Coleff-Herrera currents revisited ⋆

Alekos Vidras and Alain Yger

Abstract In the present paper, we describe the recent approach to residue currents by M. Andersson, J. E. Björk, H. Samuelsson [2, 12, 13], focusing primarily on the methods inspired by analytic continuation (which were initiated in a quite primitive form in [8]). Coleff-Herrera currents (with or without poles) play indeed a crucial role in Lelong-Poincaré type factorization formulas for integration currents on reduced closed analytic sets. As revealed by local structure theorems (which could also be understood as global when working on a complete algebraic manifold due to the GAGA principle), such objects are of algebraic nature (antiholomorphic coordinates playing basically the role of "inert" constants). Thinking about division or duality problems instead of intersection ones (especially in the "improper" setting, which is certainly the most interesting), it happens then to be necessary to revisit from this point of view the multiplicative inductive procedure initiated by N. Coleff and M. Herrera in [14], this being the main objective of this presentation. In hommage to the pioneer work of Leon Ehrenpreis, to whom we are both deeply indepted, and as a tribute to him, we also suggest a currential approach to the so-called Nœtherian operators, which remain the key stone in various formulations of Leon's Fundamental Principle.

Alekos Vidras
Department of Mathematics and Statistics, University of Cyprus, Nicosia 1678, Cyprus
e-mail: msvidras@ucy.ac.cy

Alain Yger
Institut de Mathématiques, Université de Bordeaux, 33405, Talence, France. e-mail: Alain.Yger@math.u-bordeaux1.fr

⋆ AMS classification number: 32A27,32C30.

1 From Poincaré-Leray to Coleff-Herrera construction

Let $\mathcal{X}$ be a complex n-dimensional analytic manifold. Consider $M \leq n$ closed hypersurfaces $S_1, ..., S_M$ in $\mathcal{X}$ that intersect as a non empty complete intersection, that is, the closed analytic subset $V = \bigcap_{j=1}^M S_j \subset \mathcal{X}$ is purely $(n-M)$-dimensional (all its irreducible components have complex dimension $n-M$). When $S_1, ..., S_M$ are assumed to be smooth and moreover to intersect transversally, a well known construction by J. Leray [22] (see also [1]) leads to the construction (from the cohomological point of view) of the iterated Poincaré residue morphism from $H^p(\mathcal{X} \setminus S_1 \cup \cdots \cup S_M, \mathbb{C})$ into $H^{p-M}(V, \mathbb{C})$ (paired with its dual iterated coboundary morphism) when $p \geq M$. Following a currential (instead of cohomological) point of view, the construction proposed by N. Coleff and M. Herrera in [14] allows to drop the assumption about smoothness of the S_j's and the fact they intersect transversally, keeping just (for the moment) the complete intersection hypothesis. We propose here to make explicit in this introduction the bridge between such currential construction and J. Leray's approach. In order to do that, one recalls a concept, which is of interest by itself for algebraic reasons, of *multi-logarithmic meromorphic form* ([25, 7]).

Definition 1. Let $\mathcal{X}$ and $S_1, ..., S_M, V$ be as above. A meromorphic $(p,0)$-form ω on $\mathcal{X}$ ($M \leq p \leq n$), with polar set contained in $\bigcup_{j=1}^M S_j$, is called *multi-logarithmic with respect to* $S_1, ..., S_M$ if and only if, for any $x \in V$, one can find an open neighborhood U_x of x, M holomorphic functions $s_{1,x}, ..., s_{M,x}$ in U_x such that:

- for any $j = 1, ..., M$, the hypersurface $S_j \subset \mathcal{X}$ is defined in U_x as $\{s_{j,x} = 0\}$;
- $ds_{1,x} \wedge \cdots \wedge ds_{M,x}$ is not vanishing identically on any irreducible component of V in U_x, that is the complete intersection $V \cap U_x$ is defined by the $s_{j,x}$, $j = 1, ..., M$, as a *reduced complete intersection*;
- for any $j = 1, ..., M$, the differential forms $s_{j,x}\omega$ and $s_{j,x}\, d\omega$ (or, which is equivalent, $s_{j,x}\omega$ and $ds_{j,x} \wedge \omega$) can be expressed in U_x as $\sum_{l=1}^M \omega_l$, where ω_l is a meromorphic form with polar set contained in $\bigcup_{l' \neq l} S_{l'} \cap U_x$.

Consider $\mathcal{X}$, the S_j's and ω as in Definition 1. Let $V_{\rm sing}$ be the set of singular points of V and let $U = \mathcal{X} \setminus V_{\rm sing}$. The closed hypersurfaces $\Sigma_j = S_j \cap U \subset U$, $j = 1, ..., M$ (considered as closed hypersurfaces in U) are smooth and intersect transversally in some open neighborhood $\widetilde{U} \subset U$ of $W = \bigcap_{j=1}^M \Sigma_j$. Under these conditions, one can define on the complex submanifold $W \subset U$ the Leray-Poincaré residue $\mathrm{Res}_{\Sigma_1, ..., \Sigma_M}[\omega]$ of the meromorphic form ω (considered as multi-logarithmic in $\widetilde{U}$, with respect to $\Sigma_1, ..., \Sigma_M$). Let us recall here this construction. For any $x \in W \subset V$, one can find an open neighborhood U_x in U so that $ds_{1,x} \wedge \cdots \wedge ds_{M,x}$ does not vanish identically on any irreducible component of $V \cap U_x$. If $y \in W \cap U_x$ and $ds_{1,x}(y) \wedge \cdots \wedge ds_{M,x}(y) \neq 0$, then $\{s_{j,x} = 0\}$ is necessarily a reduced equation for the complex submanifold Σ_j about y. In a neighborhood $U_{x,y} \subset U_x$ of such $y \in W \cap U_x$, $ds_{1,x} \wedge \cdots \wedge ds_{M,x}$ does not vanish and thus one can write a local division formula (iterating with respect to $j = 1, ..., M$, the division procedure for differential forms, as introduced by G. de Rham and extensively used in [22]) :

$$\omega = \Big(\bigwedge_{j=1}^{M} \frac{ds_{j,x}}{s_{j,x}}\Big) \wedge r_{x,y}[\omega] + \sigma_{x,y}[\omega]\,,$$

where the $(p-M,0)$ form $r_{x,y}[\omega]$, also denoted by $\mathrm{res}_{\Sigma_1,\dots,\Sigma_M,x}[\omega]$, and the $(p,0)$ form $\sigma_{x,y}[\omega]$, are both meromorphic, of the form $\sum_l \varphi_{l,x,y}$, $\varphi_{l,x,y}$ being a meromorphic form in $U_{x,y}$ with polar set contained in $\bigcup_{l'\neq l} S_{l'}$. The restriction of every $r_{x,y}[\omega]$ to W is a $\bar\partial$-closed, holomorphic $(p-M)$ differential form on the closed submanifold $W \cap U_{x,y}$. All such forms $\mathrm{res}_{\Sigma_1,\dots,\Sigma_M,x}[\omega]$, for $x \in W$, fit together to form a holomorphic, $\bar\partial$-closed form on the closed manifold W, which is precisely the *Poincaré-Leray residue of ω on W* and is denoted as $\mathrm{Res}_{\Sigma_1,\dots,\Sigma_M}[\omega]$. Such an holomorphic $(p-M)$-differential form on the manifold $W \subset U$ defines a (p,M) current on U :

$$\mathrm{Res}_{\Sigma_1,\dots,\Sigma_M}[\omega] \colon \varphi \in \mathcal{D}^{n-p,n-M}(U,\mathbb{C}) \mapsto \int_W \mathrm{Res}_{\Sigma_1,\dots,\Sigma_M}[\omega] \wedge \varphi\,. \qquad (1)$$

The main issue now is to extend (in some standard way) the (p,M)-current (1) to a (p,M)-current T over the whole manifold $\mathcal{X}$, such that $\mathrm{supp}\, T \subset W$ and $\bar\partial T = 0$. There are different ways of doing this, but, for reasons of algebraic nature that will be made explicit later on, the one we adopt here is based on the analytic continuation of meromorphic current valued maps. The use of this approach in different settings is the main theme of the present paper. It is based on an algorithmic construction of $\bar\partial$-closed currents sharing a common holonomicity property.

To be more specific, we consider a finite collection $f_1, f_2, \dots, f_m$ of holomorphic functions in an open set $\Omega \subset \mathbb{C}^n$, where $m \le n$, and a collection of natural numbers $q_1, q_2, \dots q_m \in \mathbb{N}$. We define now the current

$$T^f_{\underline{q},1} = \Big[\bar\partial\Big(\frac{|f_1|^{2\lambda}}{f_1^{q_1}}\Big)\Big]_{\lambda_1=0} = \bar\partial\Big[\Big(\mathbf{1} - \mathbf{1}_{[f_1=0]}\Big)\frac{1}{f_1^{q_1}}\Big]\,,$$

where $[f_1 = 0]$ denotes the principal Weil divisor $\mathrm{div}\,(f_1)$. For a holomorphic function h in Ω, there exists, by the result of C. Sabbah [26] (completed later on by A. Gyoja [19]), about any point z in Ω, a local formal Bernstein-Sato equation

$$\mathcal{Q}_z(\lambda_1,\lambda_2,\zeta,\partial/\partial\zeta)[h^{\lambda_2+1} f_1^{\lambda_1}] = \prod_\iota (\alpha_{0,\iota} + \alpha_{1,\iota}\lambda_1 + \alpha_{2,\iota}\lambda_2)\, h^{\lambda_2} f_1^{\lambda_1}\,, \qquad (2)$$

where $\alpha_{0,\iota} \in \mathbb{N}^*$, $(\alpha_{1,\iota}, \alpha_{2,\iota}) \in \mathbb{N}^2 \setminus \{(0,0)\}$. This result extends to the context of two functions a deep result due to M. Kashiwara [20]. Exploiting this local formal equation (2) in the sense of distributions in a neighborhood U_z of z, one has, by lifting the antiholomorphic polar parts, that

$$\begin{aligned} b_z(\lambda_1,\lambda_2)\Big(\frac{|h|^{2\lambda_2}}{h}|f_1|^{2\lambda_1}\Big) &= \text{(formally) } b_z(\lambda_1,\lambda_2)\,\overline{h}^{\lambda_2}\overline{f}_1^{\lambda_1}\, f_1^{\lambda_1}h^{\lambda_2-1} = \\ &= \overline{\mathcal{Q}}_z(\lambda_1,\lambda_2,\overline{\zeta},\partial/\partial\overline{\zeta})\Big[|h|^{2\lambda_2}\,\frac{\overline{h}}{h}\,|f_1|^{2\lambda_1}\Big], \end{aligned} \tag{3}$$

whenever $\operatorname{Re}\lambda_1 >> 1$, $\operatorname{Re}\lambda_2 >> 1$. Using the fact that any distribution coefficient τ of the current $T^f_{\underline{q},1}$ can be achieved through analytic continuation as $\tau = [\tau_{\lambda_1}]_{\lambda_1=0}$ (where τ_{λ_1} is a distribution coefficient of $\overline{\partial}(|f_1|^{2\lambda_1}/f_1^{q_1})$), one deduces from (3) the identity

$$b_z(0,\lambda_2)\Big(\frac{|h|^{2\lambda_2}}{h}\otimes\tau\Big) = \overline{\mathcal{Q}}_z(0,\lambda_2,\overline{\zeta},\partial/\partial\overline{\zeta})\Big[\Big(|h|^{2\lambda_2}\frac{\overline{h}}{h}\Big)\otimes\tau\Big]$$

(in the sense of distributions about z) for $\operatorname{Re}\lambda_2 >> 1$. Iterating the above identity M times, one gets

$$\begin{aligned} b_z(0,\lambda_2)\cdots b_z(0,\lambda_2+M-1)\Big(\frac{|h|^{2\lambda_2}}{h}\otimes\tau\Big) &= \\ = \overline{\mathfrak{Q}}_{z,M}(\lambda_2,\overline{\zeta},\partial/\partial\overline{\zeta})\Big[\Big(|h|^{2\lambda_2}\frac{\overline{h}^M}{h}\Big)\otimes\tau\Big]& \end{aligned} \tag{4}$$

for some differential operator $\mathfrak{Q}_{z,M}$. Provided that M is sufficiently large, one deduces from (4) that the map

$$\lambda_2 \mapsto \frac{|h|^{2\lambda_2}}{h}\,T^f_{\underline{q},1}$$

can be continued as a holomorphic map to some half-plane $\{\operatorname{Re}\lambda_2 > -\eta\}$. Furthermore, if u is an invertible holomorphic function in Ω, then any differentiation of $|u|^{2\lambda_2}$ generates λ_2 as a factor. Thus the value of the analytic continuation of

$$\lambda_2 \mapsto \frac{|uh|^{2\lambda_2}}{h}\,T^f_{\underline{q},1} = \frac{|h|^{2\lambda_2}}{h}\,|u|^{2\lambda_2}\,T^f_{\underline{q},1}$$

at $\lambda_2 = 0$ is independent of u. This is a remarkable *holonomicity property* allowing us to use the above process iteratively. In particular, the definition of

$$T^f_{\underline{q},2} = \Big[\overline{\partial}\Big(\frac{|f_2|^{2\lambda_2}}{f_2^{q_2}}\,T^f_{\underline{q},1}\Big)\Big]_{\lambda_2=0} = \Big[\overline{\partial}\Big(\frac{|f_2|^{2\lambda_2}}{f_2^{q_2}}\Big)\wedge T^f_{\underline{q},1}\Big]_{\lambda_2=0}$$

is then justified. In a similar manner, by using slightly more general form of (2), given by

$$\mathcal{Q}_z(\lambda_1,...,\lambda_m,\zeta,\partial/\partial\zeta)\Big[h^{\lambda_m+1}\prod_{j=1}^{m-1} f_j^{\lambda_j}\Big] = \Big(\prod_\iota \big(\alpha_{\iota 0}+\sum_{j=1}^{m}\alpha_{\iota j}\lambda_j\big)\Big)\, h^{\lambda_m}\prod_{j=1}^{m-1} f_j^{\lambda_j}$$

one can construct a current $T^f_{\underline{q},3}$ (for $m = 3$) by multiplying the current $T^f_{\underline{q},2}$ with a suitable meromorphic function. One continues this iteration of the analytic continuation process until the current $T^f_{\underline{q},m}$ is constructed. What is important in this approach is that it is algorithmic and essentially algebraic, because of the use of Bernstein-Sato relations. No log resolution of singularities is explicitly involved in the picture. Furthermore, this procedure mimics the Leray iterated residue construction. An interesting application of the above approach is the following :

Proposition 1. *Let $\mathcal{X}$, the S_j's, V, and ω be as before. Let $U = \mathcal{X}\setminus V_{\rm sing}$, and $\Sigma_j = S_j \cap U$ for $j = 1,...,M$. The closed hypersurfaces $\Sigma_1,...,\Sigma_M$ (in U) are smooth and intersect transversally in some open neighborhood (in U) of $W = \bigcap_{j=1}^M \Sigma_j$, which allows to define the Poincaré-Leray residue* $\mathrm{Res}_{\Sigma_1,...,\Sigma_M}[\omega]$ *as a $(p-M,0)$ holomorphic form on the closed submanifold W of U. The associated (p,M)-current* $\mathrm{Res}_{\Sigma_1,...,\Sigma_M}[\omega]$ *in U, acting as* (1)*, is the restriction of a $\overline{\partial}$-closed (p,M)-current T over $\mathcal{X}$, with* $\mathrm{Supp}\, T \subset V$.

Proof. Let $x \in V$ and U_x be the neighborhood attached to the multi-logarithmicity of ω as described in Definition 1. Since ω is a meromorphic form with polar set in $\bigcup_{j=1}^M S_j$, one can express ω in U_x as

$$\omega = \frac{\psi_x}{s_{1,x}^{q_{1,x}}\cdots s_{M,x}^{q_{M,x}}}\,,$$

where ψ_x is holomorphic in U_x. Consider the ${}'\mathcal{D}^{(p,M)}(U_x,\mathbb{C})$-valued map defined on $\{\mathrm{Re}\,\lambda_1 >> 1,...,\mathrm{Re}\,\lambda_M >> 1\}$ as

$$\begin{aligned}(\lambda_1,...,\lambda_M) \longmapsto R^{s_x,\lambda_1,...,\lambda_M}[\omega] &= \frac{(-1)^{M(M-1)/2}}{(2i\pi)^M}\Big(\bigwedge_{j=1}^M \overline{\partial}|s_{j,x}|^{2\lambda_j}\Big)\wedge\omega \\ &= \frac{1}{(2i\pi)^M}\left(\bigwedge_{j=M}^{1}\overline{\partial}\Big(\frac{|s_{j,x}|^{2\lambda_j}}{s_{j,x}^{q_{j,x}}}\Big)\right)\wedge\psi_x\,.\end{aligned}$$

The reverse order of indices expresses here the absorption of the factor $\frac{(-1)^{M(M-1)/2}}{(2i\pi)^M}$. It is known indeed from [27] that the current valued map

$$(\lambda_1,...,\lambda_M) \longmapsto R^{s_x,\lambda_1,...,\lambda_M}[\omega]$$

can be continued analytically as a function of M complex variables $(\lambda_1,...,\lambda_M)$ to $\{\mathrm{Re}\,\lambda_1 > -\eta,...,\mathrm{Re}\,\lambda_M > -\eta\}$ for some $\eta > 0$. The proof of such result relies deeply on the use of a log resolution $\widetilde{\mathcal{X}} \xrightarrow{\pi} \mathcal{X}$ such that $\pi^{-1}[\bigcup_j S_j]$ is a hypersurface with normal crossings. The approach we developed above for construction of $\overline{\partial}$-closed (p,M)-current in U_x through the iterated analytic continuation process

$$R^{s_x}[\omega] = \Big[\Big[\cdots\Big[\Big[R_x^{s_x,\lambda_1,\dots,\lambda_M}[\omega]\Big]_{\lambda_1=0}\Big]_{\lambda_2=0}\cdots\Big]_{\lambda_{M-1}=0}\Big]_{\lambda_M=0},$$

is applied at this point, taking successively λ_1 up to $\{\mathrm{Re}\,\lambda_1 > -\eta_1\}$, then λ_2 up to $\{\mathrm{Re}\,\lambda_2 > -\eta_2\}$, and so on. Note, (again) that the argument does not seem (apparently) to require the use of an appropriate log resolution to resolve singularities (namely here that of the hypersurface defined as the zero set of $hf_1\cdots f_{m-1}$), but this is indeed hidden behind the fact that there exist local Bernstein-Sato equations. This current $R^{s_x}[\omega]$ is also denoted as

$$R^{s_x}[\omega] = \left(\bigwedge_{j=1}^{M}\overline{\partial}\Big(\frac{1}{s_{j,x}^{q_{j,x}}}\Big)\right)\wedge\psi_x\,.$$

To show that all $R^{s_x}[\omega]$, for the different U_x, globalize into a $\overline{\partial}$-closed, (p,M)-current over $\mathcal{X}$, we use the holonomy of the currents under consideration. That is, for any holomorphic functions u,h, in U_x, with u non-vanishing, the current valued function

$$\lambda\in\{\mathrm{Re}\,\lambda >> 1\}\longrightarrow \frac{|uh|^{\lambda}}{h}R^{s_x}[\omega]$$

can be continued analytically into a half-plane $\{\mathrm{Re}\,\lambda > -\eta\}$, whose value at $\lambda=0$ is independent of u. The global $\overline{\partial}$-closed (p,M)-current thus obtained is denoted as $R_{[S_1]_{\rm red},\dots,[S_M]_{\rm red}}[\omega]$. This reflects the fact that it depends only on the meromorphic form ω and on the reduced cycles corresponding to the closed hypersurfaces $S_1,\dots,S_M$ (with respect to this ordering). In a neighborhood $U_{x,y}$ of some $y\in W\cap U_x$, as introduced before, the ${}'\mathcal{D}^{(p,M)}(U_{x,y},\mathbb{C})$-current-valued map

$$\begin{aligned}(\lambda_1,\dots,\lambda_M)&\in\{\mathrm{Re}\,\lambda_j>1\,;\,j=1,\dots,M\}\\ &\longmapsto \frac{1}{(2i\pi)^M}\Big(\bigwedge_{j=M}^{1}\overline{\partial}|s_{j,x}|^{2\lambda_j}\Big)\wedge\Big(\bigwedge_{j=1}^{M}\frac{ds_{j,x}}{s_{j,x}}\Big)\wedge r_{x,y}[\omega]\end{aligned}$$

can be continued as a holomorphic map to $\{\mathrm{Re}\,\lambda_j > -1\,;\,j=1,\dots,M\}$, with value at $\lambda_1=\cdots=\lambda_M=0$ the (p,M)-current

$$\varphi\in\mathcal{D}^{(n-p,n-M)}(U_{x,y},\mathbb{C})\mapsto\int_{W\cap U_x}\mathrm{Res}_{\Sigma_1,\dots,\Sigma_M}[\omega]\wedge\varphi\,.$$

Note that one has in such neighborhood $U_{x,y}$, for $\mathrm{Re}\,\lambda_j >> 1$, $j=1,\dots,M$,

$$\Big(\bigwedge_{j=M}^{1}\overline{\partial}\Big(\frac{|s_{j,x}|^{2\lambda_j}}{s_{j,x}^{q_{j,x}}}\Big)\Big)\wedge\psi_x=\Big(\bigwedge_{j=M}^{1}\overline{\partial}|s_{j,x}|^{2\lambda_j}\Big)\wedge\Big(\bigwedge_{j=M}^{1}\frac{ds_{j,x}}{s_{j,x}}\Big)\wedge r_{x,y}[\omega]$$
$$+\Big(\bigwedge_{j=M}^{1}\overline{\partial}|s_{j,x}|^{2\lambda_j}\Big)\wedge\sigma_{x,y}[\omega]\,.$$

Thus, one obtains, for any $\varphi\in\mathcal{D}^{n-p,n-m}(U_{x,y},\mathbb{C})$,

$$\Big\langle R_{[S_1]_{\rm red},...,[S_M]_{\rm red}}[\omega]\,,\,\varphi\Big\rangle=\int_{W\cap U_x}\mathrm{Res}_{\Sigma_1,...,\Sigma_M}[\omega]\wedge\varphi.$$

This comes from the fact that the ${}'\mathcal{D}^{(p,M)}(U_{x,y},\mathbb{C})$-current-valued map

$$(\lambda_1,...,\lambda_M)\longmapsto\Big(\bigwedge_{j=M}^{1}\overline{\partial}|s_{j,x}|^{2\lambda_j}\Big)\wedge\sigma_{x,y}[\omega]$$

is holomorphic in $\{\mathrm{Re}\,\lambda_j>-1\,;\,j=1,...,M\}$ and takes the value 0 at $\lambda_1=\cdots=\lambda_M=0$. Finally, using the covering of V by the U_x, $x\in V$, one concludes that the (p,M)-current defined in $U=\mathcal{X}\setminus V_{\rm sing}$ as

$$\varphi\in\mathcal{D}^{(n-p,n-M)}(U,\mathbb{C})\mapsto\int_W\mathrm{Res}_{\Sigma_1,...,\Sigma_M}[\omega]\wedge\varphi$$

can be continued as the $\overline{\partial}$-closed current $T=R_{[S_1]_{\rm red},...,[S_M]_{\rm red}}[\omega]$ over the whole manifold $\mathcal{X}$. Note that the support of T satisfies $\mathrm{supp}\,T\subset V$.

2 Regular holonomy of integration currents

Let $\mathcal{X}$ be a n-dimensional complex manifold and $V\subset\mathcal{X}$ be a closed, purely dimensional, reduced, analytic subset of codimension M. El Mir's extension theorem, [17], implies that the integration current $[V]$ is defined as the unique, positive, d-closed, (M,M)- current over $\mathcal{X}$ such that, for any test function $\varphi\in\mathcal{D}^{(n-M,n-M)}(\mathcal{X}\setminus V_{\rm sing},\mathbb{C})$,

$$\langle[V]\,,\,\varphi\rangle=\int_V\varphi=\int_{V_{\rm reg}}\varphi.$$

It is important to point out here that the closed analytic set V is considered as being embedded in the ambient manifold $\mathcal{X}$. This will be revealed to us to be important for two reasons : firstly with respect to connections between intersection and divisions problems in $\mathcal{X}$ (that one intends to study jointly), closed analytic subsets in $\mathcal{X}$ need to be understood (and studied) in terms of their defining equations. Secondly, the Coleff-Herrera sheafs of currents $\mathrm{CH}_{\mathcal{X},V}$ and $\mathrm{CH}_{\mathcal{X},V}(\cdot\,;\star S)$ that we will introduce in the two following sections are indeed sheafs of currents in $\mathcal{X}$, with support on V, which depend in a crucial way on the embedding $\iota\ : V \to \mathcal{X}$. Therefore, instead of working on the complex analytic space $(V, (\mathcal{O}_{\mathcal{X}})_{|V})$, using for example a log resolution $\widetilde{V} \xrightarrow{\pi} V$ for some closed hypersurface $H_{\rm sing}$ on V, satisfying $V_{\rm sing} \subset H_{\rm sing}$, we will work in the ambient manifold $\mathcal{X}$ and keep as far as possible to methods based on the use of Bernstein-Sato type functional equations [20, 26, 19]. We will use extensively in this section the methods introduced to prove Proposition 1. These methods allow the possibility to define (in a robust way) the exterior multiplication of the integration current $[V]$ with a semi-meromorphic form ω whose polar set intersects V along a closed analytic subset W satisfying $\dim W < \dim V$. Recall here that $'\mathcal{D}^{(p,q)}(\mathcal{X},\mathbb{C})$ denotes the space of (p,q)-currents on $\mathcal{X}$, acting on the space $\mathcal{D}^{(n-p,n-q)}(\mathcal{X},\mathbb{C})$ of smooth $(n-p,n-q)$-test forms on $\mathcal{X}$.

Proposition 2 (an holonomicity property). *Let $\mathcal{X}$ and $V \subset \mathcal{X}$ be as above. Let $h, u \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$. The $'\mathcal{D}^{(M,M)}(\mathcal{X})$-valued map*

$$(\lambda,\mu) \in \{\mathrm{Re}\,\lambda >> 1, \mathrm{Re}\,\mu >> 1\} \longmapsto |u|^{2\mu}\,\frac{|h|^{2\lambda}}{h}\,[V]$$

can be continued analytically as a holomorphic map to the product of half-planes $\{\mathrm{Re}\,\lambda > -\eta, \mathrm{Re}\,\mu > -\eta\}$ for some $\eta > 0$. Moreover, if $\overline{V \setminus \{u=0\}} = V$, then the value of this analytic continuation at $\lambda = \mu = 0$ remains unchanged if one replaces $|u|$ by 1. When $\overline{V \setminus \{h=0\}} = V$, the construction of the principal value current

$$\frac{1}{h}\,[V] := \Big[\frac{|h|^{2\lambda}}{h}\,[V]\Big]_{\lambda=0}, \tag{5}$$

is "robust" in the following sense:

$$\frac{1}{h}\,[V] = \Big[|u|^{2\mu}\,\frac{|h|^{2\lambda}}{h}\,[V]\Big]_{\lambda=\mu=0} = \Big[|u|^{2\mu}\,\frac{1}{h}\,[V]\Big]_{\mu=0} \tag{6}$$

for any holomorphic function $u \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ such that $\overline{V \setminus \{u=0\}} = V$.

Proof. The second assertion in the statement of the proposition is a consequence of the first. If $\overline{V \setminus \{u=0\}} = V$, i.e. $|u|$ does not vanish identically on any component of V (hence $[|u|^{2\mu}]_{\mu=0} \equiv 1$ almost everywhere on such component), one has

$$\Big[|u|^{2\mu}\,\frac{|h|^{2\lambda}}{h}\,[V]\Big]_{\mu=0} = \frac{|h|^{2\lambda}}{h}\,[V]$$

for $\mathrm{Re}\,\lambda >> 1$. Assume the first assertion, namely that the current-valued function (5) is holomorphic in two variables in a product of half-spaces

$\{\mathrm{Re}\,\lambda > -\eta, \mathrm{Re}\,\mu > -\eta\}$ for some $\eta > 0$. Then, following the analytic continuation in λ up to $\lambda = 0$, one gets:

$$\left[\Big[|u|^{2\mu}\frac{|h|^{2\lambda}}{h}\,[V]\Big]_{\mu=0}\right]_{\lambda=0} = \Big[|u|^{2\mu}\frac{|h|^{2\lambda}}{h}\,[V]\Big]_{\lambda=\mu=0} = \Big[\frac{|h|^{2\lambda}}{h}\,[V]\Big]_{\lambda=0}.$$

This proves the second assertion (under the assumption that the first one holds).

In order now to prove the first assertion above, let us reduce the situation to the local one, that is, when $\mathcal{X}$ is a neighborhood Ω of the origin in $\mathbb{C}^n$. One can assume that V (defined in Ω as the common zero set of holomorphic functions $v_1, ..., v_k$ in $H(\Omega)$) is the union of a finite number of irreducible components of the complete intersection $\widetilde{V} = \{f_1 = \cdots = f_M = 0\}$, with $df_1 \wedge \cdots \wedge df_M \not\equiv 0$ on each such component ([18], p. 72). Let v be a linear combination of $v_1, ..., v_k$ which does not vanish identically on any of the irreducible components of the complete intersection $\widetilde{V}$, which are not irreducible components of V. We introduce from now on the notation $\widetilde{V}^{\mathcal{X}\setminus V}$ to denote the union of the irreducible components of $\widetilde{V}$ which are not entirely contained in V. Let u_{sing} be an holomorphic function in $\mathcal{X}$ such that $\widetilde{V}_{\mathrm{sing}} \subset \{u_{\mathrm{sing}} = 0\}$ and $u_{\mathrm{sing}} \not\equiv 0$ on any irreducible component of $\widetilde{V}$. Let us introduce the differential $(M, 0)$ form

$$\omega = \frac{df_1 \wedge \cdots \wedge df_M}{f_1 \cdots f_M},$$

and the $\overline{\partial}$-closed (M, M) current

$$\frac{T^f_{\underline{1},M}}{(2i\pi)^M} \wedge df_1 \wedge \cdots \wedge df_M = \mathrm{Res}_{[f_1=0]_{\mathrm{red}},\dots,[f_M=0]_{\mathrm{red}}}[\omega],$$

where the current $T^f_{\underline{1},M}$ is defined by the iterated process

$$T^f_{\underline{1},M} = \Big[\overline{\partial}\Big(\frac{|f_M|^{2\lambda_M}}{f_M}\Big) \wedge \Big[\cdots \wedge \Big[\overline{\partial}\Big(\frac{|f_1|^{2\lambda_1}}{f_1}\Big)\Big]_{\lambda_1=0}\cdots\Big]_{\lambda_{M-1}=0}\Big]_{\lambda_M=0}$$

considered in the proof of Proposition 1 where also the notation $\mathrm{Res}_{[\cdot]}[\omega]$ was introduced. Using Bernstein-Sato equation (2) (here for $M + 4$ functions), still in its conjugate form, one can prove that the current valued function

$$(\lambda, \mu, \nu, \varpi) \mapsto |u_{\mathrm{sing}}|^{2\varpi}\,|v|^{2\nu}\,|u|^{2\mu}\,\frac{|h|^{2\lambda}}{h}\,\mathrm{Res}_{[f_1=0]_{\mathrm{red}},\dots,[f_M=0]_{\mathrm{red}}}[\omega]$$

can be continued from

$$\{(\lambda, \mu, \nu, \varpi)\,;\,\mathrm{Re}\,\lambda >> 1,\,\mathrm{Re}\,\mu >> 1,\,\mathrm{Re}\,\nu >> 1,\,\mathrm{Re}\,\varpi >> 1\}$$

to a product a half-planes

$$\{(\lambda, \mu, \nu, \varpi)\,;\, \mathrm{Re}\,\lambda > -\eta,\, \mathrm{Re}\,\mu > -\eta,\, \mathrm{Re}\,\nu > -\eta,\, \mathrm{Re}\,\varpi > -\eta\}$$

for some $\eta > 0$. Moreover, when $\mathrm{Re}\,\lambda >> 1$, $\mathrm{Re}\,\mu >> 1$, $\mathrm{Re}\,\varpi >> 1$, the value at $\nu = 0$ of

$$\nu \longmapsto |u_{\rm sing}|^{2\varpi}\,(1-|v|^{2\nu})\,|u|^{2\mu}\,\frac{|h|^{2\lambda}}{h}\,\mathrm{Res}_{[f_1=0]_{\rm red},\dots,[f_M=0]_{\rm red}}[\omega]$$

is equal to the current

$$|u_{\rm sing}|^{2\varpi}\,|u|^{2\mu}\,\frac{|h|^{2\lambda}}{h}\,[V]\,.$$

Keeping $\mathrm{Re}\,\lambda >> 1$ and $\mathrm{Re}\,\mu >> 1$ and taking the analytic continuation in ϖ up to $\varpi = 0$, we get precisely the current

$$|u|^{2\mu}\,\frac{|h|^{2\lambda}}{h}\,[V]\,.$$

3 "Holomorphic" Coleff-Herrera sheaves of currents

Given an n-dimensional analytic manifold $\mathcal{X}$, together with a closed, purely dimensional reduced analytic subset V (of codimension M), the ("holomorphic") Coleff-Herrera sheaf $\mathrm{CH}_{\mathcal{X},V}(\cdot,E)$ of E-valued $(0,M)$-currents, where $E \to \mathcal{X}$ denotes an holomorphic bundle of finite rank over $\mathcal{X}$, plays a major role in division or duality problems. The local description of its sections, together with the subsequent properties, suggest how one can profit from the $2n$ local parameters $\zeta_1,\dots,\zeta_n,\overline{\zeta}_1,\dots,\overline{\zeta}_n$ instead of just the n "holomorphic" ones $\zeta_1,\dots,\zeta_n$. Thinking heuristically, the antiholomorphic local coordinates $\overline{\zeta}_1,\dots,\overline{\zeta}_n$ remain unaffected by the holomorphic differentiations involved in the action of such currents. For example, if $\Delta_1,\dots,\Delta_M$ are Cartier divisors on $\mathcal{X}$ and $s_1,\dots,s_M$ denote corresponding holomorphic sections of the Δ_j's such that the hypersurfaces $s_j^{-1}(0)$ intersect properly (that is, define a non empty complete intersection on $\mathcal{X}$), then the usual Coleff-Herrera residue $\bigwedge_{j=1}^M \overline{\partial}(1/s_j)$ stands as a global section of the Coleff-Herrera sheaf $\mathrm{CH}_{\mathcal{X},V}(\cdot,E)$, where $E = \bigwedge_1^M \mathcal{O}_{\mathcal{X}}(-\Delta_j)$.

The concept and its importance were pointed out by J. E. Björk in [11, 12]. The original construction of global sections for such sheaves is due to N. Coleff and M. Herrera in [14]. In this section, we will recall the definition of the sheaf $\mathrm{CH}_{\mathcal{X},V}(\cdot,E)$ (following the approach of J. E. Björk, M. Andersson, H. Samuelsson [11, 12, 2, 13]), together with the local structure of its sections (which justifies their operational properties). Since our objective all along this presentation is to stick to the methods based on analytic continuation (which seems to be a natural way to introduce the objects algebraically, for example by using Bernstein-Sato functional equations as (2)), the approach we adopt here follows that developped by M. Andersson in [2].

Definition 2 (the Coleff-Herrera sheaf $\mathrm{CH}_{\mathcal{X},V}(\cdot, E)$). Let $\mathcal{X}$, V, E be as above. The ("holomorphic") *Coleff-Herrera sheaf* $\mathrm{CH}_{\mathcal{X},V}(\cdot, E)$ is the sheaf of sections of $(0, M)$ E-valued currents T on $\mathcal{X}$, with support on V, which satisfy the three following conditions :

1. For any holomorphic function u in a neighborhood of V, satisfying
$$\overline{V \setminus \{u = 0\}} = V,$$
the current-valued function
$$\lambda \in \{\mathrm{Re}\,\lambda >> 1\} \longmapsto |u|^{2\lambda}\,T$$
can be analytically continued as an holomorphic map to $\{\mathrm{Re}\,\lambda > -\eta\}$ for some $\eta > 0$, and
$$[|u|^{2\lambda}\,T]_{\lambda=0} = T$$
(that is, T satisfies the Standard Extension Property (S.E.P) with respect to its support V).
2. One has, in the sense of currents,
$$(\mathcal{I}_V)_{\mathrm{conj}}\,T \equiv 0\,,$$
where $(\mathcal{I}_V)_{\mathrm{conj}}$ denotes the complex conjugate of the ideal sheaf of sections of $\mathcal{O}_{\mathcal{X}}$ that vanish on V.
3. The current T is $\overline{\partial}$-closed.

Global sections of this sheaf, that is elements in $\mathrm{CH}_{\mathcal{X},V}(\mathcal{X}, E)$, are called E-valued *Coleff-Herrera currents* (with respect to V) on $\mathcal{X}$.

Action of adjoints of "simple" holomorphic differential operators with values in the dual bundle E^* on integration currents provides us with an example of Coleff-Herrera sheaf of currents. To be more specific :

Example 1. Let D be a Cartier divisor in $\mathcal{X}$ and U be an open subset in $\mathcal{X}$. An holomorphic differential operator with analytic coefficients Q_U : $C^\infty_{n,n-M}(U, E^*) \to C^\infty_{n-M,n-M}(U, \mathcal{O}_{\mathcal{X}}(D))$ is said to be $(n, n-M)$-*simple* in U if its splits as
$$Q_U[\varphi] = q_U[\varphi] \wedge \omega_U\,,$$
where q_U denotes an holomorphic differential operator from $C^\infty_{n,n-M}(U, E^*)$ to $C^\infty_{0,n-M}(U, E^*)$ and ω_U is an element of $\Omega^{n-M}_{\mathcal{X}}(U, E \otimes \mathcal{O}_{\mathcal{X}}(D))$, that is a global section over U of the sheaf of $E \otimes \mathcal{O}_{\mathcal{X}}(D)$-valued $(n-M)$-holomorphic forms. Let us denote as $\mathfrak{D}^{n,n-M}_{\mathcal{X}}(\cdot, E^*, D)$ the sheaf whose sections over $U \subset \mathcal{X}$ are $(n, n-M)$-simple holomorphic differential operators with analytic coefficients from $C^\infty_{n,n-M}(U, E^*)$ into $C^\infty_{n-M,n-M}(U, \mathcal{O}_{\mathcal{X}}(D))$.

If $Q_U \in \mathfrak{D}_{\mathcal{X}}^{n,n-M}(U, E^*, D)$, let Q_U^* be the adjoint operator which transforms elements from $'\mathcal{D}^{(M,M)}(U, \mathcal{O}_{\mathcal{X}}(-D))$ into elements in $'\mathcal{D}^{(0,M)}(U, E)$ as follows :

$$\Big\langle Q_U^*[T], \varphi \Big\rangle = \Big\langle T, Q_U[\varphi] \Big\rangle\,,\ \forall\, \varphi \in \mathcal{D}^{(n,n-M)}(U)\,.$$

If h_U denotes an holomorphic section of D in U such that $\overline{(V \cap U) \setminus \{h_U^{-1}(0)\}} = V \cap U$ and $Q_U \in \mathfrak{D}_{\mathcal{X}}^{n,n-M}(U, E^*)$, then the current

$$T_U = Q_U^* \Big[\frac{[V \cap U]}{h_U}\Big]$$

(where $[V \cap U]/h_U$ is defined as in (6), see Proposition 2) fulfills conditions 1 and 2 in Definition 2. This follows from the fact that the current valued function

$$\mu \in \{\mathrm{Re}\, \mu >> 1\} \longmapsto |u|^{2\mu} Q^* \Big[\frac{1}{h}\, [V]_{\mathrm{red}}\Big]\,,$$

is analytically continued to $\mathrm{Re}\, \mu > -\eta$ and that its value at $\mu = 0$ does not depend on u as soon as $\overline{V \setminus \{u = 0\}} = V$. This shows that $Q^*[\frac{1}{h}\, [V]_{\mathrm{red}}]$ satisfies both the holonomy property and the standard extension property with respect to V, exactly as $\frac{1}{h}\, [V]_{\mathrm{red}}$ does. If it is additionally $\overline{\partial}$-closed (which unfortunately cannot be read directly on the operator with meromorphic coefficients Q_U/h_U), then T_U fulfills also condition 3 in Definition 2 and therefore is a global section of the Coleff-Herrera sheaf $\mathrm{CH}_{\mathcal{X},V}(\cdot, E)$ over U.

Let U be an open subset of $\mathcal{X}$. The local structure result established in ([2, 11, 12, 13]) can be stated as follows : when $T \in {}'\mathcal{D}^{(0,M)}(U, E)$ is a $\overline{\partial}$-closed current, $T \in \mathrm{CH}_{\mathcal{X},V}(U, E)$ if and only if, for any $x \in U$, there exists a neighborhood $U_x \subset U$ of x in U, a section $Q_x \in \mathfrak{D}^{n,n-M}(U_x, E^*, \mathbb{C})$ and an holomorphic function h_x in U_x such that $\overline{V \cap U_x} \setminus \{h_x = 0\} = V \cap U_x$ and

$$T_{|U_x} = Q_x^* \Big[\frac{[V \cap U_x]}{h_x}\Big].$$

The local structure result, besides the fact that it provides a useful local representation of sections of the Coleff-Herrera sheaf $\mathrm{CH}_{\mathcal{X},V}(\mathcal{X}, E)$, also emphasizes that only holomorphic differential operators are involved in the action of such currents (which explains indeed why they do play a role of algebraic nature despite their analytic structure).

It is important also to point out that, when $\mathcal{X} = \mathbb{P}^n(\mathbb{C})$, such a local structure result reflects (thanks to the GAGA principle) into a global structure result in this algebraic setting. The matrix of differential operators $Q_{\mathcal{X},IJ,K}$ involved in the definition of $Q_{\mathcal{X}}$, when expressed in local coordinates $(\zeta_1, ..., \zeta_n)$ in some affine chart, as

$$Q_{\mathcal{X}}\Big[\Big(\sum_{\substack{J\subset\{1,\dots,n\}\\ |J|=n-M}} \varphi_I\, d\bar{\zeta}_I\Big)\wedge d\zeta_1\wedge\cdots\wedge d\zeta_n\Big] =$$

$$= \sum_{\substack{I,J\subset\{1,\dots,n\}\\ |I|=|J|=n-M}} \Big(\sum_{\substack{K\subset\{1,\dots,n\}\\ |K|=n-M}} Q_{\mathcal{X},IJ,K}(\zeta,\frac{\partial}{\partial\zeta})[\varphi_K]\Big)\, d\bar{\zeta}_J\wedge d\zeta_I\,,$$

becomes a matrix of differential operators with polynomial coefficients, while the polar factor $h_{\mathcal{X}}$ corresponds here to a polynomial section of the bundle $\mathcal{O}_{\mathcal{X}}(k)$ for some $k\in\mathbb{N}$. Such differential operators with polynomial coefficients are of course reminiscent of the *Nœtherian operators* involved in the formulation of the *Ehenpreis-Palamodov fundamental principle* [16, 24, 10, 5]. For example, when $P_1,...,P_M$ are M homogeneous polynomials in $[z_0:\cdots:z_n]$ defining a complete intersection V in $\mathbb{P}^n(\mathbb{C})$, a global section of the Coleff-Herrera sheaf $\mathrm{CH}_{\mathcal{X},V}(\cdot,\bigwedge_1^M\mathcal{O}_{\mathcal{X}}(-\deg P_j))$ can be used to test the membership to the ideal $(P_1,...,P_M)$. Note also that local structure results of this type originally go back to the work of P. Dolbeault [15].

4 “Meromorphic” Coleff-Herrera sheaves of currents

Intersection and division problems (in the case of proper intersection) are intimitately connected through the Lelong-Poincaré equation : namely, if $\Delta_1,...,\Delta_M$ are M Cartier divisors on a complex manifold $\mathcal{X}$, together with respective metrics $|\ |_j$ and holomorphic sections s_j such that the $s_j^{-1}(0)$ intersect as a non empty complete intersection $s^{-1}(0)$, then the integration current $[\mathrm{div}\,(s_1)\bullet\cdots\bullet\mathrm{div}\,(s_M)]$ (the operation between cycles being here the intersection product in the proper intersection context) factorizes as

$$[\mathrm{div}(s_1)\bullet\cdots\bullet\mathrm{div}(s_M)] = \Big(\bigwedge_{j=1}^M\overline{\partial}(1/s_j)\Big)\wedge\mathfrak{d}_1 s_1\wedge\cdots\wedge\mathfrak{d}_M s_M$$

where $\bigwedge_{j=1}^M\overline{\partial}(1/s_j)\in\mathrm{CH}_{\mathcal{X},s^{-1}(0)}(\mathcal{X},\bigwedge_1^M\mathcal{O}_{\mathcal{X}}(-\Delta_j))$ is a Coleff-Herrera current independent of the choice of the metrics $|\ |_j$ and $\mathfrak{d}_j$ stands here for the Chern connection on $(\mathcal{O}_{\mathcal{X}}(\Delta_j),|\ |_j)$ (one could in fact replace $\mathfrak{d}_j$ by the de Rham operator d since the choice of the metrics is here irrelevant). Unfortunalely, when V denotes a $(n-M)$-purely dimensional, reduced, closed analytic set in $\mathcal{X}$, the integration current $[V]$ cannot usually be factorized (locally about a point $x\in V$) as the product of a section of the Coleff-Herrera sheaf $\mathrm{CH}_{\mathcal{X},V}(\cdot,\mathbb{C})$ with a local section of the sheaf $\Omega_{\mathcal{X}}^{n-M}$ of $(n-M)$-abelian forms. A sufficient condition for this to be true is that $\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_{V,x}$ is Cohen-Macaulay (see [3]). In general (see the proof of Proposition 2), in some convenient neighborhood U_x of x, there exists a factorization $[V\cap U_x]=T_{U_x}\wedge\omega_{U_x}$, where $\omega_{U_x}\in\Omega_{\mathcal{X}}^{n-M}(U_x)$ and T_{U_x} is a section in U_x of the meromorphic

Coleff-Herrera sheaf $\mathrm{CH}_{\mathcal{X},V}(\cdot\,;\star S_x,\mathbb{C})$ defined below (S_x being here a closed hypersurface in U_x such that $\overline{(V\cap U_x)\setminus S_x} = V\cap U_x$). This motivates we enlarge the concept of Coleff-Herrera sheaf, in order to tolerate holomorphic singularities (as we proceed when we enlarge the sheaf $\mathcal{O}_{\mathcal{X}}$ of holomorphic functions in $\mathcal{X}$ by introducing the sheaf $\mathcal{M}_{\mathcal{X}}$ of meromorphic functions on $\mathcal{X}$).

Let $\mathcal{X}$, V, E be as in the previous section. We now add in our list of data a closed hypersurface S in some neighborhhood of V (in $\mathcal{X}$) such that $\overline{V\setminus S} = V$. The hypersurface S will play the role of a precribed polar set for the sections of the sheaves we are about to define.

Definition 3 (The Coleff-Herrera sheaf $\mathrm{CH}_{\mathcal{X},V}(\cdot\,;\star S,E)$). Let $\mathcal{X}$, V, E be as in Definition 2 and S be as above. The ("meromorphic") Coleff-Herrera sheaf $\mathrm{CH}_{\mathcal{X},V}(\cdot\,;\star S,E)$ is the sheaf of sections of $(0,M)$ E-valued currents on $\mathcal{X}$ ($M=\mathrm{codim}_{\mathcal{X}}V$), with support on V, which satisfy, besides conditions 1 and 2 in Definition 2, the additional condition

$$\mathrm{Supp}\,(\overline{\partial}T)\subset V\cap S. \tag{7}$$

In order to exhibit sections of meromorphic Coleff-Herrera sheaves (see Exemple 2 below), the following lemma reveals to be essential. The method we use here to prove it illustrates both the power and the flexibility of the analytic continuation method. An alternative approach (based on the regularization of currents and the use of cut-off functions) was proposed in [13].

Lemma 1. *Let V be a purely $(n-M)$-dimensional closed analytic subset in a n-dimensional complex manifold $\mathcal{X}$. Let $u,h,s\in\mathcal{O}_{\mathcal{X}}(\mathcal{X})$, and satisfy*

$$\overline{V\setminus\{h=0\}} = \overline{V\setminus\{s=0\}} = \overline{V\setminus\{u=0\}} = V$$

Let $Q\in\mathfrak{D}^{n,n-M}_{\mathcal{X}}(\mathcal{X},\mathbb{C})$ (see Example 1). The $(0,M)$ current valued map

$$(\mu,\nu)\in\{\mathrm{Re}\,\mu>>1,\,\mathrm{Re}\,\nu>>1\}\longmapsto |u|^{2\nu}\frac{|s|^{2\mu}}{s}Q^*\Big[\frac{1}{h}\,[V]\Big]$$

extends as an holomorphic map to $\{\mathrm{Re}\,\mu>-\eta,\,\mathrm{Re}\,\nu>-\eta\}$ for some $\eta>0$, whose value $\mathcal{T}$ at $\mu=\nu=0$ is independent of u. The "robust" definition of $\mathcal{T}$ makes it natural to denote it as

$$\mathcal{T}=\frac{1}{s}\,Q^*[\frac{1}{h}\,[V]].$$

The current $\mathcal{T}$ fulfills conditions 1 and 2 in Definition 2.

Proof. Let $u\in\mathcal{O}_{\mathcal{X}}(\mathcal{X})$ and μ,ν such that $\mathrm{Re}\,\nu>>1$, $\mathrm{Re}\,\mu>>1$. Then

$$\begin{aligned}
&\Big\langle |u|^{2\nu}\frac{|s|^{2\mu}}{s}\,Q^*\Big[\frac{1}{h}\,[V]\Big]\,,\,\varphi\Big\rangle = \Big\langle \frac{1}{h}\,[V]\,,\,Q(\zeta,\frac{\partial}{\partial\zeta})\Big[|u|^{2\nu}\frac{|s|^{2\mu}}{s}\,\varphi\Big]\Big\rangle\\
&= \Big\langle \frac{|u|^{2\nu}\,|s|^{2\mu}}{h}\,[V]\,,\,Q[\frac{\varphi}{s}]\Big\rangle + \mu\Big\langle \frac{|u|^{2\nu}|s|^{2\mu}}{h}\,[V]\,,\,Q_{u,s}(\mu,\nu,\zeta,\frac{\partial}{\partial\zeta})[\varphi]\Big\rangle\\
&+\nu\Big\langle \frac{|u|^{2\nu}|s|^{2\mu}}{h}\,[V]\,,\,\widetilde{Q}_{u,s}(\mu,\nu,\zeta,\frac{\partial}{\partial\zeta})[\varphi]\Big\rangle\,,
\end{aligned}$$

where $Q_{u,s}(\mu,\nu,\zeta,\partial/\partial\zeta)$ and $\widetilde{Q}_{u,s}(\mu,\nu,\zeta,\partial/\partial\zeta)$ are the meromorphic differential operators (polynomial in μ,ν) from $C^\infty_{n,n-M}(\mathcal{X})$ into $C^\infty_{n-M,n-M}(\mathcal{X})$, with polar set contained in $\{us=0\}$. One can rewrite (for some convenient $K\in\mathbb{N}$, namely the order of the differential operator Q)

$$\begin{aligned}
&\Big\langle \frac{|u|^{2\nu}|s|^{2\mu}}{h}\,[V]\,,\,Q_{u,s}(\mu,\nu,\zeta,\frac{\partial}{\partial\zeta})[\varphi]\Big\rangle\\
&= \Big\langle \frac{|u|^{2\nu}|s|^{2\mu}}{hu^K s^{K+1}}\,[V]\,,\,A_{u,s}(\mu,\nu,\zeta,\frac{\partial}{\partial\zeta})[\varphi]\Big\rangle\\
&= \Big[\Big\langle \frac{|h|^{2\lambda}}{h}\,\frac{|u|^{2\nu}|s|^{2\mu}}{u^K s^{K+1}}\,[V]\,,\,A_{u,s}(\mu,\nu,\zeta,\frac{\partial}{\partial\zeta})[\varphi]\Big\rangle\Big]_{\lambda=0},
\end{aligned}$$

where $A_{u,s}$ denotes an holomorphic differential operator (polynomial in μ,ν) from $C^\infty_{n,n-M}(\mathcal{X})$ into $C^\infty_{n-M,n-M}(\mathcal{X})$. The same reasoning holds when one replaces $Q_{u,s}$ by $\widetilde{Q}_{u,s}$ with some holomorphic differential operator $\widetilde{A}_{u,s}$ instead of $A_{u,s}$. Note also that

$$Q[\frac{\varphi}{s}] = \frac{1}{s^M}A(\zeta,\frac{\partial}{\partial\zeta})[\varphi]\,,$$

where A is an holomorphic differential operator from the space $C^\infty_{n-M,n-M}$ into $C^\infty_{n-M,n-M}$. The second assertion follows from the fact that, when $\mathrm{Re}\,\mu >> 1$, the $(0,M)$-current

$$\frac{|sh|^{2\nu}}{s}\,Q^*\Big[\frac{1}{h}\,[V]\Big]$$

is annihilated locally by $(\mathcal{I}_V)_{\mathrm{conj}}$ (since Q is an holomorphic differential operator), which remains indeed true for the current

$$\frac{1}{s}\,Q^*\Big[\frac{1}{h}\,[V]\Big] = \Bigg[\frac{|sh|^{2\nu}}{s}\,Q^*\Big[\frac{1}{h}\,[V]\Big]\Bigg]_{\nu=0}.$$

This current fulfills conditions 1 and 2 in Definition 2.

Example 2. Lemma 1 allows to revisit Example 1, introducing possible poles. Let $\mathcal{X}$, V, E, S be as in Definition 3. Let additionally D, Δ be two Cartier divisors on $\mathcal{X}$. Let $U\subset\mathcal{X}$ and h_U, s_U be respectively holomorphic sections of D and Δ in U, such that $s_U^{-1}(0) \subset S$ and $\overline{(V \cap U) \setminus h_U^{-1}(0)} = V \cap U$. Let $Q \in \mathfrak{D}_{\mathcal{X}}^{n,n-M}(U, E^*, D)$. Then the $\mathcal{O}_{\mathcal{X}}(-\Delta) \otimes E$ valued current in U

$$\mathcal{T} = \frac{1}{s_U} Q_U^* \Big[\frac{[V \cap U]}{h_U}\Big]$$

belongs to $\mathrm{CH}_{\mathcal{X},V}(U\ ; \star S, \mathcal{O}_{\mathcal{X}}(-\Delta) \otimes E)$ as soon as Q_U and h_U are such that the current $Q_U^*[[V \cap U]/h_U]$ is $\overline{\partial}$-closed.

Example 2 above provides in fact, what appears locally to be the description of sections of Coleff-Herrera sheaves, since one has the following proposition (see [13]):

Proposition 3. *Let $\mathcal{X}$ be a n-dimensional complex manifold, V be a $(n-M)$-purely dimensional closed analytic subset, S be a closed hypersurface in $\mathcal{X}$ such that $\overline{V \setminus S} = V$. Any element $\mathcal{T}$ in $\mathrm{CH}_{\mathcal{X},V}(\mathcal{X}; \star S, \mathbb{C})$ can be locally realized in an open neighborhood U_x of $x \in V$ as $\mathfrak{T} = T_s/s_s$, where T_x is a current in $\mathrm{CH}_{\mathcal{X},V}(U_x, \mathbb{C})$, $s_x \in \mathcal{O}_{\mathcal{X}}(U_x)$ satisfying $s_x^{-1}(0) \cap U_x = S \cap U_x$. This means also that one has*

$$\mathcal{T} = \frac{1}{s_x} Q_x^* \Big[\frac{1}{h_x}[V]\Big] \tag{8}$$

with $Q_x \in \mathfrak{D}_{\mathcal{X}}^{n,n-M}(U_x, \mathbb{C}, \mathbb{C})$, $h_x \in \mathcal{O}_{\mathcal{X}}(U_x)$, satisfying $\overline{(V \cap U_x) \setminus h_x^{-1}(0)} = V \cap U_x$, the current $Q_x^*[[V \cap U_x]/h_x]$ being $\overline{\partial}$-closed in U_x. Conversely, any $(0, M)$-current $\mathcal{T}$ over $\mathcal{X}$ with support contained in V, that can be locally expressed about each point $x \in V$ (in the ambient manifold $\mathcal{X}$) as (8) and is $\overline{\partial}$-closed outside S, belongs to $\mathrm{CH}_{\mathcal{X},V}(\mathcal{X}; \star S, \mathbb{C})$.

Proof. The second assertion follows from Lemma 1 since conditions 1, 2 in Definition 2 and (7) in Definition 3 can be checked locally. If $\mathcal{T} \in \mathrm{CH}_{\mathcal{X},V}(\mathcal{X}; \star S, \mathbb{C})$ and $x \in V$, $\{\sigma_x = 0\}$ being a reduced equation for S in an open neighborhood U_x of x in $\mathcal{X}$, one has $\overline{\partial}(s_x \mathcal{T}) \equiv 0$ in U_x if $s_x = \sigma_x^\gamma$ as soon as $\gamma \in \mathbb{N}$ exceeds strictly the order of $\mathcal{T}$ in $\overline{U_x}$. Therefore $s_x \mathcal{T}_{|U_x} \in \mathrm{CH}_{\mathcal{X},V}(U_x, \mathbb{C})$ (conditions 1, 2 in Definition 2 remain fulfilled, condition (7) in Definition 3 is now realized). One can check immediately that

$$\frac{1}{s_x} \times (s_x \mathcal{T}_{|U_x}) = \mathcal{T}_{|U_x}$$

(the product on the left hand side being understood as in Lemma 1), which proves that $\mathcal{T}$ can be represented as (8) in U_x.

One can adapt the proof of Lemma 1 and Proposition 3 in order to get the following result.

Proposition 4. *Let $\mathcal{X}$, V, E be as in Definition 3. Let $T \in \mathrm{CH}_{\mathcal{X},V}(\mathcal{X}, E)$, Δ be a Cartier divisor on $\mathcal{X}$, equipped with a hermitian metric $|\ |$, and s be an holomorphic section of Δ. The $(0, M)$ current-valued map*

$$\mu \in \{\mathrm{Re}\,\mu >> 1\} \longmapsto \frac{|s|^{2\mu}}{s}\, T$$

extends as an holomorphic map to $\{\mathrm{Re}\,\mu > -\eta\}$ *for some* $\eta > 0$*. Moreover, one has that*

$$\Big[\frac{|s|^{2\mu}}{s}\, T\Big]_{\mu=0} \in \mathrm{CH}_{\mathcal{X}, V^{\mathcal{X}\setminus s^{-1}(0)}}(\mathcal{X}; \star s^{-1}(0), \mathcal{O}_{\mathcal{X}}(-\Delta)\otimes E),$$

the current being independent of the choice of the metric on Δ*. Recall that* $V^{\mathcal{X}\setminus s^{-1}(0)}$ *denotes the union of irreducible components of* V *which do not lie entirely in the closed hypersurface* $s^{-1}(0)$*.*

Proof. Since it is sufficient to prove this proposition locally, one can assume that $T = Q^*[[V]/h]$, where $Q \in \mathfrak{D}_{\mathcal{X}}(\mathcal{X}, \mathbb{C})$ and $h \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ is not identically zero on any irreducible component V_ι of V which does not lie entirely in $s^{-1}(0)$. For $\mathrm{Re}\,\mu >> 1$, one has, since $1/h\, 1_{\mathcal{X}\setminus s^{-1}(0)} \cdot [V] = 1/h\, [V^{\mathcal{X}\setminus s^{-1}(0)}]$, that

$$\frac{|s|^{2\mu}}{s}\, T = \frac{|s|^{2\mu}}{s}\, Q^*\Big[\frac{1}{h}\,[V]\Big] = \frac{|s|^{2\mu}}{s}\, Q^*\Big[\frac{1}{h}\,[V^{\mathcal{X}\setminus s^{-1}(0)}]\Big].$$

We now notice that s does not vanish identically on any irreducible component of $V^{\mathcal{X}\setminus s^{-1}(0)}$, which means $\overline{V^{\mathcal{X}\setminus s^{-1}(0)} \setminus s^{-1}(0)} = V^{\mathcal{X}\setminus s^{-1}(0)}$. Proposition 4 follows immediately from Lemma 1, combined with the second assertion in Proposition 3 (replacing V by $V^{\mathcal{X}\setminus s^{-1}(0)}$).

Meromorphic E valued Coleff-Herrera currents (with respect to V, and prescribed polar set on S such that $\overline{V \setminus S} = V$) induce via the $\overline{\partial}$ operator elements in $\mathrm{CH}_{\mathcal{X}, V\cap S}(\cdot, E)$. We present here an alternative proof (based on the analytic continuation) of a key result from [13].

Theorem 1. *The* $\overline{\partial}$*-operator maps* $\mathrm{CH}_{\mathcal{X},V}(\cdot\,; \star S, E)$ *into* $\mathrm{CH}_{\mathcal{X}, V\cap S}(\cdot, E)$*.*

Remark 1. Note that the morphism above is surjective (at the level of germs at $x \in V$) as soon as $\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_{V,x}$ is Cohen-Macaulay [12].

Proof. Since one can reduce the problem to the local situation where E is trivialized, we may assume from now on that E is the trivial bundle $\mathcal{X} \times \mathbb{C}$. Let $\mathcal{T} \in \mathrm{CH}_{\mathcal{X},V}(\mathcal{X}; \star S, \mathbb{C})$. The statement in Theorem 1 amounts to check conditions 1, 2, 3 in Definition 2 locally for the current $\overline{\partial}\mathcal{T}$ (with respect to $V \cap S$). Then one can assume (see Proposition 3) that $\mathcal{X} = U$, where $U = U_x$ is an open neighborhood of a point x in V, $\mathcal{T} = 1/s\, Q^*[[V]/h]$, with $h, s \in \mathcal{O}_{\mathcal{X}}(U)$ satisfying

$$\overline{(V \cap U) \setminus \{h = 0\}} = \overline{(V \cap U) \setminus \{s = 0\}} = V \cap U,$$

and $Q \in \mathfrak{D}^{n,n-M}_{\mathcal{X}}(U, \mathbb{C}, \mathbb{C}) = \mathfrak{D}^{n,n-M}_{\mathcal{X}}(U)$. It is clear that $\overline{\partial}\mathcal{T}$ satisfies condition 3 since $\overline{\partial}^2 = 0$. Since $\mathcal{T} = [|s|^{2\mu}/s\, Q^*[[V]/h]]_{\mu=0}$ (see Lemma 1) and $Q^*[[V]/h]$ is closed (as an element in $\mathrm{CH}_{\mathcal{X},V}(U, \mathbb{C})$), one has

$$\overline{\partial}\mathcal{T} = \left[\overline{\partial}\frac{|s|^{2\mu}}{s} \wedge Q^*\Big[\frac{1}{h}\,[V]\Big]\right]_{\mu=0} = \left[\mu|s|^{2\mu}\,\frac{1}{s}\,\frac{d\overline{s}}{\overline{s}} \wedge Q^*\Big[\frac{1}{h}\,[V]\Big]\right]_{\mu=0}.$$

In order to prove that conditions 1 and 2 in Definition 2 hold for $\overline{\partial}\mathcal{T}$, it is enough to show that, when $u \in \mathcal{O}_{\mathcal{X}}(U)$ does not vanish identically on any irreducible component of $V \cap S \cap U$ (that is $\{s = u = 0\} \cap V$ is defined as a complete intersection in $V \cap U$), the $(0, M+1)$-current-valued function

$$(\mu,\nu) \in \{\mathrm{Re}\,\mu >> 1,\, \mathrm{Re}\,\nu >> 1\} \longmapsto \mu|u|^{2\nu}\,|s|^{2\mu}\,\frac{1}{s}\,\frac{d\overline{s}}{\overline{s}} \wedge Q^*\Big[\frac{1}{h}\,[V]\Big]$$

extends as an holomorphic map to $\{\mathrm{Re}\,\mu > -\eta,\, \mathrm{Re}\,\nu > -\eta\}$ for some $\eta > 0$, whose value at $\mu = \nu = 0$ is independent of u and is annihilated (as a current) by $(\mathcal{I}_{V\cap S\cap U})_{\mathrm{conj}}$. Shrinking $U = U_x$ about x, if necessary, one can assume that there exists a holomorphic differential operator $\mathcal{Q} \in \mathfrak{D}^{n,n-M-1}_{\mathcal{X}}(U,\mathbb{C},\mathbb{C})$ such that, for $\mathrm{Re}\,\mu >> 1$, $\mathrm{Re}\,\nu >> 1$, for any $\varphi \in \mathcal{D}^{(n,n-M-1)}(U,\mathbb{C})$, the following identity holds:

$$Q\Big[|u|^{2\nu}\,|s|^{2\mu}\frac{1}{s}\,\frac{d\overline{s}}{\overline{s}} \wedge \varphi\Big] = \frac{d\overline{s}}{\overline{s}} \wedge \mathcal{Q}\Big[|u|^{2\nu}\,|s|^{2\mu}\frac{1}{s}\,\varphi\Big] \quad \text{on } V_{\mathrm{reg}}.$$

This comes from consideration of the facts that multiplication with anti-holomorphic functions commutes with the action of holomorphic differential operators, and that Q splits as $Q[\varphi] = q[\varphi]\wedge\omega$, where q preserves the maximal degree of differential forms (on V_{reg}) in $d\overline{\zeta}$, and $\omega \in \Omega^{n-M}_{\mathcal{X}}(U,\mathbb{C})$. Let K be the order of $\mathcal{Q}$. There exist holomorphic differential operators $\mathcal{A}_s$, $\mathcal{A}_{s,u}$, $\widetilde{\mathcal{A}}_{s,u}$ in $\mathfrak{D}^{n,n-M-1}_{\mathcal{X}}(U,\mathbb{C},\mathbb{C})$ (the two last ones depending also polynomially on μ and ν), such that, for any $\mathrm{Re}\,\mu >> 1$, $\mathrm{Re}\,\nu >> 1$, for any $\varphi \in \mathcal{D}^{(n,n-M-1)}(U,\mathbb{C})$,

$$\begin{aligned}\mathcal{Q}\Big[|u|^{2\nu}\,|s|^{2\mu}\frac{1}{s}\,\varphi\Big] &= \frac{|u|^{2\nu}\,|s|^{2\mu}}{s^{K+1}}\,\mathcal{A}_s(\zeta,\frac{\partial}{\partial\zeta})[\varphi] \\ &+ \frac{|u|^{2\nu}\,|s|^{2\mu}}{u^K\,s^{K+1}}\Big(\mu\,\mathcal{A}_{s,u}(\mu,\nu,\zeta,\frac{\partial}{\partial\zeta}) + \nu\,\widetilde{\mathcal{A}}_{s,u}(\mu,\nu,\zeta,\frac{\partial}{\partial\zeta})\Big)[\varphi]\,.\end{aligned}$$

Consider the $(0, M+1)$-valued maps

$$\begin{aligned}(\mu,\nu) \in \{\mathrm{Re}\,\mu >> 1\,,\mathrm{Re}\,\nu >> 1\} &\longmapsto \left[\mu\frac{|u|^{2\nu}\,|s|^{2\mu}\,|h|^{2\lambda}}{s^{K+1}h}\,\mathcal{B}_s\right]_{\lambda=0} \\ (\mu,\nu) \in \{\mathrm{Re}\,\mu >> 1,\,\mathrm{Re}\,\nu >> 1\} &\longmapsto \left[\mu\frac{|u|^{2\nu}\,|s|^{2\mu}\,|h|^{2\lambda}}{u^K\,s^{K+1}h}\,\mathcal{B}_{s,u}(\mu,\nu)\right]_{\lambda=0},\end{aligned} \tag{9}$$

where

$$\langle \mathcal{B}_s, \varphi\rangle = \langle [V]_{\rm red}, \frac{d\overline{s}}{\overline{s}} \wedge \mathcal{A}_s[\varphi]\rangle$$
$$\langle \mathcal{B}_{s,u}(\mu,\nu), \varphi\rangle = \Big\langle [V]_{\rm red}\,,\, \frac{d\overline{s}}{\overline{s}} \wedge \Big(\mu\, \mathcal{A}_{s,u}(\mu,\nu,\zeta,\frac{\partial}{\partial\zeta}) + \nu\, \widetilde{\mathcal{A}}_{s,u}(\mu,\nu,\zeta,\frac{\partial}{\partial\zeta})\Big)[\varphi]\Big\rangle$$

for all $\varphi \in \mathcal{D}^{(n,n-M-1)}(U,\mathbb{C})$. We claim that both current-valued maps (9) extend as holomorphic maps to $\{{\rm Re}\,\mu > -\eta,\, {\rm Re}\,\nu > -\eta\}$ for some $\eta > 0$. Moreover, the value at $\mu = \nu = 0$ of the first of these maps is annihilated (as a current) by $(\mathcal{I}_{V\cap S\cap U})_{\rm conj}$, while the value at $\mu = \nu = 0$ of the second one equals 0.

Let us assume this claim for the moment and conclude the proof of the theorem. For ${\rm Re}\,\lambda >> 1$, ${\rm Re}\,\mu >> 1$, ${\rm Re}\,\nu >> 1$, one has

$$\begin{aligned} &\mu|u|^{2\nu}\,|s|^{2\mu}\,\frac{1}{s}\,\frac{d\overline{s}}{\overline{s}} \wedge Q^*\Big[\frac{|h|^{2\lambda}}{h}\,[V]\Big] = \\ &= \mu\frac{|u|^{2\nu}\,|s|^{2\mu}\,|h|^{2\lambda}}{s^{K+1}h}\,\mathcal{B}_s + \mu\frac{|u|^{2\nu}\,|s|^{2\mu}\,|h|^{2\lambda}}{u^K\,s^{K+1}h}\,\mathcal{B}_{s,u}(\mu,\nu)\,. \end{aligned} \tag{10}$$

Thus, the current-valued map (4), which can be rewritten because of (10) (for ${\rm Re}\,\lambda >> 1$, ${\rm Re}\,\mu >> 1$, ${\rm Re}\,\nu >> 1$) as

$$\Big[\mu\frac{|u|^{2\nu}\,|s|^{2\mu}\,|h|^{2\lambda}}{s^{K+1}h}\,\mathcal{B}_s + \mu\frac{|u|^{2\nu}\,|s|^{2\mu}\,|h|^{2\lambda}}{u^K\,s^{K+1}h}\,\mathcal{B}_{s,u}(\mu,\nu)\Big]_{\lambda=0},$$

extends as a holomorphic function of (μ,ν) to $\{{\rm Re}\,\mu > -\eta,\, {\rm Re}\,\nu > -\eta\}$ for some $\eta > 0$, the value at $\mu = \nu = 0$ being equal to

$$\Big[\Big[\mu\frac{|s|^{2\mu}\,|h|^{2\lambda}}{s^{M+1}h}\,\mathcal{B}_s\Big]_{\lambda=0}\Big]_{\mu=0},$$

which is independent of u and annihilated (as a current) by $(\mathcal{I}_{V\cap S\cap U})_{\rm conj}$. This proves that $\overline{\partial}\mathcal{T}$ fulfills conditions 1 and 2 in Definition 2.

Proving the claim clearly amounts to prove that for any positive integers σ,τ, the $(M,M+1)$ current-valued map

$$\{{\rm Re}\,\mu >> 1,\, {\rm Re}\,\mu >> 1\} \longmapsto \Big[\mu\,\frac{|u|^{2\nu}|s|^{2\mu}|h|^{2\lambda}}{u^\tau s^\sigma h}\,\frac{d\overline{s}}{\overline{s}} \wedge [V]\Big]_{\lambda=0}$$

extends as an holomorphic map to $\{{\rm Re}\,\lambda > -\eta,\, {\rm Re}\,\mu > -\eta\}$ for some $\eta > 0$, whose value at $\mu = \nu = 0$ is annihilated by $(\mathcal{I}_{V\cap S\cap U})_{\rm conj}$. In order to do that, we need to introduce a smooth log resolution $\mathcal{V} \xrightarrow{\pi} V$ for the closed hypersurface $W = V \cap \{\zeta\,;\, h(\zeta)s(\zeta)u(\zeta) = 0\} \subset V$. That is, $\mathcal{V}$ is an $(n-M)$-dimensional complex manifold, π is a proper surjective holomorphic map such that the closed analytic subset $\mathcal{W}$ (obtained as the union of $\pi^{-1}(W)$ with the set of points in $\mathcal{V}$ about which π is not a local isomorphism) is a closed hypersurface in $\mathcal{V}$ with normal crossings. Such a log resolution can be obtained applying Hironaka theorem. Let $\iota_V \ : V \to \mathcal{X}$ be the inclusion embedding. For any $\varphi \in \mathcal{D}^{(n-M,n-M-1)}(U,\mathbb{C})$, one can rewrite, using the properness of π and a (sufficiently refined) partition of unity (V_ι, ρ_ι) subordinated to the support of $(\iota_V \circ \pi)^*[\varphi]$,

$$\Big\langle \mu \, \frac{|u|^{2\nu}|s|^{2\mu}|h|^{2\lambda}}{u^\tau s^\sigma h} \, \frac{d\overline{s}}{\overline{s}} \wedge [V]_{\rm red} \, , \, \varphi \Big\rangle$$

as a sum of contributions of the form

$$\mu \int_{V_\iota} \frac{|u_\iota \xi_\iota^{\gamma_\iota}|^{2\nu} |s_\iota \xi_\iota^{\beta_\iota}|^{2\mu} |h_\iota \xi_\iota^{\alpha_\iota}|^{2\lambda}}{u_\iota^\tau s_\iota^\sigma h_\iota \xi_\iota^{\tau\gamma_\iota + \sigma\beta_\iota + \alpha_\iota}} \Big(\frac{d\overline{s}_\iota}{\overline{s}_\iota} + \sum_{j=1}^{n-M} \beta_{\iota,j} \frac{d\overline{\xi}_{\iota,j}}{\overline{\xi}_{\iota,j}} \Big) \wedge \rho_\iota(\xi_\iota) \, (\iota_V \circ \pi)^*[\varphi](\xi_\iota) \, , \tag{11}$$

where $\xi_\iota = (\xi_{\iota,1}, ..., \xi_{\iota,n-M})$ denote centered local coordinates in V_ι, $u_\iota, s_\iota, h_\iota$ are invertible functions in V_ι, and $\xi_\iota^{\gamma_\iota}, \xi_\iota^{\beta_\iota}, \xi_\iota^{\alpha_\iota}$ are monomial functions in the centered coordinates $(\xi_{\iota,1}, ..., \xi_{\iota,n-M})$ with respective multi exponents $\gamma_\iota, \beta_\iota, \alpha_\iota \in \mathbb{N}^{n-M}$. The function

$$(\lambda,\mu,\nu) \longmapsto \mu \int_{V_\iota} \frac{|u_\iota \xi_\iota^{\gamma_\iota}|^{2\nu} |s_\iota \xi_\iota^{\beta_\iota}|^{2\mu} |h_\iota \xi_\iota^{\alpha_\iota}|^{2\lambda}}{u_\iota^\tau s_\iota^\sigma h_\iota \xi_\iota^{\tau\gamma_\iota + \sigma\beta_\iota + \alpha_\iota}} \, \frac{d\overline{s}_\iota}{\overline{s}_\iota} \wedge \rho_\iota(\xi_\iota) \, (\iota_V \circ \pi)^*[\varphi](\xi_\iota) \tag{12}$$

clearly extends as an holomorphic function of (λ,μ,ν) to a product of half planes $\{{\rm Re}\, \lambda > -\eta, \, {\rm Re}\, \mu > -\eta, \, {\rm Re}\, \nu > -\eta\}$ for some $\eta > 0$, whose value at $\lambda = \mu = \nu = 0$ equals to 0. The reason is that the singularities under the integral in (12) are only *holomorphic* singularities. The same remains true if $\varphi = \overline{h}\psi$, where $h \in \mathcal{I}_{V\cap S\cap U}$, since in this case any $\xi_{\iota,j}$, $j = 1, ..., n-m$, such that $\beta_{\iota,j} \neq 0$ divides $\pi^* h$, which implies that all *antiholomorphic* singularities in the term under the integral in (11) are thus canceled. It remains to study the meromorphic analytic continuation (as a function of (λ,μ,ν)) of

$$(\lambda,\mu,\nu) \longmapsto \mu \int_{V_\iota} \frac{|u_\iota \xi_\iota^{\gamma_\iota}|^{2\nu} |s_\iota \xi_\iota^{\beta_\iota}|^{2\mu} |h_\iota \xi_\iota^{\alpha_\iota}|^{2\lambda}}{u_\iota^\tau s_\iota^\sigma h_\iota \xi_\iota^{\tau\gamma_\iota + \sigma\beta_\iota + \alpha_\iota}} \, \frac{d\overline{\xi}_{\iota,j}}{\overline{\xi}_{\iota,j}} \wedge \rho_\iota \, (\iota_V \circ \pi)^*[\varphi] \tag{13}$$

for $j \in \{1, ..., n-M\}$ such that $\beta_{\iota,j} > 0$. Using integration by parts, one can rewrite (13) (when ${\rm Re}\, \lambda >> 1$, ${\rm Re}\, \mu >> 1$, ${\rm Re}\, \nu >> 1$) as

$$\begin{aligned} &\frac{\mu}{\alpha_{\iota,j}\lambda + \beta_{\iota,j}\mu + \gamma_{\iota,j}\nu} \times \\ &\times \int_{V_\iota} \frac{|u_\iota \xi_\iota^{\gamma_\iota}|^{2\nu} |s_\iota \xi_\iota^{\beta_\iota}|^{2\mu} |h_\iota \xi_\iota^{\alpha_\iota}|^{2\lambda}}{u_\iota^\tau s_\iota^\sigma h_\iota \xi_\iota^{\tau\gamma_\iota + \sigma\beta_\iota + \alpha_\iota}} \, d\overline{\xi}_{\iota,j} \wedge \frac{\partial}{\partial \overline{\xi}_{\iota,j}} \Big(\rho_\iota \, (\iota_V \circ \pi)^*[\varphi] \Big) \, . \end{aligned}$$

We need here to distinguish two more cases.

- If $\gamma_{\iota,j} = 0$, the function

$$(\mu,\nu) \longmapsto \Big[\mu \int_{V_\iota} \frac{|u_\iota \xi_\iota^{\gamma_\iota}|^{2\nu} |s_\iota \xi_\iota^{\beta_\iota}|^{2\mu} |h_\iota \xi_\iota^{\alpha_\iota}|^{2\lambda}}{u_\iota^\tau s_\iota^\sigma h_\iota \xi_\iota^{\tau\gamma_\iota+\sigma\beta_\iota+\alpha_\iota}} \frac{d\bar{\xi}_{\iota,j}}{\bar{\xi}_{\iota,j}} \wedge \rho_\iota\,(\iota_V\circ\pi)^*[\varphi]\Big]_{\lambda=0}$$

$$= \Big[\frac{\mu}{\alpha_{\iota,j}\lambda + \beta_{\iota,j}\mu} \times$$

$$\times \int_{V_\iota} \frac{|u_\iota \xi_\iota^{\gamma_\iota}|^{2\nu} |s_\iota \xi_\iota^{\beta_\iota}|^{2\mu} |h_\iota \xi_\iota^{\alpha_\iota}|^{2\lambda}}{u_\iota^\tau s_\iota^\sigma h_\iota \xi_\iota^{\tau\gamma_\iota+\sigma\beta_\iota+\alpha_\iota}}\, d\bar{\xi}_{\iota,j} \wedge \frac{\partial}{\partial\bar{\xi}_{\iota,j}}\Big(\rho_\iota\,(\iota_V\circ\pi)^*[\varphi]\Big)\Big]_{\lambda=0}$$

$$= \frac{1}{\beta_{\iota,j}}\Big[\int_{V_\iota} \frac{|u_\iota \xi_\iota^{\gamma_\iota}|^{2\nu} |s_\iota \xi_\iota^{\beta_\iota}|^{2\mu} |h_\iota \xi_\iota^{\alpha_\iota}|^{2\lambda}}{u_\iota^\tau s_\iota^\sigma h_\iota \xi_\iota^{\tau\gamma_\iota+\sigma\beta_\iota+\alpha_\iota}}\, d\bar{\xi}_{\iota,j} \wedge \frac{\partial}{\partial\bar{\xi}_{\iota,j}}\Big(\rho_\iota\,(\iota_V\circ\pi)^*[\varphi]\Big)\Big]_{\lambda=0}$$

extends as an holomorphic function to $\{\mathrm{Re}\,\mu > -\eta,\ \mathrm{Re}\,\nu > -\eta\}$ for some $\eta > 0$.

- If $\gamma_{\iota,j} > 0$, one uses a primitive form of a *Whitney division lemma*, a clever trick introduced by H. Samuelsson in [27]. The hyperplane of coordinates $\{\xi_{\iota,j} = 0\} \cap V_\iota$ lies in the closed analytic set $\{(\iota_V\circ\pi)^*[s] = (\pi\circ\iota_V)^*[u] = 0\} \cap V_\iota$, whose image by π is included in the $(n-M-2)$- dimensional closed analytic subset of U defined as $\{u = 0\} \cap S \cap V \cap U$. Since any differential form $d\bar{\zeta}_I$, $|I| = n-M-1$, has a vanishing pullback to $S \cap \{u = 0\} \cap V \cap U$ for dimension reasons, the $(0, n-M-1)$-differential form $(\iota_V\circ\pi)^*[d\bar{\zeta}_I]$ has a vanishing pullback to $\{\xi_{\iota,j} = 0\} \cap V_\iota$, which means that $(\iota_V\circ\pi)^*[d\bar{\zeta}_I](\xi_\iota) = \bar{\xi}_{\iota,j}\,\overline{\omega}_I(\xi_\iota)$ for some $(0, n-M-1)$-smooth form $\overline{\omega}_I$ in V_ι. Then $\bar{\xi}_{\iota,j}$ divides $(\iota_V\circ\pi)^*[\varphi]$ in V_ι, which implies that antiholomorphic singularities under the integral in (13) are canceled. Therefore (13) extends as an holomorphic function of (λ,μ,ν) to a product of half planes $\{\mathrm{Re}\,\lambda > -\eta,\ \mathrm{Re}\,\mu > -\eta,\ \mathrm{Re}\,\nu > -\eta\}$ for some $\eta > 0$.

This completes the proof of the claim, and thus of the theorem.

Proposition 4, together with Proposition 1, implies the following : if $\mathcal{X}, V, E, \Delta, s$ are given as in Proposition 4, then, for all open subsets $U \subset \mathcal{X}$,

$$\bar{\partial}\Big(\Big[\frac{|s|^{2\mu}}{s}\,T\Big]_{\mu=0}\Big) \in \mathrm{CH}_{\mathcal{X}, V^{\mathcal{X}\setminus s^{-1}(0)}\cap s^{-1}(0)}(U, \mathcal{O}_{\mathcal{X}}(-\Delta)\otimes E) \tag{14}$$

whenever $T \in \mathrm{CH}_{\mathcal{X},V}(U,E)$. Note that $V^{\mathcal{X}\setminus s^{-1}(0)} \cap s^{-1}(0)$ is either purely $(n-M-1)$-dimensional or empty, in which last case (14) is somehow irrelevant since the current on the left hand side is 0. We will need in the next section the following result, which is by far more involved, that we formulate here without proof (see [9] for a detailed proof).

Proposition 5. *Let $\mathcal{X}, V, E, \Delta, s$ be as given in Proposition 4. Let S be a hypersurface in $\mathcal{X}$ such that $\overline{V \setminus S} = V$ and $\mathcal{T} \in \mathrm{CH}_{\mathcal{X},V}(\mathcal{X}\ ;\star S, E)$. The $(0, M+1)$ current-valued map*

$$\nu \in \{\mathrm{Re}\,\nu >> 1\} \longmapsto \overline{\partial}\Big(\frac{|s|^{2\nu}}{s}\Big) \wedge \mathcal{T}$$

extends as an holomorphic function to $\{\mathrm{Re}\,\nu > -\eta\}$ *for some* $\eta > 0$*. Moreover, one has*

$$\Big[\overline{\partial}\Big(\frac{|s|^{2\nu}}{s}\Big) \wedge \mathcal{T}\Big]_{\nu=0} \in \mathrm{CH}_{\mathcal{X},V^{\mathcal{X}\setminus s^{-1}(0)}\cap s^{-1}(0)}\Big(\mathcal{X}\ ;\star\Sigma_{S,s}, \mathcal{O}_{\mathcal{X}}(-\Delta)\otimes E\Big),$$

where $\Sigma_{S,s}$ *denotes any closed hypersurface in a neighborhhod of* V *in* $\mathcal{X}$*, such that*

$$\overline{(V^{\mathcal{X}\setminus s^{-1}(0)} \cap s^{-1}(0)) \setminus \Sigma_{S,s}} = V^{\mathcal{X}\setminus s^{-1}(0)} \cap s^{-1}(0)$$

and $\Sigma_{S,s} \supset S^{V\setminus s^{-1}(0)} \cap s^{-1}(0)$*,* $S^{V\setminus s^{-1}(0)}$ *being the union of all components of* S *whose intersection with* V *does not lie entirely in* $V^{\mathcal{X}\setminus s^{-1}(0)} \cap s^{-1}(0)$*.*

5 Essential intersection and Coleff-Herrera original construction

Let $\mathcal{X}, V, E, S$ be as in Proposition 5. Let also Δ_1 be a Cartier divisor on $\mathcal{X}$, equipped with a hermitian metric $|\ |$ and s_1 be an holomorphic section of Δ_1. Propositions 4 and 5 imply that any global section $\mathcal{T} \in \mathrm{CH}_{\mathcal{X},V}(\mathcal{X}\ ;\star S, E)$ splits into the sum of an element from $\mathrm{CH}_{\mathcal{X},V^{s_1^{-1}(0)}}(\mathcal{X}\ ;\star S, E)$ and an element in $\mathrm{CH}_{\mathcal{X},V^{\mathcal{X}\setminus s_1^{-1}(0)}}(\mathcal{X}\ ;\star S, E)$. That is

$$\mathcal{T} = \Big[(1-|s_1|^{2\lambda_1})\,\mathcal{T}\Big]_{\lambda_1=0} + \Big[|s_1|^{2\lambda_1}\,\mathcal{T}\Big]_{\lambda_1=0} = \mathcal{T}_{|s_1^{-1}(0)} + \mathcal{T}_{\mathcal{X}\setminus s_1^{-1}(0)} \qquad (15)$$

(see also [4]). We remark that this splitting is independent of the choice of the metric on Δ_1. To be more specific, suppose that

$$\mathcal{T} = \frac{1}{s} Q^{*}\Big[\frac{[V]}{h}\Big],$$

where s is a holomorphic section of a Cartier divisor Δ, h is a holomorphic section of a Cartier divisor D, and $Q \in \mathfrak{D}_{\mathcal{X}}^{n,n-M}(\mathcal{X}, E^{*}\otimes \mathcal{O}_{\mathcal{X}}(\Delta), D)$ (see Examples 1 and 2). Then, for any test function in $C^{\infty}_{n,n-M}(\mathcal{X}, E^{*})$, one has

$$\begin{aligned}&\Big\langle \Big[(1-|s_1|^{2\lambda_1})\mathcal{T}\Big]_{\lambda_1=0},\varphi\Big\rangle = \\ &= \Bigg[\Bigg[\Bigg[\Bigg[\int_V \frac{|h|^{2\mu}}{h}\,Q^*\Big[\frac{|s|^{2\nu}}{s}(1-|s_1|^{2\lambda_1})\,\varphi\Big]\Bigg]_{\lambda=0}\Bigg]_{\mu=0}\Bigg]_{\lambda_1=0} \\ &= \Bigg[\Bigg[\int_{V^{s_1^{-1}(0)}} \frac{|h|^{2\mu}}{h}\,Q^*\Big[\frac{|s|^{2\nu}}{s}\,\varphi\Big]\Bigg]_{\lambda=0}\Bigg]_{\mu=0}.\end{aligned}$$

Furthermore

$$\begin{aligned}&\Big[\overline{\partial}\Big(\frac{|s_1|^{2\lambda_1}}{s_1}\Big)\wedge\mathcal{T}\Big]_{\lambda_1=0} = \Big[\overline{\partial}\Big(\frac{|s_1|^{2\lambda_1}}{s_1}\Big)\wedge\mathcal{T}_{|\mathcal{X}\setminus s_1^{-1}(0)}\Big]_{\lambda_1=0} \\ &\in \mathrm{CH}_{\mathcal{X},V_1}\big(\mathcal{X}\ ;\star\mathrm{Pol}_1,\mathcal{O}_{\mathcal{X}}(-\Delta_1)\otimes E\big),\end{aligned} \tag{16}$$

where V_1 stands for the closed analytic set $\overline{V^{\mathcal{X}\setminus s_1^{-1}(0)}}\cap s_1^{-1}(0)$ and Pol_1 denotes a closed hypersurface in $\mathcal{X}$ satisfying $\overline{V_1\setminus\mathrm{Pol}_1}=V_1$. Note that (15) can be understood as the pendant (at the level of meromorphic Coleff-Herrera currents) of the *gap sheaf operation* in intersection theory (see e.g. [23]). Namely, the splitting of the cycle $[V]$ corresponding to V as the sum $[V]^{s_1^{-1}(0)}$ of its components whose supports lie completely in the hypersurface $s_1^{-1}(0)$, and the sum $[V]^{\mathcal{X}\setminus s_1^{-1}(0)}$ of the other ones. On the other hand, the wedge product operation (16) can be understood as the pendant (at the level of Coleff-Herrera currents) of the *proper intersection product* between two cycles whose corresponding supports $\overline{V^{\mathcal{X}\setminus s_1^{-1}(0)}}$ and $s_1^{-1}(0)$ intersect properly.

Given an ordered collection $\Delta_1,...,\Delta_m$ of Cartier divisors, with $m\leq n-M$, together with respective holomorphic sections $s_1,...,s_m$, the operation (16) can be iterated because of the iterative process, initiated with $\mathcal{T}_0=\mathcal{T}$:

$$\mathcal{T}_{j+1}=\Big[\overline{\partial}\Big(\frac{|s_j|^{2\lambda_j}}{s_j}\Big)\wedge\mathcal{T}_j\Big]_{\lambda_j=0}=\Big[\overline{\partial}\Big(\frac{|s_j|^{2\lambda_j}}{s_j}\Big)\wedge\mathcal{T}_{j\,|\mathcal{X}\setminus s_j^{-1}(0)}\Big]_{\lambda_j=0},\ 0\leq j<m.$$

When this procedure is carried up to the end, one gets

$$\mathcal{T}_m\in\mathrm{CH}_{\mathcal{X},(V\cap s_1^{-1}(0)\cap\cdots\cap s_m^{-1}(0))_{\mathrm{ess}}}\Big(\mathcal{X}\ ;\star\mathrm{Pol}_m,\bigwedge_1^m\mathcal{O}_{\mathcal{X}}(-\Delta_j)\otimes E\Big),$$

where $V\cap s_1^{-1}(0)\cap\cdots\cap s_m^{-1}(0)=V_{\mathrm{ess}}[s]$ stands for the *essential intersection* (see e.g. [14]) of V respect to the ordered sequence of hypersurfaces $s_1^{-1}(0),...,s_m^{-1}(0)$. If $\mathcal{T}_0=T_0\in\mathrm{CH}_{\mathcal{X},V}(\mathcal{X},E)$, then the current $\mathcal{T}_m$ is a global section of the Coleff-Herrera sheaf $\mathrm{CH}_{\mathcal{X},V_{\mathrm{ess}}[s]}(\cdot,\bigwedge_1^m\mathcal{O}_{\mathcal{X}}(-\Delta_j)\otimes E)$.

One could consider as well (as in [21]) the $\bigwedge_1^m\mathcal{O}(-\Delta_j)$-valued current $\mathcal{R}^{s_1,...,s_m}\wedge[V]$ (which is $\overline{\partial}$-closed) obtained, starting from $\mathcal{R}^{\{\}}\wedge[V]=[V]$, through the inductive procedure

$$\mathcal{R}^{s_1,\dots,s_{j+1}} \wedge [V] = \Big[\overline{\partial}\Big(\frac{|s_j|^{2\lambda_j}}{s_j}\Big) \wedge \mathcal{R}^{s_1,\dots,s_j} \wedge [V]\Big]_{\lambda_j=0}, 0 \leq j < m. \quad (17)$$

This point of view was introduced in a slightly different form in [14]. The authors consider there a $(p,0)$ semi-meromorphic form ω on a complex space $(V,\mathcal{O}_V)$, with poles along the union of a finite number of reduced hypersurfaces $S_1,...,S_m$ of V (taken in a prescribed order). They construct on $(V,\mathcal{O}_V)$ a (m,p)-residue current $R_{S_1,\dots,S_m}[\omega]$ with support the essential intersection $(S_1 \cap \cdots \cap S_m)_{\mathrm{ess}}$. Note that the residual objects defined in [14] are intrinsic with respect to the complex space $(V,\mathcal{O}_V)$, that is, independent of the embedding $\iota : V \to \mathcal{X}$. The construction proposed here and that in [14] are of course related : besides the fact that our currents are treated here as $(M+k, M+p)$ currents, $0 \leq k \leq m$, in the ambient manifold $\mathcal{X}$ instead of (m,p) currents on the complex analytic space V, the main difference between the two approaches is that the singularities $1/s_j$ in (17) are isolated from local expressions for the denominator of ω.

In the particular case where there exist holomorphic bundles $E_1,...,E_L$ on $\mathcal{X}$ such that the integration current $[V]$ can be expressed as

$$[V] = \sum_{l=1}^{L} T_{l,0} \wedge \omega_l,$$

where $T_{l,0} \in \mathrm{CH}_{\mathcal{X},V}(\mathcal{X},E_l)$ and $\omega_l \in \Omega_{\mathcal{X}}^{n-M}(\mathcal{X},E_l^*)$ (which occurs for example, when one restricts $\mathcal{X}$ to some relatively compact open subset. When $\mathcal{X}$ is Stein and $\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_{V,x}$ is Cohen-Macaulay about each point $x \in V$, see [3], Example 1), then one can factorize $\mathcal{R}^{s_1,\dots,s_m} \wedge [V]$ as

$$\mathcal{R}^{s_1,\dots,s_m} \wedge [V] = \sum_{l=1}^{L} T_{l,m} \wedge \omega_l, \quad (18)$$

where each $T_{l,m}$ is some $(\bigwedge_1^m \mathcal{O}_{\mathcal{X}}(-\Delta_j)) \otimes E_l$ valued Coleff-Herrera currents (with respect to $V_{\mathrm{ess}}[s]$) which is a pole-free Coleff-Herrera current. Factorization (18) remains valid in general, but one needs to tolerate then poles in the Coleff-Herrera sections $T_{l,m}$.

In conclusion, we claim that the results presented here (within the robust frame of analytic continuation), together with the geometric formalism of intersection theory (where the role of integration currents on cycles is played by global sections of Coleff-Herrera sheaves), should be a starting point to attack division or duality problems with methods inspired by those used in intersection theory.

References

1. Aizenberg, L.A., Yuzhakov, A.P. : Integral representation and residues in Multidimensional complex analysis. Amer. Math. Soc. Providence, RI, (1983)
2. Andersson, M. : Uniqueness and factorization of Coleff-Herrera currents. Ann. Fac. Sci. Toulouse Math. Sér. 6, **18**, no. 4 , 651–661 (2009)
3. Andersson, M. : Coleff-Herrera currents, duality, and Noetherian operators. Preprint Gothenburg (2009), available at arXiv : 0902.3064
4. Andersson, M., Wulcan, E. : Decomposition of residue currents. J. Reine Angew. Math. **638**, 103 –118 (2010).
5. Andersson, M., Wulcan, E. : Residue currents with prescribed annihilator ideals. Ann. Sci. École Norm. Sup., (4) **40**, no. 6, 985–1007 (2007)
6. Bernstein, I. N. : The analytic continuation of generalized functions with respect to a parameter. Functional Analysis and its applications **6**, 273–285 (1972)
7. Aleksandrov, A., Tsikh, A. : Théorie des résidus de Leray et formes de Barlet sur une intersection compléte singuliére. C. R. Acad. Sci. Paris, **333**, Série I, 1–6 (2001)
8. Berenstein, C. A., Gay, R., Vidras, A., Yger, A.: Residue currents and Bézout identities. Progress in Mathematics 114, Birkhäuser, (1993)
9. Berenstein, C. A., Vidras, A., Yger, A. : Multidimensional residue theory and applications, *manuscript in preparation*.
10. Björk, J. E. : Rings of of differential operators, North-Holland, (1979)
11. Björk, J. E. : Residue calculus and $\mathcal{D}$-modules on complex manifolds. Preprint, Stockholm University, (1996)
12. Björk, J. E. : Residues and $\mathcal{D}$-modules. In : O.A. Laudal, Piene R. (eds.) The legacy of Niels Henrik Abel, pp. 605–651, Springer-Verlag, Berlin (2004)
13. Björk, J. E., Samuelsson, H. : Regularizations of residue currents. J. Reine Angew. Math. **649**, 33–54 (2010)
14. Coleff, N., Herrera, M. : Les courants résiduels associés à une forme méromorphe. Lecture Notes in Math. 633, Springer-Verlag, Berlin, New-York (1978)
15. Dolbeault, P. : On the structure of residual currents. Several complex variables (Stockholm, 1987/1988), Math. Notes **38**, 258–273 Princeton Univ. Press, Princeton, NJ (1993)
16. Ehrenpreis, L. : Fourier Analysis in several complex variables. Wiley-Interscience, New-York, (1970)
17. El Mir, H. : Sur le prolongement des courants positifs fermés. Acta Math. **153**, 1–45 (1984)
18. Grauert, H., Remmert, R. : Coherent analytic sheaves, Grunlehren der math. Wissenschaften **265**, Springer-Verlag, Berlin, 1984.
19. Gyoja, A. : Bernstein-Sato's polynomial for several analytic functions. J. Math. Kyoto Univ. **33**, no. 2, 399–411 (1993)
20. Kashiwara, M. : $B-$functions and holonomic systems, Inventiones Math. **38**, no. 1, 33–53 (1976/77)
21. Lärkäng, R., Samuelsson, H. : Various approaches to products of residue currents. Preprint Gothenburg (2009), available at arXiv : 1005-2056
22. Leray, J. : Le calcul différentiel et intégral sur une variété analytique complexe, Problème de Cauchy III, Bull. Soc. Math. France **87**, 81–180 (1959)
23. Massey D. B. : Numerical control over Complex Analytic Singularities, Memoirs of the American Mathematical Society **163**, no. 778 (2003)
24. Palamodov, V. P., : Linear Operators with Constant Coefficients. Springer-Verlag, New-York (1970)
25. Poly, J. B. : Sur un théorème de J. Leray en théorie des résidus. C. R. Acad. Sci. Paris Sr. A-B **274**, A171–A174 (1972)
26. Sabbah, C. : Proximité évanescente II, Equations fonctionelles pour plusieurs fonctions analytiques. Compositio Math **64**, no. 2, 213–241 (1987)

27. Samuelsson, H. : Analytic continuation of residue currents. Ark. Mat. **47**, no.1, 127–141 (2009)