Preliminary material
Residue calculus in one variable

1 An analytical approach

The aim of this paragraph is to illustrate an analytical approach to the notion of residue; we shall also give some applications.

Let U be an open connected set of $\mathbb{C}, \alpha$ a point of U, and h an holomorphic function on $U \setminus \{\alpha\}$; also, let $r > 0$ be small enough as to ensure that the disk $D = D(\alpha, r)$ is relatively compact in U.

For all $\zeta \in D(\alpha, r)$, $\zeta \neq \alpha$, we can write

$$h(\zeta) = \sum_{j \in \mathbb{Z}} a_j (\zeta - \alpha)^n \tag{1.1}$$

where the so-called Laurent coefficients are given by

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} h(\alpha + re^{i\theta}) e^{-in\theta} d\theta \tag{1.2}$$

(note that the a_n do not depend on r).

Let $A = \{n \in \mathbb{Z} : a_n \neq 0\}$: then the *order* of h at α is defined as the following element $\theta_\alpha(h) \in \mathbb{Z} \cup \{-\infty, +\infty\}$:

$$\theta_\alpha(h) = \begin{cases} +\infty & \text{if } A = \emptyset \\ \operatorname{Inf} A & \text{if } A \neq \emptyset \text{ and is bounded from below} \\ -\infty & \text{if } A \neq \emptyset \text{ and is not bounded from below.} \end{cases} \tag{1.3}$$

The singularity of h is called *removable* if $\theta_\alpha(h) \geq 0$ or $\theta_\alpha(h) = +\infty$. In such a case, there exists a unique holomorphic extension g of h on U. In particular if $\theta_\alpha(h) = +\infty$, the analytic continuation principle ensures that $g \equiv 0$ on U.

If $\theta_\alpha(h)$ is a positive (resp. negative) integer, we say that α is a *zero (resp. pole)* of order $\theta_\alpha(h)$ (resp. $-\theta_\alpha(h)$) for h.

The singularity is *essential* if $\theta_\alpha(h) = -\infty$.

If $\theta_\alpha(h) = n \in \mathbb{Z}$, for any $\zeta \in D \setminus \{\alpha\}$, we can write $h(\zeta) = (\zeta - \alpha)^n u(\zeta)$, where u is an holomorphic function on D which does not vanish at α.

If U is an open connected set of $\mathbb{C}$, A is a closed discrete subset of U, and h an holomorphic function in $U \setminus A$, we shall say that h is meromorphic in U if, for any α in A, $\theta_\alpha(h)$ is either a relative integer or $+\infty$.

Let h be a meromorphic function on a simply connected domain U. Let α be a pole of h, and let $\epsilon_\alpha > 0$ be small enough as to have $\overline{D}(\alpha, \epsilon_\alpha) \subset U$. Then the *residue* of $h(\zeta)d\zeta$ at the pole α is, by definition, the quantity

$$\operatorname{Res}\left[h(\zeta)d\zeta, \alpha\right] := \frac{1}{2\pi i} \int_{|\zeta - \alpha| = \epsilon_\alpha} h(\zeta) d\zeta\,. \tag{1.4}$$

Denoting by γ a continuous loop in U which does not contain any pole of h, and writing $\eta_\alpha(\gamma) = \frac{1}{2i\pi}\int_\gamma \frac{d\zeta}{\zeta - \alpha}$ for the index of γ about α, we have the classical residue formula

$$\frac{1}{2\pi i}\int_\gamma h(\zeta)d\zeta = \sum_{\substack{\alpha\in U \\ \alpha \text{ pole of } h}} \eta_\alpha(\gamma)\,\mathrm{Res}\,[h(\zeta)d\zeta, \alpha]\,. \tag{1.5}$$

(1.1) Remark — From the homological equivalence

$$\gamma \cong \sum_{\substack{\alpha\in U \\ \alpha \text{ pole of } h}} \eta_\alpha(\gamma)\partial D(\alpha, \epsilon_\alpha)\,,$$

(where $\partial D(\alpha, \epsilon_\alpha)$ has the direct trigonometric orientation), it follows that

$$\int_\gamma h(\zeta)d\zeta = \sum_{\substack{\alpha\in U \\ \alpha \text{ pole of } h}} \eta_\alpha(\gamma)\int_{|\zeta-\alpha|=\epsilon_\alpha} h(\zeta)d\zeta$$

and, since

$$\int_{|\zeta-\alpha|=\epsilon_\alpha} h(\zeta)d\zeta = 2\pi i\,\mathrm{Res}\,[h(\zeta)d\zeta, \alpha]\,,$$

we recover (1.5).

(1.2) Remark — It is also interesting to point out that the symbol $\mathrm{Res}\,[h(\zeta)d\zeta, \alpha]$ does not have any real connection with integration; we just use the integration symbol in order to materialize the duality homology-cohomology. More precisely, if c is a cycle (i.e., a smooth chain with vanishing boundary), and ω is a cocycle (i.e., a C^1 closed differential form), then the duality $< c, \omega >$ is expressed by

$$< c, \omega > := \int_c \omega\,.$$

Before continuing the exposition of the theory, let us recall how one can classically perform computations of residues.

Let D be a disk centered at α, let h be an holomorphic function in $D \setminus \{\alpha\}$ which has a simple pole at α, i.e.,

$$h(\zeta) = \frac{1}{\zeta - \alpha} f(\zeta)$$

with f holomorphic in a neighborhood of α and $f(\alpha) \neq 0$. The Taylor expansion of $f(\zeta)$, $f(\zeta) = \sum_{j\geq 0} a_j(\zeta - \alpha)^j$ gives us $a_0 = f(\alpha)$ for the coefficient of $\frac{1}{\zeta-\alpha}$ in the Laurent expansion of h.

If α is a multiple pole of h, with order m, we can avoid to compute the Laurent development of h about α. In such a case, we have the following formula.

$$\mathrm{Res}\,[h(\zeta)d\zeta, \alpha] := \frac{1}{(m-1)!}\frac{d^{m-1}}{d\zeta^{m-1}}\left((\zeta - \alpha)^m f(\zeta)\right)\big|_{\zeta=\alpha}\,, \tag{1.6}$$

(note anyway that computing directly the Laurent development of h about α is usually more efficient from the computational point of view). In fact if

$$h(\zeta) = \sum_{j=-m}^{\infty} a_j(\zeta - \alpha)^j ,$$

we can consider the function

$$\zeta \mapsto g(\zeta) = h(\zeta)(\zeta - \alpha)^m = \sum_{j=-m}^{\infty} a_j(\zeta - \alpha)^{j+m}$$

so

$$a_{-1} = \mathrm{Res}[hd\zeta, \alpha] = \lim_{\zeta\to\alpha} \frac{1}{(m-1)!}\frac{d^{m-1}g}{d\zeta^{m-1}}(\zeta)$$

Next we describe some applications of residues calculus in a global context.

Let a be any complex number; the explicit computation of the residue of the function $\dfrac{h'}{h-a}$, together with the residue theorem, enables us to compute the number of zeroes and poles of a meromorphic function. More precisely, if h is a nonconstant meromorphic function in an open set U, Γ is the oriented boundary of a compact set $K \subset U$, if h does not have poles on Γ, a is a complex number outside $h(\Gamma)$ and f is an holomorphic function in D, the integral

$$\frac{1}{2\pi i}\int_\Gamma f(\zeta)\frac{h'(\zeta)}{h(\zeta)-a}d\zeta$$

equals the difference between the sum of the values of f at all zeroes of $h - a$ in K (counted with their multiplicities) and the sum of the values of f at all poles of h in K (counted also with their orders). In particular, if $f(\zeta) = 1$ we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{h'(\zeta)}{h(\zeta)-a}d\zeta = A - B$$

where A denotes the sum of the multiplicities of the distinct roots of the equation $h(z)-a = 0$ contained in K, and B is the sum of the orders of the poles of h contained in K.

Let f be a continuous function on $\gamma + i\mathbb{R}$, with $\gamma \in \mathbb{R}$.

We set, whenever the limit below exists,

$$\mathrm{PV}\int_{\gamma+i\mathbb{R}} f(\zeta)t^{-\zeta}d\zeta := \lim_{\rho\to\infty}\int_{\gamma-i\rho}^{\gamma+i\rho} f(\zeta)t^{-\zeta}d\zeta\,.$$

Let us recall the classical Jordan's lemma:

(1.1) Lemma — *Let f be a continuous function on $\gamma + i\mathbb{R}$, with $\gamma \in \mathbb{R}$.*

Suppose that f can be extended to a meromorphic function on the half-plane $\mathrm{Re}(\zeta) < \gamma$, *which is continuous up to the boundary and such that, for some increasing sequence (ρ_n) of strictly positive numbers, we have*

$$\lim_{n\to\infty} \max_{\substack{|\zeta-\gamma|=\rho_n \\ \mathrm{Re}(\zeta)<\gamma}} \{|f(\zeta)|\} = 0\,.$$

Then, if $0 < t < 1$ *and if*

$$\mathrm{PV}\int_{\gamma+i\mathbf{R}} f(\zeta)t^{-\zeta}d\zeta$$

exists, we have

$$\mathrm{PV}\int_{\gamma+i\mathbf{R}} f(\zeta)t^{-\zeta}d\zeta = 2\pi i \lim_{n\mapsto+\infty} \sum_{\substack{\alpha \text{ pole of } f\\ \mathrm{Re}(\alpha)<\gamma \\ |\alpha-\gamma|<\rho_n}} \mathrm{Res}[f(\zeta)t^{-\zeta}d\zeta, \alpha]\,.$$

Proof — For $\rho > 0$, we have

$$\int_{\gamma-i\rho}^{\gamma+i\rho} f(\zeta)t^{-\zeta}d\zeta = t^{-\gamma}\int_{\gamma-i\rho}^{\gamma+i\rho} t^{-(\zeta-\gamma)}f(\zeta)d\zeta\,.$$

For any ρ_n, we consider the loop obtained by a vertical segment $[\gamma-i\rho_n, \gamma+i\rho_n]$ followed by a half-circle

$$\gamma_{\rho_n} := \left\{\gamma+\rho_n e^{i\theta}; \frac{\pi}{2} \le \theta \le \frac{3}{2}\pi\right\}\,.$$

Cauchy's theorem gives

$$\int_{\gamma-i\rho_n}^{\gamma+i\rho_n} f(\zeta)t^{-(\zeta-\gamma)}d\zeta + \int_{\gamma_{\rho_n}} f(\zeta)t^{-(\zeta-\gamma)}d\zeta =$$
$$= 2\pi i \sum_{\substack{\alpha \text{ pole of } f\\ \mathrm{Re}(\alpha)<\gamma \\ |\alpha-\gamma|<\rho_n}} \mathrm{Res}[f(\zeta)t^{\gamma-\zeta}d\zeta, \alpha]\,.$$

It remains to show that $\int_{\gamma_{\rho_n}} f(\zeta)t^{-(\zeta-\gamma)}d\zeta$ vanishes as $n \to +\infty$.

Indeed

$$\int_{\gamma_{\rho_n}} f(\zeta)t^{\gamma-\zeta}d\zeta = \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} f(\gamma+\rho_n e^{i\theta})t^{-\rho_n e^{i\theta}} i\rho_n e^{i\theta}d\theta =$$
$$= \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} f(\gamma+\rho_n e^{i\theta})e^{-(\log t)\rho_n(\cos\theta+i\sin\theta)} i\rho_n e^{i\theta}d\theta =$$
$$= \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} f(\gamma+\rho_n e^{i\theta})e^{|\log t|\rho_n(\cos\theta+i\sin\theta)} i\rho_n e^{i\theta}d\theta$$

So

$$\left|\int_{\gamma_{\rho_n}} f(\zeta)t^{\gamma-\zeta}d\zeta\right| \le \rho_n \max_{\substack{|\zeta-\gamma|=\rho_n\\ \mathrm{Re}(\zeta)\le\gamma}} \{|f(\zeta)|\} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{-|\log t|\rho_n \sin\theta}d\theta.$$

From the standard estimate

$$\sin\theta \ge \frac{2\theta}{\pi} \text{ for } \theta \in \left[0, \frac{\pi}{2}\right],$$

it follows that

$$\rho_n \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{-|\log t|\rho_n \sin\theta}d\theta \le \rho_n \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{\frac{-2|\log t|\rho_n}{\pi\theta}}d\theta$$

remains bounded when $n \to \infty$. The conclusion of the lemma follows. ∎

(1.3) REMARK — In the special case $t = \frac{1}{e}$, Jordan's lemma gives

$$\mathrm{PV} \int_{\gamma + i\mathbb{R}} f(\zeta) e^{\zeta} d\zeta = 2\pi i \sum_{\substack{\alpha \text{ pole of } f \\ \mathrm{Re}(\alpha) < \gamma,\ |\alpha - \gamma| < \rho_n}} \mathrm{Res}[f(\zeta) e^{\zeta} d\zeta, \alpha]\,.$$

Of course, Jordan's lemma still holds when f is a continuous function on the real axis that can be extended to a meromorphic function in the half-plane $\mathrm{Im}(\zeta) \geq 0$, continuous up to the boundary and satisfying in this half-plane the condition $\lim\limits_{n\to\infty} \max\limits_{|\zeta| = \rho_n} |f(\zeta)| = 0$ for a sequence of radii ρ_n, $n \geq 0$.

In this case, and whenever the integral

$$\mathrm{PV} \int_{-\infty}^{+\infty} f(t) e^{it} dt$$

exists, we obtain

$$\mathrm{PV} \int_{-\infty}^{+\infty} f(t) e^{it} dt = 2\pi i \lim_{n \mapsto \infty} \sum_{\substack{\alpha \text{ pole of } f \\ \mathrm{Im}(\alpha) > 0 \\ |\alpha| < \rho_n}} \mathrm{Res}[f(\zeta) e^{i\zeta} d\zeta, \alpha]\,.$$

Moreover, if f is an even real fonction on $\mathbb{R}$ that satisfies these properties and tends to zero at infinity on $\mathbb{R}$, one has, for any $\omega > 0$,

$$\int_0^{\infty} f(t) \cos \omega t dt = \mathrm{Re} \left\{ \pi i \lim_{n \mapsto \infty} \sum_{\substack{\alpha \text{ pole of } f \\ \mathrm{Im}(\alpha) > 0 \\ |\alpha| < \rho_n}} \mathrm{Res}[f(\zeta) e^{i\omega\zeta} d\zeta, \alpha] \right\}\ ;$$

when f is an odd real function on $\mathbb{R}$ that satisfies these properties and tends to zero at infinity on $\mathbb{R}$, one has, for any $\omega > 0$,

$$\int_0^{\infty} f(t) \sin \omega t dt = \mathrm{Im} \left\{ \pi i \lim_{n \mapsto \infty} \sum_{\substack{\alpha \text{ pole of } f \\ \mathrm{Im}(\alpha) > 0 \\ |\alpha| < \rho_n}} \mathrm{Res}[f(\zeta) e^{i\omega\zeta} d\zeta, \alpha] \right\}\ .$$

Now we have a result that will be very important for us later about the behaviour of the so-called inverse Mellin transform.

Let θ be any function defined on $]0, \infty[$, taking complex values, which is locally integrable on this interval and with bounded support.

We suppose there are two constants c and $\gamma_0 > 0$ such that, near the origin, we have

$$|\theta(t)| \leq c t^{-\gamma_0} \text{ for } t > 0\,.$$

Then we can define the Mellin transform M_θ of the function θ as follows: for any complex number λ with $\mathrm{Re}(\lambda) > \gamma_0$,

$$M_\theta(\lambda) = \lambda \int_0^{+\infty} t^{\lambda - 1} \theta(t) dt\,.$$

It is easy to prove that the function M_θ is holomorphic into the half-plane $\mathrm{Re}(\lambda) > \gamma_0$.

The problem is how to recognize some properties of the function θ (essentially related to its behavior near the origin) from the study of its Mellin transform M_θ.

The Fourier inversion formula provides some way to recover the function θ from the knowledge of its transform M_θ.

Let γ be a real number, $\gamma > \gamma_0$ such that

$$\omega \mapsto f(\omega) = M_\theta(\gamma + i\omega)$$

belongs to the space $L^2(\mathbb{R})$.

We can write, thanks to the change of variables $t = e^s$,

$$f(\omega) = (\gamma + i\omega) \int_{-\infty}^{+\infty} e^{i\omega s} e^{\gamma s} \theta(e^s) ds\,.$$

Then the function $\omega \mapsto f(\omega)/(\gamma + i\omega)$ appears as the Fourier transform of the function $s \mapsto e^{\gamma s}\theta(e^s)$, which belongs to the space $L^1(\mathbb{R})$.

The Fourier inversion formula (applied to the function $\omega \mapsto f(\omega)/(\gamma+i\omega)$ which belongs at the space $L^1(\mathbb{R})$), implies that the function $s \mapsto e^{\gamma s}\theta(e^s)$ is continuous in $\mathbb{R}$ and that, for any $s \in \mathbb{R}$, for any $\gamma > \gamma_0$,

$$e^{\gamma s}\theta(e^s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M_\theta(\gamma + i\omega)}{\gamma + i\omega} e^{-is\omega} d\omega,$$

which is equivalent, for any real number $\gamma > \gamma_0$, to

$$\theta(t) = \frac{1}{2i\pi} \int_{\gamma + i\mathbb{R}} \frac{M_\theta(\lambda)}{\lambda} t^{-\lambda} d\lambda\,.$$

This is known as the Mellin inversion formula.

(1.1) Proposition — *Suppose there exist two constants $C > 0$ and $\eta > 0$ such that the Mellin Transform extends to an holomorphic function f in the half plane $\mathrm{Re}(\lambda) > -\eta$, such that the following conditions hold: $|f(\lambda)| \leq C$ for $-\frac{\eta}{2} \leq \mathrm{Re}(\lambda) \leq \gamma_0 + \eta$ and*

$$\int_{\mathbb{R}} \frac{|f(\gamma + i\omega)|}{1 + |\omega|} d\omega < +\infty$$

for

$$-\frac{\eta}{2} \leq \gamma \leq \gamma_0 + \eta\,.$$

Then the function θ extends to $[0, \infty[$ to a fonction which is continuous at the origin and such that $\theta(0) = f(0)$.

Proof — Let t be a real strictly positive number. The function $\lambda \mapsto f(\lambda)\dfrac{t^{-\lambda}}{\lambda}$ is meromorphic in the half plane $\mathrm{Re}(\lambda) > -\eta$, with an unique (and simple) pole at the origin, with residue $f(0)$.

By the Mellin inversion formula, one has, for $\gamma = \gamma_0 + \eta$,

$$\frac{1}{2i\pi}\int_{\gamma+i\mathbf{R}} \frac{f(\lambda)}{\lambda} t^{-\lambda} d\lambda = \theta(t)\,.$$

(the second hypothesis implies the convergence of the integral and the validity of the Mellin inversion formula). One can use the residue theorem as follows. Let Γ_ρ be the loop which is the concatenation of $\Gamma_{\rho 1} = [\gamma - i\rho, \gamma + i\rho]$, $\Gamma_{\rho 2} = [\gamma + i\rho, -\eta/2 + i\rho]$, $\Gamma_{\rho 3} = [-\eta/2 + i\rho, -\eta/2 - i\rho]$, $\Gamma_{\rho 4} = [-\eta/2 - i\rho, \gamma - i\rho]$ (for $\rho > 0$). One has, for $t > 0$,

$$\frac{1}{2i\pi}\int_{\Gamma_\rho} \frac{f(\lambda)t^{-\lambda}}{\lambda} d\lambda = f(0)\,.$$

It is easy to see that the integrals on $\Gamma_{\rho 2}$ and $\Gamma_{\rho 4}$ of the differential form $f(\lambda)t^{-\lambda}d\lambda/\lambda$ tend to zero when ρ tends to infinity. The integral along $\Gamma_{\rho 1}$ tends to $\theta(t)$ when ρ tends to $+\infty$. Therefore, one has

$$\theta(t) = \frac{1}{2i\pi}\int_{-\frac{\eta}{2}+i\mathbf{R}} \frac{f(\lambda)}{\lambda} t^{-\lambda} d\lambda + f(0)\,.$$

Since the integral in the formula above is bounded by (constant) $t^{\eta/2}$, we have the result we needed to prove. ∎

(1.4) Remark — If $f = M_\theta$ is an holomorphic function in the right half-plane $\mathrm{Re}(\zeta) > -\eta$ for some $\eta > 0$ and satisfies, for some constants $C > 0, R > 0$,

$$\begin{cases} |f(\zeta)| \le C \text{ for } \frac{-\eta}{2} \le \mathrm{Re}(\zeta) \le \gamma_0 + \eta, \qquad |\operatorname{Im}\zeta| \ge R \\ \int_{|\omega|>R} \frac{|f(\gamma+i\omega)|}{1+|\omega|} d\omega \le \infty \text{ for } \frac{-\eta}{2} \le \gamma \le \gamma_0 + \eta\,, \end{cases}$$

but admits poles in the right half-plane $\mathrm{Re}(\zeta) > -\eta$ (of course only in the strip $|\operatorname{Im}\zeta| \le R$), then the same reasoning shows that for all $\gamma \in]-\frac{\eta}{2}, \gamma_0]$, such that f does not have poles on $\gamma + i\mathbf{R}$,

$$\theta(t) = \sum_{\substack{\alpha \text{ pole of } F \\ \mathrm{Re}(\alpha)>\gamma}} \mathrm{Res}[t^{-\zeta}f(\zeta)d\zeta/\zeta, \alpha] + \frac{1}{2i\pi}\int_{\gamma+i\mathbf{R}} f(\zeta)t^{-\zeta}\frac{d\zeta}{\zeta} \tag{1.7}$$

We may also observe that

$$\lim_{t\to 0} \frac{\int_{\gamma+i\mathbf{R}} f(\zeta)t^{-\zeta}\frac{d\zeta}{\zeta}}{\sum\limits_{\substack{\alpha \text{ pole of } f \\ \mathrm{Re}(\alpha)>\gamma}} \mathrm{Res}[t^{-\zeta}f(\zeta)d\zeta/\zeta, \alpha]} = 0\,;$$

so we can think of (1.7) as an asymptotic development for θ when $t \to 0$ (the precision of this expansion increases as γ moves towards the left).

The following example is an application of Remark 1.4.

Let $\zeta \mapsto \Gamma(\zeta)$, with $\mathrm{Re}(\zeta) > 0$, be the Gamma function defined by

$$\Gamma(\zeta) = \int_0^\infty e^{-t} t^{\zeta-1} dt \,. \tag{1.8}$$

In this positive half-plane Γ is an analytic function. Integration by parts provides the following functional equation: for any ζ in the half-plane $\mathrm{Re}\,\zeta > 0$,

$$\Gamma(\zeta+1) = \int_0^\infty e^{-t} t^{\zeta} dt = \left[-e^{-t} t^{\zeta}\right]_0^\infty + \int_0^\infty e^{-t} \zeta t^{\zeta-1} dt = \zeta\Gamma(\zeta)\,;$$

this functional equation allows us to continue Γ to the whole complex plane as a meromorphic function with poles at $0, -1, -2, \ldots$; one has also $\Gamma(n+1) = n!$ if n is a positive integer.

(1.2) PROPOSITION — *Let $t > 0$, β be a complex number such that $\mathrm{Re}(\beta) > 0$ and $0 < \gamma < \mathrm{Re}(\beta)$. Then, we have the formula*

$$(1+t)^{-\beta} = \frac{1}{2i\pi\Gamma(\beta)} \int_{\gamma+i\mathbf{R}} \Gamma(\zeta)\Gamma(\beta-\zeta) t^{-\zeta} d\zeta. \tag{1.9}$$

PROOF — In order to prove this formula, we notice that the function

$$\zeta \mapsto \Gamma_\beta(\zeta) = \Gamma(\zeta) \cdot \Gamma(\beta - \zeta)$$

is a meromorphic function in $\mathbb{C}$, with poles are $0, -1, -2, \ldots$ and $\beta, \beta+1, \ldots$. On the other hand, Stirling's formula

$$\Gamma(\zeta) \sim \sqrt{2\pi} e^{-\zeta} \zeta^{\zeta - \frac{1}{2}}$$

ensures the rapid decrease of Γ_β at infinity along any vertical line in the complex plane. Let n be a strictly positive integer and $\eta \in]0, 1/2[$ such that Γ_β has no poles on the line $-n - \eta + i\mathbb{R}$. If one uses Cauchy's formula as in the proof of Proposition 1.1, one has

$$\begin{aligned}
&\frac{1}{2i\pi\Gamma(\beta)} \int_{\gamma+i\mathbf{R}} \Gamma(\zeta)\Gamma(\beta-\zeta) t^{-\zeta} d\zeta = \\
&= \frac{1}{2i\pi\Gamma(\beta)} \int_{-n-\eta+i\mathbf{R}} \Gamma(\zeta)\Gamma(\beta-\zeta) t^{-\zeta} d\zeta + \\
&\qquad + \sum_{k=0}^{n} \mathrm{Res}[\Gamma(\zeta)\Gamma(\beta-\zeta) t^{-\zeta} d\zeta, -k] = \\
&= \frac{1}{2i\pi\Gamma(\beta)} \int_{-n-\eta+i\mathbf{R}} \Gamma(\zeta)\Gamma(\beta-\zeta) t^{-\zeta} dt + 1 + \\
&\qquad + \sum_{k=1}^{n} \frac{(-\beta)(-\beta-1)\ldots(-\beta-k+1)t^k}{k!}\,.
\end{aligned}$$

As we notice immediately

$$R_n(t) = 1 + \sum_{k=1}^{n} \frac{(-\beta)(-\beta-1)\ldots(-\beta-k+1)t^k}{k!}$$

represents the principal part in the Taylor development at order n for the function $t \mapsto (1+t)^{-\beta}$ about $t=0$. If we denote, for $t>0$,

$$\Psi(t) = \frac{1}{2i\pi\Gamma(\beta)} \int_{\gamma+i\mathbf{R}} \Gamma(\zeta)\Gamma(\beta-\zeta)t^{-\zeta}d\zeta\,,$$

it follows from

$$\left|\frac{1}{2i\pi\Gamma(\beta)} \int_{-n-\eta+i\mathbf{R}} \Gamma(\zeta)\Gamma(\beta-\zeta)t^{-\zeta}d\zeta\right| \leq C_n t^{n+\eta}$$

that, when t tends to 0,

$$\Psi(t) - R_n(t) = o(t^n)$$

for any $n \in \mathbb{N}^*$. If one uses the functional equation of Γ, one has, for any $n \in \mathbb{N}^*$,

$$\Gamma(-n-\eta+it) = \frac{\Gamma(1-\eta+it)}{\prod\limits_{k=0}^{n}(-k-\eta+it)} \tag{1.10}$$

and

$$\Gamma(\beta+n+\eta-it) = \Gamma(\beta+\eta-it)\prod_{k=0}^{n-1}(\beta+k+\eta-it)\,. \tag{1.11}$$

For any $\epsilon>0$, there exists $T(\epsilon)$ in $\mathbb{N}^*$ such that, for any $(t,k)\in\mathbf{R}\times\mathbb{N}$ such that $|t|+k\geq T(\epsilon)$, one has

$$\left|\frac{\eta-it+\beta+k}{\eta-it+k}\right| \leq (1+\epsilon)\,.$$

If one uses formulas (1.10) and (1.11), one obtains, as soon as $n>T(\epsilon)$, the estimate

$$\begin{aligned}\int_{|t|\geq T(\epsilon)} |\Gamma_\beta(-n-\eta+it)|dt &\leq (1+\epsilon)^{n-1}\int_{|t|\geq T(\epsilon)} \frac{|\Gamma(\beta+n-it)\Gamma(1-\eta+it)|}{|\eta-it+n|}dt\\ &\leq C(\epsilon)(1+\epsilon)^{n-1}\,.\end{aligned}$$

Therefore, one has, for $n>T(\epsilon)$, the more precise estimate

$$\left|\frac{1}{2i\pi\Gamma(\beta)} \int_{-n-\eta+i\mathbf{R}} \Gamma(\zeta)\Gamma(\beta-\zeta)t^{-\zeta}d\zeta\right| \leq C(\epsilon)t^{n+\eta}(1+\epsilon)^n\,,$$

which allows to conclude that

$$\lim_{n\mapsto\infty}\left|\frac{1}{2i\pi\Gamma(\beta)} \int_{-n-\eta+i\mathbf{R}} \Gamma(\zeta)\Gamma(\beta-\zeta)t^{-\zeta}d\zeta\right| = 0$$

whenever $t(1+\epsilon)<1$. Therefore, for any such t, one has $\lim\limits_{n\mapsto\infty} R_n(t) = \Psi(t)$ and the proposition is proved. This concludes the proof of the proposition when $0<t<1$. In order to prove the result when $t>1$, we just need to move the contour of integration to the right instead of moving it to the left. In order to get the result for $t=1$, it is enough to apply Lebesgue's theorem in order to verify that the function Ψ is continuous at $t=1$. □

(1.5) REMARK — If t_1, t_2 are two strictly positive numbers, one has, for any complex number β such that $\mathrm{Re}\,\beta > 0$ and any γ such that $0 < \gamma < \mathrm{Re}\,\beta$,

$$(t_1 + t_2)^{-\beta} = t_2^{-\beta}(1 + \frac{t_1}{t_2})^{-\beta} = \frac{1}{2i\pi\Gamma(\beta)} \int_{\gamma+i\mathbf{R}} \Gamma(s)\Gamma(\beta - s) t_1^{-s} t_2^{s-\beta} ds\,.$$

This mecanism can be iterated as follows: if $\mathrm{Re}\,\beta > 0$, if $t_1, \dots, t_p$ are p strictly positive numbers, one has, for any $(\gamma_1, \dots, \gamma_{p-1}) \in]0, \infty[^{p-1}$ such that $\gamma_1 + \ldots + \gamma_{p-1} < \mathrm{Re}\,\beta$,

$$(t_1 + \cdots + t_p)^{-\beta} = \frac{1}{(2i\pi)^{p-1}\Gamma(\beta)} \int_{\gamma_1+i\mathbf{R}} \cdots \int_{\gamma_{p-1}+i\mathbf{R}} \Gamma_p^*(\zeta) t_1^{-\zeta_1} \cdots t_{p-1}^{-\zeta_{p-1}} t_p^{\zeta^*} d\zeta_1 \cdots d\zeta_{p-1}\,, \tag{1.12}$$

where

$$\Gamma_p^*(\zeta) = \Gamma(\zeta_1) \cdots \Gamma(\zeta_{p-1}) \Gamma(\beta - \zeta_1 - \cdots - \zeta_{p-1})\,,\ \zeta^* = \sum_{k=1}^{p-1} \zeta_k - \beta\,.$$

Such a formula (1.12) transforms the additive operation between the t_j (namely $(t_1 + \cdots + t_p)^{-\beta}$) into a multiplicative one (namely $t_1^{-\zeta_1} \cdots t_{p-1}^{-\zeta_{p-1}} t_p^{\zeta^*}$ in the integrant). From this point of view, it plays a quite interesting algebraic role and will have consequences for us later, despite its analytic aspect. One can view such a formula (1.12) as a continuous version of the binomial formula (with negative exponent).

2 The geometric point of a view

We consider a meromorphic function $h = f/s$, where f and s are holomorphic in some domain U in $\mathbb{C}$, and let α be a zero of s: we denote by χ any biholomorphic transform between a neighborhood V_α of α in the $\zeta-$plane and a disk centered at $\chi(\alpha) = 0$ in the $w-$plane such that

$$s(\chi^{-1}(w)) = w^m\,,$$

where m is the multiplicity of α as a zero of s.

It is clear that for any $\epsilon > 0$ sufficiently small,

$$\{|s|^2 = \epsilon\} \cap V_\alpha = \chi^{-1}(\{|w|^2 = \epsilon\})$$

is a cycle homologous to some circular loop $\partial D(\alpha, \epsilon_\alpha)$; therefore, we have immediately

$$\begin{aligned} \mathrm{Res}[h(\zeta)d\zeta, \alpha] &= \frac{1}{2\pi i} \int_{\{|s|^2=\epsilon\}} \frac{f(\zeta)}{s(\zeta)} d\zeta = \\ &= \frac{1}{2\pi i} \frac{1}{\epsilon} \int_{\{|s|^2=\epsilon\}} f(\zeta)\overline{s(\zeta)} d\zeta = \\ &= \lim_{\epsilon \to 0} \left(\frac{1}{2\pi i} \frac{1}{\epsilon} \int_{\{|s|^2=\epsilon\}} f(\zeta)\overline{s(\zeta)} d\zeta \right) \end{aligned}\,. \tag{1.13}$$

Now let $s = (s_1, \dots, s_k)$ be a family of holomorphic functions in the domain U.

Let α be a common zero of the functions $s_1, \dots, s_k$ and m be the common multiplicity of α as a zero of $s_1, \dots, s_k$, that is the minimum of the multiplicities $m_1, \dots, m_k$ of α respectively as a zero of the functions $s_1, \dots, s_k$.

Then for any j in $\{1,\dots,k\}$, we may write

$$s_j(\zeta) = (\zeta-\alpha)^m g_j(\zeta)$$

where $g = (g_1,\dots,g_k)$ is a family of holomorphic functions in U such that at least one of the elements does not vanish at α.

We can write $\|s\|^2 = \sum_{j=1}^{k} \overline{s}_j s_j$, $\|g\|^2 = \sum_{j=1}^{k} \overline{g}_j g_j$, so that $\|s\|^2 = |(\zeta-\alpha)|^{2m}\|g\|^2$. The function $\|s\|^2$ has an isolated zero at the point α while the function $\|g\|^2$ is strictly positive in a neighborhood of the point α.

We define formally two differential forms of bidegree (1,0) in U by setting

$$\partial \log(\|s\|^2) = \|s\|^{-2}\partial\|s\|^2$$
$$\partial \log(\|g\|^2) = \|g\|^{-2}\partial\|g\|^2$$

We can observe that the first one is smooth in $U \setminus \{s_1 = \ldots = s_k = 0\}$, while second one is smooth in $U \setminus \{g_1 = \cdots = g_k = 0\}$, and in particular in a neighborhood of the point α in U. Note also that

$$\partial \log(\|s\|^2) = \frac{m d\zeta}{\zeta - a} + \partial \log\|g\|^2 \,.$$

Let us define a real change of variables by setting

$$w = (\zeta-\alpha)\|g\|^{\frac{1}{m}} \,.$$

This induces a real diffeomorphism between some open neighborhood V_α of α and some open disk D about the origin in the w-plane. We denote as χ the inverse application, that is

$$\zeta = \alpha + wA(w)\,,\ w \in D\,,$$

where

$$A(w) = \|g(\chi(w)\|^{-1/m}\,.$$

We have

$$\begin{cases} d\zeta = A(w)dw + wdA(w) \\ d\overline{\zeta} = A(w)d\overline{w} + \overline{w}dA(w) \end{cases}.$$

and therefore

$$\begin{aligned}\chi^*(\partial\log\|s\|^2) &= \frac{m(A(w)dw + wdA(w))}{wA(w)} + \chi^*(\partial\log\|g\|^2) = \\ &= \frac{mdw}{w} + \omega\,,\end{aligned}$$

where ω is a smooth form in D.

For any test function in the space $\mathcal{D}(V) = \mathcal{D}^{(0,0)}(V)$ of smooth complex valued functions with compact support in V, for any $\epsilon > 0$, let

$$\begin{aligned}\theta(\varphi;\epsilon) &= \frac{1}{2i\pi}\int_{\|s\|^2=\epsilon} \varphi(\zeta)\partial\log\|s(\zeta)\|^2 = \\ &= \frac{1}{2i\pi}\int_{\|w\|^{2m}=\epsilon} (\varphi\circ\chi)\chi^*(\partial\log\|s\|^2)\,.\end{aligned}$$

In according to the previous remarks, we have

$$\lim_{\epsilon\to 0^+}\theta(\varphi;\epsilon) = m\varphi(\alpha)\,,$$

which means that, for any such test function φ,

$$m\varphi(a) = \lim_{\epsilon\to 0^+}\frac{1}{2i\pi}\int_{\|s\|^2=\epsilon}\frac{\varphi\partial\|s\|^2}{\|s\|^2} = \lim_{\epsilon\to 0^+}\frac{1}{2i\pi\epsilon}\int_{\|s\|^2=\epsilon}\varphi\partial\|s\|^2\,. \tag{1.14}$$

We can now look at the problem from the semi-local point of view. If $s_1,\dots,s_k$ are holomorphic functions in U, one can define the integration current on the zero dimensional analytic set $\Delta = \{s_1 = \cdots = s_k = 0\}$, taking into account the multiplicities, as the linear functional δ_Δ on $\mathcal{D}^{(0,0)}(U)$ such that

$$\delta_\Delta(\varphi) = \sum_{\alpha\in U,\ s(\alpha)=0} m(\alpha)\varphi(\alpha)\,,\ \ \varphi\in\mathcal{D}^{(0,0)}(U)\,,$$

where we denote as $m(\alpha)$ the minimum of the multiplicities of α as a zero of $s_1,\dots,s_k$.

We can state the following lemma

(2.1) Lemma — *Let $s=(s_1,\dots,s_k)$ be a family of k holomorphic functions in some domain U in the complex plane. The s_j define effective divisors in U, namely $D_1,\dots,D_k$. For any test function φ in $\mathcal{D}^{(0,0)}(U)$, we have, if $\Delta = D_1\cap\ldots\cap D_k$,*

$$\begin{aligned}\delta_\Delta(\varphi) &= \lim_{\epsilon\to 0^+}\frac{1}{2i\pi}\int_{\|s\|^2=\epsilon}\varphi\partial\log\|s\|^2 = \\ &= \lim_{\tau\to 0^+}\frac{\tau}{2i\pi}\int_U\varphi\frac{\overline{\partial}\|s\|^2\wedge\partial\|s\|^2}{\|s\|^2(\|s\|^2+\tau)^2}\end{aligned} \tag{1.15}$$

Proof — The proof is rather simple. If φ is a test function, we can consider the function

$$\epsilon\to\theta(\varphi;\epsilon) = \frac{1}{2i\pi}\int_{\|s\|^2=\epsilon}\varphi\partial\log\|s\|^2\,.$$

It is clear that

$$\lim_{\epsilon\to 0^+}\theta(\varphi;\epsilon) = 0$$

if the support of φ does not contain a common zero of $(s_1,\dots,s_k)$. On the other hand, we have seen (see (1.14)) that if the support of φ lies in a neighborhood of such a common zero α, one has

$$\lim_{\epsilon\to 0^+}\theta(\varphi;\epsilon) = m(\alpha)\varphi(\alpha) = \delta_\Delta(\varphi)\,.$$

In order to recover the second formula in (1.15) we just notice that, for any integer $p\geq 1$, one has, for $\tau>0$

$$\tau\int_0^\infty\frac{t^{p-1}}{(t+\tau)^{p+1}}dt = \frac{1}{p}\,. \tag{1.16}$$

Moreover, if σ is a locally integrable function on $[0,\infty[$, which is continuous at the origin and compactly supported, then, it follows from (1.16) that

$$\sigma(0) = \lim_{\tau\to 0^+} \tau \int_0^\infty \frac{t^{p-1}\sigma(t)}{(t+\tau)^{p+1}}dt\,.$$

Apply this with $p=1$ and $\sigma = \theta(\varphi;\cdot)$, where φ is a test function in U which support lies in an arbitrary small neighborhood of a common zero α of $s_1,\dots,s_k$. We get

$$\delta_\Delta(\varphi) = \theta(\varphi;0) = \lim_{\tau\to 0^+} \tau \int_0^\infty \frac{\theta(\varphi;t)}{(t+\tau)^2}dt\,.$$

It follows from Fubini's theorem that

$$\tau \int_0^\infty \frac{\theta(\varphi;t)}{(t+\tau)^2} = \frac{\tau}{2i\pi}\int_U \varphi \frac{\overline{\partial}\|s\|^2\wedge\partial\|s\|^2}{\|s\|^2(\tau+\|s\|^2)^2}\,. \tag{1.17}$$

Let us prove that with more details; we can use the change of variables we used before, so that

$$\zeta = \alpha + w\|g\|^{-1/m},\ w\in D,\ z\in U\,,$$

where $m = m(\alpha)$. This leads to

$$\begin{aligned}\overline{\partial}\|s\|^2 &= m|w|^{2m-2}w d\overline{w}\\ \partial\|s\|^2 &= m|w|^{2m-2}\overline{w}dw\,.\end{aligned}$$

Using the change of variables,

$$\begin{aligned}&\frac{\tau}{2i\pi}\int_V \varphi\frac{\overline{\partial}\|s\|^2\wedge\partial\|s\|^2}{\|s\|^2(\tau+\|s\|^2)^2} =\\ =&\frac{\tau m^2}{2i\pi}\int_D \frac{(\varphi\circ\chi)|w|^{2(2m-2)+1}d\overline{w}\wedge dw}{|w|^{2m}(\tau+|w|^{2m})^2} =\\ =&\frac{\tau m^2}{2i\pi}\int_D (\varphi\circ\chi)(w)|w|^{2m-3}\frac{d\overline{w}\wedge dw}{(\tau+|w|^{2m})^2}\end{aligned}$$

Let us use polar coordinates $w = r^{\frac{1}{m}}e^{i\xi}$, $r>0, \xi\in[0,2\pi]$, so that $|w|^{2m} = r$ and

$$d\overline{w}\wedge dw = \frac{i}{m}\left(r^{\frac{1}{2m}}\right)^{2(1-m)} e^{i\xi}dr\wedge d\xi\,.$$

This implies

$$\begin{aligned}&\frac{\tau m^2}{2i\pi}\int_D (\varphi\circ\chi)(w)|w|^{2m-3}\frac{d\overline{w}\wedge dw}{(\tau+|w|^{2m})^2} =\\ &= \frac{\tau m}{2i\pi}\int_0^{+\infty}\frac{\left(r^{\frac{1}{2m}}\right)^{2m-3+2-2m}}{(\tau+r)^2}\left(\int_0^{2\pi}(\varphi\circ\chi)\left(r^{\frac{1}{2m}}e^{i\xi}\right)ie^{i\xi}d\xi\right)dr =\\ =&\tau\int_0^{+\infty}\frac{\theta(\varphi;t)}{(\tau+t)^2}dt\,.\end{aligned}$$

This proves (1.16) and so we have proved the second formula in (1.14) when the support of φ lies in an arbitrary small neighborhood of some zero α of s. If the support of φ does not contain any point in $\{s_1 = \ldots = s_k = 0\}$, it is clear that

$$\lim_{\tau\to 0^+} \frac{\tau}{2i\pi}\int_U \varphi \frac{\overline{\partial}\|s\|^2 \wedge \partial\|s\|^2}{\|s\|^2(\|s\|^2+\tau)^2} = 0 = \delta_\Delta(\varphi)\,.$$

The proof of the lemma is complete. ∎

In the situation of lemma 2.1, one can associate to the system $(s_1, \ldots, s_k)$ a collection of $(0,1)$ currents in U (that is continuous linear functionals on the space $\mathcal{D}^{(1,0)}(U)$ of smooth $(1,0)$ differential forms with compact support in U), called residual currents, which action can be expressed in terms of the functions $s_1, \ldots, s_k$.

If $[s] = [s_1, \ldots, s_k]$, the current $T_j^{[s]}$, $1 \le j \le k$ is by definition the $(0,1)$ current acting on a smooth $(1,0)-$test form with compact support in U as

$$T_j^{[s]}(\varphi) := \lim_{\epsilon\to 0^+} \frac{1}{2i\pi}\frac{1}{\epsilon}\int_{\|s\|^2=\epsilon} \overline{s_j(\zeta)}\varphi(\zeta)d\zeta\,.$$

Of course, we need to prove than this action makes sense and defines a $(0,1)$ current in U. In order to do that, we compute first the Mellin transform (in the sense we defined above) of the function

$$\theta_j(\varphi;\epsilon) : \epsilon \mapsto \frac{1}{2i\pi}\frac{1}{\epsilon}\int_{\|s\|^2=\epsilon} \overline{s_j(\zeta)}\varphi(\zeta)d\zeta\,.$$

When the support of φ lies in some arbitary small neighborhood of a zero α of s, the computation of this Mellin transform can be performed using Fubini's theorem and a local change of coordinates as before. It is always possible to reduce our problem to this situation, since

$$\lim_{\epsilon\to 0^+} \frac{1}{2i\pi}\frac{1}{\epsilon}\int_{\|s\|^2=\epsilon} \overline{s_j(\zeta)}\varphi(\zeta)d\zeta = 0$$

when the support of φ does not contain a zero of s. In this case, the Mellin transform is the function

$$M_j^{[s]}(\varphi;\cdot) : \lambda \mapsto \frac{1}{2i\pi}\int_U \|s\|^{2(\lambda-1)}\frac{\overline{s}_j d\|s\|^2}{\|s\|^2}\wedge\varphi d\zeta = \frac{1}{2i\pi}\int_U \|s\|^{2(\lambda-1)}\frac{\overline{s}_j\overline{\partial}\|s\|^2}{\|s\|^2}\wedge\varphi d\zeta\,.$$

Let us assume then that the support of φ lies in a small neighborhood of α, such that $s_j(\alpha) = 0$, $j = 1, \ldots, k$.

We want to show in this case that the function $M_j^{[s]}(\varphi;\cdot)$ can be analytically continued as a meromorphic function in the complex plane which is holomorphic in some half plane $\operatorname{Re}\lambda > -\eta$, with $\eta > 0$. We use again the local real change of variables $w = (\zeta-\alpha)\|g\|^{\frac{1}{m}}$,

$m = m(\alpha)$, which leads, since $d|w|^{2m} = m|w|^{2m-2}(wd\overline{w} + \overline{w}dw)$, to

$$
\begin{aligned}
M_j(\lambda) &= \frac{\lambda}{2i\pi}\int_V \|s\|^{2(\lambda-1)}\overline{s}_j \frac{d\|s\|^2}{\|s\|^2} \wedge \varphi d\zeta = \\
&= \frac{\lambda}{2i\pi}\int_D |w|^{2m(\lambda-1)}\overline{w}^m \frac{d|w|^{2m}}{|w|^{2m}} \wedge \chi^*\left(\frac{\overline{g}_j}{\|g\|^2}\varphi d\zeta\right) = \\
&= \frac{\lambda}{2i\pi}\int_D |w|^{2(m\lambda-m-1)}\overline{w}^m (wd\overline{w} + \overline{w}dw) \wedge (\alpha_j dw + \beta_j dw) = \\
&= \frac{m\lambda}{2i\pi}\int_V |w|^{2(m\lambda-m-1)}\overline{w}^m (w\alpha_j - \overline{w}\beta_j) d\overline{w} \wedge dw = \\
&= \frac{m\lambda}{2i\pi}\int_V w^{m(\lambda-1)}\overline{w}^{m\lambda-1}\alpha_j d\overline{w} \wedge dw - \frac{m\lambda}{2i\pi}\int_V w^{m(\lambda-1)-1}\overline{w}^{m\lambda}\beta_j d\overline{w} \wedge dw = \\
&= \lambda(M_{j,1}(\lambda) - M_{j,2}(\lambda))
\end{aligned}
$$

if we write

$$\chi^*\left(\frac{\overline{g}_j}{\|g\|^2}\right) = \alpha_j dw + \beta_j d\overline{w}\,,$$

and

$$
\begin{aligned}
M_{j,1}(\lambda) &= \frac{m}{2i\pi}\int_V w^{m(\lambda-1)}\overline{w}^{m\lambda-1}\alpha_j d\overline{w} \wedge dw \\
M_{j,2}(\lambda) &= \frac{m}{2i\pi}\int_V w^{m(\lambda-1)-1}\overline{w}^{m\lambda}\beta_j d\overline{w} \wedge dw\,.
\end{aligned}
$$

We now use the results described in the Appendix about homogeneous distributions. Since $m(\lambda-1) + (m\lambda-1) = 2m\lambda - m - 1$ and $m(\lambda-1) - (m\lambda-1) = 1-m$, the possible poles for the function $M_{j,1}$ are the points λ such that $2m\lambda - m - 1 = -|1-m| - 2l$, $l \in \mathbb{N}^*$, that is $\lambda = \frac{(1-l)}{m}$, $l \in \mathbb{N}^*$, and there is in particular a simple pole at $\lambda = 0$. Since $m(\lambda-1) - 1 + m\lambda = 2m\lambda - m - 1$ and $m(\lambda-1) - 1 - m\lambda = -m-1$, the possible poles for the function $M_{j,2}$ are the points λ such that $2m\lambda - m - 1 = -m - 1 - 2l$, $l \in \mathbb{N}^*$, that is $\lambda = \frac{-l}{m}$, $l \in \mathbb{N}^*$. Therefore, the function M_j can be continued as a meromorphic function with poles in the half plane $\operatorname{Re}\zeta \le 1/m$, and we have $M_j(0) = \operatorname{Res}_0 M_{j,1}$.

We can even be more precise and get (using the results in the Appendix)

$$M_j(0) = M_{j,1}(0) = \operatorname{Res}_0(M_{j,1}) = \frac{1}{m!}\frac{\partial^{m-1}}{\partial\zeta^{m-1}}\Big(\varphi\frac{\overline{g}_j}{\|g\|^2}\Big)(\alpha)\,.$$

With the same of variables, we get easily, if the support of the test form φ lies in an arbitrary small neighborhood of the zero α of s that the meromorphic continuation of M_j is rapidly decreasing at infinity on all vertical lines in the complex plane. Therefore, it follows from Proposition 1.1 that

$$\lim_{\epsilon\to 0^+}\frac{1}{2i\pi}\frac{1}{\epsilon}\int_{\|s\|^2=\epsilon}\overline{s_j(\zeta)}\varphi(\zeta)d\zeta = \frac{1}{m!}\frac{\partial^{m-1}}{\partial\zeta^{m-1}}\Big(\varphi\frac{\overline{g}_j}{\|g\|^2}\Big)(\alpha)\,. \tag{1.18}$$

This justifies our definition of the currents $T_j^{[s]}$. Note that, if $k=1$, formula (1.18) fits with formula (1.13) when φ is holomorphic near any zero of s. We have the formula

$$T_j^{[s]}(\varphi) = \sum_{\alpha,\ s(\alpha)=0}\frac{1}{m(\alpha)!}\frac{\partial^{m(\alpha)-1}}{\zeta^{m(\alpha)-1}}\Big(\varphi\frac{\overline{g}_j}{\|g\|^2}\Big)(\alpha)\,. \tag{1.19}$$

(2.1) REMARK — The key point here is that the action of all these residue currents involve only computations of holomorphic derivatives, not of antiholomorphic ones.

Let $D_1, \dots, D_k$ be k effective divisors in U and $s_1, \dots, s_k$ global sections for the ideal sheaves that are associated respectively with $D_1, \dots, D_k$. Taking into account Lemma 2.1, we have at our disposal some kind of factorization formula for the integration current, depending of course of the choice of the sections, namely

$$\delta_{D_1 \cap \dots \cap D_k} = \sum_{j=1}^{k} T_j^{[s]} \wedge ds_j \,. \tag{1.20}$$

In the particular case $k = 1$, the behavior of the residual current $T^{[s_1]}$ with respect to a change of section for the divisor D_1 is governed by what we will call the Transformation Law, which in the one variable case can be written as

$$T^{[s_1]}(\varphi) = T^{[as_1]}(a\varphi) \tag{1.21}$$

for any holomorphic function a in U which does not vanish on $\{s_1 = 0\}$ (note that in fact (1.21) remains valid without this last restriction, which tells us much more). When a is invertible in a neighborhood of $\{s_1 = 0\}$, we have also

$$T^{[as_1]}(\varphi) = T^{[s_1]}\left(\frac{\varphi}{a}\right) \,. \tag{1.22}$$

A second very important remark about these currents is that (as this can be immediately seen from (1.19)), one has, in the sense of currents,

$$s_i T_j^{[s]} \equiv 0, \qquad 1 \le i, j \le k \,,$$

which means

$$T_j^{[s]}(s_i\varphi) = 0, \qquad 1 \le i, j \le k \,, \text{ for any } \varphi \in \mathcal{D}^{(1,0)}(U) \,. \tag{1.23}$$

In the particular case $k = 1$, the residue current $T^{[s]}$ is a current which is $\overline{\partial}-$closed. In fact, in this case, one has the formula

$$T^{[s]} = \overline{\partial}\, \mathrm{PV}\, [1/s_1]$$

where the distribution PV $[1/s_1]$ (PV for Principal Value) acts on any test function $\psi \in \mathcal{D}(U)$ as

$$\mathrm{PV}\, [1/s_1](\psi) := \lim_{\epsilon \to 0} \frac{1}{2i\pi} \frac{1}{\epsilon} \int_{U \cap \{|s|^2 \ge \epsilon\}} \psi d\overline{\zeta} \wedge d\zeta \,.$$

This means also, thinking about Mellin transforms, that PV $[1/s_1](\psi)$ is the value at $\lambda = 0$ of the analytic continuation of

$$\lambda \mapsto \frac{1}{2i\pi} \int_U \frac{\|s_1\|^{2\lambda}}{s_1} \psi \overline{d\zeta} \wedge d\zeta \,.$$

This will justify a notation we will frequently use later on:

$$T^{[s_1]} = \overline{\partial}[1/s_1] \ (\text{or } \overline{\partial}(1/s_1))\,.$$

Note that this Principal Value current (or here distribution) is related to the generator s_1 of the ideal corresponding to the divisor D_1, not to the divisor itself (as the integration current is).

A major stumbling block about the currents $T_j^{[s]}$ when $k > 1$ is that they are not $\overline{\partial}$-closed. Nevertheless, we still can define Principal Value distributions, namely ${}^j\mathrm{PV}^{[s]}$, $j = 1, \ldots, k$, which actions on a test function $\psi \in \mathcal{D}(U)$ are defined as

$${}^j\mathrm{PV}^{[s]} = \frac{1}{2i\pi}\left(\int_U \|s\|^{2(\lambda-1)}\overline{s}_j \psi d\overline{\zeta} \wedge d\zeta\right)_{\lambda=0}$$

(this means one takes first the analytic continuation, then the value at the origin). An immediate computation shows that, for $1 \le j \le k$, one has the relation (in the sense of currents)

$$\overline{\partial}({}^j\mathrm{PV}^{[s]}) = T_j^{[s]} + {}^jS^{[s]},$$

where the action of the $(0,1)$ current ${}^jS^{[s]}$ [(1)] on a $(1,0)$ smooth test form with compact support in U is given by

$${}^jS^{[s]}(\varphi) = \frac{1}{2i\pi}\left(\int_U \|s\|^{2(\lambda-2)}\sum_{l\neq j} s_l(\overline{s_l ds_j - s_j ds_l}) \wedge \varphi\right)_{\lambda=0}$$

We leave as an exercise to check that the meromorphic function of λ involved in the definition of ${}^jS^{[s]}$ is holomorphic in some half-plane $\mathrm{Re}\,\lambda > -\eta$, with $\eta > 0$. This can be done using the results in the Appendix about homogeneous distributions.

One can also perform formal multiplications between Principal Value distributions and currents of the form $T_j^{[s]}$ or ${}^jS^{[s]}$. Such operations will be very useful for us later. We will define for example

$${}^lT_j^{[s]}$$

as the $(0,1)$ current whose action on the $(1,0)$ smooth forms φ is given by

$${}^lT_j^{[s]}(\varphi) = \frac{1}{2i\pi}\left(\lambda\int_U \|s\|^{4(\lambda-1)}\overline{s_j s_l}\frac{\overline{\partial}\|s\|^2}{\|s\|^2} \wedge \varphi\right)_{\lambda=0},$$

or

$${}^{l,j}S^{[s]}$$

as the $(0,1)$ current whose action on the $(1,0)$ smooth forms φ is given by

$${}^{l,j}S^{[s]}(\varphi) = \frac{1}{2i\pi}\left(\int_U \|s\|^{2(\lambda-3)}\overline{s_l}\sum_{i\neq j} s_i(\overline{s_i ds_j - s_j ds_i})\varphi\right)_{\lambda=0}.$$

---

(1) We will adopt the following convention for notations: upper-left indices will correspond to principal value currents, lower-right indices to residual currents.

It is important to note here that all these currents depend in a crucial way of the sequence $s_1, \dots, s_k$. It is easy to check that, for any $j, l, q \in \{1, \dots, k\}$, one has

$$s_q^2 \, {}^l T_j^{[s]} \equiv 0 \,,$$

which justifies that all currents ${}^l T_j^{[s]}$ (as all $T_j^{[s]}$), can be considered as residual currents. On the other hand, currents of the form ${}^j \mathrm{PV}^{[s]}$, ${}^j S^{[s]}$, or ${}^{l,j} S^{[s]}$ will be Principal Value currents; they are smooth outside the zero set of $\|s\|^2$ but their support is the whole domain U. This process can be iterated further up; for example, we will denote as ${}^{l,j}\mathrm{PV}^{[s]}$ the principal value distribution obtained as

$$(\|s\|^{4(\lambda-1)} \overline{s}_l \overline{s}_j)_{\lambda=0}$$

(such a product of Principal Values is commutative, which of course is not the case for the products between Principal Values and Residue currents we defined before). Note that all these constructions are based on the Gelfand's approach (using analytic continuation of f^λ, where f is a positive real analytic function) for the division problem in the theory of distributions.

Let us now look at some of the global aspects of this definition. From the geometric point of view, formula (1.22) and property (1.23) (applied only if $k = 1$, since there is unfortunately -up to now!- no Transformation Law valid when $k > 1$) allows us, given an effective Cartier divisor $\mathbf{D} = (U_i, s^{(i)})$ on a Riemann surface Σ, to define the residual current $\mathcal{R}_{\mathbf{D}}$ as the collection $\mathcal{R}_{\mathbf{D}}$ of all currents $T^{[s^{(i)}]} \in {}'\mathcal{D}^{(0,1)}(U_i)$ for i any arbitrary index, and consider it as a geometric object (that is an object depending on the divisor and not on the local sections). In a more general context (local complete intersection in a finite dimensional complex manifold), this is well described in [DP]. If $\mathcal{N}_{\mathbf{D}}^*$ is the sheaf of sections[(1)] of the conormal bundle attached to the divisor $\mathbf{D}$ and ${}'\mathcal{D}_{\mathbf{D}}^{(0,1)}$ is the sheaf of currents annihilated (as currents) by the ideal $\mathcal{I}_{\mathbf{D}}$ (or, to be more precise, the sheaf of ideals $\mathcal{I}$) corresponding to the divisor $\mathbf{D}$), then $\mathcal{R}_{\mathbf{D}}$ is a global section (on Σ, or on the open subset U where the divisor is defined) of the sheaf

$$\det \mathcal{N}_{\mathbf{D}}^* \otimes {}'\mathcal{D}_{\mathbf{D}}^{(0,1)} \,.$$

In this setting, the factorization formula

$$\delta_D = \mathcal{R}_{\mathbf{D}} \wedge ds \tag{1.24}$$

makes sense globally, if we think about ds as a Jacobian factor (in fact $ds_i = ds^{(i)}$ in the chart $(U_i, s^{(i)})$), which can be interpreted from the geometric point of view as a global section of the sheaf $\Omega_{\mathbf{D}|\Sigma} \otimes \det \mathcal{N}_{\mathbf{D}}$. We refer to [DP] for more details on this approach.

In fact, for arithmetic purposes (see for example [L], chapter 1, 2), it may be useful to consider on a complex manifold not only the notion of divisor, as we did above, but the notion of Hermitian line bundle. An holomorphic line bundle on the manifold Σ is in fact a covering of Σ with open subsets $U_\iota, \iota \in S$, plus a cocycle $(g_{\iota\iota'}), \iota, \iota' \in A$, where $g_{\iota\iota'}$ is an holomorphic non vanishing function in $U_\iota \cap U_{\iota'}$ and the $g_{\iota\iota'}$ satisfy the cocycle conditions (sce for example [GH] or [GA], chapter 2 for these notions). Any divisor $\mathbf{D}$ defines in a natural

(1) The reader not familiar with sheaf theory or differential geometry may skip this brief geometric section or consult some standard reference, for example [GH] or [GA], chapter 2. Anyway, this approach will be developped later on in the multivariable context.

way a line bundle $[\mathbf{D}]$. An hermitian metric on the line bundle $(U_{\iota\iota'}, g_{\iota\iota'})$ is a collection of strictly positive functions $(\rho_\iota), \rho_\iota$ being defined in U_ι, such that, for any ι, ι', one has in the intersection $U_\iota \cap U'_\iota$,

$$|g_{\iota\iota'}|^2 = \frac{\rho_\iota}{\rho'_\iota} \tag{1.25}$$

It is clear (from (1.25)) that, given some hermitian metric on a line bundle, the differential form defined as

$$dd^c \log(\rho_\iota)$$

in each U_ι, $\iota \in A$, is in fact globally defined; it is called the first Chern form $c_1(\rho)$ of the hermitian metric ρ. If s in a section for the divisor $\mathbf{D}$, then the locally integrable function

$$\mathbf{G} := -\log\left(\frac{|s_\iota(\zeta)|^2}{\rho_\iota(\zeta)}\right)\,,\ \zeta \in U_\iota\,,$$

(which is globally defined, because of the compatibility conditions (1.25)), satisfies, in the sense of currents, the Lelong-Poincaré equation

$$dd^c\mathbf{G} + \delta_{\mathbf{D}} = c_1(\rho)\,, \tag{1.26}$$

where $d^c = (i/2\pi)\overline{\partial}$. This extends (from the local point of view to the global one) the classical result, which follows from formula (1.15), which tells us that, if s is an holomorphic function in a domain U of $\mathbb{C}$, one has

$$\lim_{\tau \to 0^+} dd^c \log(|s|^2 + \tau) = \delta_{\{s=0\}}$$

or, more briefly

$$dd^c \log \|s\|^2 = \delta_{\{s=0\}}\,.$$

The Lelong-Poincaré equation plays a crucial role in intersection theory; for example, in arithmetic intersection theory (see [L] or [Sou]), one defines a multiplication operation between pairs ($[\mathbf{D}]$,G), where G is a Green current for $[\mathbf{D}]$, that is a solution of the equation

$$dd^cG + \delta_{\mathbf{D}} = \text{ smooth form.}$$

The additional idea we would like to develop is this course is the role of the equation

$$\overline{\partial}\Theta = T + \text{ correcting term,}$$

where T is a residual current, in the theory of division. The factorization formula

$$\delta_{\mathbf{D}} = \mathcal{R}_{\mathbf{D}} \wedge ds$$

mentionned above when $\mathbf{D}$ is an effective divisor on a Riemann surface, or the factorization formula

$$\delta_{D_1 \cap \cdots \cap D_k} = \sum_{j=1}^k T_j^{[s_1,\dots,s_k]} \wedge ds_j$$

valid when the D_j are effective divisors in some domain in $\mathbb{C}$ and the s_j global sections of the respective ideal sheafs, will motivate such an objective.

There is also another important geometric approach for this notion of residue. This is the approach which was proposed by Leray in a series of deep papers about the Cauchy problem (see [Le]). Of course, in the one variable case, it is hard to have a correct vision of what happens. So, we will consider for one time the $n-$dimensional situation. Assume that we bave a smooth hypersurface $\mathcal{S}$ in a complex n dimensional variety $\mathcal{X}$, defined locally (let say in a chart U) by some equation $s = 0$. The smoothness assumption will correspond here to the hypothesis $s = 0 \Rightarrow ds \neq 0$. Given a $l-$cycle in $\mathcal{X}$, that is a $l-$chain γ such that $\partial\gamma = 0$ (where ∂ is the boundary operator[(1)]), and a l smooth cocycle ω (that is a $d-$closed differential form) in $\mathcal{X} \setminus \mathcal{S}$, one would like to compute

$$\int_\gamma \omega\,.$$

Also, thinking in more geometric terms, a related question is to compute the homology and the cohomology of $\mathcal{X} \setminus \mathcal{S}$. If we had a precise description of these homology and cohomology groups, we should be able to compute $< \gamma, \omega >$, taking particular representatives respectively in the homology or cohomology classes for γ or ω. We can also make the question more precise if we assume ω is a (p, q) cocycle; in this case our problem is to give a description for the cohomology groupe $H^{p,q}(\mathcal{X} \setminus \mathcal{S})$. For example, in the one variable case, we know a basis for the 1-homology of $\mathcal{X} \setminus \mathcal{S}$ when $\mathcal{X}$ is an open subset U in $\mathbb{C}$ such that $U \cup B$ remains open if B is any hole in U (see for example [BG], chapter I). lf ω is a $(1, 0)-$cocycle (that is a (1, 0) smooth form such that $\overline{\partial}\omega = 0$) in $U \setminus \mathcal{S}$, then $\omega = h(\zeta)d\zeta$, where h is holomorphic in $U \setminus \mathcal{S}$ and formula (1.5) (at least when U is simply connected, otherwise one has also to take into account the residues corresponding to the holes) answers the problem about the computation of the integral. What Leray constructed is, for each integer l, $0 \leq l \leq 2(n-1)$, a coboundary morphism ρ_l from the homology group $H_l(\mathcal{S})$ (this makes sense since $\mathcal{S}$ is a smooth manifold with complex dimension $n-1$) into the homology group $H_{l+1}(\mathcal{X} \setminus \mathcal{S})$, together with a residual morphism $\mathrm{Res}_l^{[\mathcal{S}]}$ from the cohomology group $H^{l+1}(\mathcal{X} \setminus \mathcal{S})$ into $H^l(\mathcal{S})$. Let us describe here this geometric construction.

• The morphism ρ_l (for $l \leq 2(n-1)$) in constructed as follows: a $l+1$ cycle $\tilde{\gamma}$ in $\mathcal{X} \setminus \mathcal{S}$ is in the class $\rho_l(\dot{\gamma})$, where $\dot{\gamma}$ is the class of a $l-$cycle on $\mathcal{S}$, if and only if $\tilde{\gamma}$ is homologous in $\mathcal{X} \setminus \mathcal{S}$ to the boundary of a $l+2$ cycle τ in $\mathcal{X}$ which intersects $\mathcal{S}$ along the cycle γ in a transversal way. The fact that one can construct such a $l+2$ cycle τ in a consequence of the existence (at least locally) of a retraction from a tubular neighborbood of $\mathcal{S}$ on $\mathcal{S}$ (remember that $\mathcal{S}$ is smooth). For example, in the case $n = 1$, and $\mathcal{X}$ is a domain U, the image of the class of the cycle $\sum\limits_{j=1}^{N} m_j\alpha_j$, where the α_j are isolated points in U, is just the homology class of the cycle $\sum\limits_{j=1}^{N} m_j\gamma_{\alpha_j}$, where, for any $j \in \{1, \ldots, N\}$, γ_{α_j} is the boundary of a small closed disk in U containing α_j and none of the $\alpha_{j'}$ for $j' \neq j$.

• The construction of the residual morphism $\mathrm{Res}_l^{[\mathcal{S}]}$, $0 \leq l \leq 2(n-1)$ is based on a division lemma for differential forms, that we will state here.

(1) An elementary and brief summary about De Rham's complex and De Rham's theorem, close to point of view here, can be found in the introductive chapter of [AY].

(2.2) Lemma — *Let U be some open set in $\mathbb{C}^n$ and s be a function holomorphic in U such that $s = 0 \Rightarrow ds \neq 0$. Let l be an integer between 0 and $2n-1$. Let φ is any $l+1$ closed differential form in $U \setminus \{s=0\}$, which is semi-meromorphic, which means there exists some exponent $\mu \in \mathbb{N}$ such that $s^\mu \varphi$ is the restriction to $U \setminus \{s=0\}$ of a smooth form. Then, if $\mu \geq 1$, one can always lower the order of the pole, that is there exist two smooth differential forms in U, ψ and θ, with respective degrees l and $l+1$ such that*

$$\varphi = \frac{ds \wedge \psi}{s^\mu} + \frac{\theta}{s^{\mu-1}}$$

in $U \setminus \{s=0\}$. Moreover, when $\mu = 1$, the restriction of the differential form ψ to the manifold $\{s=0\}$ is closed as a l-differential form on $\{s=0\}$.

Proof — We have in U, since $d\varphi = 0$,

$$d(s^\mu \varphi) = \mu s^{\mu-1} ds \wedge \varphi\,,$$

so that

$$ds \wedge d(s^\mu \varphi) = 0\,.$$

Since one can play with s as if it was a local coordinate (remember $ds \neq 0$ when $s=0$), there exists a smooth form θ_1 on U such that

$$d(s^\mu \varphi) = ds \wedge \theta_1\,.$$

Then

$$ds \wedge \left(s^\mu \varphi - s\frac{\theta_1}{\mu}\right) = \frac{s}{\mu}(\mu ds \wedge s^{\mu-1}\varphi - ds \wedge \theta_1) = 0,$$

The same linear algebra argument leads to

$$s^\mu \varphi - s\frac{\theta_1}{\mu} = ds \wedge \psi\,,$$

where ψ is smooth in U; then, finally

$$\varphi = \frac{ds \wedge \psi}{s^\mu} + \frac{\theta_1}{\mu s^{\mu-1}}$$

and we are done. When $\mu = 1$, we have

$$\varphi = \frac{ds}{s} \wedge \psi + \theta\,.$$

Then, since φ is closed

$$0 = -\frac{ds}{s} \wedge d\psi + d\theta = ds\,,$$

which implies

$$ds \wedge d\psi = 0$$

on $\{s=0\}$, which concludes the proof of the lemma. $\square$

If we transpose the problem on a smooth n-dimensional complex manifold $\mathcal{X}$, $\mathcal{S}$ being an hypersurface of $\mathcal{X}$, this lemma can be used in order to show that, in the cohomology class (in $H^*(\mathcal{X} \setminus \mathcal{S})$) of any semi-meromorphic φ defined and closed in $\mathcal{X} \setminus \mathcal{S}$, with pole or order $\mu > 1$ on $\mathcal{S}$, one can find a representative φ_1 with pole of order at most 1 on $\mathcal{S}$ (this can be done as an exercice about Stokes's formula.) If φ is a $l+1$ form, where l is an integer between 0 and $2(n-1)$, the cohomology class (in $H^l(\mathcal{S})$) of the restriction $\psi_{1|\mathcal{S}}$ of any element ψ_1, associated to this representative φ_1 by Lemma 2.2, depends only of the cohomology class of φ. We define this way the residual maps $\mathrm{Res}_l^{[\mathcal{S}]}$, $l = 0, \dots, 2(n-1)$.

$$\mathrm{Res}_l^{[\mathcal{S}]}(\dot{\varphi}) := \dot{\psi}_{1|\mathcal{S}} \text{ in } H_l(\mathcal{S})\,.$$

The residue formula of Leray is now the following: for any cycle ξ in $H_l(\mathcal{S})$, where l is any integer between 0 and $2(n-1)$, for any cocycle φ in $H^{l+1}(\mathcal{X} \setminus \mathcal{S})$, such that φ is a semi-meromorphic (and closed) differential form in $\mathcal{X}$ with poles on $\mathcal{S}$, one has the duality formula

$$\int_\xi \mathrm{Res}_l^{[\mathcal{S}]}(\dot{\varphi}) = \frac{1}{2i\pi} \int_{\rho_l(\xi)} \varphi\,.$$

This is a geometric way to understand Cauchy's formula. Of course, the fact that $\mathcal{S}$ is smooth is essential in this theory. Note that in this approach, the notion of residue is completely geometric; compare to what we have done before, the residue does not act anymore on smooth forms (as it was the case in the formalism of currents), but on semi-meromorphic forms. For example, in the one variable situation (let us say in some domain U of $\mathbb{C}$, where s defines an effective divisor), one has

$$\mathrm{Res}_0^{[\mathcal{S}]}(\varphi/s^\mu) = T^{[s^\mu]}(\varphi)\,,$$

for any test smooth form φ with compact support in U. An additional restriction in Leray theory is therefore that it is a cohomological theory, where the differential forms which one uses have usually to be closed. The formalism of currents is at least a good parade for this restriction.

3 The algebraic point of view

The relation between residues and interpolation is a well known fact, which goes back to the work of Lagrange.

Let $z_1, \dots z_{n+1}$ be $n+1$ distinct points in the complex plane, and $w_1, \dots, w_{n+1}$, $n+1$ complex numbers: there exists a unique polynomial $p(z)$ of degree n such that

$$p(z_1) = w_1, \dots, p(z_{n+1}) = w_{n+1}\,.$$

The polynomial $p(z)$ (called the Lagrange interpolation polynomial) is a linear combination of the polynomials p_j, $j = 1, \dots, n+1$, of degree n which respectively take the value 1 at the point z_j and vanish at the points z_l, $l \neq j$.

So we have

$$p(z) = \sum_{j=1}^{n+1} w_j \frac{s(z)}{(z - z_j)s'(z_j)} \tag{1.27}$$

where $s(z) = (z - z_1) \dots (z - z_{n+1})$.

(3.1) Remark — The division by $z-z_j$ in the general term in the above sum is well defined because

$$\lim_{z\to z_j}\frac{s(z)}{(z-z_j)s'(z_j)}=\lim_{z\to z_j}\frac{s(z)-s(z_j)}{s'(z_j)(z-z_j)}=1\,,\ j=1,\dots,n+1\,.$$

Let h be an holomorphic function in an open bounded domain U with piecewise C^1 boundary, such that h is continuous in $\overline{U}$; the polynomial $p(z)$ of degree n, which interpolates the values of h at the $n+1$ points in U, $z_1,z_2,\dots,z_{n+1}$ (non necessarily distinct), is given by Hermite's formula

$$p(z)=\frac{1}{2\pi i}\int_{\partial U}h(\zeta)\frac{s(\zeta)-s(z)}{s(\zeta)(\zeta-z)}d\zeta \tag{1.28}$$

where $s(z)=(z-z_1)\dots(z-z_{n+1})$ and we have, for any $z\in U$, by means of Cauchy's integral formula, the following integral representation formula for the error term

$$h(z)-p(z)=\frac{s(z)}{2\pi i}\int_{\partial U}\frac{h(\zeta)d\zeta}{s(\zeta)(\zeta-z)}\,. \tag{1.29}$$

We can restate this in a different way: if we consider an arbitrary holomorphic function s in U, continuous in $\overline{U}$ and nonvanishing on the boundary of U, we have, for any function h holomorphic in U and continuous up to the boundary,

$$\begin{aligned} h(z)&=\frac{1}{2\pi i}\int_{\partial U}h(\zeta)\frac{s(\zeta)-s(z)}{s(\zeta)(\zeta-z)}d\zeta+\frac{1}{2\pi i}s(z)\int_{\partial U}\frac{h(\zeta)}{s(\zeta)(\zeta-z)}d\zeta=\\ &=\sum_{\substack{\alpha\text{ zero of } s\\ \alpha\in U}}\operatorname{Res}\left[\frac{s(\zeta)-s(z)}{s(\zeta)(\zeta-z)}h(\zeta)d\zeta,\alpha\right]+\frac{1}{2\pi i}s(z)\int_{\partial U}\frac{h(\zeta)}{s(\zeta)(\zeta-z)}d\zeta=\\ &=<\overline{\partial}\Big(\frac{1}{s}\Big)(\zeta),\ \frac{s(\zeta)-s(z)}{(\zeta-z)}d\zeta>+\frac{1}{2\pi i}s(z)\int_{\partial U}\frac{h(\zeta)}{s(\zeta)(\zeta-z)}d\zeta\,. \end{aligned} \tag{1.30}$$

This classical formula (usually known as Lagrange's or Kronecker's formula) can be extended to the case when the single function s is replaced by a system of functions $s_1,\dots,s_k$. Namely, we have the following

(3.1) Proposition — *Let U be some bounded domain in $\mathbb{C}$ with piecewise C^1 boundary and $s_1,\dots,s_k$, k functions holomorphic in U, continuous in $\overline{U}$, and such that $\|s\|^2$ does not vanish on ∂U. Any function h which is holomorphic in U and continuous in $\overline{U}$ can be represented in U as*

$$\begin{aligned} h(z)&=\frac{1}{2i\pi}\int_{\partial U}\left(\frac{\sum_{l=1}^{k}s_l(z)\overline{s_l(\zeta)}}{\|s(\zeta)\|^2}\right)^2\frac{h(\zeta)d\zeta}{\zeta-z}+\\ &+2\sum_{l=1}^{k}\sum_{j=1}^{k}s_l(z)<{}^{l}T_j^{[s_1\dots,s_k]}(\zeta)+{}^{l,j}S^{[s_1,\dots,s_k]}(\zeta),h(\zeta)\frac{s_j(\zeta)-s_j(z)}{\zeta-z}d\zeta>+\\ &+\sum_{j=1}^{k}\langle T_j^{[s_1,\dots,s_k]}(\zeta),h(\zeta)\frac{s_j(z)-s_j(\zeta)}{z-\zeta}d\zeta\rangle. \end{aligned} \tag{1.31}$$

In particular, on has the Kronecker's interpolation formula

$$h(z) \equiv \sum_{j=1}^{k} \langle T_j^{[s_1,\dots,s_k]}(\zeta), h(\zeta)\frac{s_j(z)-s_j(\zeta)}{z-\zeta} d\zeta\rangle \mod (s_1,\dots,s_k)\,, \tag{1.32}$$

where $(s_1,\dots,s_k)$ denotes the ideal generated by the s_j in the Banach space $B(U)$ of the functions holomorphic in U which are continuous up to the boundary.

Proof (of the proposition) — The key idea of the proof is the use of some *weighted* version of the Cauchy-Pompeïu formula (in fact, just an avatar of Green-Riemann's formula). If ψ is a function C^1 in $\overline{U'}$, where U' is a bounded domain with piecewise C^1 boundary, then, one has, for any $z \in U'$,

$$\psi(z) = \frac{1}{2i\pi}\int_{\partial U'} \frac{\psi(\zeta)d\zeta}{\zeta - z} - \frac{1}{2i\pi}\int\!\!\int_{U'} \frac{\overline{\partial}\psi \wedge d\zeta}{\zeta - z}\,. \tag{1.33}$$

We take U' relatively compact in U, which is a close approximation of U from the inside, so that certainly $\|s\|^2$ does not vanish in $\overline{U}\setminus U'$. Then, we consider, for z fixed in U', and λ a complex parameter such that $\mathrm{Re}\,(\lambda) \gg 1$,

$$\psi_{z,\lambda}\colon\ \zeta \mapsto h(\zeta)\Big(1 - \|s(\zeta)\|^{2\lambda} + \|s(\zeta)\|^{2(\lambda-1)}\big(\sum_{j=1}^{k}\overline{s_j(\zeta)}s_j(z)\big)\Big)^2.$$

This function satisfies $\psi_{z,\lambda}(z) = h(z)$ and, since we can rewrite it as

$$h(\zeta)\left(1 + (z-\zeta)\|s(\zeta)\|^{2(\lambda-1)}\Big(\sum_{j=1}^{k}\overline{s_j(\zeta)}\big(\frac{s_j(z)-s_j(\zeta)}{z-\zeta}\big)\Big)\right)^2,$$

we have

$$\overline{\partial}(\psi_{z,\lambda}d\zeta) = 2h(\zeta)(z-\zeta)\Big(1 - \|s(\zeta)\|^{2\lambda} + \|s(\zeta)\|^{2(\lambda-1)}\big(\sum_{j=1}^{k}\overline{s_j(\zeta)}s_j(z)\big)\Big)\overline{\partial}Q_\lambda(z,\zeta)\,,$$

where

$$Q_\lambda(z,\zeta) = \|s(\zeta)\|^{2(\lambda-1)}\Big(\sum_{j=1}^{k}\overline{s_j(\zeta)}\big(\frac{s_j(z)-s_j(\zeta)}{z-\zeta}d\zeta\big)\Big)\,.$$

If we apply formula (1.33) with such a function $\psi_{z,\lambda}$, we get

$$\begin{aligned} &h(z) - \frac{1}{2i\pi}\int_{\partial U}\left(\frac{\sum_{l=1}^{k} s_l(z)\overline{s_l(\zeta)}}{\|s(\zeta)\|^2}\right)^2\frac{h(\zeta)d\zeta}{\zeta - z} = \\ &= \frac{1}{i\pi}\int\!\!\int_U h(\zeta)\Big(1 - \|s(\zeta)\|^{2\lambda} + \|s(\zeta)\|^{2(\lambda-1)}\big(\sum_{j=1}^{k}\overline{s_j(\zeta)}s_j(z)\big)\Big)\overline{\partial}Q_\lambda(z,\zeta)\wedge d\zeta\,. \end{aligned} \tag{1.34}$$

We consider then the analytic continuation of both sides of the identity (1.34) as functions of λ. It follows from the results in the preceeding section that the two meromorphic functions that one obtains that way are holomorphic at the origin. The equality between their values at $\lambda = 0$, together with the definition of the action of the currents $T_j^{[s]}$, ${}^{l}T_j^{[s]}$, ${}^{l,j}S^{[s]}$, leads to the representation formula (1.31). ∎

(3.2) Remark — We will frequently use (in the spirit of the algebraic theory of residues that will be the guideline of these notes), the notations for the residual currents:

$$\mathrm{Res}\begin{bmatrix}\varphi(\zeta)\\ s_j(\zeta)\\ s_1(\zeta),\dots,s_k(\zeta)\end{bmatrix} = T_j^{[s_1,\dots,s_k]}(\varphi),\ \varphi\in\mathcal{D}^{(1,0)}(U)$$

in the case of several functions ($k>1$) or

$$\mathrm{Res}\begin{bmatrix}\varphi(\zeta)\\ s(\zeta)\end{bmatrix} =< \overline{\partial}(1/s(\zeta)),\varphi(\zeta)>,\ \varphi\in\mathcal{D}^{(1,0)}(U)\,,$$

in the case of one function ($k=1$). This will lead to the following more algebraic formulation of the semilocal Kroneker's formula (1.32), as for example

$$h(z)\equiv\sum_{j=1}^{k}\mathrm{Res}\begin{bmatrix}h(\zeta)\frac{s_j(z)-s_j(\zeta)}{z-\zeta}d\zeta\\ s_j(\zeta)\\ s_1(\zeta),\dots,s_k(\zeta)\end{bmatrix}\quad\mathrm{mod}\ \ (s_1,\dots,s_k)$$

in the case $k>1$.

We will derive some interesting remark from Proposition 3.1, connected with the crucial role played here by the holomorphic $(0,1)$ differential forms δ_j, $j=1,\dots k$, depending on two variables (z,ζ) and defined by

$$(\zeta,z)\mapsto\delta_j(z,\zeta):=\frac{s_j(\zeta)-s_j(z)}{\zeta-z}d\zeta,\qquad j=1,\dots,k\,.$$

The collection of all such forms will be denoted as the collection of $1-$Bézoutians attached to the system $(s_1,\dots,s_k)$. When z is a point in U such that

$$\|s(z)\|<\min_{\zeta\in\partial U}\|s(\zeta)\|\,,$$

one can rewrite the boundary integral in formula (1.31) as [1]

(1.35)

$$\frac{1}{2i\pi}\sum_{j=1}^{k}\int_{\partial U}h(\zeta)\overline{s_j(\zeta)}\left(\frac{\sum_{l=1}^{k}s_l(z)\overline{s_l(\zeta)}}{\|s(\zeta)\|^2}\right)^2\frac{\delta_j(z,\zeta)}{<\overline{s(\zeta)},s(\zeta)-s(z)>}=$$

$$=\frac{1}{2i\pi}\sum_{j,l_1,l_2}s_{l_1}(z)s_{l_2}(z)\left(\int_{\partial U}h(\zeta)\|s(\zeta)\|^{4(\lambda-1)}\overline{s_js_{l_1}s_{l_2}}\ \frac{\delta_j(z,\zeta)}{\|s(\zeta)\|^2\left(1-\frac{<\overline{s(\zeta)},s(\zeta)>}{\|s(\zeta)\|^2}\right)}\right)_{\lambda=0}=$$

$$=\frac{1}{2i\pi}\sum_{j,l_1,l_2}\sum_{q=0}^{\infty}\left(\int_{\partial U}h(\zeta)\|s(\zeta)\|^{2(2\lambda-q-3)}\overline{s_js_{l_1}s_{l_2}}<\overline{s(\zeta)},s(z)>^q\ \delta_j(z,\zeta)\right)_{\lambda=0}s_{l_1}(z)s_{l_2}(z).$$

---

[1] The bracket $<a,b>$ between two vectors in C^k denotes here also the bilinear form $\sum a_jb_j$; we will also take as usual s as an abridged notation for vector $(s_1,\dots,s_k)$.

Using Stokes's formula inside U for $\mathrm{Re}(\lambda) \gg 1$ in order to transform any coefficient

$$\int_{\partial U} h(\zeta)\|s(\zeta)\|^{2(2\lambda-q-3)}\overline{s_j s_{l_1} s_{l_2}} < \overline{s(\zeta)}, s(z) >^q \delta_j(z,\zeta)$$

in the development (1.35) above, then taking the analytic continuation (as functions of λ) and evaluating the value at $\lambda = 0$, we get from (1.35) that

$$\begin{aligned}&\frac{1}{2i\pi}\int_{\partial U}\Big(\frac{\sum_{l=1}^k s_l(z)\overline{s_l(\zeta)}}{\|s(\zeta)\|^2}\Big)^2\frac{h(\zeta)d\zeta}{\zeta-z} = \\ &= \sum_{j=1}^k \sum_{\substack{\underline{l}\in \mathbf{N}^k \\ l_1+\cdots+l_k\geq 2}} < \overline{\partial}({}^{j,\underline{l}}\mathrm{PV}^{[s]}), h(\zeta)\delta_j(z,\zeta) > s_1^{l_1}(z)\cdots s_k^{l_k}(z)\,,\end{aligned}$$

where the ${}^{j,\underline{l}}\mathrm{PV}^{[s]}$ are principal value distributions constructed using multiplications operations as in the previous section; since the singular support of such distribution is set of common zeroes of the s_j, the brackets $< \cdot, \cdot >$ are all well defined. Therefore, formula (1.31) can be written, for such z, as

$$\begin{aligned}h(z) &= \sum_{j=1}^k \sum_{\substack{\underline{l}\in \mathbf{N}^k \\ l_1+\cdots+l_k\geq 2}} < \overline{\partial}({}^{j,\underline{l}}\mathrm{PV}^{[s]}), h(\zeta)\delta_j(z,\zeta) > s_1^{l_1}(z)\cdots s_k^{l_k}(z) + \\ &+ 2\sum_{l=1}^k\sum_{j=1}^k s_l(z) < {}^lT_j^{[s_1\ldots,s_k]}(\zeta) + {}^{l,j}S^{[s_1,\ldots,s_k]}(\zeta), h(\zeta)\delta_j(z,\zeta) > + \\ &+ \sum_{j=1}^k \langle T_j^{[s_1,\ldots,s_k]}(\zeta), h(\zeta)\delta_j(z,\zeta)\rangle.\end{aligned} \tag{1.36}$$

It is clear in this formula that the Bézoutians play a crucial role since they allow us to compute (in terms of the action on h times them of Principal Value or Residue currents) an expansion formula for h with respect to the ideal generated by the s_j in the subset of U where $\|s\|$ is strictly smaller than $\min_{\partial U}\|s\|$.

When $k = 1$, this division formula (1.36) is much simpler. One has in this case,

$$\delta(\zeta, z) = \frac{s(\zeta)-s(z)}{\zeta - z}d\zeta\,.$$

So, if z is a point in U such that $|s(z)| < \min_{\partial U}|s|$, one has, with the observation

$$\frac{1}{s(\zeta)-s(z)} = \frac{1}{s(\zeta)}\cdot\frac{1}{1-\frac{s(z)}{s(\zeta)}} = \frac{1}{s(\zeta)}\sum_{k=0}^\infty\Big(\frac{s(z)}{s(\zeta)}\Big)^k\ ,\ \zeta\in\partial U\,,$$

that one can represent at this point z any function holomorphic in U and continuous in $\overline{U}$ as

$$\begin{aligned}h(z) &= \frac{1}{2\pi i}\sum_{k=0}^\infty\Big(\int_{\partial U} h(\zeta)\frac{\delta(\zeta,z)}{s^{k+1}(\zeta)}\Big)s^k(z) = \\ &= \sum_{k=0}^\infty < \overline{\partial}\Big(\frac{1}{s^{k+1}}\Big)(\zeta), h(\zeta)\delta(z,\zeta) > s^k(z)\,.\end{aligned} \tag{1.37}$$

We now want to analyse what happens in the algebraic case, that is when s is a polynomial. In this case, we have the very important proposition:

(3.2) Proposition (Abel-Jacobi) — *Let P, $Q \in \mathbb{C}[X]$. Let γ be a loop which turns once, in anti-clockwise sense, around the zeroes of P. If*

$$P(X) = a_0 X^d + \cdots + a_d$$
$$Q(X) = b_0 X^q + \cdots + a_q$$

and

$$R(X) = A_0 X^{d-1} + \cdots + A_d$$

is the rest of the Euclidean division of Q by P, then we have

$$\frac{1}{2i\pi}\int_\gamma \frac{Q(\zeta)}{P(\zeta)} d\zeta = \frac{A_0}{a_0}\,. \tag{1.38}$$

In particular, if $q \leq d-2$, then

$$\frac{1}{2i\pi}\int_\gamma \frac{Q(\zeta)}{P(\zeta)} d\zeta = 0\,. \tag{1.39}$$

Proof — The loop can be chosen to be a circle $C(0,R)$, where R is large enough so that the open disk of radius R contains all zeroes of P. Then the Laurent expansion of Q/P in the region $|z| > \max\{|\alpha|;\ Q(\alpha) = 0\}$ is, up to a polynomial (the quotient of the division $Q : P$),

$$\frac{A_0\zeta^{d-1} + \cdots}{a_0\zeta^d + \cdots} = \frac{A_0}{a_0}\frac{1}{\zeta} + \frac{\alpha_2}{\zeta^2} + \frac{\alpha_3}{\zeta^3} + \cdots$$

which implies

$$\frac{1}{2i\pi}\int_{C(0,R)} \frac{Q(\zeta)}{P(\zeta)} d\zeta = \frac{A_0}{a_0}\,.$$

The second assertion is obvious, because in this case $A_0 = 0$ (the rest is the polynomial Q itself). $\square$

If we assume that U contains all zeroes of P, then, it follows from Abel-Jacobi formula that the residue symbols

$$< \overline{\partial}\left(\frac{1}{P^{k+1}}\right), Q(\zeta)\delta(z,\zeta) >,\ k \in \mathbb{N}\,,$$

are zero as soon as $(k+1)\deg P \geq \deg Q + (\deg P - 1) + 2$, that is $k \deg s \geq \deg Q + 1$.

The implies that, in such a case, formula (1.37) (when $s = P$ and $h = Q$) can be truncated and becomes, if one use the standard global algebraic notation

$$\operatorname{Res}\begin{bmatrix} A(X)dX \\ B(X) \end{bmatrix}$$

as the total sum of residues of the rational function A/B when A, B are polynomials, the algebraic identity

$$Q(Y) = \sum_{k \deg P \leq \deg Q} \mathrm{Res} \begin{bmatrix} Q(X) \frac{P(X)-P(Y)}{X-Y} dX \\ P^{k+1}(X) \end{bmatrix} P^k(Y)\,. \tag{1.40}$$

Now, we use formula (1.40) to provide a substitute for the Euclidean division algorithm, for example in the search for a Bézout identity. Assume that $P_1, \ldots, P_m$ are m polynomials in one variable, without any common zero, such that $\deg P_1 > 0$.

Then, we have, from (1.40)

$$1 \equiv \mathrm{Res} \begin{bmatrix} \frac{P_1(X)-P_1(Y)}{X-Y} dX \\ P_1(X) \end{bmatrix}. \tag{1.41}$$

Certainly, we can find, since $P_1, P_2, \ldots, P_m$ have *no* common zeroes, a linear combination $P = \lambda_2 P_2 + \cdots + \lambda_m P_m$ which does not vanish at the zeroes of P_1. Then, one can rewrite (1.41) as

$$1 \equiv \mathrm{Res} \begin{bmatrix} \frac{P(X)(P_1(X)-P_1(Y))}{P(X)(X-Y)} dX \\ \\ P_1(X) \end{bmatrix}.$$

Now, rewrite

$$\frac{P(X)(P_1(X)-P_1(Y))}{P(X)(X-Y)} = \frac{1}{P(X)} \begin{vmatrix} \dfrac{P_1(X)-P_1(Y)}{X-Y} & \dfrac{P(X)-P(Y)}{X-Y} \\ 0 & P(X) \end{vmatrix}$$

$$= \frac{1}{P(X)} \begin{vmatrix} \dfrac{P_1(X)-P_1(Y)}{X-Y} & \dfrac{P(X)-P(Y)}{X-Y} \\ P_1(Y)-P_1(X) & P(Y) \end{vmatrix}.$$

We have then, since the local residues (with respect to P_1) of $R(X)P_1(X)$, where R is a rational function with no poles on $\{P_1 = 0\}$, are all zero,

$$1 \equiv \mathrm{Res} \begin{bmatrix} \dfrac{1}{P(X)} \begin{vmatrix} \dfrac{P_1(X)-P_1(Y)}{X-Y} & \dfrac{P(X)-P(Y)}{X-Y} \\ P_1(Y) & P(Y) \end{vmatrix} dX \\ P_1(X) \end{bmatrix} =$$

$$= P(Y)\,\mathrm{Res} \begin{bmatrix} \dfrac{P_1(X)-P_1(Y)}{(X-Y)P(X)} dX \\ \\ P_1(X) \end{bmatrix} - P_1(Y)\,\mathrm{Res} \begin{bmatrix} \dfrac{P(X)-P(Y)}{(X-Y)P(X)} dX \\ \\ P_1(X) \end{bmatrix}$$

This is a Bézout identity similar to the one one could obtain with the Euclidean division algorithm.

Let us summarize the situation, before studying residues in several complex variables. Up to this point, we have introduced three different definitions, in the local, semi-local, global case.

• **In the local case.**

Let $\mathcal{O}$ be the set of germs at the origin, that is the set of the equivalence classes $(U, s)_\sim$ where U is an open neighborhood of zero and s is a holomorphic function on U, with respect the equivalence relation $(U, s) \sim (V, \sigma)$ if there exists an open neighborhood W of zero in $U \cap V$ such that $s_{|W} = \sigma_{|W}$.

Let $\dot{s}$ be a germ in $\mathcal{O}$ and $\dot{\varphi}$ a germ of $C^\infty(1,0)$ form at the origin.

The residue symbol is the local residue at the origin,

$$< \overline{\partial}(1/\dot{s}),\, \dot{\varphi} >_0 = < \overline{\partial}\left(\frac{1}{s}\right), \varphi > ,$$

where φ is a representant of the germ $\dot{\varphi}$ and s a representant of the germ $\dot{s}$ such that the origin is the only zero of s in the support of φ. The residue symbol does not depend of course of the choice of the representants.

• **In the semi-local case.**

This case has been in this chapter the most usual for us. We have a notion of residual current, which provides a semilocal notion in some open bounded set U. If $s \in H(U)$, the ring of the analytic functions on U, the associated residue current acts on a test form in $\mathcal{D}^{(1,0)}(U)$ as

$$< \overline{\partial}\left(\frac{1}{s}\right), \varphi >_U \ .$$

(the index U here just specifies that the duality bracket corresponds to the action of a current in U). It will be more convenient for us to suppose that s extends to a function continuous on $\overline{U}$, that is an element of the Banach space $B(U)$ of the functions holomorphic in U and continuous up to the boundary (equipped with the sup norm).

•**In the global case.**

If $P, Q \in \mathbb{K}[X]$ where $\mathbb{K}$ is any subfield of $\mathbb{C}$, the global notion of total sum residues is given by

$$\mathrm{Res}\begin{bmatrix} Q(X)dX \\ P(X) \end{bmatrix} \in \mathbb{K} ,$$

defined as the sum of all local residues of the rational function P/Q at all zeroes of P. We have even extended this notion to the case when Q is a rational function with no poles on the set $\{P = 0\}$. In all these examples, the quotient space by the principal ideal generated by $\dot{s}$, s or P is $\mathbb{P} = \mathcal{O}/\dot{s}\mathcal{O}$ (in the local case), $\mathbb{P} = B(U)/sB(U)$ (in the semilocal case), $\mathbb{P} = \mathbb{K}[X]/P\mathbb{K}[X]$ (in the global case); in any case $\mathbb{P}$ is a $\mathbb{C}$ or $\mathbb{K}$ finite dimensional vectorial space.

In each of these examples $\mathrm{Hom}_{\mathbf{K}}(\mathbb{P}, \mathbb{K})$, ($\mathbb{K} = \mathbb{C}$ in the two first cases, $\mathbb{K}$ a subfield of $\mathbb{C}$ in the last one) can be equipped with a structure of left $\mathbb{P}$-module with the action

$$a \cdot \Theta(b) := \Theta(ab)\,, \quad a,\, b \in \mathbb{P},\ \Theta \in \mathrm{Hom}_{\mathbf{K}}(\mathbb{P}, \mathbb{K})\,.$$

A consequence of the Lagrange interpolation formula is that this left $\mathbb{P}$-module is in fact generated *by one element*. Any element $\Theta \in \mathrm{Hom}_{\mathbf{K}}(\mathbb{P}, \mathbb{K})$ can be written as

$$\Theta = \left\{\text{class of } \Theta_z\left(\frac{s(z) - s(\zeta)}{z - \zeta}\right)\right\}\ \mathrm{Res}^{[s]}$$

where

$$\mathrm{Res}^{[s]}(\dot{h}) := < \overline{\partial}\left(\frac{1}{s}\right)(\zeta), h(\zeta)d\zeta >_0$$

in the first case,

$$\mathrm{Res}^{[s]}(\dot{h}) := < \overline{\partial}\left(\frac{1}{s}\right)(\zeta), h(\zeta)d\zeta >_U$$

in the second case, and

$$\mathrm{Res}^{[P]}(\dot{Q}) = \mathrm{Res}\begin{bmatrix} Q(X)dX \\ P(X) \end{bmatrix}$$

in the last one. Moreover, the Lagrange interpolation formula shows that this generator $\mathrm{Res}^{[s]}$ generates $\mathrm{Hom}_{\mathbf{K}}(\mathbb{P}, \mathbb{K})$ as a *free* $\mathbb{K}-$module. Such an $\mathbb{K}-$algebra $\mathbb{P}$, where $\mathrm{Hom}_{\mathbf{K}}(\mathbb{P}, \mathbb{K})$ is generated by *one* element and is a free modulus with rank one, is a *Gorenstein Algebra*. The linear form $\mathrm{Res}^{[s]}$ on $\mathbb{P} \times \mathbb{P}$ is such that the bilinear form

$$(a, b) \mapsto \mathrm{Res}^{[s]}(ab)$$

is *non degenerated*. Such a linear form is called a residue for the Gorenstein algebra. We have therefore shown here that our notion of residue fits completely, in any of our three examples, with this notion of residue for a Gorenstein algebra.

Let us say a few words respect to the case $k > 1$. We know that any ideal (either in $\mathcal{O}$, in $B(U)$, in $\mathbb{K}[X]$) is principal, so that given $s_1, \ldots, s_k$ a collection of elements in one of these rings, one can find an element s_0 which generates the ideal $(s_1, \ldots, s_k)$. The quotient $\mathbb{P}$ of the ring by $(s_1, \ldots, s_k)$ is locally Cohen-Macaulay and the algebra $\mathbb{P}$ is Gorenstein. The residues $T_j^{[s_1,\ldots,s_k]}$, induce generators for $\mathrm{Hom}(\mathbb{P}, \mathbb{K})$ as a $\mathbb{P}$-module, as in the case $k = 1$, since we have the generalized version of Kronecker's formula (1.32). The advantage of this list of elements $T_j^{[s_1,\ldots,s_k]}$, $j = 1, \ldots, k$ instead of the single generator $\mathrm{Res}^{[s_0]}$ is that they can be constructed *directly* from the data $s_1, \ldots, s_k$, without the search for a generator for the ideal. When we will work in higher dimensions (dimension n for example), where all ideals are not principal anymore, (even, we are not generally in a locally Cohen-Macaulay situation), a formula of the form (1.32) remains valid, except that the substitutes for the $T_j^{[s]}$ define only elements in $\mathrm{Hom}(\widetilde{\mathbb{P}}, \mathbb{K})$, where $\widetilde{\mathbb{P}}$ is the quotient of the ring by the integral closure of the n-th power of the ideal. Kronecker's formulas (1.31) or (1.32) are imperfect, in the sense that they do not lead to a good division procedure with remainder term. Neththeless, we will see that this approach remains quite useful in the non complete intersection case, or the non locally Cohen-Macaulay case.