m-1}^{q(k)}$.) Now we set

$$\Phi_k(z') = \overline{A_k(z')}\, \chi(|z'|^2)$$

where χ is a smooth approximation of the characteristic function of the small ball $\{|z'| < \delta\}$. It is clear that one can choose χ such that integral (5.7) is not zero. $\diamondsuit$

5.4 Holomorphic forms on A as residues of logarithmic forms

Let us start with the simplest case when A is a reduced variety of codimension one, i.e. a divisor $D = \{f = 0\}$ in $\mathbb{C}^n$. Here it is easy to see that any element

$$\psi = h(z)\, dz/df = h(z) dz_1 \wedge \ldots \wedge dz_n/df$$

of the Grothendieck dualizing module is a regular meromorphic form on A, i.e a meromorphic form in the ambient space whose restriction to $\operatorname{reg} D$ is holomorphic. Indeed, the set $\operatorname{reg} D$ is covered by the open sets

$$U_i = \{z \in D : \frac{\partial f}{\partial z_i} \neq 0\},\ \ i = 1, ..., n$$

and one has

$$\psi|_{U_i} = (-1)^{n-i} \frac{h(z)\, dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n}{\frac{\partial f}{\partial z_i}(z)}\Bigg|_D .$$

These restrictions are compatible, which means they define a global differential form on $\operatorname{reg} D$. At the same time we see that ψ coincides with the Poincaré residue of the meromorphic form

$$\omega = \frac{h(z)\, dz_1 \wedge \ldots \wedge dz_n}{f(z)}.$$

We will see this is not indeed an accident and that for any complete intersection A, elements $\psi \in \omega^q(A)$ are realized as residues of so-called logarithmic forms. Such forms were introduced in [78] and [79] in order to generalize Leray's residue theory on polar sets with singularities. Thus J.-B. Poly [78] remarked that the Leray residue (in the case of a smooth polar divisor) is well defined for any (not necessarily d-closed) differential form ω, as soon as ω and its differential $d\omega$ have a simple pole along the hypersurface D. The forms ω with such properties were called *logarithmic forms*. Following the paper [2], we introduce the notion of multi-logarithmic differential form. Such a meromorphic differential form has poles along a divisor $D = \bigcup_{i=1}^m D_i$ such that $A = \bigcap_{i=1}^m D_i$ is a reduced complete intersection. We then define the residue form of a multi-logarithmic differential form as a generalization of the corresponding definitions of J. Leray [61], J.-B. Poly [78] and K. Saito [79]. The main statement asserts that there is a natural residue morphism which maps the complex of the multi-logarithmic differential forms onto the complex of differential forms which are holomorphic on A.

Let A be a reduced complete intersection in the domain $U \subset \mathbb{C}^n$, realized as the intersection

$$A = D_1 \cap \ldots \cap D_m$$

of divisors $D_i = \{f_i = 0\}$. Let $\Omega_U^\bullet = (\Omega_U^s, d)_{s=0,1,\ldots}$ be the de Rham complex of germs of holomorphic differential forms on U. Let

$$\widehat{D_j} = D_1 \cup \ldots \cup D_{j-1} \cup D_{j+1} \cup \ldots \cup D_m, \quad j = 1, \ldots, m\,,$$

and $\Omega_U^s(\star\widehat{D_j})$ be the $\mathcal{O}_U$-module of meromorphic differential forms of degree s consisting of all the differential $(s,0)$-forms with polar divisor $\widehat{D_j}$. Write $\widehat{D_1} = \emptyset$ for $m = 1$, so that $\Omega_U^s(\star\widehat{D_1}) = \Omega_U^s$.

Proposition 5.2 [2] *Let ω be a meromorphic differential s-form on U, such that $s \geq m$, with poles along the divisor $D = D_1 \cup \ldots \cup D_m$. The following conditions are equivalent :*

- *i)* $\quad f_j\,\omega \in \sum\limits_{i=1}^m \Omega_U^s(\star\widehat{D_i})\,, \quad f_j\,d\omega \in \sum\limits_{i=1}^m \Omega_U^{s+1}(\star\widehat{D_i}), \quad j = 1, \ldots, m\,;$
- *ii) There is a holomorphic function g which is not equal identically to zero on every irreducible component of A, a holomorphic $s-m$-form ξ and a meromorphic s-form $\eta \in \sum_{i=1}^m \Omega_U^s(\star\widehat{D_i})$ on U such that*

$$g\,\omega = \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_m}{f_m} \wedge \xi + \eta\,.$$

As function g in the property *ii)*, one can use the universal denominator for A. Any meromorphic differential s-form ω on U satisfing one of the equivalent conditions *i)* or *ii)* is called a *logarithmic form*, since after multiplication by the universal denominator g, one can divide it by the logarithmic differential form

$$\frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_m}{f_m}$$

modulo form η (which gives no contribution to the residue of ω). We shall denote the $\mathcal{O}_{U,x}$-module of germs of multi-logarithmic s-forms at x as $\Omega_{U,x}^s(\log A)$ and the space of section of the corresponding sheaf as $\Omega_U^s(\log A)$.

Definition 5.3 *Keeping to the notations used in proposition* 5.2, *ii), the restriction on* $A = D_1 \cap \ldots \cap D_m$ *of the form* ξ/g *is called the residue form of* ω*, that is :*

$$res\ \omega = \left.\frac{\xi}{g}\right|_A .$$

In case the D_i are smooth divisors in general position, one can choose the universal denominator as $g \equiv 1$, which implies that $res\ \omega$ coincides in that case with the residue form in Leray-Norguet theory. It is not difficult to check the independence of $res\ \omega$ respect to the choice of the universal denominator. Remark here that the universal denominator originally introduced for weakly holomorphic functions keeps its main property when used for differential forms, namely :

"*after multiplication of* ω *by the universal denominator, one can forget divisors may be singular and work only with holomorphic forms in the ambient space.*"

Theorem 5.4 [2] *In case* A *is a reduced complete intersection of pure codimension* m*, there is a natural one-to-one correspondence*

$$res\,[\Omega_U^{q+m}(\log A)] \longleftrightarrow \omega^q(A)$$

between the set of residues of multi-logarithmic $q+m$*-forms and holomorphic* q *forms on* A*.*

In order to prove this theorem we need to summarize the main facts about residue currents which were recalled in subsection 2.?. Let us assume that the set defined by the system of equations

$$\{g = f_1 = \ldots = f_m = 0\}$$

in some open set of $\mathbb{C}^n$ is a *complete intersection* and that $f_1, ..., f_m$ define A (as a reduced complete intersection). Then the residue current R_f and its principal values $P_g R_f$ respect to g have the following properties :

- 1) $R_f[\omega] = P_g R_f[g\,\omega]$;
- 2) $P_g R_f\Big[\frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_m}{f_m} \wedge \frac{\xi}{g}\Big] = \left.\Big[\frac{\xi}{g}\Big]\right|_A = \Big[\frac{\xi}{g}\Big] \wedge [A]$, where $[A]$ is the integration current (holomorphic chain) on A ; in particular, when ξ and g are identically equal to the constant 1, then

$$R_f\Big[\frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_k}{f_m}\Big] = df_1 \wedge \ldots df_m \wedge \bar{\partial}\left[\frac{1}{f_1}\right] \wedge \ldots \wedge \bar{\partial}\left[\frac{1}{f_m}\right] = [A]\ ;$$

- 3) $f_j P_g R_f = 0, \quad j = 1, \ldots, m$;
- 4) if U is a domain of holomorphy (or a Stein manifold) then for $v \in \mathcal{O}_U$, one has

$$vR_f \equiv 0 \Longleftrightarrow v \in (f_1, \ldots, f_m)\, \mathcal{O}_U \ ;$$

in particular, this property is valid in the local rings $\mathcal{O}_{U,z}, \ z \in U$.

Proof of theorem 5.4. Let $\omega \in \Omega^{q+m}_{U,0}(\log A)$, so that

$$\omega = \frac{\alpha}{f_1 \cdots f_m}, \quad \text{and} \quad g\,\omega = \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_m}{f_m} \wedge \xi + \eta,$$

where α, ξ are holomorphic and η satisfies condition $ii)$ of proposition 5.2. In view of property 1) one has

$$R_f[\omega] = P_g R_f[g\omega] = P_g R_f\left[\frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_m}{f_m} \wedge \frac{\xi}{g}\right] + P_g R_f\left[\frac{\eta}{g}\right].$$

The second summand corresponds to the current which is trivial by property 3). Property 2) implies

$$R_f[\omega] = \left[\frac{\xi}{g}\right]\Big|_A.$$

Hence the current $\left[\frac{\xi}{g}\right]\Big|_A$ defined as the residue $res\,\omega = \frac{\xi}{g}\Big|_A$ of the multi-logarithmic form ω coincides with the residue current $R_A[\omega]$ which is $\bar\partial$-closed.

Conversely, under the same assumptions, let ξ/g be a meromorphic form on A, so that the corresponding current

$$\Big\langle [\psi], \varphi \Big\rangle = \Big\langle \left[\frac{\xi}{g}\right]\Big|_A, \varphi \Big\rangle = \lim_{\varepsilon \to 0} \int_{A \cap \{|g| > \varepsilon\}} \frac{\xi}{g} \wedge \varphi, \quad \varphi \in \mathcal{D}^{\bullet}(U)$$

is $\bar\partial$-closed. Then, in view of property 2),

$$0 = \bar\partial \left[\frac{\xi}{g}\right]\Big|_A = \bar\partial\Big(\left[\frac{\xi}{g}\right] \wedge [A]\Big).$$

Since $\bar\partial[A] = 0$, it follows from Leibnitz rule and property 2) that

$$0 = \bar\partial\left[\frac{\xi}{g}\right] \wedge [A] = \bar\partial\left[\frac{\xi}{g}\right] \wedge df_1 \wedge \ldots \wedge df_m \wedge \bar\partial\left[\frac{1}{f_1}\right] \wedge \ldots \wedge \bar\partial\left[\frac{1}{f_m}\right].$$

Property 4) yields

$$df_1 \wedge \ldots \wedge df_m \wedge \xi \in (g, f_1, \ldots, f_m)\, \Omega_U^{q+m},$$

that is,

$$df_1 \wedge \ldots \wedge df_m \wedge \xi = \alpha g - f_1\eta_1 - \ldots - f_m\eta_m\,, \tag{5.8}$$

where α, $\eta_1, \ldots, \eta_m \in \Omega_U^{q+m}$ are holomorphic forms on the domain $U \subset \mathbb{C}^n$. Here property 4) was used to compute the coefficients of the form $\xi \wedge df$. Let us consider the meromorphic form $\omega = \alpha / f_1 \cdots f_m$; we have

$$g\cdot\omega = g\cdot\frac{\alpha}{f_1 \cdots f_m} = \frac{df \wedge \xi + f_1\eta_1 + \ldots + f_m\eta_m}{f_1 \cdots f_m} = \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_m}{f_m} \wedge \xi + \eta,$$

where

$$\eta = \sum_{i=1}^{m} \frac{f_i\eta_i}{f_1 \cdots f_m} \in \sum_{i=1}^{m} \Omega_{U,0}^{q+m}(\star\widehat{D}_i).$$

This completes the proof of theorem 5.4. $\diamondsuit$

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